A Local Limit Theorem for Robbins-Monro Procedure

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Abstract

The Robbins-Monro algorithm is a recursive, simulation-based stochastic procedure to approximate the zeros of a function that can be written as an expectation. It is known that under some technical assumptions, a Gaussian convergence can be established for the procedure. Here, we are interested in the local limit theorem, that is, quantifying this convergence on the density of the involved objects. The analysis relies on a parametrix technique for Markov chains converging to diffusions, where the drift is unbounded.

1 Introduction and Assumptions

This paper is devoted to the study of a Local Limit Theorem for a Robbins Monro procedure. These algorithms have first been introduced in [7] to approximate the solution of an equation $h(\theta) = 0$, where h can be written as an expectation. Since then, extensive literature have been published on the subject, but to the best of our knowledge, the local limit theorem has never been obtained. In this work, we limit ourselves to a simpler version of the one dimensional Robbins-Monro algorithms that already shows the technical difficulties we have to overcome in order to obtain the local limit theorem. We refer to the monograph from Benveniste Metivier and Priouret [1] or Nevelson and Khas'minskii [6] for a general presentation of these algorithms and a review of the literature.

We fix probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where all the random variables we consider below are defined. Let $(\gamma_k)_{k\geq 0}$ be a decreasing time step that will be specified later, and $(\eta_k)_{k\geq 0}$ a collection of independent and identically distributed random variables. We define the following recursive procedure:

$$\theta_{n+1} = \theta_n - \gamma_{n+1} \cdot \sigma(m(\theta_n) - \eta_{n+1}),\ \theta_0 \in \mathbb{R}, \tag{1.1}$$

This algorithm is a special case of the general Robbins-Monro procedure which writes:

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, \eta_{n+1}),$$

and where the innovations η_k can have a Markovian structure (instead of *i.i.d.* assumed here). Generally, this procedure is used to approximate the zeros of the function: $h(\theta) = \mathbb{E}[H(\theta, \eta)]$, where η has the same distribution as η_k.

Even though the general theory extends to the case of multiple zeros, in this paper, we assume that h has only one zero, θ^* (i.e. $h(\theta^*) = 0$). The general assumption on the step sequence $(\gamma_k)_{k\geq 0}$ is usually the following:

$$\sum_{k\geq 0} \gamma_k = +\infty, \quad \sum_{k\geq 0} \gamma_k^2 < +\infty.$$

Under these assumption, it can be shown that the convergence:

$$\theta_n \underset{n\to+\infty}{\longrightarrow} \theta^*,$$

holds almost surely. This convergence is exactly a Law of Large numbers in the case $H(\theta, x) = \sigma(m(\theta) - x)$, and it is therefore natural to ask about a Central Limit Theorem. Following the procedure described in [1], we therefore look for a suitable renormalization for the process and investigate a convergence in Law.

In a general context, it can be shown that the procedure (1.1) tends to follow the solution of the Ordinary Differential Equation (ODE):

$$\frac{d}{dt}\bar{\theta}_t = h(\bar{\theta}_t),\ \bar{\theta}_0 = \theta_0,\ \text{where we recall that: } h(\theta) = \mathbb{E}[H(\theta, \eta)]. \tag{1.2}$$

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Thus, fluctuations of the algorithm are to be considered with respect to the solution $(\bar{\theta}_t)_{t\geq 0}$ of the ODE (1.2).

Precisely, we consider a *shift* in the indexation of the procedure that will allows us to consider $(\theta_n)_{n\geq 0}$ close to stationarity. Let $N \in \mathbb{N}$, and consider a sequence $\left(\theta_n^N\right)_{n\geq 0} = (\theta_{N+n})_{n\geq 0}$, of shifted Robbins-Monro algorithms. These algorithms satisfy the following recurrence equation:

$$\begin{cases} \theta_{n+1}^N = \theta_n^N - \gamma_{n+1}^N H(\theta_n^N, \eta_{n+1}^N) \\ \theta_0^N \in \mathbb{R}. \end{cases} \tag{1.3}$$

where $\eta_{n+1}^N = \eta_{N+n+1}$, and $\gamma_{n+1}^N = \gamma_{N+n+1}$. Set now:

$$t_0^N = 0,\ t_1^N = \gamma_1^N,\ t_2^N = \gamma_1^N + \gamma_2^N,\ \ldots,\ t_k^N = \gamma_1^N + \cdots + \gamma_k^N,$$

and for an arbitrary terminal time $T > 0$, we set:

$$M(N) = \inf\{k \in \mathbb{N}\ ;\ t_k^N \geq T\}.$$

We consider the re-normalized process:

$$U_t^N := \sum_{k=0}^{+\infty} \frac{\theta_k^N - \bar{\theta}_{t_k^N}}{\sqrt{\gamma_k^N}} \mathbf{1}_{\{t_k^N \leq t < t_{k+1}^N\}}. \tag{1.4}$$

Expressing the dynamics of this Markov chain we obtain (see Proposition 2.1 below):

$$U_{t_{k+1}^N}^N = U_{t_k^N}^N + \left(\alpha_{t_k^N}^N - \sigma\sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}} \int_0^1 m'(\bar{\theta}_{t_k^N} + \delta U_{t_k^N}^N \sqrt{\gamma_k^N}) d\delta\right) U_{t_k^N}^N \gamma_{k+1}^N + \sqrt{\gamma_{k+1}^N}\xi^{\bar{\theta}_{t_k^N}} + o(1),$$

where the $o(1)$ is deterministic and goes to zero as N goes to infinity. Also, the step sequence $(\gamma_k^N)_{k\geq 1}$ can be chosen so that $\alpha^N \to \frac{1}{2}$. We know from the literature that the process $(U_t^N)_{t\geq 0}$ converges weakly to $(X_t)_{t\geq 0}$ the solution of the SDE:

$$dX_t = \left(-am'(\bar{\theta}_t) + \frac{1}{2}\right) X_t dt + \sigma dW_t. \tag{1.5}$$

We point out that the coefficients of the limiting SDE can be seen as the point-wise limit of the coefficients in the dynamics of the Markov chain. We refer to Benveniste *et al.* [1] or Kushner and Yin [5] for the proof of this convergence. In this article, we are interested in quantifying the convergence of $(U_t^N)_{t\geq 0}$ towards $(X_t)_{t\geq 0}$, on the densities of the involved objects. However, as we will make clear below, the fact that the convergence is point-wise is problematic for our approach, also, we introduce a cut-off $a_N \to +\infty$, and define:

$$F_N(t, z) = \left(\alpha_t^N - \sigma\sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}} \int_0^1 m'\left(\bar{\theta}_t + \delta sign(z)\Big(|z| \wedge a_N\Big)\sqrt{\gamma_k^N}\right) d\delta\right)$$

where implicitly, $k = k(t)$ is the index such that $t_k^N \leq t < t_{k+1}^N$. This new function F_N is essentially the drift term in the dynamics of $(U_{t_k^N}^N)_{t_k^N > 0}$ above with the cut-off a_N, and allows us to use the perturbation technique developed in Konakov Kozhina and Menozzi [4].

1.1 List of assumptions

We list here all the assumptions needed on our model.

A-1 (**Smoothness condition**) The function $m(x)$ has four derivatives and these derivatives are bounded in a neighborhood of the image of $\bar{\theta}_t$, for $t \in [0, T]$, the solution of (1.2).

A-2 (**Attractivity condition**) We have the attractivity condition:

$$-\sigma m'(\theta^*) + \frac{1}{2} \leq 0.$$

A-3 (**Condition on the partitions**) The sequence $(\gamma_k)_{k\geq 0}$ is such that

$$\sum_{k\geq 1} \gamma_k = +\infty, \quad \sum_{k\geq 1} \gamma_k^2 < +\infty.$$

We denote $\gamma_k^N = \gamma_{k+N}$ and set:

$$t_0^N = 0,\ t_1^N = \gamma_1^N,\ t_2^N = \gamma_1^N + \gamma_2^N,\ \ldots,\ t_k^N = \gamma_1^N + \cdots + \gamma_k^N.$$

and for an arbitrary terminal time $T > 0$, we denote:

$$M(N) = \inf\{k \in \mathbb{N}\ ;\ t_k^N > T\}.$$

Finally, the sequence of partitions $t_1^N < \cdots < t_{M(N)}^N$ of the interval $[0, T]$ is chosen in such a way that, for sufficiently large N, there exists a constant $c > 1$, for all $0 \le i \le j \le M(N)$:

$$c^{-1} \le \frac{\gamma_i^N}{\gamma_j^N} \le c.$$

A-4 (**Condition on the innovations**) The innovations $\eta, \eta_1, \eta_2 \ldots$ are independent and identically distributed. We also assume that they have a common density we denote ρ that satisfies:

- Centered :
$$\int_{\mathbb{R}} z\rho(z)dz = 0,$$
- Variance :
$$\int_{\mathbb{R}} z^2\rho(z)dz = 1,$$
- Smoothness: there is an index $M > S + 1$, $S > 8$ such that for all $1 \le \nu \le 5$:
$$\left|\frac{\partial^\nu}{\partial z^\nu}\rho(z)\right| dz \le C\frac{1}{1 + |z|^M},\quad C > 0.$$

A-5 (**Cut-off**) We introduce a cut-off level $(a_N)_{N\ge1}$, such that

$$a_N \underset{N\to+\infty}{\longrightarrow} +\infty,\quad a_N\sqrt{\gamma_0^N} \underset{N\to+\infty}{\longrightarrow} 0$$

In the rest of this paper, these assumptions are always in force, except when explicitly stated otherwise.
Example: Let us mention that the sequence $\gamma_k = \frac{1}{k\ln(k)}$ satisfies these conditions.

Remark 1.1. In our specific case, it is an exercise to show that the algorithm (1.1) converges to θ^* almost surely and that the central limit theorem holds. One can for instance refer to the monograph by Benveniste, Métivier and Priouret [1], specifically Chapter 4 section 4.5 and check that the list of assumptions is verified in our case. Note also that the assumption that the innovations are centered and with variance 1 is not constraining due to the linear feature of our model.

Our main result is the following.

Theorem 1.1. *Fix a terminal time T. There exists a sequence of stochastic processes $(V_t^N)_{t\ge0}$ such that for some constant $C > 0$,*

$$\mathbb{P}\left(\sup_{k\in[\![1,M(N)]\!]} |U_{t_k^N}^N - V_{t_k^N}^N| > C\sqrt{\gamma_0^N}\right) \le \frac{C}{a_N\sqrt{\gamma_0^N}}.$$

Moreover, denoting p_N the density of V_t^N and p the density of the Gaussian diffusion (1.5) *below, and q_N the density of the cut-off diffusion X_t^N defined in* (3.15) *below, there exist a constant $C > 0$, for all $t_k^N \le T - \delta$, $\delta > 0$ for all $(x, y) \in K_x \times K_y$ with K_x, K_y compact sets, and a constant C depending on T, δ, K_x, K_y such that:*

$$\left|(p_N - q_N)(t_k^N, T, x, y)\right| \le C\sqrt{\gamma_0^N}\frac{\sqrt{T - t_k^N}^{-1}}{\left(1 + \dfrac{|\hat{\theta}_{t_k^N,T}^N(y) - x|}{\sqrt{T - t_k^N}}\right)^{S-7}},$$

and

$$\left|(p - q_N)(t_k^N, T, x, y)\right| \le Ca_N\sqrt{\gamma_0^N}g_C(T - t_k^N, \theta_{t_k^N,T}(y) - x),$$

where for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$, $g_c(t,z) := \frac{c}{\sqrt{t}} \exp\left(-c\frac{|z|^2}{t}\right)$, *and* $\theta_{t,T}(y)$ *is the solution of the ODE:*

$$\frac{d}{dt} z_t = \left(-\sigma m'(z_t) + \frac{1}{2}\right) z_t, \ z_T = y,$$

and $\hat{\theta}^N_{t_k^N,T}(y)$ *is the backward Euler scheme:*

$$\begin{cases} x^N_{t^N_{k+1}} = x^N_{t^N_k} + F_N(t^N_{k+1}, x^N_{t^N_{k+1}}) x^N_{t^N_k} \gamma^N_{k+1}, \\ x^N_T = y \end{cases}$$

where F_N *is defined in* (3.11) *below. Also,* $a_N \to +\infty$ *is a cut-off threshold such that* $a_N \sqrt{\gamma_0^N} \underset{N\to+\infty}{\longrightarrow} 0$

The multidimensional case and more general models will be considered in a separate publication. The proof of this result is based on an extension of two results by Konakov and Mammen [3], and Konakov Kozhina and Menozzi [4], and rely the parametrix expansions for the involved objects.

We decided to present only the linear case here for simplicity in the computations. This model is already highlighting the main difficulties, namely the unbounded feature of the drift in the considered equations and the fact that the convergence of the coefficients is only point-wise.

This article is organized as follow. In Section. 2, we derive the dynamics for the renormalized process and the limiting SDE. In Section 3, we set up the parametrix technique to obtain an explicit representation for the densities, and compare the expansions in order to derive the theorem.

2 Dynamics for the renormalized process and limiting SDE

In this section, we derive the dynamics for the Markov chain and the limiting SDE. We recall that we are interested in the algorithm:

$$\theta^N_{n+1} = \theta^N_n + \gamma_{n+1} \cdot \sigma\Big(m(\theta^N_n) - \eta_{k+1}\Big), \quad \theta^N_0 \in \mathbb{R},$$

where $a \in \mathbb{R}$, and the innovations η_k are *i.i.d.*. From the general theory (see e.g. [1]), we know that the central limit theorem is obtained by considering the following renormalization:

$$U^N_t = \sum_{k=0}^{+\infty} \frac{\theta^N_k - \bar{\theta}_{t^N_k}}{\sqrt{\gamma^N_k}} \mathbf{1}_{\{t^N_k \le t < t^N_{k+1}\}}.$$

This is due to the fact that the algorithm tends to follow the solution of the ODE $\frac{d}{dt}\bar{\theta}_t = h(\bar{\theta}_t)$.

Proposition 2.1. *Set for all* $\theta \in \mathbb{R}$, $\xi^\theta = H(\theta,\eta) - h(\theta)$, *where* η *has the same distribution as the innovations* η_k. *The Markov chain* $(U^N_{t^N_k})$ *has the following dymanics:*

$$U^N_{t^N_{k+1}} = U^N_{t^N_k} + \left(\alpha^N_{t^N_k} - \sigma\sqrt{\frac{\gamma^N_k}{\gamma^N_{k+1}}} \int_0^1 m'(\bar{\theta}_{t^N_k} + \delta U^N_{t^N_k}\sqrt{\gamma^N_k}) d\delta\right) U^N_{t^N_k} \gamma^N_{k+1} + \sqrt{\gamma^N_{k+1}} \xi^{\bar{\theta}_{t^N_k}} + \sqrt{\gamma^N_{k+1}} \left(h(\bar{\theta}_{t_k}) - \frac{\bar{\theta}_{t_{k+1}} - \bar{\theta}_{t_k}}{\gamma^N_{k+1}}\right), \tag{2.6}$$

where $\sqrt{\gamma^N_{k+1}}\left(h(\bar{\theta}_{t_k}) - \frac{\bar{\theta}_{t_{k+1}} - \bar{\theta}_{t_k}}{\gamma^N_{k+1}}\right) \underset{N\to+\infty}{\longrightarrow} 0$ *and is deterministic. Also,*

$$\alpha^N_{t_k} = \frac{\sqrt{\gamma^N_k} - \sqrt{\gamma^N_{k+1}}}{(\gamma^N_{k+1})^{3/2}} \underset{N\to+\infty}{\longrightarrow} \bar{\alpha} = \frac{1}{2}.$$

Proof. We write its dynamics as a Markov chain. To alleviate the notations, we denote t_k and η_k instead of t^N_k

and η_k^N. We have:

$$\begin{aligned}
U_{t_{k+1}}^N &= \frac{\theta_{k+1}^N - \bar{\theta}_{t_{k+1}}}{\sqrt{\gamma_{k+1}^N}} \\
&= \frac{\theta_k^N + \gamma_{k+1}^N H(\theta_k^N, \eta_{k+1}) - \bar{\theta}_{t_{k+1}}}{\sqrt{\gamma_{k+1}^N}} \\
&= \frac{\theta_k^N - \bar{\theta}_{t_k}}{\sqrt{\gamma_k^N}} \frac{\sqrt{\gamma_k^N}}{\sqrt{\gamma_{k+1}^N}} + \sqrt{\gamma_{k+1}^N} H(\theta_k^N, \eta_{k+1}) - \frac{\bar{\theta}_{t_{k+1}} - \bar{\theta}_{t_k}}{\sqrt{\gamma_{k+1}^N}} \\
&= U_{t_k}^N \frac{\sqrt{\gamma_k^N}}{\sqrt{\gamma_{k+1}^N}} + \sqrt{\gamma_{k+1}^N} H(\bar{\theta}_{t_k} + U_{t_k}^N \sqrt{\gamma_k^N}, \eta_{k+1}) - \frac{\bar{\theta}_{t_{k+1}} - \bar{\theta}_{t_k}}{\sqrt{\gamma_{k+1}^N}}.
\end{aligned}$$

Now, we can write:

$$\begin{aligned}
U_{t_k}^N \frac{\sqrt{\gamma_k^N}}{\sqrt{\gamma_{k+1}^N}} &= U_{t_k}^N + U_{t_k}^N \frac{\sqrt{\gamma_k^N} - \sqrt{\gamma_{k+1}^N}}{\sqrt{\gamma_{k+1}^N}} \\
&= U_{t_k}^N + \alpha_{t_k}^N U_{t_k}^N \gamma_{k+1}^N,
\end{aligned}$$

where we recall $\alpha_{t_k}^N = \frac{\sqrt{\gamma_k^N} - \sqrt{\gamma_{k+1}^N}}{(\gamma_{k+1}^N)^{3/2}} \underset{N\to+\infty}{\longrightarrow} \bar{\alpha} = \frac{1}{2}$ by assumption. The dynamics of U^N becomes:

$$\begin{aligned}
U_{t_{k+1}}^N &= U_{t_k}^N + \alpha_{t_k}^N U_{t_k}^N \gamma_{k+1}^N + \sqrt{\gamma_{k+1}^N}\Big(H(\bar{\theta}_{t_k} + U_{t_k}^N \sqrt{\gamma_k^N}, \eta_{k+1}) - H(\bar{\theta}_{t_k}, \eta_{k+1})\Big) \\
&\quad + \sqrt{\gamma_{k+1}^N}\Big(H(\bar{\theta}_{t_k}, \eta_{k+1}) - h(\bar{\theta}_{t_k})\Big) + \sqrt{\gamma_{k+1}^N}\left(h(\bar{\theta}_{t_k}) - \frac{\bar{\theta}_{t_{k+1}} - \bar{\theta}_{t_k}}{\gamma_{k+1}^N}\right).
\end{aligned}$$

Recalling that for all $\theta \in \mathbb{R}$, $\xi^\theta = H(\theta, \eta) - h(\theta)$, we get:

$$\begin{aligned}
U_{t_{k+1}}^N &= U_{t_k}^N + \alpha_{t_k}^N U_{t_k}^N \gamma_{k+1}^N \\
&\quad + \sqrt{\gamma_{k+1}^N}\Big(H(\bar{\theta}_{t_k} + U_{t_k}^N \sqrt{\gamma_k^N}, \eta_{k+1}) - H(\bar{\theta}_{t_k}, \eta_{k+1})\Big) + \sqrt{\gamma_{k+1}^N}\xi^{\bar{\theta}_{t_k}} \\
&\quad + \sqrt{\gamma_{k+1}^N}\left(h(\bar{\theta}_{t_k}) - \frac{\bar{\theta}_{t_{k+1}} - \bar{\theta}_{t_k}}{\gamma_{k+1}^N}\right).
\end{aligned}$$

Now, we observe that:

$$\sqrt{\gamma_{k+1}^N}\left(h(\bar{\theta}_{t_k}) - \frac{\bar{\theta}_{t_{k+1}} - \bar{\theta}_{t_k}}{\gamma_{k+1}^N}\right) \underset{N\to+\infty}{\longrightarrow} 0.$$

Besides, we can use Taylor's formula:

$$H(\bar{\theta}_{t_k} + U_{t_k}^N \sqrt{\gamma_k^N}, \eta_{k+1}) - H(\bar{\theta}_{t_k}, \eta_{k+1}) = \left(\int_0^1 \partial_1 H(\bar{\theta}_{t_k} + U_{t_k}^N \sqrt{\gamma_k^N}\delta, \eta_{k+1}) d\delta\right) U_{t_k}^N \sqrt{\gamma_k^N}.$$

Here, ∂_1 represents the derivative with respect to the first argument of $H(\theta, x)$.

