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SHORT COMMUNICATIONS

New Method for Constructing Chernoff Functions

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Abstract—We state and, for the first time, prove a theorem in the theory of strongly continuous operator semigroups. This theorem, which has essentially been suggested by O.G. Smolyanov, in particular, enables one to reduce solving the Schrödinger equation to solving the heat equation.

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A *Feynman formula* is a representation of the solution of an evolution equation (or a related object) or any function in the form of the limit of a sequence of multiple integrals as the multiplicity tends to infinity. Multiple integrals of finite multiplicity occurring in a Feynman formula are approximations to a Feynman path integral (i.e., an integral of "infinite multiplicity"), which explains the interest in Feynman formulas by specialists in mathematical and theoretical physics.

Smolyanov [1] suggested using the Chernoff theorem [2] to represent the solution of the Cauchy problem for differential equations of the evolution type in the form of a Feynman formula. This type of equations includes the heat equation, the Schrödinger equation, and other equations of the form $\partial u(t,x)/\partial t = Lu(t,x)$, where L is a polynomial in the operator of the differentiation with respect to the variable x. Smolyanov's scientific group applied this approach to various classes of equations, including equations on manifolds [3] and graphs [4], equations with time-dependent right-hand side [5, 6], equations with a variable x in an infinite-dimensional space [7, 8], equations with a positive integer degree of the Laplacian on the right-hand side [9, 10], equations arising in connection with tau-quantization [19], and other classes of equations.

Alongside with the application of this approach, notions related to the Chernoff theorem were rethought, developed, and refined. This includes new notions such as Chernoff equivalence, the Chernoff function of an operator (i.e., a function Chernoff equivalent to the semigroup generated by the operator), the Butko–Schilling–Smolyanov theorem on the composition of Chernoff functions [20], Chernoff tangency, a theorem on the relationship between the Chernoff functions of the heat equation and the Schrödinger equation [14, 15], and the Smolyanov theorem on Chernoff tangency, which is considered in the present paper. The last two theorems, the second of which develops the first one, in particular provide a method for constructing quasi-Feynman formulas. Each of the above-mentioned notions admits two statements, one for arbitrarily small time (i.e., for $t \in [0,\delta)$ with an arbitrary $\delta > 0$) and the other with unbounded time (i.e., for $t \geq 0$).

The aim of the present paper is to state the Smolyanov theorem in full detail and provide a complete proof. See the paper [15], the survey papers [11–13], and the textbooks [16, 17] for the terminology used here and for an introduction to the problems considered here. In what follows, we denote the space of all linear bounded operators on a Banach space X by $\mathcal{L}(X)$.

Let us introduce the main definition.

Definition. Let X be a Banach space, and let $\delta > 0$; in addition, let $G : [0,\delta) \to \mathcal{L}(X)$ be a given function (or, which is the same, a family $(G(t))_{t\in[0,\delta)}$ of operators in $\mathcal{L}(X)$), and let $L : \mathrm{Dom}(L) \to X$ be a given closed linear operator with domain $\mathrm{Dom}(L) \subset X$. We say that the function G is *Chernoff tangent* to the operator L if the following conditions are satisfied.

(CT1). The mapping $t \mapsto G(t)f \in X$ is continuous on $[0,\delta)$ for each $f \in X$.

(CT2). $G(0) = I$; i.e., $G(0)f = f$ for each $f \in X$.

(CT3). There exists a dense linear subspace $D \subset X$ such that there exists a limit

$$\lim_{t\to 0}(G(t)f - f)/t =: G'(0)f$$

for every $f \in D$.

(CT4). The operator $(G'(0), D)$ is closable, and the closure is $(L, \mathrm{Dom}(L))$.

If a function G is Chernoff tangent to an operator L, then the expression $\lim_{n\to\infty} G(t/n)^n u_0$ can be referred to as a *formal Chernoff solution* of the Cauchy problem $u_t' = Lu$, $u(0) = u_0$. If, in addition, there exists a C_0-semigroup $(e^{tL})_{t\geq 0}$, and $\|G(t)\| \leq 1 + \alpha t$, then, by the Chernoff theorem, this expression defines a function u that is indeed a solution (see [16, 17]) of the above-mentioned Cauchy problem. The Smolyanov theorem on the Chernoff tangency is stated in the following form.