Thus, we have the dynamics for U^N:

$$\begin{aligned}
U_{t_{k+1}}^N = U_{t_k}^N + \left(\alpha_{t_k}^N + \sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}} \int_0^1 \partial_1 H(\bar{\theta}_{t_k} + U_{t_k}^N \sqrt{\gamma_k^N}\delta, \eta_{k+1}) d\delta\right) U_{t_k}^N \gamma_{k+1}^N \\
+ \sqrt{\gamma_{k+1}^N}\xi^{\bar{\theta}_{t_k}} + o(1), \qquad (2.7)
\end{aligned}$$

where $o(1)$ is non random and goes to zero as N goes to infinity. Let us point out that we can derive equation (2.7) for a general Robbins-Monro algorithm. However, the dependency in η_{k+1} of the "drift term" can be a problem. We can avoid it by limiting ourselves to the linear case. Indeed, plugging $H(\theta, x) = -\sigma(m(\theta) - x)$, we have:

$$\begin{aligned} H(\bar{\theta}_{t_k} + U^N_{t_k}\sqrt{\gamma^N_k}, \eta_{k+1}) - H(\bar{\theta}_{t_k}, \eta_{k+1}) &= -\sigma m(\bar{\theta}_{t_k} + U^N_{t_k}\sqrt{\gamma^N_k}) + \sigma m(\bar{\theta}_{t_k}) \\ &= -\int_0^1 \sigma m'(\bar{\theta}_{t_k} + \delta U^N_{t_k}\sqrt{\gamma^N_k}) U^N_{t_k}\sqrt{\gamma^N_k} d\delta. \end{aligned}$$

Hence, the "drift term" does not depends on η_{k+1} anymore. Another consequence of our choice of the linear model is from the definition of ξ^θ. Observe that:

$$\xi^\theta = H(\theta, \eta) - h(\theta) = \sigma\big(\eta - m(\theta)\big) - \sigma\big(\mathbb{E}(\eta) - m(\theta)\big) = \sigma\big(\eta - \mathbb{E}(\eta)\big).$$

Thus, in this case $\xi^{\bar{\theta}_{t_k}} := \xi_{k+1}$ are *i.i.d.*, centered and with variance σ^2, and most importantly, does not depend on $\bar{\theta}_{t_k}$. and we recover the dynamics:

$$\begin{aligned} U^N_{t_{k+1}} = U^N_{t_k} + \left(\alpha^N_{t_k} - \sigma\sqrt{\frac{\gamma^N_k}{\gamma^N_{k+1}}}\int_0^1 m'(\bar{\theta}_{t_k} + \delta U^N_{t_k}\sqrt{\gamma^N_k}) d\delta\right) U^N_{t_k}\gamma^N_{k+1} \\ + \sqrt{\gamma^N_{k+1}}\xi_{k+1} + o(1). \end{aligned}$$

□

In the next section, we use a Parametrix expansion to give an explicit expression of the density of the Markov chain $U^N_{t_k}$. Building from equation (2.6), we can guess the expression for the limiting SDE. Summing (2.6) for $k = 0$ to $n-1$, we get:

$$\begin{aligned} U^N_{t_n} - U^N_{t_0} = \sum_{k=0}^{n-1}\left(\alpha^N_{t_k} - \sigma\sqrt{\frac{\gamma^N_k}{\gamma^N_{k+1}}}\int_0^1 m'(\bar{\theta}_{t_k} + U^N_{t_k}\sqrt{\gamma^N_k}\delta) d\delta\right) U^N_{t_k}\gamma^N_{k+1} \\ + \sum_{k=0}^{n-1}\sqrt{\gamma^N_{k+1}}\xi_k + o(1). \end{aligned}$$

From the last equation, we can identify the limiting SDE, with heuristic arguments. We already discussed that $\alpha^N_{t_k} \to \frac{1}{2}$ as N goes to infinity. Besides, for all $x \in \mathbb{R}$ fixed, if $t_k = t^N_k \to t$, we see that

$$-\sigma m'\left(\bar{\theta}_{t_k} + x\sqrt{\gamma^N_k}\delta\right) \underset{N\to+\infty}{\longrightarrow} -\sigma m'(\bar{\theta}_t). \tag{2.8}$$

Finally, the independent variables ξ_k converge to a Brownian motion because of the renormalisation. We can thus guess the limit to be equation (1.5) defined above:

$$dX_t = \left(-\sigma m'(\bar{\theta}_t) + \frac{1}{2}\right) X_t dt + \sigma dW_t. \tag{2.9}$$

Remark 2.1. Observe that the general case where $\xi = \xi^\theta$, we have to consider the sum of $\sqrt{\gamma^N_{k+1}}\xi^{\bar{\theta}_{t_k}}$ who do not have the same distribution. It can be shown that in this case, denoting $R(\theta) = Var(\xi^\theta)$, the convergence holds:

$$\sum_{k=0}^{n-1}\sqrt{\gamma^N_{k+1}}\xi^{\bar{\theta}_{t_k}} \underset{N\to+\infty}{\longrightarrow} \int_t^s \sqrt{R(\bar{\theta}_u)} dB_u$$

The actual proof for the convergence of U^N to X is given in Benveniste *et al.* [1], see Theorem 12, Chapter 4 (p.328). It relies on proving tightness for the law of U^N, and uniqueness to the martingale problem associated with the limiting diffusion. Pointing out the major difficulties here, we see that for the limiting SDE, the drift is unbounded and the convergence of the drift (2.8) only holds for fixed $x \in \mathbb{R}$.

To conclude this section, let us point out that since the limiting SDE is linear and the diffusion coefficient is constant, we can solve it explicitly by introducing the resolvant of the ODE $\dot{x}_t = (-am'(\bar{\theta}_t) + \frac{1}{2})x_t$. Indeed, following the notations in Theorem 1.1, let us denote $(\theta_{s,t}(x))_{s\geq 0}$ the solution of this SDE such that $x_t = x$. Note that for an arbitrary Robbins Monro algorithm, from the general theory of ODEs, $\theta_{s,t}$ is a matrix. Then, using Itô's formula, we have that:

$$X^{t,x}_s = \theta_{s,t}(x) + \int_t^s \theta_{s,u}\sigma dW_u.$$

Consequently, if $p(t,s,x,y) = \frac{d}{dy}\mathbb{P}(X_s \in dy | X_t = x)$, then, p is a gaussian density and the following density estimate holds:

$$\frac{C_1}{\sqrt{s-t}}\exp\left(-C_1\frac{|y-\theta_{s,t}(x)|^2}{s-t}\right) \leq p(t,s,x,y) \leq \frac{C_2}{\sqrt{s-t}}\exp\left(-C_2\frac{|y-\theta_{s,t}(x)|^2}{s-t}\right),$$

where C_1 and C_2 are positive constants. Nevertheless, we chose to write the density as a parametrix expansion, as it will be easier to compare it to the density of the Markov chain.

3 The Parametrix Setting

Fix a time horizon T, and let us denote by $p(t,s,x,z)$ the transition density of $(X_t)_{t\leq T}$. The goal is to quantify the difference between the transition density of the diffusion $(X_t)_{t\leq T}$ and the Markov chain $(U^N_{t^N_k})_{k\in[\![1,N]\!]}$.

However, the unbounded feature of the drifts in (2.6) and (1.5) and the fact that the convergence (2.8) above is point-wise is problematic. To deal with this, we introduce the following modified Markov chain. For all $x \in \mathbb{R}$ fixed, we denote:

$$G_N(t^N_k, x) = \left(\alpha^N_{t^N_k} - \sigma\sqrt{\frac{\gamma^N_k}{\gamma^N_{k+1}}}\int_0^1 m'\left(\bar{\theta}_{t^N_k} + \delta x\sqrt{\gamma^N_k}\right)d\delta\right) \underset{N\to+\infty}{\longrightarrow} -am'\left(\bar{\theta}_t\right) + \frac{1}{2}.$$

We circle the problem of the point-wise convergence by changing the drift term:

$$V^N_{t^N_{k+1}} = V^N_{t^N_k} + F_N(t^N_k, V^N_{t^N_k})V^N_{t^N_k}\gamma^N_{k+1} + \sqrt{\gamma^N_{k+1}}\xi_{k+1}, \tag{3.10}$$

where we define:

$$F_N(t^N_k, V^N_{t^N_k}) = \left(\alpha^N_{t^N_k} - \sigma\sqrt{\frac{\gamma^N_k}{\gamma^N_{k+1}}}\int_0^1 m'\left(\bar{\theta}_{t^N_k} + \delta sign(V^N_{t^N_k})\left(|V^N_{t^N_k}| \wedge a_N\right)\sqrt{\gamma^N_k}\right)d\delta\right).$$

In the rest of the paper, we will be led to consider F_N for continuous time and arbitrary spacial point, writing:

$$F_N(t,x) = \left(\alpha^N_t - \sigma\sqrt{\frac{\gamma^N_k}{\gamma^N_{k+1}}}\int_0^1 m'\left(\bar{\theta}_t + \delta sign(x)\left(|x| \wedge a_N\right)\sqrt{\gamma^N_k}\right)d\delta\right), \tag{3.11}$$

where implicitly, $k = k(t)$ is the index for which $t^N_k \leq t < t^N_{k+1}$. Put simply, we consider $F_N(u,x) = G_N(u, sign(x)\left(|x| \wedge a_N\right))$ instead of G_N in the drift, and where $(a_N)_{N\in\mathbb{N}}$ a sequence of positive integers tending to $+\infty$, such that $a_N\sqrt{\gamma^N_k} \to 0$ when $N \to +\infty$. Notice that as a_N grows bigger, the dynamics of (2.6) and (3.10) coincide, up to a deterministic term that goes to zero. Specifically, we have:

Lemma 3.1. *The sequences $(V^N_t)_{t\geq 0}$ and $(U^N_t)_{t\geq 0}$ with $U^N_{t^N_0} = V^N_{t^N_0} = x$, where x is chosen in a compact set K_x are close in the following sense:*

$$\mathbb{P}\left(\sup_{k\in[\![1,M(N)]\!]}|U^N_{t^N_k} - V^N_{t^N_k}| > C\sqrt{\gamma^N_0}\right) \leq \frac{C}{a_N\sqrt{\gamma^N_0}}, \tag{3.12}$$

where $C = C(T, K_x) > 0$.

Proof. Firstly, we need to quantify the rate of convergence of the $o(1)$ in (2.7). We define:

$$\beta^N_{k+1} := \sqrt{\gamma^N_{k+1}}\left(h(\bar{\theta}_{t^N_k}) - \frac{\bar{\theta}_{t^N_{k+1}} - \bar{\theta}_{t^N_k}}{\gamma^N_{k+1}}\right),$$

where $(\bar{\theta}_t)_{t\geq 0}$ is the solution of the ODE:

$$\frac{d}{dt}\bar{\theta}_t = \sigma\Big(m(\bar{\theta}_t) - \mathbb{E}(\eta)\Big),\ \bar{\theta}_0 = \theta_0.$$

Using Taylor's formula, and the ODE to express the successive the derivatives of $(\bar{\theta}_t)_{t\geq 0}$, we get:

$$\begin{aligned}\frac{\bar{\theta}_{t_{k+1}^N}-\bar{\theta}_{t_k^N}}{\gamma_{k+1}^N}-h(\bar{\theta}_{t_k^N}) &= \frac{\sigma^2}{2}m'(\bar{\theta}_{t_k^N})(\bar{\theta}_{t_k^N}-\mathbb{E}(\eta))\gamma_{k+1}^N \\ &\quad+\frac{\sigma^2}{6}\Big(m''(\bar{\theta}_{t_k^N})+am(\bar{\theta}_{t_k^N})^2\Big)(m(\bar{\theta}_{t_k^N})-\mathbb{E}(\eta))(\gamma_{k+1}^N)^2+O((\gamma_{k+1}^N)^3)\end{aligned}$$

Thus, we have:

$$\begin{aligned}\beta_{k+1}^N=\frac{\sigma^2}{2}m'(\bar{\theta}_{t_k^N})(\bar{\theta}_{t_k^N}-\mathbb{E}(\eta))(\gamma_{k+1}^N)^{\frac{3}{2}}+\frac{\sigma^2}{6}\Big(m''(\bar{\theta}_{t_k^N})+\sigma m(\bar{\theta}_{t_k^N})^2\Big)(m(\bar{\theta}_{t_k^N})-\mathbb{E}(\eta))(\gamma_{k+1}^N)^{\frac{5}{2}} \\ +O((\gamma_{k+1}^N)^{\frac{7}{2}}).\end{aligned}$$

Note that since the successive derivatives of m are bounded, we have:

$$\sum_{k=0}^{M(N)}|\beta_{k+1}^N|\leq C\frac{\sigma^2}{2}\sqrt{\gamma_1^N}\int_0^T|m'(\bar{\theta}_t)(m(\bar{\theta}_t)-\mathbb{E}(\eta))|dt\leq C\sqrt{\gamma_0^N}, \tag{3.13}$$

We introduce the exit time:

$$\tau_{a_N}:=\inf\{k\in[\![1,M(N)]\!],\ |V_{t_k^N}^N|\geq a_N\}.$$

Notice that by definition, for all $k\leq\tau_{a_N}$, we have $G_N(t_k^N,V_{t_k^N}^N)=F_N(t_k^N,V_{t_k^N}^N)$. In particular, since we start with $U_{t_0}^N=V_{t_0}^N$, we have:

$$\begin{aligned}U_{t_1}^N &= U_{t_0}^N+G_N(t_0^N,U_{t_0^N})U_{t_0}^N\gamma_1^N+\sqrt{\gamma_1^N}\xi_1+\beta_1^N \\ &= V_{t_0}^N+F_N(t_0^N,V_{t_0}^N)V_{t_0}^N\gamma_1^N+\sqrt{\gamma_1^N}\xi_1+\beta_2^N=V_{t_1}^N+\beta_1^N.\end{aligned}$$

Besides, the derivative of $x\mapsto F_N(t,x)x$ is bounded by some constant L independent of t (see the proof of Lemma 3.6 below). We prove by induction the following:

$$\left|U_{t_k^N}^N-V_{t_k}^N\right|\leq\prod_{i=1}^{k-1}(1+L\gamma_{i+1}^N)(|\beta_1^N|+\cdots+|\beta_k^N|), \tag{3.14}$$

where by definition, $\prod_{i=1}^0=1$. For $k=1$, it follows from the identity $U_{t_1}^N=V_{t_1}^N+\beta_1^N$. Assume now that (3.14) holds for for $k\geq 1$. We write for $k+1$:

$$\begin{aligned}U_{t_{k+1}}^N &= U_{t_k}^N+G_N(t_k^N,U_{t_k}^N)U_{t_k}^N\gamma_{k+1}^N+\sqrt{\gamma_{k+1}^N}\xi_{k+1}+\beta_{k+1}^N \\ V_{t_{k+1}}^N &= V_{t_k^N}^N+G_N(t_k^N,V_{t_k^N}^N)V_{t_k}^N\gamma_{k+1}^N+\sqrt{\gamma_{k+1}^N}\xi_{k+1}\end{aligned}$$

where $\beta_{k+1}^N=\sqrt{\gamma_{k+1}^N}\left(h(\bar{\theta}_{t_k^N})-\frac{\bar{\theta}_{t_{k+1}^N}-\bar{\theta}_{t_k^N}}{\gamma_{k+1}^N}\right)$. Taking the difference, we have:

$$\left|U_{t_{k+1}}^N-V_{t_{k+1}}^N\right|\leq|U_{t_k}^N-V_{t_k}^N|+|G_N(t_k^N,U_{t_k}^N)U_{t_k}^N-G_N(t_k^N,V_{t_k}^N)V_{t_k}^N|\,\gamma_{k+1}^N+|\beta_{k+1}^N|$$

Using the induction hypothesis, and the Lipschitz property of $x\mapsto G_N(t,x)x$, we have:

$$\begin{aligned}\left|U_{t_{k+1}^N}^N-V_{t_{k+1}}^N\right| &\leq (1+L\gamma_{k+1}^N)\prod_{i=1}^{k-1}(1+L\gamma_{i+1}^N)(|\beta_1^N|+\cdots+|\beta_k^N|)+|\beta_{k+1}^N| \\ &\leq \prod_{i=1}^{k}(1+L\gamma_i^N)(|\beta_1^N|+\cdots+|\beta_k^N|).\end{aligned}$$

Thus, we have obtained (3.14) for $k\leq\tau_{a_N}$. Now, from (3.13) and (3.14), it follows that:

$$\sup_{k\leq\tau_{a_N}}\left|U_{t_k^N}^N-V_{t_k}^N\right|\leq C\sqrt{\gamma_0^N}\left((1+L\gamma_0^N)^{\frac{1}{L\gamma_0^N}}\right)^{TL}\leq C\sqrt{\gamma_0^N}.$$

and consequently, we have:

$$\mathbb{P}\left(\sup_{k\in[\![1,M(N)]\!]}|U^N_{t^N_k}-V^N_{t^N_k}|>C\sqrt{\gamma^N_0}\right)\leq\mathbb{P}\Big(\tau_{a_N}\leq M(N)\Big),$$

and it remains us to show that $\mathbb{P}(\tau_{a_N}\leq M(N))$ goes to zero. Starting from the dynamics (3.10), by an immediate induction, we have:

$$\begin{aligned}V^N_{t^N_k} &= \prod_{j=0}^{k-1}\Big(1+F_N(t^N_j,V^N_{t^N_j})\gamma^N_j\Big)\,x+\sqrt{\gamma^N_0}\prod_{j=1}^{k-1}\Big(1+F_N(t^N_j,V^N_{t^N_j})\gamma^N_j\Big)\,\xi_1\\ &\quad+\cdots+\sqrt{\gamma^N_{k-2}}\Big(1+F_N(t^N_{k-1},V^N_{t^N_{k-1}})\gamma^N_{k-1}\Big)\,\xi_{k-1}+\sqrt{\gamma^N_{k-1}}\xi_k\end{aligned}$$

Recall that $F_N(t^N_{k-1},V^N_{t^N_{k-1}}):=G_N\left(t^N_{k-1},sign(V^N_{t^N_{k-1}})\left(\left|V^N_{t^N_{k-1}}\right|\wedge a_N\right)\right)$ and that $\sqrt{\gamma^N_0}a_N\to 0$ as $N\to\infty$. Also, by assumption, $m'(\cdot)$ is bounded in a neighborhood of $\bar{\theta}_t$, so that $F_N(t^N_{k-1},V^N_{t^N_{k-1}})$ is bounded by a constant C for all sufficiently large N and $k=0,1,...$We have

$$|V^N_{t^N_k}|\geq a_N\Longrightarrow[|\xi_1|+|\xi_2|+...+|\xi_k|]\geq\frac{a_N}{\sqrt{\gamma^N_0}\left(1+C\gamma^N_0\right)^k}-\frac{|x|}{\sqrt{\gamma^N_0}}.$$

Hence, using that for a sequence of integers $0<M(N)\leq\frac{T}{\gamma^N_0}$ we have when N goes to infinity:

$$\left(1+C\gamma^N_0\right)^{M(N)}=e^{M(N)\ln(1+C\gamma^N_0)}\leq e^{CT}.$$

we obtain

$$\begin{aligned}P\left(\tau_{a_N}\leq M(N)\right) &= P\Big(\exists k\in[\![1,M(N)]\!]:|V^N_{t^N_k}|\geq a_N\Big)\\ &\leq P\left(|\xi_1|+|\xi_2|+...+\left|\xi_{M(N)}\right|\geq\frac{a_N-|x|\left(1+C\gamma^N_0\right)^{M(N)}}{\sqrt{\gamma^N_0}\left(1+C\gamma^N_0\right)^{M(N)}}\right)\\ &\leq P\left(|\xi_1|+|\xi_2|+...+\left|\xi_{M(N)}\right|\geq\frac{Ca_N}{\sqrt{\gamma^N_0}}\right).\end{aligned}$$

We recall that $x\in K_x$, with K_x a compact set. By the Doob maximal inequality for submartingales,

$$P\Big(\tau_{a_N}\leq M(N)\Big)\leq C\frac{M(N)\mathbb{E}|\xi_1|}{a_N}\sqrt{\gamma^N_0}\leq\frac{C}{a_N\sqrt{\gamma^N_0}}.$$

We finally obtain:

$$P\left(\sup_{k\in[\![1;M(N)]\!]}\left|U^N_{t^N_k}-V^N_{t^N_k}\right|>C\sqrt{\gamma^N_0}\right)\leq\frac{C}{a_N\sqrt{\gamma^N_0}}.$$

□

Ideally, it would have been preferable to obtain a local limit theorem for the convergence of $(U^N_{t_k})_{k\in[\![1,M(N)]\!]}$ to $(X_t)_{t\in[0,T]}$, as exposed in Konakov *et al.* [4], as $N\to+\infty$. However, in this case, the unbounded feature of the drift, and the fact that the convergence (2.8) happens for fixed $x\in\mathbb{R}$ is problematic. Nevertheless, from the above estimate (3.12), we understand that it is justified to investigate a local limit theorem for V^N_t converging to X_t, as the step size γ^N_k tends to zero. We denote $p_N(t^N_i,t^N_j,x,z)$ the transition density of $V^N_{t_k}$:

$$p_N(t^N_i,t^N_j,x,z)dz=\mathbb{P}\left(V^N_{t^N_j}\in dz|V^N_{t^N_i}=x\right).$$

To prove Theorem 1.1, we introduce the intermediate diffusion equation:

$$dX^N_t=F_N(t,X^N_t)X^N_t dt+\sigma dW_t. \tag{3.15}$$

Observe that from now on, N is a fixed integer large enough so that the Markov chains $(U_t^N)_{t\le T}$ and $(V_t^N)_{t\le T}$ are close with great probability (cf equation (3.12)). We denote q_N the transition density of $(X_t^N)_{t\le T}$. We split the difference:

$$(p-p_N)(t,T,x,y)=(p-q_N)(t,T,x,y)+(q_N-p_N)(t,T,x,y),$$

and we investigate the two contributions separately. Specifically, in Section 3.1, using a parametrix approach, we quantify the distance $|p-q_N|$, and in Section 3.2, we turn to the Markov chains transition densities $|q_N-p_N|$, whose analysis rely on a generalisation of a result of Konakov and Mammen [3] to the case of an unbounded drift.

3.1 The diffusive part

In this section, we prove the following Lemma:

Lemma 3.2. *Assume $a_N\sqrt{\gamma_0^N}\underset{N\to+\infty}{\longrightarrow}0$. Fix a time horizon $T>0$. There exists a constant $C=:C(T)>0$, for all $t\le T$, for all $(x,y)\in\mathbb{R}\times\mathbb{R}$,*

$$\big|(p-q_N)(t,T,x,y)\big|\le Ca_N\sqrt{\gamma_0^N}\Lambda_{T-t}^N(y)g_C(T-t,\theta_{t,T}(y)-x),$$

where $g_c(t,z)$ is defined in the statement of Theorem 1.1, and

$$\Lambda_{T-t}^N(y)=1+\sqrt{T-t}(1+|y|^2)+|y|e^{C(T-t)a_N^2\gamma_0^N|y|^2}\Big(\sqrt{T-t}+(T-t)\big(1+|y|^2\big)\Big).$$

Remark 3.1. For all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, we will use the following notation throughout this paper:

$$g_\sigma(t,x)=\frac{1}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2\sigma^2 t}\right),\ \text{ or } g_C(t,x)=\frac{1}{C\sqrt{t}}\exp\left(-\frac{x^2}{Ct}\right),$$

to denote a Gaussian-like density. We point out however that for any given C, this function does not necessarily integrates to one. Moreover, we can write

$$g_C(t,s,x,y):=g_C(s-t,y-x)=\frac{1}{C\sqrt{s-t}}e^{-\frac{|x-y|^2}{C(s-t)}},$$

and we have the following property: for all $C_1,C_2>0$, there exists $C_3,C>0$ such that for all $t,u,s\in\mathbb{R}_+$ and all $x,y\in\mathbb{R}$,

$$g_{C_3}(t,s,x,y)\le C\int_{\mathbb{R}}g_{C_1}(t,u,x,z)g_{C_2}(u,s,z,y)dz.$$

Remark 3.2. The presence of the term $\Lambda_{T-t}^N(y)$ above comes from our approach. First, since we chose to work with an arbitrary time $T<+\infty$, we have to keep track of the different exponents in time. Observe that if we fixed $T<1$, the expression above can be simplified significantly.

Secondly, the factor $|y|^2$ can be absorbed in the Gaussian density, but the estimate we obtain is less precise, so we decided not to.

Thirdly, the only problematic contribution is $e^{C(T-t)a_N^2\gamma_0^N|y|^2}$, and term stems from the control of the distance between the solutions of the two ODEs we consider (see Lemma 3.3 and 3.4). Nevertheless, since we assumed $a_N^2\gamma_0^N\to0$ as N goes to infinity, we do get the expected rate on the difference of the densities.

Finally, notice that if we set y in a compact set K_y, the corrective term $\Lambda_{T-t}^N(y)$ can be absorbed in some constant depending on K_y, and we recover the usual rate of convergence.

Proof. We recall here that:

$$dX_t^N=F_N(t,X_t^N)X_t^Ndt+\sigma dW_t,$$

where:

$$\begin{aligned}F_N(t,x)&=\left(\alpha_t^N-\sigma\sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}}\int_0^1 m'\left(\bar\theta_t+\delta sign(x)\Big(|x|\wedge a_N\Big)\sqrt{\gamma_k^N}\right)d\delta\right)\\ &\underset{N\to+\infty}{\longrightarrow}-am'(\bar\theta_t)+\frac{1}{2}.\end{aligned}$$

We also recall that $k = k(t)$ is the index for which $t_k^N \leq t < t_{k+1}^N$. Thanks to the cut-off a_N, this convergence is now uniform with respect to x. Thus, we are now formally looking for a stability result for the density when the drift is not bounded. In other words, we need to control the distance between the transition density of $(X_t^N)_{t\geq 0}$ and the transition density of the limiting SDE:

$$dX_t = \left(-am'(\bar{\theta}_t) + \frac{1}{2}\right) X_t + \sigma dW_t.$$

This proof is inspired from Konakov *et al.* [4], in the case of an unbounded drift. To quantify the distance between p and q_N, we rely on the parametrix representations. Since the drifts coefficients are unbounded, we have to freeze along the solutions of the ODE. Fix a terminal time $T > 0$ and a terminal position $y \in \mathbb{R}$. We consider $\theta_{t,T}(y)$ and $\theta_{t,T}^N(y)$ the backward flows respectively solutions of the following ODEs:

$$\begin{cases} \frac{d}{dt} z_t = \left(-am'(\bar{\theta}_t) + \frac{1}{2}\right) z_t, \\ z_T = y, \end{cases} \qquad \begin{cases} \frac{d}{dt} z_t^N = F_N(t, z_t^N) z_t^N, \\ z_T^N = y. \end{cases}$$

To approximate $(X_t)_{t\leq T}$, we consider the following frozen process:

$$\tilde{X}_s^{T,y} = x + \int_t^s \left(-am'\big(\theta_{u,T}(y)\big) + \frac{1}{2}\right) \theta_{u,T}(y) du + \sigma W_{s-t}.$$

This frozen process is a Gaussian process, and due to positivity of σ, its density exists and we denote it by:

$$\frac{d}{dz} \mathbb{P}(\tilde{X}_s^{T,y} \in dz | \tilde{X}_t^{T,y} = x) = \tilde{p}^{T,y}(t, s, x, z).$$

Observe that due to the choice of the freezing parameters we have the following identity:

$$\tilde{p}^{T,y}(t, T, x, y) = \frac{1}{\sigma\sqrt{2\pi(T-t)}} \exp\left(\frac{|\theta_{t,T}(y) - x|^2}{2\sigma^2(T-t)}\right).$$

From this point forward, we will use the following abuse of notation $\tilde{p}(t, T, x, y) = \tilde{p}^{T,y}(t, T, x, y)$. Similarly, to approximate $(X_t^N)_{t\leq T}$, we consider the following frozen processes:

$$\tilde{X}_s^{T,y,N} = x + \int_t^s F_N\big(u, \theta_{u,T}^N(y)\big)\theta_{u,T}^N(y) du + \sigma W_{s-t},$$

We denote its density by

$$\frac{d}{dz} \mathbb{P}(\tilde{X}_s^{T,y,N} \in dz | \tilde{X}_t^{T,y,N} = x) = \tilde{q}_N^{T,y}(t, s, x, z),$$

and we once again have the identity:

$$\tilde{q}_N^{T,y}(t, T, x, y) = \frac{1}{\sigma\sqrt{2\pi(T-t)}} \exp\left(\frac{|\theta_{t,T}^N(y) - x|^2}{2\sigma^2(T-t)}\right).$$

Again, we will denote $\tilde{q}_N(t, T, x, y) := \tilde{q}_N^{T,y}(t, T, x, y)$ for convenience. It is elementary to prove the convergence of the parametrix series in this specific case, especially since the diffusion coefficients are constants. We thus write

$$(p - q_N)(t, T, x, y) = \big(\tilde{p} - \tilde{q}_N\big)(t, T, x, y) + \sum_{r=1}^{+\infty} \Big(\tilde{p} \otimes H^{(r)} - \tilde{q}_N \otimes H_N^{(r)}\Big)(t, T, x, y),$$

where $f \otimes g(t, T, x, y) = \int_t^T \int_{R^d} f(t, u, x, z) g(u, T, z, y) du dz$ and

$$H(t, T, x, y) = \big(L_t - \tilde{L}_t\big)\tilde{p}(t, T, x, y), \ H_N(t, T, x, y) = \big(L_t^N - \tilde{L}_t^N\big)\tilde{q}_N(t, T, x, y),$$

where L_t, $\tilde{L}_t$, L_t^N, and $\tilde{L}_t^N$ represents the generators of $(X_t)_{t\geq 0}$, $(\tilde{X}_t^{T,y})_{t\geq 0}$, $(X_t^N)_{t\geq 0}$ and $(\tilde{X}_t^{T,y,N})_{t\geq 0}$ respectively. We will now proceed to give an estimate on each term in the above sum. We have the following Lemma:

Lemma 3.3. *For all $t \leq T$, there exists $C > 0$ such that for all $x, y \in \mathbb{R}$, we have:*

$$\big|(\tilde{p} - \tilde{q}_N)(t, T, x, y)\big| \leq C\sqrt{(T-t)} a_N \sqrt{\gamma_0^N} |y| e^{C(T-t)^2 a_N^2 \gamma_0^N |y|^2} g_C(T-t, \theta_{t,T}(y) - x).$$

Proof. We write using the finite increments theorem:

$$\begin{aligned}(\tilde{p}-\tilde{q}_N)(t,T,x,y) &= \frac{1}{\sigma\sqrt{2\pi(T-t)}}\left(\exp\left(\frac{|\theta_{t,T}(y)-x|^2}{2\sigma^2(T-t)}\right)-\exp\left(\frac{|\theta^N_{t,T}(y)-x|^2}{2\sigma^2(T-t)}\right)\right)\\ &= \int_0^1 g'_\sigma(T-t,\lambda\theta_{t,T}(y)+(1-\lambda)\theta^N_{t,T}(y)-x)\big(\theta_{t,T}(y)-\theta^N_{t,T}(y)\big)d\lambda.\end{aligned}$$

Formally, since g_σ is a Gaussian density, it is known that its derivative yields a singularity in $(T-t)^{-1/2}$, thus, we see that to control the difference $|\tilde{p}-\tilde{q}_N|$, we need a control of $|\theta_{t,T}(y)-\theta^N_{t,T}(y)|$, which is done in the following Lemma:

Lemma 3.4. *For all $0\le t\le T$ there exits a constant $C>0$ such that:*

$$|\theta_{t,T}(y)-\theta^N_{t,T}(y)|\le C(T-t)a_N\sqrt{\gamma^N_0}|y|.$$

Besides, for some positive C_T, the following estimate holds:

$$|\theta_{t,T}(y)|+|\theta^N_{t,T}(y)|\le C_T|y|,$$

Similarly, for all $0\le t^N_k\le T$ there exits a constant $C>0$ such that:we have:

$$|\theta_{t^N_k,T}(y)-\hat{\theta}^N_{t^N_k,T}(y)|\le C(T-t^N_k)a_N\sqrt{\gamma^N_0}|y|. \tag{3.16}$$

Besides, for some positive C_T, the following estimate holds:

$$|\hat{\theta}^N_{t^N_k,T}(y)|\le C_T|y|,$$

The proof of this result relies on general results on stability for ODEs and we postpone this proof to the end of this section. We obtain that there exists $C>0$ such that:

$$\begin{aligned}\big|\tilde{p}-\tilde{q}_N\big|(t,T,x,y)&\le\int_0^1\big|g'_\sigma(T-t,\lambda\theta_{t,T}(y)+(1-\lambda)\theta^N_{t,T}(y)-x)\big|\big|\theta_{t,T}(y)-\theta^N_{t,T}(y)\big|d\lambda\\ &\le C_T(T-t)a_N\sqrt{\gamma^N_0}|y|\int_0^1 d\lambda\frac{1}{\sqrt{T-t}}\big|g_C(T-t,\lambda\theta_{t,T}(y)+(1-\lambda)\theta^N_{t,T}(y)-x)\big|.\end{aligned}$$

Now, we write using Young's inequality:

$$\begin{aligned}|\lambda\theta_{t,T}(y)+(1-\lambda)\theta^N_{t,T}(y)-x|^2 &= |\theta_{t,T}(y)-x+(1-\lambda)\left[\theta^N_{t,T}(y)-\theta_{t,T}(y)\right]|^2\\ &\ge \frac{1}{2}|\theta_{t,T}(y)-x|^2-C^2(T-t)^2a_N^2\gamma^N_0|y|^2.\end{aligned}$$

Plugging the above inequality into the Gaussian exponent yields:

$$\begin{aligned}&g_C(T-t,\lambda\theta_{t,T}(y)+(1-\lambda)\theta^N_{t,T}(y)-x)\\ =\;&\frac{1}{C\sqrt{2\pi(T-t)}}\exp\left(-\frac{|\lambda\theta_{t,T}(y)+(1-\lambda)\theta^N_{t,T}(y)-x|^2}{2(T-t)C}\right)\\ \le\;&\frac{1}{C\sqrt{2\pi(T-t)}}\exp\left(-\frac{|\theta_{t,T}(y)-x|^2}{4C(T-t)}\right)\times\exp\left(C^2(T-t)^2\gamma^N_0a_N^2|y|^2\right)\end{aligned}$$

Finally, we thus obtain the following upper bound. There are $C,C_1>0$ such that:

$$|\tilde{p}-\tilde{q}_N|(t,T,x,y)\le C_1\sqrt{(T-t)}a_N\sqrt{\gamma^N_0}|y|e^{C(T-t)^2a_N^2\gamma^N_0|y|^2}g_C(T-t,\theta_{t,T}(y)-x).$$

□

Remark 3.3 (Additional factor for the frozen densities)**.**

We point out that the additional term in $|y|e^{C(T-t)^2a_N^2\gamma^N_0|y|^2}$ actually comes from the point-wise convergence (2.8). Observe that if the convergence (2.8) was actually uniform, from that point forward, we could have reproduced the proof of the stability result [4] in the case of an unbounded drift. Also, we point out that the above proof also shows that in the argument of the Gaussian density $g_C(T-t,\cdot)$, we can put indifferently $\theta_{t,T}(y)$ or $\theta^N_{t,T}(y)$ at the price of the additional factor $e^{C(T-t)a_N^2\gamma_0^N|y|^2}$. That is, we actually have:

$$\frac{1}{2}e^{-\frac{C}{2}(T-t)a_N^2\gamma_0^N|y|^2}g_{\frac{C}{2}}(T-t,\theta^N_{t,T}(y)-x)\le g_C(T-t,\theta_{t,T}(y)-x)$$
$$\le 2e^{C(T-t)a_N^2\gamma_0^N|y|^2}g_{2C}(T-t,\theta^N_{t,T}(y)-x). \quad (3.17)$$

The rest of the proof thus consists in controlling the iterated convolutions with the additional factor $|y|e^{C(T-t)a_N^2\gamma_0^N|y|^2}$. Finally, since by assumption, $a_N\sqrt{\gamma_0^N}\underset{\mathbb{N}\to+\infty}{\longrightarrow}0$, we see that for large N the correction becomes negligible.