Theorem. *Let a complex-valued function R be holomorphic in a neighborhood of the point $1 \in \mathbb{C}$; further, assume that $R(1) = 1$, $R'(1) = \alpha \in \mathbb{C}$, and $\alpha \neq 0$. In addition, let X be complex Banach space, let $\delta > 0$ be a number, and let a function $F : [0,\delta) \to \mathcal{L}(X)$ be Chernoff tangent to a closed operator $(L, \mathrm{Dom}(L))$ and satisfy the estimate $\|F(t)\| \leq 1+\omega t$ for all $t \in [0,\delta)$. Then there exists a number $\delta_1 > 0$ such that the function $F_R(t) = R(F(t))$ is well defined for all $t \in [0,\delta_1)$; moreover, the function F_R is Chernoff tangent to the operator αL.*

Remark 1. It is assumed without further stipulations in the statement and proof of the theorem that all notions are used for the case of arbitrarily small time. The statement and proof of the theorem in the case of an unbounded time can readily be constructed by analogy.

Proof of the theorem. Let us verify the four conditions in the definition of Chernoff tangency of the function F_R and the operator αL starting from condition (CT1). Since the function R is holomorphic in a neighborhood of unity with $R(1) = 1$ and $R'(1) = \alpha$, it follows that there exists a $\beta > 0$ such that the Taylor series expansion

$$R(z) = 1 + \alpha(z-1) + \sum_{k=2}^{\infty} r_k(z-1)^k = 1 + \alpha(z-1) + (z-1)^2\Psi(z)$$

holds for $|z-1| \leq \beta$, where the series is absolutely convergent for $|z-1| \leq \beta$ and $|\Psi(z)| \leq a$ for some $a > 0$. Since $\|F(t)\| \leq 1 + \omega t$, it follows that if $\delta_1 = \min(\delta, \beta/\omega)$, then the inequality $0 \leq t < \delta_1$ implies that $\|F(t)\| \leq 1+\beta$, which allows one to define the function $F_R(t)$ by the formula

$$F_R(t) = R(F(t)) = \lim_{n\to\infty}\left(I + \alpha(F(t) - I) + \sum_{k=2}^{n} r_k(F(t)-I)^k\right).$$

For a given $t \in [0,\delta_1)$, the operators in the prelimit expression are linear and continuous, and the numerical series $\sum_{k=2}^{\infty} r_k(z-1)^k$ is absolutely convergent; therefore, so is the series of linear bounded operators, and since X is a Banach space, we conclude that the operator series is convergent in the norm of $\mathcal{L}(X)$.

We have thereby shown that the function $F_R : [0,\delta_1) \to \mathcal{L}(X)$ is well defined. The continuity of the function $t \mapsto F_R(t)x$ for each $x \in X$ follows from the continuity of the function $\mathcal{L}(X) \ni A \mapsto R(A) \in \mathcal{L}(X)$ in the norm of $\mathcal{L}(X)$ and the continuity of the function $t \mapsto F(t)x$ for each $x \in X$, which, in turn, is part of condition (CT1) for F. Thus, we have proved that condition (CT1) holds for F_R.

Condition (CT2) for F_R readily follows from condition (CT2) for F and the chain of relations $F_R(0) = R(F(0)) = R(I) = I$.

Condition (CT3) for F claims that there exists a dense linear subspace $D \subset X$ such that $F(t)x = x + tLx + o(t)$ for every $x \in D$. This, together with the relation

$$F_R(t) = I + \alpha(F(t) - I) + \Psi(F(t))(F(t) - I)^2,$$

implies that condition (CT3) holds for F_R. Indeed,

$$\begin{aligned}\frac{F_R(t)x - x}{t} &= \alpha\frac{x + tLx + o(t) - x}{t} + \Psi(F(t))(F(t) - I)\frac{F(t)x - x}{t}\\ &= \alpha Lx + o(1) + \Psi(F(t))(F(t) - I)(Lx + o(1)) = \alpha Lx + o(1)\end{aligned}$$

for arbitrary $x \in D$. The last relation follows from the estimate

$$\|\Psi(F(t))(F(t) - I)(Lx + o(1))\| \le \|\Psi(F(t))\|\|(F(t) - I)(Lx)\| + \|\Psi(F(t))\|\,\|(F(t) - I)(o(1))\|$$