We now turn to the estimate on the parametrix kernels. Observe that we can write:

$$\begin{aligned}\sum_{r=1}^{+\infty}\Big(\tilde p\otimes H^{(r)}-\tilde q_N\otimes H_N^{(r)}\Big)(t,T,x,y) &= \tilde p\otimes\sum_{r=1}^{+\infty}H^{(r)}(t,T,x,y)-\tilde q_N\otimes\sum_{r=1}^{+\infty}H_N^{(r)}(t,T,x,y)\\ &=: \tilde p\otimes\Phi(t,T,x,y)-\tilde q_N\otimes\Phi_N(t,T,x,y),\end{aligned}$$

where we wrote $H^{(r)}=H^{(r-1)}\otimes H$ and $H_N^{(r)}=H_N^{(r)}\otimes H_N$, with the convention that $H^{(0)},H_N^{(0)}=Id$. Now, we can split as usual:

$$\tilde p\otimes\Phi(t,T,x,y)-\tilde q_N\otimes\Phi_N(t,T,x,y)=\tilde p\otimes\big(\Phi-\Phi_N\big)(t,T,x,y)+\big(\tilde p-\tilde q_N\big)\otimes\Phi_N(t,T,x,y).$$

We have the following lemma:

Lemma 3.5. *For all $t\le T$, there exits $C_T>0$ such that for all $x,y\in\mathbb{R}$, the following estimates holds:*

$$|\Phi|(t,T,x,y)\le C_T\frac{1}{\sqrt{T-t}}g_C(T-t,\theta_{t,T}(y)-x), \quad (3.18)$$
$$|\Phi_N|(t,T,x,y)\le C_T\frac{1}{\sqrt{T-t}}g_C(T-t,\theta^N_{t,T}(y)-x), \quad (3.19)$$

$$|\Phi_N-\Phi|(t,T,x,y)\le Ca_N\sqrt{\gamma_0^N}\Lambda^N_{T-t}(y)g_C(T-t,\theta_{t,T}(y)-x). \quad (3.20)$$

Proof. The first two estimates are a general estimate and derive from the controls on the parametrix kernels H,H_N. We give here some ideas of the proof. The two terms are treated similarly, we focus on the first estimate. First, observe that there exists a constant $C>0$ such that we have the upper bound for the parametrix kernel:

$$|H|(t,T,x,y)\le\frac{C_T}{\sqrt{T-t}}g_C(T-t,\theta_{t,T}(y)-x).$$

We deduce by induction the following estimate:

$$|H^{(k)}|(t,T,x,y)\le C_T(T-t)^{\frac{k}{2}-1}\prod_{r=1}^{k}B\left(\frac{r}{2},\frac{1}{2}\right)g_C(T-t,\theta_{t,T}(y)-x),$$

where $B(\cdot,\cdot)$ denotes the Beta function. Notice that from the properties of the Beta function, we have:

$$\prod_{r=1}^{k}B\left(\frac{r}{2},\frac{1}{2}\right)=\frac{\Gamma(1/2)^{k+1}}{\Gamma\left(\frac{k+1}{2}\right)}=\frac{\sqrt{\pi}^{k+1}}{\Gamma\left(\frac{k+1}{2}\right)}.$$

The announced estimate for $|\Phi|$ follows by taking the sum of the right hand side.

The third estimate is more involved, as we have to deal with the sensitivity in N. We start by obtaining a similar bound on $H-H_N$. We write:

$$\begin{aligned}(H-H_N)(t,T,x,y) &= \big(L_t-\tilde L_t\big)\tilde p(t,T,x,y)-\big(L_t^N-\tilde L_t^N\big)\tilde q_N(t,T,x,y)\\ &= \big(L_t-\tilde L_t-L_t^N+\tilde L_t^N\big)\tilde p(t,T,x,y)+\big(L_t^N-\tilde L_t^N\big)(\tilde p-\tilde q_N)(t,T,x,y)\\ &=: I+I\!I, \quad (3.21)\end{aligned}$$

where we recall the expressions of the generators for a test function φ:

$$\begin{aligned}
L_t\varphi(x) &= \left(-\sigma m'(\bar{\theta}_t) + \frac{1}{2}\right) x \cdot \varphi'(x) + \frac{1}{2}\sigma^2\varphi''(x),\\
\tilde{L}_t\varphi(x) &= \left(-\sigma m'(\bar{\theta}_t) + \frac{1}{2}\right) \theta_{t,T}(y) \cdot \varphi'(x) + \frac{1}{2}\sigma^2\varphi''(x),\\
L_t^N\varphi(x) &= F_N(t,x)x \cdot \varphi'(x) + \frac{1}{2}\sigma^2\varphi''(x),\\
\tilde{L}_t^N\varphi(x) &= F_N(t,\theta_{t,T}^N(y))\theta_{t,T}^N(y) \cdot \varphi'(x)\rangle + \frac{1}{2}\sigma^2\varphi''(x).
\end{aligned}$$

We will need the following estimate:

Lemma 3.6. *There exists $C_T > 0$ such that:*

$$\begin{aligned}
&\left|\Big(-am'(\bar{\theta}_t) + \frac{1}{2}\Big)\big(x - \theta_{t,T}(y)\big) - \Big(xF_N(t,x) - F_N(t,\theta_{t,T}^N(y))\theta_{t,T}^N(y)\Big)\right|\\
&\qquad \le C a_N\sqrt{\gamma_0^N}\Big(|x-\theta_{t,T}(y)| + (T-t)\big(|y|+|y|^2\big)\Big) \qquad (3.22)
\end{aligned}$$

Proof. We start the proof here by showing an auxiliary result, namely that $x \mapsto F_N(t,x)x$ is Lipschitz (uniformly in $t \le T$). The idea is to bound the first derivative. Note that we have to be careful here, since F_N is not exactly differentiable at the cut-off point. We have:

$$\Big(F_N(t,x)x\Big)' = F_N'(t,x)x + F_N(t,x).$$

Now, since we assumed m' to be bounded in a tubular neighborhood of $\bar{\theta}_t$, we have that $F_N(t,x) \le C$, for some constant $C > 0$. Thus, the derivative is bounded tubular neighborhood of $\bar{\theta}_t$ when $F_N'(t,x)x$ is. Observe carefully now that:

$$(F_N)'(t,x) = -\sigma\frac{\sqrt{\gamma_k^N}}{\sqrt{\gamma_{k+1}^N}}\int_0^1 m''\big(\bar{\theta}_t + sign(x)\big(|x|\wedge a_N\big)\sqrt{\gamma_0^N}\delta\big)\sqrt{\gamma_0^N}\delta d\delta \mathbf{1}_{\{|x|\le a_N\}}. \qquad (3.23)$$

Thus, thanks to our cut-off, we actually have:

- either $|x| \ge a_N$, and in which case $F_N'(t,x) = 0$,
- or $|x| \le a_N$, but then, since m'' is bounded, we can write:

$$|F_N'(t,x)x| \le C|x| \le C \times a_N\sqrt{\gamma_0^N}.$$

Thus, in both cases, the derivative of $F_N(t,x)x$ is bounded and we can conclude that $x \mapsto F_N(t,x)x$ is Lipschitz.

We recall that F_N is defined in (3.11), and we denote for convenience $A(\bar{\theta}_t) = -\sigma m'(\bar{\theta}_t) + \frac{1}{2}$. We write:

$$\begin{aligned}
&\left|\Big(-\sigma m'(\bar{\theta}_t) + \frac{1}{2}\Big)\big(x-\theta_{t,T}(y)\big) - \Big(F_N(t,x)x - F_N(t,\theta_{t,T}^N(y))\theta_{t,T}^N(y)\Big)\right|\\
\le\ &\left|A(\bar{\theta}_t)\big(x-\theta_{t,T}(y)\big) - \Big(F_N(t,x)x - F_N(t,\theta_{t,T}(y))\theta_{t,T}(y)\Big)\right|\\
&+\left|F_N\big(t,\theta_{t,T}^N(y)\big)\theta_{t,T}^N(y) - F_N\big(t,\theta_{t,T}^N(y)\big)\theta_{t,T}(y)\right|\\
&+\left|F_N\big(t,\theta_{t,T}^N(y)\big)\theta_{t,T}(y) - F_N\big(t,\theta_{t,T}(y)\big)\theta_{t,T}(y)\right| =: I_1 + I_2 + I_3.
\end{aligned}$$

We start with I_2, writing:

$$I_2 = \left|F_N\big(t,\theta_{t,T}^N(y)\big)\Big(\theta_{t,T}^N(y) - \theta_{t,T}(y)\Big)\right| \le |F_N(t,\theta_{t,T}^N(y))| \times \left|\theta_{t,T}^N(y) - \theta_{t,T}(y)\right|.$$

Now, since m' is bounded in a tubular neighborhood of $\bar{\theta}_t$, $|F_N(t,\theta_{t,T}^N(y))| \le C$ and using Lemma 3.4, we obtain:

$$I_2 \le C(T-t)a_N\sqrt{\gamma_0^N}|y|.$$

For I_3, we have:

$$I_3 = \left|\Big(F_N\big(t, \theta^N_{t,T}(y)\big) - F_N\big(t, \theta_{t,T}(y)\big)\Big)\theta_{t,T}(y)\right| \le \left|F_N(t, \theta^N_{t,T}(y)) - F_N(t, \theta_{t,T}(y))\right| \times |\theta_{t,T}(y)|.$$

Since we assumed m'' to be bounded in a tubular neighborhood around $\bar{\theta}_t$, we have from (3.23) that F_N' is bounded. We have to be careful at this point, as $x \wedge a_N$ is not smooth, only but piece-wise differentiable. Anyhow, the point that we can bound the derivative of F_N still stands and we have directly the announced estimate for this term. Thus, $F_N(t, \cdot)$ is Lipschitz, so that:

$$\begin{aligned} I_3 &\le \left|F_N(t, \theta^N_{t,T}(y)) - F_N(t, \theta_{t,T}(y))\right| |\theta_{t,T}(y)| \le C|\theta^N_{t,T}(y) - \theta_{t,T}(y)||\theta_{t,T}(y)| \\ &\le C(T-t)a_N\sqrt{\gamma_0^N}|y|^2, \end{aligned}$$

using again Lemma 3.4 and the definition of $\theta_{t,T}(y)$. Let us now turn to I_1 above. Set $\tilde{F}_N(t,x) = F_N(t,x)x$, we use the finite increment theorem:

$$\tilde{F}_N(t,x) - \tilde{F}_N(t, \theta_{t,T}(y)) = \int_0^1 \tilde{F}_N'\big(t, \lambda x + (1-\lambda)\theta_{t,T}(y)\big)(x - \theta_{t,T}(y))d\lambda.$$

We use the notation F_N' to denote the derivative with respect to the x variable to lighten the notations. Thus, we get for I_1:

$$I_1 \le \left|\int_0^1 d\lambda\Big(A(\bar{\theta}_t) - \tilde{F}_N'\big(t, \lambda x + (1-\lambda)\theta_{t,T}(y)\big)\Big)\right| \times |x - \theta_{t,T}(y)|.$$

Now, since $\tilde{F}_N'(t,x) = (F_N)'(t,x)x + F_N(t,x)$, we write:

$$\begin{aligned} I_1 &\le \left|\int_0^1 d\lambda\Big(A(\bar{\theta}_t) - F_N(t, \lambda x + (1-\lambda)\theta_{t,T}(y))\Big)\right| \times |x - \theta_{t,T}(y)| \\ &\quad + \left|\int_0^1 d\lambda (F_N)'\big(t, \lambda x + (1-\lambda)\theta_{t,T}(y)\big)(\lambda x + (1-\lambda)\theta_{t,T}(y))\right| \times |x - \theta_{t,T}(y)|. \end{aligned}$$

Using again the control (3.23) above, we get that for all $x \in \mathbb{R}$, $|F_N'(t,x)x| \le Ca_N\sqrt{\gamma_0^N}$. Thus, we have:

$$\begin{aligned} &\left|\int_0^1 d\lambda (F_N)'\big(t, \lambda x + (1-\lambda)\theta_{t,T}(y)\big)(\lambda x + (1-\lambda)\theta_{t,T}(y))\right| \times |x - \theta_{t,T}(y)| \\ &\qquad \le Ca_N\sqrt{\gamma_0^N}|x - \theta_{t,T}(y)|. \end{aligned}$$

Now, from estimate (3.30) in the proof of Lemma 3.4 below, we know that for all $x \in \mathbb{R}$,

$$\left|\left(F_N(t,x) - \left(-am'(\bar{\theta}_t) + \frac{1}{2}\right)\right)\right| \le C|x| \wedge a_N\sqrt{\gamma_0^N} \le Ca_N\sqrt{\gamma_0^N},$$

which gives for the first contribution above:

$$\left|\int_0^1 \Big(A(\bar{\theta}_t) - F_N(t, \lambda x + (1-\lambda)\theta_{t,T}(y))\Big)\right| \times |x - \theta_{t,T}(y)| \quad \le \quad Ca_N\sqrt{\gamma_0^N}|x - \theta_{t,T}(y)|.$$

Consequently, we obtain for this term:

$$I_1 \le Ca_N\sqrt{\gamma_0^N}|x - \theta_{t,T}(y)|.$$

Putting together the estimates for I_1, I_2 and I_3 proves (3.22).

□

Returning to the control of (3.21), we start with I above. In this case, we have:

$$\begin{aligned} |I| &= \left|(L_t - \tilde{L}_t - L_t^N + \tilde{L}_t^N)\tilde{p}(t,T,x,y)\right| \\ &\le |\nabla_x \tilde{p}(t,T,x,y)| \left|\left(-am'(\bar{\theta}_t) + \frac{1}{2}\right)(x - \theta_{t,T}(y)) - xF_N(t,x) + \theta^N_{t,T}(y)F_N(t, \theta^N_{t,T}(y))\right|. \end{aligned}$$

Then, we deduce the following estimate for I, using again estimate (3.22), that there exists $C > 0$ such that:

$$|I| \le C a_N \sqrt{\gamma_0^N}\Big(|x-\theta_{t,T}(y)| + (T-t)\big(|y|+|y|^2\big)\Big)\frac{1}{\sqrt{T-t}} g_C(T-t,\theta_{t,T}(y)-x).$$

Now, observe that there exists $C' > 0$ such that

$$\frac{|x-\theta_{t,T}(y)|}{\sqrt{T-t}} g_C(T-t,\theta_{t,T}(y)-x) \le g_{C'}(T-t,\theta_{t,T}(y)-x).$$

Observe also that we have:

$$(T-t)\big(|y|+|y|^2\big)\frac{1}{\sqrt{T-t}} g_C(T-t,\theta_{t,T}(y)-x) = \sqrt{T-t}\big(|y|+|y|^2\big) g_C(T-t,\theta_{t,T}(y)-x)$$

Consequently, we obtain the following estimate for some $C > 0$:

$$|I| \;\le\; C a_N \sqrt{\gamma_0^N}\Big(1+\sqrt{T-t}(|y|+|y|^2)\Big) g_C(T-t,\theta_{t,T}(y)-x)$$

Let us now turn to $I\!I$ in (3.21) above. Then, we can bound :

$$\begin{aligned}
|I\!I| &:= \Big|\big(L_t^N - \tilde{L}_t^N\big)(\tilde{p}-\tilde{q}_N)(t,T,x,y)\Big| \\
&= |\langle \nabla_x(\tilde{p}-\tilde{q}_N)(t,T,x,y), F_N(t,x)x - F_N(t,\theta_{t,T}^N(y))\theta_{t,T}^N(y)\rangle| \\
&\le \big|\nabla_x(\tilde{p}-\tilde{q}_N)(t,T,x,y)\big| \times |F_N(t,x)x - F_N(t,\theta_{t,T}^N(y))\theta_{t,T}^N(y)|.
\end{aligned}$$

Using Lemma 3.6, we have:

$$\begin{aligned}
&|F_N(t,x)x - F_N(t,\theta_{t,T}^N(y))\theta_{t,T}^N(y)| \\
\le\; &\left|\Big(-am'(\bar{\theta}_t)+\frac{1}{2}\Big)\big(x-\theta_{t,T}(y)\big) - \Big(xF_N(t,x) - F_N(t,\theta_{t,T}^N(y))\theta_{t,T}^N(y)\Big)\right| \\
&+\left|\Big(-am'(\bar{\theta}_t)+\frac{1}{2}\Big)\big(x-\theta_{t,T}(y)\big)\right| \\
\le\; & C a_N \sqrt{\gamma_0^N}\Big(|x-\theta_{t,T}(y)| + (T-t)\big(|y|+|y|^2\big)\Big) + C|x-\theta_{t,T}(y)|.
\end{aligned} \tag{3.24}$$

Consequently, we have the estimate:

$$\begin{aligned}
|I\!I| \;\le\; & \big|\nabla_x(\tilde{p}-\tilde{q}_N)(t,T,x,y)\big| C a_N \sqrt{\gamma_0^N}\Big(|x-\theta_{t,T}(y)| + (T-t)\big(|y|+|y|^2\big)\Big) \\
&+C|x-\theta_{t,T}(y)|\big|\nabla_x(\tilde{p}-\tilde{q}_N)(t,T,x,y)\big|.
\end{aligned}$$

Now, to estimate the gradient term, we write:

$$\begin{aligned}
\nabla_x(\tilde{p}-\tilde{q}_N)(t,T,x,y) &= \nabla_x\left(g_\sigma(T-t,\theta_{t,T}(y)-x) - g_\sigma(T-t,\theta_{t,T}^N(y)-x)\right) \\
&= \int_0^1 g_\sigma''(T-t,\lambda\theta_{t,T}(y)+(1-\lambda)\theta_{t,T}^N(y)-x)\big(\theta_{t,T}(y)-\theta_{t,T}^N(y)\big)d\lambda.
\end{aligned}$$

Bounding $|\theta_{t,T}(y)-\theta_{t,T}^N(y)|$ with Lemma 3.4, and reproducing the arguments at the end of the proof of Lemma 3.3 to estimate g_σ'', we obtain that there exists a constants $C, C' > 0$ such that:

$$\big|\nabla_x(\tilde{p}-\tilde{q}_N)(t,T,x,y)\big| = C' a_N \sqrt{\gamma_0^N}|y|e^{C(T-t)a_N^2\gamma_0^N|y|^2} g_C(T-t,\theta_{t,T}^N(y)-x).$$

Observe that we actually put $\theta_{t,T}^N(y)$ instead of $\theta_{t,T}(y)$ in the argument of the Gaussian density in order to match the contribution given by the difference of the generators $L_t^N - \tilde{L}_t^N$. Consequently, we have the upper bound:

$$\begin{aligned}
|I\!I| \;\le\; & C(a_N\sqrt{\gamma_0^N})^2|y|e^{C(T-t)a_N^2\gamma_0^N|y|^2} g_C(T-t,\theta_{t,T}^N(y)-x)\Big(|x-\theta_{t,T}(y)| + (T-t)\big(|y|+|y|^2\big)\Big) \\
&+C|x-\theta_{t,T}(y)|a_N\sqrt{\gamma_0^N}|y|e^{C(T-t)a_N^2\gamma_0^N|y|^2} g_C(T-t,\theta_{t,T}^N(y)-x) \\
\le\; & C a_N\sqrt{\gamma_0^N}|y|e^{C(T-t)a_N^2\gamma_0^N|y|^2} g_C(T-t,\theta_{t,T}^N(y)-x)\Big(\sqrt{T-t} + T-t + (T-t)\big(|y|+|y|^2\big)\Big).
\end{aligned}$$

Now, adding the estimates for I and $I\!I$, we get:

$$\begin{aligned}
& |H - H_N|(t,T,x,y) \\
\leq\ & Ca_N\sqrt{\gamma_0^N}\Big(1+\sqrt{T-t}(|y|+|y|^2)\Big)g_C(T-t,\theta_{t,T}(y)-x) \\
& + Ca_N\sqrt{\gamma_0^N}|y|e^{C(T-t)a_N^2\gamma_0^N|y|^2}g_C(T-t,\theta_{t,T}^N(y)-x)\Big(\sqrt{T-t}+(T-t)\big(1+|y|+|y|^2\big)\Big) \\
=\ & Ca_N\sqrt{\gamma_0^N}g_C(T-t,\theta_{t,T}(y)-x) \\
& \times\Big(1+\sqrt{T-t}(|y|+|y|^2)+|y|e^{C(T-t)a_N^2\gamma_0^N|y|^2}\Big(\sqrt{T-t}+(T-t)\big(1+|y|^2\big)\Big)\Big)
\end{aligned}$$

To simplify the notations, let us define

$$\Lambda_{T-t}^N(y) = 1+\sqrt{T-t}(|y|+|y|^2)+|y|e^{C(T-t)a_N^2\gamma_0^N|y|^2}\Big(\sqrt{T-t}+(T-t)\Big)\big(1+|y|^2\big),$$

so that we have the estimate:

$$(H-H_N)(t,T,x,y) \leq Ca_N\sqrt{\gamma_0^N}\Lambda_{T-t}^N(y)g_C(T-t,\theta_{t,T}(y)-x). \tag{3.25}$$

To obtain estimate (3.20), use estimate (3.25) to control the successive convolutions. We proceed prove the following estimate by induction.

$$\big|(H^{(k)}-H_N^{(k)})(t,T,x,y)\big| \leq Ca_N\sqrt{\gamma_0^N}\Lambda_{T-t}^N(y)(T-t)^{\frac{k-1}{2}}\prod_{r=1}^{k}B\left(\frac{r}{2},\frac{1}{2}\right)g_C(T-t,\theta_{t,T}(y)-x). \tag{3.26}$$

For $k=1$, this is estimate (3.25) and has already been established. Assume it holds for $k\geq 1$. We write:

$$(H^{(k+1)}-H_N^{(k+1)})(t,T,x,y) = \big(H^{(k)}-H_N^{(k)}\big)\otimes H(t,T,x,y)+H_N^{(k)}\otimes\big(H-H_N\big)(t,T,x,y) = I+I\!I.$$

For the contribution I above, we use the induction hypothesis and we derive:

$$\begin{aligned}
|I| \leq Ca_N\sqrt{\gamma_0^N}\prod_{r=1}^{k}B\left(\frac{r}{2},\frac{1}{2}\right)\int_t^T(u-t)^{\frac{k-1}{2}}\int_{\mathbb{R}^d}\Lambda_{u-t}^N(z)g_C(u-t,\theta_{t,u}(z)-x) \\
\times\frac{1}{\sqrt{T-u}}g_C(T-u,\theta_{u,T}(y)-z)dudz.
\end{aligned}$$

We split the space the integration over z as follows:

$$\begin{aligned}
D_1 &= \{z\in\mathbb{R}\ ;\ \Lambda_{u-t}^N\big(z-\theta_{t,T}(y)\big)\geq\Lambda_{T-t}^N(y)\} \\
D_2 &= \{z\in\mathbb{R}\ ;\ \Lambda_{u-t}^N\big(z-\theta_{t,T}(y)\big)<\Lambda_{T-t}^N(y)\}.
\end{aligned}$$

Now, observe that when $z\in D_2$, we have:

$$\Lambda_{u-t}^N(z)=\Lambda_{u-t}^N(z-\theta_{t,T}(y)+\theta_{t,T}(y))\leq C_T\Big(\Lambda_{u-t}^N\big(z-\theta_{t,T}(y)\big)+\Lambda_{T-t}^N(y)\Big)\leq C_T\Lambda_{T-t}^N(y).$$

Thus, we can bound the $\Lambda_{u-t}^N(z)$ contribution by $\Lambda_{T-t}^N(y)$ and take it out of the integral:

$$\begin{aligned}
|I_{|D_2}| \leq\ & Ca_N\sqrt{\gamma_0^N}\Lambda_{T-t}^N(y)\prod_{r=1}^{k}B\left(\frac{r}{2},\frac{1}{2}\right)\int_t^T(u-t)^{\frac{k-1}{2}}\int_{D_2}g_C(u-t,\theta_{t,u}(z)-x) \\
& \times\frac{1}{\sqrt{T-u}}g_C(T-u,\theta_{u,T}(y)-z)dudz.
\end{aligned}$$

Now, we can bound the convolution of Gaussian densities:

$$\int_{D_2}g_C(u-t,\theta_{t,u}(z)-x)g_C(T-u,\theta_{u,T}(y)-z)dz\leq Cg_C(T-t,\theta_{t,T}(y)-x),$$

and the time integral increase the Beta function by the right amount. For $z \in D_1$, observe that we have:

$$\Lambda^N_{u-t}(z) = \Lambda^N_{u-t}(z - \theta_{t,T}(y) + \theta_{t,T}(y)) \leq C_T\Big(\Lambda^N_{u-t}\big(z - \theta_{t,T}(y)\big) + \Lambda^N_{T-t}(y)\Big) \leq C_T \Lambda^N_{u-t}\big(z - \theta_{t,T}(y)\big).$$

Therefore:

$$\begin{aligned}|I_{|D_1}| \leq C a_N \sqrt{\gamma^N_0} \prod_{r=1}^k B\left(\frac{r}{2}, \frac{1}{2}\right) \int_t^T (u-t)^{\frac{k-1}{2}} \int_{\mathbb{R}} g_C(u-t, \theta_{t,u}(z) - x) \\ \times \frac{\Lambda^N_{u-t}\big(z - \theta_{t,T}(y)\big)}{\sqrt{T-u}} g_C(T-u, \theta_{u,T}(y) - z) du dz.\end{aligned}$$

The important thing to notice here is that up to a reversal of the flow, the same argument appears both in the Gaussian density and in $\Lambda^N_{u-t}\big(z - \theta_{t,T}(y)\big)$. We now claim that the following estimate holds for γ^N_0 small enough:

$$\int_{\mathbb{R}} g_C(u-t, \theta_{t,u}(z) - x) \Lambda^N_{u-t}\big(z - \theta_{t,T}(y)\big) g_C(T-u, \theta_{u,T}(y) - z) dz \leq g_{C'}(T-t, \theta_{t,T}(y) - x). \tag{3.27}$$

Indeed, observe that

$$\begin{aligned} & e^{C(u-t)a_N^2\gamma_0^N|z-\theta_{u,T}(y)|^2} g_C(T-u, \theta_{u,T}(y) - z) \\ = \; & e^{C(u-t)a_N^2\gamma_0^N|z-\theta_{u,T}(y)|^2} \frac{1}{C\sqrt{2\pi(T-u)}} \exp\left(-\frac{|z - \theta_{u,T}(y)|^2}{C(T-u)}\right) \\ = \; & \frac{1}{C\sqrt{2\pi(T-u)}} \exp\left(-|z-\theta_{u,T}(y)|^2\left(\frac{1}{C(T-u)} - C(u-t)a_N^2\gamma_0^N\right)\right) \\ \leq \; & C g_{C'}(T-u, \theta_{u,T}(y) - z), \end{aligned}$$

since we assumed $a_N\sqrt{\gamma^N_0} \to 0$ when N goes to infinity. Besides, for $m = 1, 2, 3$ we can bound up to a modification of C:

$$|z - \theta_{u,T}(y)|^m g_C(T-u, \theta_{u,T}(y) - z) \leq (T-u)^{m/2} g_C(T-u, \theta_{u,T}(y) - z) \leq C_T g_C(T-u, \theta_{u,T}(y) - z).$$

Thus, the claim (3.27) follows from the convolution of Gaussian densities, and again, the integral in time increase the Beta function by the right amount. Thus, we have the required estimate on I. We now turn to $I\!I$ above. Using the induction hypothesis for $k = 1$ and the estimate on $H^{(k)}_N$, we get:

$$\begin{aligned} I\!I = C_T \int_t^T du \int_{\mathbb{R}} dz (u-t)^{\frac{k}{2}-1} \prod_{r=1}^k B\left(\frac{r}{2}, \frac{1}{2}\right) g_C(u-t, \theta^N_{t,u}(z) - x) \\ \times a_N \sqrt{\gamma^N_0} \Lambda^N_{T-u}(y) g_C(T-u, \theta_{u,T}(y) - z). \end{aligned}$$

In this case, we claim that:

$$\int_{\mathbb{R}} dz g_C(u-t, \theta^N_{t,u}(z) - x) g_C(T-u, \theta_{u,T}(y) - z) \leq e^{C(T-t)a_N^2\gamma_0^N|y|^2} g_C(T-t, \theta_{t,T}(y) - x) \tag{3.28}$$

which readily gives the announced estimate, up to a modification of the constant C. To prove the above inequality, we bound using (3.17):

$$g_C(T-u, \theta_{u,T}(y) - z) \leq 2e^{C(T-t)a_N^2\gamma_0^N|y|^2} g_{\frac{C}{2}}(T-u, \theta^N_{u,T}(y) - z).$$

Now, we use the fact that Gaussian densities are closed under convolution, i.e. the following holds:

$$\int_{\mathbb{R}} dz g_C(u-t, \theta^N_{t,u}(z) - x) g_C(T-u, \theta^N_{u,T}(y) - z) \leq C g_{C'}(T-t, \theta^N_{t,T}(y) - x).$$

We thus get:

$$\int_{\mathbb{R}} dz g_C(u-t, \theta^N_{t,u}(z) - x) g_C(T-u, \theta_{u,T}(y) - z) \leq e^{C(T-t)a_N^2\gamma_0^N|y|^2} g_C(T-t, \theta^N_{t,T}(y) - x).$$

We conclude using once again (3.17) to replace $\theta^N_{t,T}(y)$ by $\theta_{t,T}(y)$ in the above inequality. Finally, to get estimate (3.20) and complete the proof of Lemma 3.5, it remains to sum (3.26) for $k \geq 1$.

□

To complete the proof of Lemma 3.2, we recall that

$$(p-q_N)(t,T,x,y) = \big(\tilde p-\tilde q_N\big)(t,T,x,y) + \sum_{r=1}^{+\infty}\Big(\tilde p\otimes H^{(r)} - \tilde q_N\otimes H_N^{(r)}\Big)(t,T,x,y),$$

thus, we can write:

$$\begin{aligned}\big|p-q_N\big|(t,T,x,y) &\le \big|\tilde p-\tilde q_N\big|(t,T,x,y) + \tilde p\otimes\big|\Phi-\Phi_N\big|(t,T,x,y)\\ &\quad +\big|\tilde p-\tilde q_N\big|\otimes|\Phi_N|(t,T,x,y)\\ &=: I + \mathit{II} + \mathit{III}.\end{aligned}$$

Now, the contribution I above clearly yields the announced estimate (see Lemma 3.3 above). For the contribution III, we use Lemma 3.3 to estimate the difference of the frozen densities and Lemma 3.5, to control Φ_N. As discussed above, we bound:

$$|z|e^{C(u-t)a_N^2\gamma_0^N|z|^2} \le C(|\theta^N_{u,T}(y)-z|+|y|)e^{C(T-t)a_N^2\gamma_0^N|\theta^N_{u,T}(y)-z|^2}e^{C(T-t)a_N^2\gamma_0^N|y|^2},$$

and use (3.17), and (3.27) to derive:

$$\begin{aligned}&\big|\tilde p-\tilde q_N\big|\otimes|\Phi_N|(t,T,x,y)\\ &\le C\int_t^T\int_{\mathbb R} a_N\sqrt{\gamma_0^N}\sqrt{(u-t)}|z|e^{C(u-t)a_N^2\gamma_0^N|z|^2}g_C(u-t,\theta_{t,u}(z)-x)\\ &\qquad\times\frac{1}{\sqrt{T-u}}g_C(T-u,\theta^N_{u,T}(y)-z)dudz\\ &\le Ca_N\sqrt{\gamma_0^N}e^{C(T-t)a_N^2\gamma_0^N|y|^2}\int_t^T\sqrt{u-t}\int_{\mathbb R}|\theta^N_{u,T}(y)-z|e^{C(T-t)a_N^2\gamma_0^N|\theta_{u,T}(y)-z|^2}\\ &\qquad\times g_C(u-t,\theta_{t,u}(z)-x)\frac{1}{\sqrt{T-u}}g_C(T-u,\theta^N_{u,T}(y)-z)dudz\\ &+Ca_N\sqrt{\gamma_0^N}|y|e^{C(T-t)a_N^2\gamma_0^N|y|^2}\int_t^T\sqrt{u-t}\int_{\mathbb R}e^{C(T-t)a_N^2\gamma_0^N|\theta_{u,T}(y)-z|^2}g_C(u-t,\theta_{t,u}(z)-x)\\ &\qquad\times\frac{1}{\sqrt{T-u}}g_C(T-u,\theta^N_{u,T}(y)-z)dudz\end{aligned}$$

Finally, using (3.28), we get:

$$\begin{aligned}&\big|\tilde p-\tilde q_N\big|\otimes|\Phi_N|(t,T,x,y)\\ &\le Ca_N\sqrt{\gamma_0^N}(1+|y|)e^{C(T-t)a_N^2\gamma_0^N|y|^2}(T-t)B\left(\frac32,\frac12\right)g_C(T-t,\theta_{t,T}(y)-x)\\ &\le Ca_N\sqrt{\gamma_0^N}\Lambda^N_{T-t}(y)(T-t)B\left(\frac32,\frac12\right)g_C(T-t,\theta_{t,T}(y)-x).\end{aligned}$$

Finally, for the contribution II above, we write using equation (3.20):

$$\begin{aligned}&\tilde p\otimes\big|\Phi-\Phi_N\big|(t,T,x,y)\\ &\le C\int_t^T\int_{\mathbb R}g_\sigma(u-t,\theta_{t,u}(z)-x)a_N\sqrt{\gamma_0^N}\Lambda^N_{T-u}(y)g_C(T-u,\theta_{u,T}(y)-z)dudz,\end{aligned}$$

which directly gives the announced control (see estimate (3.27)). Piecing estimates for I, II and III together yields the announced estimates. □

Proof of Lemma 3.4. We now turn to estimating $|\theta_{t,T}(y)-\theta^N_{t,T}(y)|$. To simplify the notations, we set $z_t^N = \theta^N_{t,T}(y)$, and $z_t=\theta_{t,T}(y)$. Observe first that the following estimate holds:

$$|z_t^N|\le C_T|y|.$$

Indeed, we have the following inequality:

$$|z_t^N| \leq |y| + \int_t^T \left|F_N(t, z_u^N)\right||z_u^N|du.$$

Now, since for all $x \in \mathbb{R}$, the quantity $|F_N(t,x)|$ is bounded, using Gronwall's Lemma, we readily derive:

$$|z_t^N| \leq C|y|e^{K(T-t)} = C_T|y|.$$

Note that we have the representations:

$$\begin{aligned} z_t^N &= y - \int_t^T \left(-\sigma m'(\bar{\theta}_u) + \frac{1}{2}\right) z_u^N du - \int_t^T \left(F_N(u, z_u^N) - \left(-\sigma m'(\bar{\theta}_u) + \frac{1}{2}\right)\right) z_u^N du, \\ z_t &= y - \int_t^T \left(-\sigma m'(\bar{\theta}_u) + \frac{1}{2}\right) z_u du. \end{aligned}$$