and the inequality $|\Psi(z)| \le a$ proved above, the relation $\|(F(t) - I)(Lx)\| = o(1)$ is a consequence of condition (CT1) for F, and the inequality $\|F(t)\| \le 1 + \omega t$ implies the inequality

$$\|(F(t) - I)(o(1))\| \le (1 + \omega t + 1)\|o(1)\| = o(1).$$

Condition (CT4) for F implies that (L, D) is closable and the closure is $(L, \mathrm{Dom}(L))$. Since $\mathrm{Dom}(L) = \mathrm{Dom}(\alpha L)$, we see that this is equivalent to the closability of $(\alpha L, D)$ with closure $(\alpha L, \mathrm{Dom}(L))$. The proof of the theorem is complete.

Remark 2. The theorem proved above generalizes part of Theorem 3.1 proved in [15]. In Theorem 3.1, $R(z) = e^{i(z-1)}$, $\alpha = i$, the operator L is an arbitrary self-adjoint operator with dense domain on a Hilbert space, and F is an arbitrary function ranging in the space of bounded self-adjoint operators and Chernoff tangent to L. The theorem proved above restates an assertion (cf. [18, p. 256]) suggested by Smolyanov as a generalization of one part of Theorem 3.1. The multiplication by a complex number with unit absolute value (including i) is geometrically interpreted as a rotation, which is why this result is called the Smolyanov theorem on a rotation of the Chernoff tangency.

Remark 3. A quasi-Feynman formula is an expression containing multiple integrals of arbitrarily large multiplicity. Unlike Feynman formulas, quasi-Feynman formulas can contain summation or other operations on the right-hand side. Quasi-Feynman formulas are longer than Feynman ones but are easier to prove (see [15]).

If $F(t)$ is an integral operator for each $t \in [0, \delta_1)$, then the operator $F(t)^n$ is given by an n-fold multiple integral. If the operator $(\alpha L, \mathrm{Dom}(\alpha L))$ is the generator of a C_0-semigroup $(e^{t\alpha L})_{t\ge 0}$ and $\|F_R(t)\| \le \omega_1 t$ for all $t \in [0, \delta_1)$ for some given number $\omega_1 > 0$, then, by the Chernoff theorem,

$$e^{t\alpha L}x = \lim_{n\to\infty}\left(F_R\left(\frac{t}{n}\right)\right)^n x = \lim_{n\to\infty}\left(I + \alpha\left(F\left(\frac{t}{n}\right) - I\right) + \sum_{k=2}^{\infty} r_k\left(F\left(\frac{t}{n}\right) - I\right)^k\right)^n x$$

for every $x \in X$, where the convergence is in the norm of X. The expression on the right-hand side contains definite integrals of arbitrarily large multiplicity and hence is a quasi-Feynman formula.

Remark 4. To solve the equation $u_t' = Lu$, one usually expands the operator L into the sum $L = A + B$ of generators of semigroups and uses the Trotter product formula

$$e^{(A+B)t} = \lim_{n\to\infty}(e^{At/n}e^{Bt/n}).$$

In particular, if A is the operator of multiplication by a function, $(Af)(x) = V(x)f(x)$, then the semigroup has the form $(W(t)f)(x) = e^{tV(x)}f(x)$. However, if the Chernoff theorem is used instead of the Trotter theorem, then the function $z \mapsto e^z$ can be replaced by any holomorphic function $h(z)$ satisfying the conditions $h(0) = h'(0) = 1$. Indeed, consider the operator-valued function K defined by the relation $(K(t)f)(x) = h(tV(x))f(x)$. Both functions W and K are Chernoff tangent to A, which can be verified directly or by reference to the theorem proved above. In this case, K is not a semigroup, but if h is a bounded function [for example, $h(z) = 1 + \arctan z$], then, even for an unbounded function V, the operator $K(t)$ is the operator of multiplication by the bounded function $x \mapsto h(tV(x))$. An example of use of this approach (in the case of a bounded function V) can be found in the paper [21], whose results can be treated as the construction of a quasi-Feynman formula with generalized functions in the integrand of a multiple integral with growing multiplicity. Related problems were discussed in [22–28].

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