Thus, taking the difference yields:

$$z_t^N - z_t = -\int_t^T \left(-\sigma m'(\bar{\theta}_u) + \frac{1}{2}\right) \left(z_u^N - z_u\right) du - \int_t^T \left(F_N(u, z_u^N) - \left(-\sigma m'(\bar{\theta}_u) + \frac{1}{2}\right)\right) z_u^N du. \tag{3.29}$$

Observe now that for all $x \in \mathbb{R}$, we can bound

$$\left|\left(F_N(u,x) - \left(-\sigma m'(\bar{\theta}_u) + \frac{1}{2}\right)\right)\right| \leq C(|x| \wedge a_N)\sqrt{\gamma_0^N} \leq Ca_N\sqrt{\gamma_0^N}. \tag{3.30}$$

Indeed, we write for all $x \in \mathbb{R}$:

$$\begin{aligned} & F_N(u,x) - \left(-\sigma m'(\bar{\theta}_u) + \frac{1}{2}\right) \\ = & \left(\alpha_u^N - \sigma\sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}} \int_0^1 m'\left(\bar{\theta}_u + \delta sign(x)\Big(|x| \wedge a_N\Big)\sqrt{\gamma_k^N}\right) d\delta\right) - \left(-\sigma m'(\bar{\theta}_u) + \frac{1}{2}\right) \\ = & -\sigma\sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}} \left(\int_0^1 m'\left(\bar{\theta}_u + \delta sign(x)\Big(|x| \wedge a_N\Big)\sqrt{\gamma_k^N}\right) - m'(\bar{\theta}_u) d\delta\right) \\ & -\sigma\left(\sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}} - 1\right) m'(\bar{\theta}_u) + \left(\alpha_u^N - \frac{1}{2}\right). \end{aligned}$$

Since m' is Lipschitz, we have

$$|m'\left(\bar{\theta}_u + \delta sign(x)\Big(|x| \wedge a_N\Big)\sqrt{\gamma_k^N}\right) - m'(\bar{\theta}_u)| \leq C(|x| \wedge a_N)\sqrt{\gamma_k^N} \leq Ca_N\sqrt{\gamma_0^N}.$$

On the other hand, we recall:

$$\alpha_{t_k}^N = \frac{\sqrt{\gamma_k^N} - \sqrt{\gamma_{k+1}^N}}{(\gamma_{k+1}^N)^{3/2}} \underset{N \to +\infty}{\longrightarrow} \bar{\alpha} = \frac{1}{2},$$

thus, since m' is bounded, we have:

$$\begin{aligned} & \left|-\sigma\left(\sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}} - 1\right) m'(\bar{\theta}_u) + \left(\alpha_u^N - \frac{1}{2}\right)\right| \\ \leq & \left|-\sigma\left(\sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}} - 1\right) m'(\bar{\theta}_u)\right| + \left|\frac{\sqrt{\gamma_k^N} - \sqrt{\gamma_{k+1}^N}}{(\gamma_{k+1}^N)^{3/2}} - \frac{1}{2}\right| \\ \leq & C\left|\sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}} - 1\right| + \left|\frac{\sqrt{\gamma_k^N} - \sqrt{\gamma_{k+1}^N}}{(\gamma_{k+1}^N)^{3/2}} - \frac{1}{2}\right|. \end{aligned}$$

Thus, expanding $\sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}}$ and $\frac{\sqrt{\gamma_k^N}-\sqrt{\gamma_{k+1}^N}}{(\gamma_{k+1}^N)^{3/2}}$ we derive that:

$$\left|-\sigma\left(\sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}}-1\right)m'(\bar{\theta}_u)+\left(\alpha_u^N-\frac{1}{2}\right)\right|\le C\sqrt{\gamma_0^N}.$$

Consequently, we have the upper bound:

$$\left|F_N(t,x)-\left(-\sigma m'(\bar{\theta}_u)+\frac{1}{2}\right)\right|\le Ca_N\sqrt{\gamma_0^N}.$$

Plugging this inequality in (3.29) yields:

$$\begin{aligned}|z_t^N-z_t| &\le \int_t^T\left|-\sigma m'(\bar{\theta}_u)+\frac{1}{2}\right|\left|z_u^N-z_u\right|du\\ &\quad+\int_t^T\left|F_N(t,z_u^N)-\left(-\sigma m'(\bar{\theta}_u)+\frac{1}{2}\right)\right||z_u^N|du\\ &\le \int_t^T\left|-\sigma m'(\bar{\theta}_u)+\frac{1}{2}\right|\left|z_u^N-z_u\right|du+C\int_t^T a_N\sqrt{\gamma_0^N}|z_u^N|du\\ &\le \int_t^T\left|-\sigma m'(\bar{\theta}_u)+\frac{1}{2}\right|\left|z_u^N-z_u\right|du+C(T-t)a_N\sqrt{\gamma_0^N}|y|,\end{aligned}$$

where to get the last inequality, we recall that for all $u\le T$, we have $|z_u^N|\le C_T|y|$. Therefore, applying Gronwall's Lemma yields:

$$|z_t^N-z_t|\le Ce^{C(T-t)}(T-t)a_N\sqrt{\gamma_0^N}|y|=C_T(T-t)a_N\sqrt{\gamma_0^N}|y|.$$

This completes the proof for the continuous objects. The discrete part is obtained as a corollary of the previous result, writing:

$$|\theta_{t_k^N,T}(y)-\hat{\theta}^N_{t_k^N,T}(y)|\le|\theta_{t_k^N,T}(y)-\theta^N_{t_k^N,T}(y)|+|\theta^N_{t_k^N,T}(y)-\hat{\theta}^N_{t_k^N,T}(y)|.$$

The first contribution is estimated using the proved part of Lemma 3.4. The second contribution is nothing but the classical error for the Backward Euler scheme, which is well known to be estimated as:

$$|\theta^N_{t_k^N,T}(y)-\hat{\theta}^N_{t_k^N,T}(y)|\le C_T\gamma_0^N|y|.$$

We get the conclusion by putting the worst estimate. Besides, the fact that $|\hat{\theta}^N_{t_k^N,T}(y)|\le C_T|y|$ can be obtained by induction, directly from the expression of $\hat{\theta}^N_{t_k^N,T}(y)$, since it satisfies the recursion:

$$\begin{cases}x^N_{t^N_{k+1}}=x^N_{t_k^N}+F(t^N_{k+1},x^N_{t^N_{k+1}})x^N_{t^N_{k+1}}\gamma^N_{k+1}.\\ x_T^N=y\end{cases}$$

□

Remark 3.4 (Importance of the cut-off in F_N)**.** We point out that in the above proof, we actually proved the following inequality $\forall x\in\mathbb{R}$:

$$\left|\left(F_N(t,x)-\left(-\sigma m'(\bar{\theta}_t)+\frac{1}{2}\right)\right)\right|\le C|x|\sqrt{\gamma_0^N}.$$

Without the cut-off, this would give the following bound $|z_t^N-z_t|\le C_T(T-t)\sqrt{\gamma_0^N}|y|^2$, which cannot be handled with the arguments developed here.

3.2 The Local Limit Theorem for the Markov Chain

In this section, we deal with the discretization of the diffusion introduced above. The Markov chains we define below are defined for times on the grid $\{t_k^N,k\in[\![1,M(N)]\!]\}$. However, we fixed a time horizon $T>0$ above and there is no reason why this time is on that grid. Nevertheless, we keep the notation T where we actually mean T_N, the closest point of the grid on the left of T, to shorten the already long notations.

We investigate the distance between the transition densities of the Markov chain $(V^N_{t^N_k})_{t^N_k\le T}$ and the diffusion $(X^N_t)_{t\le T}$. The analysis follows the arguments presented in Konakov and Mammen [3], but in the case of an unbounded drift coefficient. The strategy of the proof is the same, but due to the unbounded nature of the drift, some adjustments are necessary. We detail those below.

Lemma 3.7. *Fix a time horizon $T>0$. There exists a positive integer $S>0$, such that for all $t^N_k\le T$, for all $(x,y)\in K_x\times K_y$, with K_x,K_y compact sets, and a constant $C>0$ depending on the compact sets such that:*

$$\left|(q_N-p_N)(t^N_k,T,x,y)\right|\le Ca_N\sqrt{\gamma^N_0}\frac{(T-t^N_k)^{-\frac12}}{\left(1+\frac{|\theta_{t^N_k,T}(y)-x|}{\sqrt{T-t^N_k}}\right)^{S-7}}.$$

The proof of this result is globally the same as for the diffusive part, in that we compare the two parametrix series. Just like for the diffusive case, to handle the unbounded drift, we freeze the coefficients along the solutions of the ODE associated. We keep the notations of the last paragraph. To approximate

$$dX^N_t=F_N(t,X^N_t)X^N_t+\sigma dW_t,$$

we considered the following frozen processes:

$$\tilde X^{T,y,N}_s=x+\int_t^s F_N\left(u,\theta^N_{u,T}(y)\right)\theta^N_{u,T}(y)du+\sigma W_{s-t},$$

and we once again have the identity:

$$\tilde q^{T,y}_N(t,T,x,y)=g_\sigma(T-t,\theta^N_{t,T}(y)-x).$$

Similarly, to approximate the Markov chain:

$$V^N_{t^N_{k+1}}=V^N_{t^N_k}+F_N(t^N_k,V^N_{t^N_k})V^N_{t^N_k}\gamma^N_{k+1}+\sqrt{\gamma^N_{k+1}}\xi_{k+1},$$

we consider $\hat\theta^N_{t_k,T}(y)$ the backward Euler scheme for the ODE:

$$\begin{aligned}x^N_{t^N_{k+1}}&=x^N_{t^N_k}+F_N(t^N_{k+1},x^N_{t^N_{k+1}})x^N_{t^N_{k+1}}\gamma^N_{k+1},\\ x^N_T&=y,\end{aligned}$$

and define the frozen Markov chain:

$$\tilde V^{T,y,N}_{t^N_{k+1}}=\tilde V^{T,y,N}_{t^N_k}+F_N(t^N_{k+1},\hat\theta^N_{t^N_{k+1},T}(y))\hat\theta^N_{t^N_{k+1},T}(y)\gamma^N_{k+1}+\sqrt{\gamma^N_{k+1}}\xi_{k+1}.$$

This Markov chain admits a transition density:

$$\frac{d}{dz}\mathbb P(\tilde V^{T,y,N}_{t^N_j}\in dz|\tilde V^{T,y,N}_{t^N_k}=x)=\tilde p^{T,y}_N(t^N_k,t^N_j,x,z).$$

Observe that we can rewrite the frozen Markov chain:

$$\tilde V^{T,y}_{t^N_j}=x+\hat\theta_{t^N_j,T}(y)-\hat\theta^N_{t^N_k,T}(y)+\sum_{i=k}^{j-1}\sqrt{\gamma^N_{i+1}}\xi_{i+1},\ \tilde V^{T,y,N}_{t^N_k}=x,$$

consequently, once again, we have for $t^N_j=T$:

$$\tilde p^{T,y}_N(t_k,T,x,y)=p_{S_N}\left(\hat\theta^N_{t_k,T}(y)-x\right),$$

if p_{S_N} denotes the density of the sum *i.i.d.* variables $\sum_{i=k}^{j-1}\sqrt{\gamma^N_{i+1}}\xi_{i+1}$. Defining now the one step generators, for a test function φ:

$$\begin{aligned}\mathcal L_N\varphi(t^N_i,t^N_j,x,y)&=\frac{1}{t^N_{i+1}-t^N_i}\int_{\mathbb R}\Big(\varphi(t^N_{i+1},t^N_j,z,y)-\varphi(t^N_{i+1},t^N_j,x,y)\Big)p_N(t^N_i,t^N_{i+1},x,z)dz\\&=\frac{1}{t^N_{i+1}-t^N_i}\mathbb E\Big(\varphi(t^N_{i+1},t^N_j,V^N_{t^N_{i+1}},y)-\varphi(t^N_{i+1},t^N_j,V^N_{t^N_i},y)\Big|V^N_{t^N_i}=x\Big),\\ \tilde{\mathcal L}_N\varphi(t^N_i,t^N_j,x,y)&=\frac{1}{t^N_{i+1}-t^N_i}\int_{\mathbb R}\Big(\varphi(t^N_{i+1},t^N_j,z,y)-\varphi(t^N_{i+1},t^N_j,x,y)\Big)\tilde p^{T,y}_N(t^N_i,t^N_{i+1},x,z)dz\\&=\frac{1}{t^N_{i+1}-t^N_i}\mathbb E\Big(\varphi(t^N_{i+1},t^N_j,\tilde V^{T,y,N}_{t^N_{i+1}},y)-\varphi(t^N_{i+1},t^N_j,\tilde V^{T,y,N}_{t^N_i},y)\Big|\tilde V^{T,y,N}_{t^N_i}=x\Big).\end{aligned}$$

Using the Markov property repeatedly, we obtain the following representation for the density of the Markov Chain $(V^N_{t_k})_{t_k\le T}$:

$$p_N(t_k,T,x,y)=\sum_{r=0}^{N}\tilde p_N\otimes_N\mathcal{K}_N^{[r]}(t_k,T,x,y),$$

where we denote $\otimes_N$ the discretized time-space convolution:

$$f\otimes_N g(t_i^N,t_j^N,x,y)=\sum_{k=i}^{j-1}\gamma_{k+1}^N\int_{\mathbb{R}}dzf(t_i^N,t_k^N,x,z)g(t_k^N,t_j^N,z,y).$$

We recall that $\gamma^N_{k+1}=t^N_{k+1}-t^N_k$, so that the first sum can be seen as a discretized time integral. Also, we denoted $\mathcal{K}_N^{[r]}=K_N^{[r-1]}\otimes_N\mathcal{K}_N$, $\mathcal{K}_N^{[0]}=Id$, and

$$\mathcal{K}_N(t,T,x,y)=\Big(\mathcal{L}_N-\tilde{\mathcal{L}}_N\Big)\tilde p_N^{T,y}(t,T,x,y).$$

From the results of the last section, we know that the parametrix series representation holds for the diffusion and so to estimate $|q_N-p_N|$, we write:

$$(q_N-p_N)(t_k,T,x,y)=\sum_{r=0}^{+\infty}\tilde q_N\otimes H_N^{(r)}(t_k,T,x,y)-\sum_{r=0}^{N}\tilde p_N\otimes_N\mathcal{K}_N^{[r]}(t_k,T,x,y).$$

Now, we somehow have to compare the two expansions. To that end, we introduce a series of intermediate steps, summarized in the flowchart below.

$$q_N(t_k,T,x,y)=\sum_{r=0}^{+\infty}\tilde q_N\otimes H_N^{(r)}(t_k,T,x,y)$$

Step 1: $\downarrow$ Replace $\otimes$ with $\otimes_N$: discretisation of the time integral

$$\sum_{r=0}^{+\infty}\tilde q_N\otimes_N H_N^{[r]}(t_k,T,x,y)$$

Step 2: $\downarrow$ Truncate the rest of the series: $\sum_{r\ge N}^{+\infty}|\tilde q_N\otimes H_N^{(r)}|\le O(\gamma_0^N)g_C$

$$\sum_{r=0}^{N}\tilde q_N\otimes_N H_N^{[r]}(t_k,T,x,y)$$

Step 3: $\downarrow$ Replace H_N with K_N+M_N

$$\sum_{r=0}^{N}\tilde q_N\otimes_N(K_N+M_N)^{[r]}(t_k,T,x,y)$$

Step 4: $\downarrow$ Replace $\tilde q_N$ with $\tilde p_N$: control via classical Edgeworth expansions

$$\sum_{r=0}^{N}\tilde p_N\otimes_N(K_N+M_N)^{[r]}(t_k,T,x,y)$$

Step 5: $\downarrow$ Replace K_N+M_N with $\mathcal{K}_N$

$$\sum_{r=0}^{N}\tilde p_N\otimes_N\mathcal{K}_N^{[r]}(t_k,T,x,y)=p_N(t_k,T,x,y)$$

Where we defined $K_N(t_k,T,x,y)=(L_{t_k}-\tilde L_{t_k})\tilde p_N(t_k,T,x,y)$, and M_N is a remainder (see (3.32)).

Step one deals with the discretisation of the time integral and is proved exactly as in Konakov and Mammen [3]. Thus we omit this step here. Step two deals with the rest of the convergent parametrix series, and can be obtained as a corollary of the results of the last section. Step 4 is done using the classical Edgeworth expansions, and is a consequence of Theorem 19.3 in Battacharaya and Rao [2], with the slight modification that the arguments in the densities involve the transport part (see Lemma 3.13 below). Finally, Steps 3 and 5 are very similar and are dealt with Lemma 3.8 below. Note that the proof of Step 5 is trivial once the controls of Lemma 3.8 are obtained. There are two main difficulties for these steps. The first one is to understand how the one-step generators $\mathcal{L}_N$ and $\tilde{\mathcal{L}}_N$ relate to the generators of the diffusions L_t and $\tilde L_t$. The second difficulty is to control the iterated convolutions.

3.2.1 The discrete generators

Before we dive into the proof of steps 3 to 5, we take a few lines here to understand the discrete generators. To compute the one-step generator, we need an expression for the transition density of the Markov chain $(V_{t_k}^N)_{k\geq 1}$ for one step. Using the recurrence relation:

$$\begin{aligned}\mathbb{P}\Big(V_{t_{k+1}^N}^N \leq y | V_{t_k^N}^N = x\Big) &= \mathbb{P}\Big(V_{t_k^N}^N + F_N(t_k^N, V_{t_k^N}^N)V_{t_k^N}^N\gamma_{k+1}^N + \sqrt{\gamma_{k+1}^N}\xi_{k+1} \leq y | V_{t_k^N}^N = x\Big) \\ &= \mathbb{P}\Big(\xi_{k+1} \leq \frac{y-(x+F_N(t_k^N,x)x\gamma_{k+1}^N)}{\sqrt{\gamma_{k+1}^N}}\Big).\end{aligned}$$

Consequently, we see that:

$$p_N(t_k^N, t_{k+1}^N, x, y) = \frac{1}{\sqrt{\gamma_{k+1}^N}} p_\xi\left(\frac{y-(x+F_N(t_k^N,x)x\gamma_{k+1}^N)}{\sqrt{\gamma_{k+1}^N}}\right). \tag{3.31}$$

Note that we can do the same computations $(\tilde{V}_{t_k^N}^{T,y,N})_{t_k^N\leq T}$, for the frozen Markov chain, to get the expression:

$$\tilde{p}_N(t_k^N, t_{k+1}^N, x, y) = \frac{1}{\sqrt{\gamma_{k+1}^N}} p_\xi\left(\frac{y-(x+F_N\big(t_{k+1}^N, \hat{\theta}_{t_{k+1}^N,T}^N(y)\big)\hat{\theta}_{t_{k+1}^N,T}^N(y)\gamma_{k+1}^N)}{\sqrt{\gamma_{k+1}^N}}\right).$$

Next, when taking the difference $\mathcal{L}_N - \tilde{\mathcal{L}}_N$, one can notice that we can rewrite:

$$\begin{aligned}&(\mathcal{L}_N - \tilde{\mathcal{L}}_N)\varphi(t_k^N, t_l^N, x, y) \\ =\ & \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}}\Big(\varphi(t_{k+1}^N, t_l^N, z, y) - \varphi(t_{k+1}^N, t_l^N, x, y)\Big)p_N(t_k^N, t_{k+1}^N, x, z)dz \\ & - \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}}\Big(\varphi(t_{k+1}^N, t_l^N, z, y) - \varphi(t_{k+1}^N, t_l^N, x, y)\Big)\tilde{p}_N(t_k^N, t_{k+1}^N, x, z)dz \\ :=\ & \Big((\mathcal{L}_N - \tilde{\mathcal{L}}_N)_1 - (\mathcal{L}_N - \tilde{\mathcal{L}}_N)_2\Big)\varphi(t_k^N, t_l^N, x, y).\end{aligned}$$

That way, the arguments in the test function φ only change on the first spacial component, and we can expand $\varphi(t_{k+1}^N, t_l^N, z, y) - \varphi(t_{k+1}^N, t_l^N, x, y)$ using Taylor's Formula.

$$\begin{aligned}\varphi(t_{k+1}^N, t_l^N, z, y) - \varphi(t_{k+1}^N, t_l^N, x, y) = \partial_x\varphi(t_{k+1}^N, t_l^N, x, y)(z-x) \\ +\frac{1}{2}\partial_x^2\varphi(t_{k+1}^N, t_l^N, x, y)(z-x)^2 + \int_0^1 \partial_x^3\varphi(t_{k+1}^N, t_l^N, x+\lambda(z-x), y)(z-x)^3\frac{(1-\lambda)^2}{2}d\lambda.\end{aligned}$$

Now, for $(\mathcal{L}_N - \tilde{\mathcal{L}}_N)_1\varphi(t_k^N, t_l^N, x, y)$, we integrate the expansion against $p_N(t_k^N, t_{k+1}^N, x, z)$ to get:

$$\begin{aligned}&(\mathcal{L}_N - \tilde{\mathcal{L}}_N)_1\varphi(t_k^N, t_l^N, x, y) \\ =\ & \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}}\Big(\varphi(t_{k+1}^N, t_l^N, z, y) - \varphi(t_{k+1}^N, t_l^N, x, y)\Big)p_N(t_k^N, t_{k+1}^N, x, z)dz \\ =\ & \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}}\partial_x\varphi(t_{k+1}^N, t_l^N, x, y)(z-x)p_N(t_k^N, t_{k+1}^N, x, z)dz \\ & + \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}}\frac{1}{2}\partial_x^2\varphi(t_{k+1}^N, t_l^N, x, y)(z-x)^2 p_N(t_k^N, t_{k+1}^N, x, z)dz \\ & + \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}}\int_0^1 \partial_x^3\varphi(t_{k+1}^N, t_l^N, x+\lambda(z-x), y)(z-x)^3\frac{(1-\lambda)^2}{2}p_N(t_k^N, t_{k+1}^N, x, z)dzd\lambda\end{aligned}$$

Let us deal with the rest later. For the first two terms, the idea is to relate it to the moments of the innovations

ξ, using in particular (3.31). We have:

$$\begin{aligned}
& \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}} \partial_x\varphi(t_{k+1}^N, t_l^N, x, y)(z-x)p_N(t_k^N, t_{k+1}^N, x, z)dz \\
= \ & \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}} \partial_x\varphi(t_{k+1}^N, t_l^N, x, y)(z-x)\frac{1}{\sqrt{\gamma_{k+1}^N}}p_\xi\left(\frac{z-(x+F_N(t_k^N, x)x\gamma_{k+1}^N)}{\sqrt{\gamma_{k+1}^N}}\right)dz \\
= \ & \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}} \partial_x\varphi(t_{k+1}^N, t_l^N, x, y)\Big(\sqrt{\gamma_{k+1}^N}\tilde z + F_N(t_k^N, x)x\gamma_{k+1}^N\Big)p_\xi(\tilde z)d\tilde z \\
= \ & \partial_x\varphi(t_{k+1}^N, t_l^N, x, y)\cdot F_N(t_k^N, x)x,
\end{aligned}$$

where we recall that the innovations ξ have zero mean. Similarly, for the second term, we get:

$$\begin{aligned}
& \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}} \frac{1}{2}\partial_x^2\varphi(t_{k+1}^N, t_l^N, x, y)(z-x)^2p_N(t_k^N, t_{i+1}^N, x, z)dz \\
= \ & \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}} \frac{1}{2}\partial_x^2\varphi(t_{k+1}^N, t_l^N, x, y)(z-x)^2\frac{1}{\sqrt{\gamma_{k+1}^N}}p_\xi\left(\frac{z-(x+F_N(t_k^N, x)x\gamma_{k+1}^N)}{\sqrt{\gamma_{k+1}^N}}\right)dz \\
= \ & \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}} \frac{1}{2}\partial_x^2\varphi(t_{k+1}^N, t_l^N, x, y)\Big(\sqrt{\gamma_{k+1}^N}\tilde z + F_N(t_k^N, x)x\gamma_{k+1}^N\Big)^2p_\xi(\tilde z)d\tilde z
\end{aligned}$$

Expanding the square and using the fact that the innovations have zero mean and variance σ, we get:

$$\begin{aligned}
& \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}} \frac{1}{2}\partial_x^2\varphi(t_{k+1}^N, t_l^N, x, y)(z-x)^2p_N(t_k^N, t_{k+1}^N, x, z)dz \\
= \ & \frac{\sigma^2}{2}\partial_x^2\varphi(t_{k+1}^N, t_l^N, x, y) + \gamma_{k+1}^N\frac{1}{2}\partial_x^2\varphi(t_{k+1}^N, t_l^N, x, y)\cdot(F_N(t_k^N, x)x)^2.
\end{aligned}$$

The important remark here is that we recover the infinitesimal generator L_t from the first two terms in the expansion of φ. Finally, for the remainder in the expansion of φ, we just change variables:

$$\begin{aligned}
& \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}}\int_0^1 \partial_x^3\varphi(t_{k+1}^N, t_l^N, x+\lambda(z-x), y)(z-x)^3\frac{(1-\lambda)^2}{2}d\lambda p_N(t_k^N, t_{k+1}^N, x, z)dz \\
= \ & \frac{1}{\gamma_{k+1}^N}\int_{\mathbb{R}}\int_0^1 \partial_x^3\varphi(t_{k+1}^N, t_l^N, x+\lambda(z-x), y)(z-x)^3\frac{(1-\lambda)^2}{2}d\lambda \\
& \times\frac{1}{\sqrt{\gamma_{k+1}^N}}p_\xi\left(\frac{z-(x+F_N(t_k^N, x)x\gamma_{k+1}^N)}{\sqrt{\gamma_{k+1}^N}}\right)dz \\
= \ & \int_{\mathbb{R}}\int_0^1 \partial_x^3\varphi\left(t_{k+1}^N, t_l^N, x+\lambda\Big(\sqrt{\gamma_{k+1}^N}\tilde z + F_N(t_k^N, x)x\gamma_{k+1}^N\Big), y\right)\left(\tilde z + F_N(t_k^N, x)x\sqrt{\gamma_{k+1}^N}\right)^3 \\
& \times\frac{(1-\lambda)^2}{2}d\lambda p_\xi(\tilde z)\sqrt{\gamma_{k+1}^N}d\tilde z
\end{aligned}$$

The same computations can be done on $(\mathcal{L}_N - \tilde{\mathcal{L}}_N)_2$, for which we have the frozen density $\tilde p_N$ instead of p_N. In this case, we use the fact that

$$\tilde p_N(t_k^N, t_{k+1}^N, x, z)dz = \frac{1}{\sqrt{\gamma_{k+1}^N}}p_\xi\left(\frac{z-(x+F_N(t_{k+1}^N, \hat\theta_{t_{k+1},T}^N(y))\hat\theta_{t_{k+1},T}^N(y)\gamma_{k+1}^N)}{\sqrt{\gamma_{k+1}^N}}\right).$$

Therefore we get:

$$\begin{aligned}
& (\mathcal{L}_N - \tilde{\mathcal{L}}_N)\varphi(t_k^N, t_l^N, x, y) \\
= \ & \Big(L_{t_k^N}^N - \tilde L_{t_{k+1}^N}^N\Big)\varphi(t_{k+1}^N, t_l^N, x, y) \\
& +\gamma_{k+1}^N\frac{1}{2}\partial_x^2\varphi(t_{k+1}^N, t_l^N, x, y)\cdot\Big((F_N(t_k^N, x)x)^2 - (F_N(t_{k+1}^N, \hat\theta_{t_{k+1}^N,T}^N(y))\hat\theta_{t_{k+1}^N,T}^N(y))^2\Big) \\
& +R_\varphi(t_k^N, t_l^N, x, y),
\end{aligned}$$

where we denote $R_\varphi(t_k^N, t_l^N, x, y)$ the difference of the remainders. Let us obtain a nicer expression for the remainder. For the sake of clarity, define for a while

$$\Psi(t,u,z) = \partial_x^3\varphi\left(t_{k+1}^N, t_l^N, x + \lambda\Big(\sqrt{\gamma_{k+1}^N} z + F_N(t,u)u\gamma_{k+1}^N\Big), y\right)\left(z + F_N(t,u)u\sqrt{\gamma_{k+1}^N}\right)^3.$$

We also drop the tilda in the integration variable, we have:

$$\begin{aligned}
R_\varphi(t_k^N, t_l^N, x, y) &= \int_{\mathbb{R}}\int_0^1 \Big(\Psi(t_k^N, x, z) - \Psi(t_{k+1}^N, \hat\theta^N_{t_{k+1}^N,T}(y), z)\Big)\frac{(1-\lambda)^2}{2} p_\xi(z)\sqrt{\gamma_{k+1}^N}\, dz d\lambda\\
&= \int_{\mathbb{R}}\int_0^1 \Big(\Psi(t_k^N, x, z) - \Psi(t_k^N, \hat\theta^N_{t_{k+1}^N,T}(y), z)\Big)\frac{(1-\lambda)^2}{2} p_\xi(z)\sqrt{\gamma_{k+1}^N}\, dz d\lambda\\
+ \int_{\mathbb{R}}\int_0^1 &\Big(\Psi(t_k^N, \hat\theta^N_{t_{k+1}^N,T}(y), z) - \Psi(t_{k+1}^N, \hat\theta^N_{t_{k+1}^N,T}(y), z)\Big)\frac{(1-\lambda)^2}{2} p_\xi(z)\sqrt{\gamma_{k+1}^N}\, dz d\lambda.
\end{aligned}$$

Thus, using an additional Taylor formula:

$$\begin{aligned}
&R_\varphi(t_k^N, t_l^N, x, y) =\\
&\sqrt{\gamma_{k+1}^N}\int_{\mathbb{R}}\int_0^1\int_0^1 \partial_u\Psi\Big(t_k^N, \mu x + (1-\mu)\hat\theta^N_{t_{k+1}^N,T}(y), z\Big)(x - \hat\theta^N_{t_{k+1}^N,T}(y))\\
&\times\frac{(1-\lambda)^2}{2} p_\xi(z) dz d\lambda d\mu\\
-(\gamma_{k+1}^N)^{\frac{3}{2}}&\int_{\mathbb{R}}\int_0^1\int_0^1 \partial_t\Psi\Big(\mu t_k^N + (1-\mu)t_{k+1}^N, \hat\theta^N_{t_{k+1}^N,T}(y), z\Big)\frac{(1-\lambda)^2}{2} p_\xi(z) dz d\lambda d\mu\\
&= I + II.
\end{aligned}$$

It remains us to compute $\partial_u\Psi$ and $\partial_t\Psi$:

$$\begin{aligned}
\partial_u\Psi &= \partial_u\left\{\partial_x^3\varphi\left(t_{k+1}^N, t_l^N, x + \lambda\Big(\sqrt{\gamma_{k+1}^N} z + F_N(t_k^N,u)u\gamma_{k+1}^N\Big), y\right)\left(z + F_N(t_k^N,u)u\sqrt{\gamma_{k+1}^N}\right)^3\right\}\\
&= \partial_x^4\varphi\left(t_{k+1}^N, t_l^N, x + \lambda\Big(\sqrt{\gamma_{k+1}^N} z + F_N(t_k^N,u)u\gamma_{k+1}^N\Big), y\right)\\
&\quad\times\lambda\gamma_{k+1}^N\Big(\partial_u F_N(t_k^N,u)u + F_N(t_k^N,u)\Big)\Big(z + F_N(t_k^N,u)u\sqrt{\gamma_{k+1}^N}\Big)^3\\
&\quad+\partial_x^3\varphi\left(t_{k+1}^N, t_l^N, x + \lambda\Big(\sqrt{\gamma_{k+1}^N} z + F_N(t_k^N,u)u\gamma_{k+1}^N\Big), y\right)\\
&\quad\times 3\Big(z + F_N(t_k^N,u)u\sqrt{\gamma_{k+1}^N}\Big)^2\sqrt{\gamma_{k+1}^N}\Big(\partial_u F_N(t_k^N,u)u + F_N(t_k^N,u)\Big).
\end{aligned}$$

We plug this expression into the one for R_φ. To lighten the notations, we denote for a while $\mu x + (1-\mu)\hat\theta^N_{t_{k+1}^N,T}(y) = \Theta$. We get:

$$\begin{aligned}
I &= \sqrt{\gamma_{k+1}^N}\int_{\mathbb{R}}\int_0^1\int_0^1\Bigg[\partial_x^4\varphi\left(t_{k+1}^N, t_l^N, x + \lambda\Big(\sqrt{\gamma_{k+1}^N} z + F_N(t_k^N,\Theta)\Theta\gamma_{k+1}^N\Big), y\right)\\
&\quad\times\lambda\gamma_{k+1}^N\Big(\partial_u F_N\Big(t_k^N,\Theta\Big)\Theta + F_N(t_k^N,\Theta)\Big)\Big(z + F_N\Big(t_k^N,\Theta\Big)\sqrt{\gamma_{k+1}^N}\Theta\Big)^3\\
&\quad+\partial_x^3\varphi\left(t_{k+1}^N, t_l^N, x + \lambda\Big(\sqrt{\gamma_{k+1}^N} z + F_N\Big(t_k^N,\Theta\Big)\Theta\gamma_{k+1}^N\Big), y\right)\\
&\quad\times 3\left(z + F_N\Big(t_k^N,\Theta\Big)\Theta\sqrt{\gamma_{k+1}^N}\right)^2\sqrt{\gamma_{k+1}^N}\times\Big(\partial_u F_N\Big(t_k^N,\Theta\Big)\Theta + F_N\Big(t_k^N,\Theta\Big)\Big)\Bigg]\\
&\quad\times\frac{(1-\lambda)^2}{2}\Big(x - \hat\theta^N_{t_{k+1}^N,T}(y)\Big)p_\xi(z) dz d\lambda d\mu.
\end{aligned}$$

We now turn to $I\!I$ above. We have for $\partial_t \Psi$:

$$\begin{aligned}
\partial_t \Psi &= \partial_x^4 \varphi\left(t_{k+1}^N, t_l^N, x+\lambda\Big(\sqrt{\gamma_{k+1}^N} z + F_N(t,u)u\gamma_{k+1}^N\Big), y\right) \\
&\quad \times \partial_t F_N(t,u)u\gamma_{k+1}^N\Big(z+F_N(t,u)u\sqrt{\gamma_{k+1}^N}\Big)^3 \\
&\quad +3\partial_x^3 \varphi\left(t_{k+1}^N, t_l^N, x+\lambda\Big(\sqrt{\gamma_{k+1}^N} z + F_N(t,u)u\gamma_{k+1}^N\Big), y\right) \\
&\quad \times\Big(z+F_N(t,u)u\sqrt{\gamma_{k+1}^N}\Big)^2 \partial_t F_N(t,u)u\sqrt{\gamma_{k+1}^N}.
\end{aligned}$$

Again, plugging this value in $I\!I$ the definition of the remainder above yields:

$$\begin{aligned}
I\!I &= -(\gamma_{k+1}^N)^{\frac{3}{2}} \int_{\mathbb{R}}\int_0^1\int_0^1 \frac{(1-\lambda)^2}{2} dz d\lambda d\mu p_\xi(z) \\
&\quad \times\Big[\partial_x^4 \varphi\left(t_{k+1}^N, t_l^N, x+\lambda\Big(\sqrt{\gamma_{k+1}^N} z + F_N(t,u)u\gamma_{k+1}^N\Big), y\right) \\
&\quad \times \partial_t F_N(t,u)u\gamma_{k+1}^N\Big(z+F_N(t,u)u\sqrt{\gamma_{k+1}^N}\Big)^3 \\
&\quad +3\partial_x^3 \varphi\left(t_{k+1}^N, t_l^N, x+\lambda\Big(\sqrt{\gamma_{k+1}^N} z + F_N(t,u)u\gamma_{k+1}^N\Big), y\right) \\
&\quad \times\Big(z+F_N(t,u)u\sqrt{\gamma_{k+1}^N}\Big)^2 \partial_t F_N(t,u)u\sqrt{\gamma_{k+1}^N}\Big],
\end{aligned}$$

with $t = \mu t_k^N + (1-\mu)t_{k+1}^N$ and $u = \hat{\theta}^N_{t_{k+1}^N,T}(y)$.

We keep this expression for the remainder. Observe that we use it substituting $\tilde{p}_N$, $\tilde{q}_N$ instead of φ. Obviously, a derivation yields a singularity, and the goal is to compensate the singularity. If we look closely at the expression above, we see that for the first contribution, a derivative of order 4 is taken. This leads to a singularity of order $(1/\gamma_{k+1}^N)^2$. One of those is compensated by the γ_{k+1}^N in the expression above, and we can also compensate another $\sqrt{\gamma_{k+1}^N}$ by writing $x - \hat{\theta}^N_{t_k,T}(y)$ at the appropriate time scale (see the arguments exposed in the proof of Lemma 3.5). Notice that the last compensation is precisely due to our choice to freeze the coefficients along the Euler Scheme. This finally leads to a singularity of order $1/\sqrt{\gamma_{k+1}^N}$, which is summable.

Besides, this expression for the difference of the discrete generators will allow us to give an explicit expression for the remainder M_N first introduced in Step 3 above. First, let us recall that:

$$\mathcal{K}_N(t_k^N, t_j^N, x, y) = \underbrace{\Big(\mathcal{L}_N - \tilde{\mathcal{L}}_N\Big)}_{\substack{\text{One-step} \\ \text{Generators}}} \underbrace{\tilde{p}_N(t_k^N, t_j^N, x, y),}_{\substack{\text{Frozen} \\ \text{Markov Chain}}}$$

and:

$$K_N(t_k^N, t_j^N, x, x) = \underbrace{\Big(L^N_{t_k^N} - \tilde{L}^N_{t_{k+1}^N}\Big)}_{\substack{\text{Generators} \\ \text{Diffusion}}} \underbrace{\tilde{p}_N(t_k^N, t_j^N, x, y).}_{\substack{\text{Frozen} \\ \text{Markov Chain}}}$$

Moreover, using the expression for the difference of the discrete generators $\mathcal{L}_N - \tilde{\mathcal{L}}_N$, we have:

$$\begin{aligned}
&\mathcal{K}_N(t_k^N, t_l^N, x, y) \\
=\ & K_N(t_{k+1}^N, t_l^N, x, y) \\
&+\gamma_{k+1}\frac{1}{2}\partial_x^2 \tilde{p}_N(t_{k+1}^N, t_l^N, x, y)\cdot\Big((F_N(t_k^N, x)x)^2 - (F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))\hat{\theta}^N_{t_{k+1}^N,T}(y))^2\Big) \\
&+R_{\tilde{p}_N}(t_k^N, t_l^N, x, y).
\end{aligned}$$

Therefore, we have the explicit expression for the remainder term introduced in step 3. Namely, we set:

$$\begin{aligned}
& M_N(t_k^N, t_l^N, x, y) \\
= \; & \gamma_{k+1}^N \frac{1}{2}\partial_x^2 \tilde p_N(t_{k+1}^N, t_l^N, x, y) \cdot \Big((F_N(t_k^N, x)x)^2 - (F_N(t_{k+1}^N, \hat\theta^N_{t_{k+1}^N,T}(y))\hat\theta^N_{t_{k+1}^N,T}(y))^2 \Big) \\
& + R_{\tilde p_N}(t_k^N, t_l^N, x, y). \qquad (3.32)
\end{aligned}$$

We now give some preliminary estimates on the kernels $\mathcal{K}_N, K_N$ and the remainder M_N. The following estimates holds.

Lemma 3.8. *For all $0 \le t_k^N \le T$, all $x, y \in \mathbb{R}$, there exists a constant $C > 0$ such that:*

$$\begin{aligned}
\Big(|\mathcal{K}_N| + |K_N|\Big)(t_k^N, T, x, y) &\le C \frac{1 + a_N\sqrt{\gamma_0^N}|y|^{S+2}}{\sqrt{T - t_k^N}} \frac{(T - t_k^N)^{-1/2}}{\left(1 + \frac{|x - \hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}}, \\
|M_N|(t_k^N, T, x, y) &\le \frac{C\sqrt{\gamma_0^N}}{\sqrt{T - t_k^N}} \frac{(T - t_k^N)^{-\frac12}}{\left(1 + \frac{|x - \hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}} \Big(1 + a_N\sqrt{\gamma_0^N}|y|^{S+2}\Big).
\end{aligned}$$

Besides, if $y \in K$ where K is a compact set, then, there exists C depending on K such that:

$$\begin{aligned}
\Big(|\mathcal{K}_N| + |K_N|\Big)(t_k^N, T, x, y) &\le C \frac{1}{\sqrt{T - t_k^N}} \frac{(T - t_k^N)^{-1/2}}{\left(1 + \frac{|x - \hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}}, \\
|M_N|(t_k^N, T, x, y) &\le \frac{C\sqrt{\gamma_0^N}}{\sqrt{T - t_k^N}} \frac{(T - t_k^N)^{-\frac12}}{\left(1 + \frac{|x - \hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}}.
\end{aligned}$$

Proof. The control on K_N is obtained in a similar fashion as in the previous section, only we replace $\tilde q_N$ with $\tilde p_N$, but the difference here is that the time parameter is not the same. Before diving in the proof of Lemma 3.8, we thus give an additional argument to deal with the different time parameters.

$$\Big(L^N_{t_k^N} - \tilde L^N_{t_{k+1}^N}\Big)\varphi(x) = \Big(L^N_{t_k^N} - \tilde L^N_{t_k^N}\Big)\varphi(x) + \Big(\tilde L^N_{t_k^N} - \tilde L^N_{t_{k+1}^N}\Big)\varphi(x).$$

For the first part, we can see that the singularity induced by the derivation is the same as in the previous section, but the argument of the density is $x - \hat\theta^N_{t_k^N,T}(y)$ instead of $x - \theta^N_{t_k^N,T}(y)$. The estimate is thus obtained using Lemma 3.4 above, specifically, the control of the cost of replacing $\hat\theta^N_{t_k^N,T}(y)$ with $\theta^N_{t_k^N,T}(y)$.

Turning to the second part, by definition, we have, for all test function φ:

$$\Big(\tilde L^N_{t_{k+1}^N} - \tilde L^N_{t_k^N}\Big)\varphi(x) = \Big[F_N(t_{k+1}^N, \hat\theta^N_{t_{k+1}^N,T}(y))\hat\theta^N_{t_{k+1}^N,T}(y) - F_N(t_k^N, \hat\theta^N_{t_k^N,T}(y))\hat\theta^N_{t_k^N,T}(y)\Big]\varphi'(x).$$

To estimate the pre-factor, we split as follows:

$$\begin{aligned}
& F_N(t_{k+1}^N, \hat\theta^N_{t_{k+1}^N,T}(y))\hat\theta^N_{t_{k+1}^N,T}(y) - F_N(t_k^N, \hat\theta^N_{t_k^N,T}(y))\hat\theta^N_{t_k^N,T}(y) \\
= \; & F_N(t_{k+1}^N, \hat\theta^N_{t_{k+1}^N,T}(y))\hat\theta^N_{t_{k+1}^N,T}(y) - F_N(t_{k+1}^N, \hat\theta^N_{t_k^N,T}(y))\hat\theta^N_{t_k^N,T}(y) \\
& + F_N(t_{k+1}^N, \hat\theta^N_{t_k^N,T}(y))\hat\theta^N_{t_k^N,T}(y) - F_N(t_k^N, \hat\theta^N_{t_k^N,T}(y))\hat\theta^N_{t_k^N,T}(y) = I + I\!I.
\end{aligned}$$

For I above, we use the uniform (in time) Lipschitz property of $x \mapsto F_N(t,x)x$ to write:

$$I \le C|\hat\theta^N_{t_{k+1}^N,T}(y) - \hat\theta^N_{t_k^N,T}(y)| \le C\gamma_{k+1}^N|y|,$$

by definition of the Euler scheme. For the second term, we write:

$$I\!I \le C\Big|F_N(t_{k+1}^N, \hat\theta^N_{t_k^N,T}(y)) - F_N(t_k^N, \hat\theta^N_{t_k^N,T}(y))\Big||y| \le C\gamma_{k+1}^N|y|.$$

Consequently, we have:

$$F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))\hat{\theta}^N_{t_{k+1}^N,T}(y) - F_N(t_k^N, \hat{\theta}^N_{t_k^N,T}(y))\hat{\theta}^N_{t_k^N,T}(y) \leq \gamma_{k+1}^N|y|, \tag{3.33}$$

Thus, the following estimate holds:

$$\Big(\tilde{L}^N_{t_{k+1}^N} - \tilde{L}^N_{t_k^N}\Big)\varphi(x) \leq C\gamma_{k+1}^N|y|\varphi'(x). \tag{3.34}$$

Now, replacing the test function with the corresponding density, we wee that the coefficient γ_{k+1}^N can be used to compensate the singularity induced by the derivation.

The control on $\mathcal{K}_N$ is deduced from the one of K_N and from a control of the remainder M_N, which is the main difficulty of this lemma. Recall M_N is made from two terms that we control separately. Set for a while

$$M_N^1(t_k^N, T, x, y) = \frac{\gamma_{k+1}^N}{2}\partial_x^2\tilde{p}_N(t_{k+1}^N, T, x, y)\cdot\Big((F_N(t_k^N, x)x)^2 - (F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))\hat{\theta}^N_{t_{k+1}^N,T}(y))^2\Big),$$

we claim that the following estimate holds:

$$M_N^1(t_k^N, T, x, y) \leq \frac{C\gamma_0^N}{\sqrt{T - t_{k+1}^N}}\frac{(T-t_k^N)^{-1/2}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N,T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-4}}(1 + |y|^2). \tag{3.35}$$

To get to this control, notice that we have a second derivative on the density. From the control of Battacharaya and Rao [2], we can deduce the following estimate:

$$\Big|\partial_x^2\tilde{p}_N(t_{k+1}^N, T, x, y)\Big| \leq \frac{C}{T - t_k^N}\frac{(T-t_k^N)^{-1/2}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_{k+1}^N,T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-2}}.$$

See also estimate (3.15) in Konakov Mammen [3] for additional details. We now turn to the control of the multiplier term $(F_N(t_k^N, x)x)^2 - (F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))\hat{\theta}^N_{t_{k+1}^N,T}(y))^2$. Notice that we can write:

$$\begin{aligned}
&(F_N(t_k^N, x)x)^2 - (F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))\hat{\theta}^N_{t_{k+1}^N,T}(y))^2 \\
= \;& \Big(F_N(t_k^N, x)x + F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))\hat{\theta}^N_{t_{k+1}^N,T}(y)\Big) \\
&\times\Big(F_N(t_k^N, x)x - F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))\hat{\theta}^N_{t_{k+1}^N,T}(y)\Big).
\end{aligned}$$

Now, from the controls of the previous section, specifically, the fact that $x \mapsto F_N(t,x)x$ is Lipschitz, and estimate (3.33) above, we have:

$$|F_N(t_k^N, x)x - F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))\hat{\theta}^N_{t_{k+1}^N,T}(y)| \quad\leq\quad C\Big(|x - \hat{\theta}^N_{t_{k+1}^N,T}(y)| + \gamma_{k+1}^N|y|\Big).$$

Now, for the second factor, we have:

$$\begin{aligned}
&|F_N(t_k^N, x)x + F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))\hat{\theta}^N_{t_{k+1}^N,T}(y)| \\
\leq\;& |F_N(t_k^N, x)(x - \hat{\theta}^N_{t_k,T}(y))| + |F_N(t_k^N, x)\hat{\theta}^N_{t_k,T}(y)| + |F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))\hat{\theta}^N_{t_{k+1}^N,T}(y)|.
\end{aligned}$$

Recall that we assumed F_N to be bounded, and bounding $\hat{\theta}^N_{t_k,T}(y)$ using Lemma 3.4, we can write:

$$|F_N(t_k^N, x)x + F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))\hat{\theta}^N_{t_{k+1}^N,T}(y)| \leq C_T\Big(|x - \hat{\theta}^N_{t_{k+1},T}(y)| + |y|\Big).$$

Combining the last two estimates, we get:

$$\begin{aligned}
&\big|F_N(t_k^N, x)^2x^2 - F_N(t_{k+1}^N, \hat{\theta}^N_{t_{k+1}^N,T}(y))^2\hat{\theta}^N_{t_{k+1}^N,T}(y)^2\big| \\
&\leq C\Big(|x - \hat{\theta}^N_{t_{k+1}^N,T}(y)|^2 + |x - \hat{\theta}^N_{t_{k+1}^N,T}(y)||y| + \gamma_{k+1}^N|y|^2\Big).
\end{aligned}$$

Thus, the estimate for M_N^1, the first term in the expression of M_N is:

$$\begin{aligned} M_N^1(t_k^N,T,x,y) &\le \frac{C\gamma_0^N}{T-t_k^N}\Big(|x-\hat\theta^N_{t_{k+1}^N,T}(y)|^2+|x-\hat\theta^N_{t_{k+1}^N,T}(y)||y|+\gamma_{k+1}^N|y|^2\Big)\\ &\quad\times\frac{(T-t_k^N)^{-1/2}}{\left(1+\frac{|x-\hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-2}}. \end{aligned}$$

Now, the idea is to pair $(T-t_{k+1}^N)^{-1}$ with $|x-\hat\theta^N_{t_{k+1}^N,T}(y)|$ to cancel the singularity by loosing a power in the polynomial estimate. Namely, we get:

$$\begin{aligned} M_N^1(t_k^N,T,x,y) &\le C\gamma_0^N\Big(\frac{|x-\hat\theta^N_{t_{k+1}^N,T}(y)|^2}{T-t_k^N}+\frac{1}{\sqrt{T-t_k^N}}\frac{|x-\hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T-t_k^N}}|y|\Big)\\ &\quad\times\frac{(T-t_k^N)^{-1/2}}{\left(1+\frac{|x-\hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-2}}. \end{aligned}$$

Let us investigate each term individually. First, we have:

$$C\gamma_0^N\left(\frac{|x-\hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^2\times\frac{(T-t_k^N)^{-1/2}}{\left(1+\frac{|x-\hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-2}}\le C\gamma_0^N\times\frac{(T-t_k^N)^{-1/2}}{\left(1+\frac{|x-\hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-4}}.$$

Next, we have:

$$\frac{1}{\sqrt{T-t_k^N}}\frac{|x-\hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T-t_k^N}}|y|\cdot\frac{(T-t_k^N)^{-1/2}}{\left(1+\frac{|x-\hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-2}}\le\frac{1}{\sqrt{T-t_k^N}}\frac{(T-t_k^N)^{-1/2}}{\left(1+\frac{|x-\hat\theta^N_{t_k^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-3}}|y|.$$

Also, we have $\frac{\gamma_{k+1}^N|y|^2}{T-t_k^N}\le C|y|^2$. To get the estimate for M_N^1, we bound it by the worst term appearing above, that is, we have:

$$M_N^1(t_k^N,T,x,y)\le\frac{C\gamma_0^N}{\sqrt{T-t_k^N}}\frac{(T-t_k^N)^{-1/2}}{\left(1+\frac{|x-\hat\theta^N_{t_{k+1}^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-4}}(1+|y|^2).$$

Observe that by choosing y in a compact set, we can remove the last multiplier $1+|y|^2$ up to a modification of the constant C.

We now turn to $R_{\tilde p_N}$. Recall we denoted above $\mu x+(1-\mu)\hat\theta^N_{t_{k+1}^N,T}(y)=\Theta$. Set for a while

$$\begin{aligned} R^4_{\tilde p_N}(z,\lambda,\mu) &:= \partial_x^4\tilde p_N\left(t_{k+1}^N,T,x+\lambda\Big(\sqrt{\gamma_{k+1}^N}z+F_N(t_k^N,\Theta)\Theta\gamma_{k+1}^N\Big),y\right)\\ &\quad\times\lambda\gamma_{k+1}^N\Big(\partial_uF_N\Big(t_k^N,\Theta\Big)\Theta+F_N\big(t_k^N,\Theta\big)\Big)\\ &\quad\times\Big(z+F_N\Big(t_k^N,\Theta\Big)\sqrt{\gamma_{k+1}^N}\Theta\Big)^3, \end{aligned}$$

and

$$\begin{aligned} R^3_{\tilde p_N}(z,\lambda,\mu) &:= \partial_x^3\tilde p_N\left(t_{k+1}^N,T,x+\lambda\Big(\sqrt{\gamma_{k+1}^N}z+F_N\Big(t_k^N,\Theta\Big)\Theta\gamma_{k+1}^N\Big),y\right)\\ &\quad\times 3\left(z+F_N\Big(t_k^N,\Theta\Big)\big(\mu x+(1-\mu)\hat\theta^N_{t_{k+1}^N,T}(y)\big)\sqrt{\gamma_{k+1}^N}\right)^2\\ &\quad\sqrt{\gamma_{k+1}^N}\times\Big(\partial_uF_N\Big(t_k^N,\Theta\Big)\Theta+F_N\Big(t_k^N,\Theta\Big)\Big), \end{aligned}$$

so that in the remainder $R_{\tilde{p}_N}$, the term I writes:

$$I=\sqrt{\gamma_{k+1}^N}\int_{\mathbb{R}}\int_0^1\int_0^1\left(R_{\tilde{p}_N}^4+R_{\tilde{p}_N}^3\right)(z,\lambda,\mu)\times\frac{(1-\lambda)^2}{2}\Big(x-\hat{\theta}^N_{t_{k+1}^N,T}(y)\Big)p_\xi(z)dzd\lambda d\mu.$$

Notice that for clarity purposes, we denoted $R_{\tilde{p}_N}^4$ the term corresponding to the fourth derivative, and $R_{\tilde{p}_N}^3$ the one for the third derivative. We start with $R_{\tilde{p}_N}^4$ which is the most singular term. For simplicity, let us denote for now

$$\begin{aligned}
u &= (\hat{\theta}^N_{t_{k+1}^N,T}(y)-x)\Big[\ 1+\lambda\mu\gamma_{k+1}^N F_N\big(t_k^N,\mu x+(1-\mu)\hat{\theta}^N_{t_{k+1}^N,T}(y)\big)\Big]\\
v &= \lambda\sqrt{\gamma_{k+1}^N}z+\gamma_{k+1}^N\lambda F_N\big(t_k^N,\mu x+(1-\mu)\hat{\theta}^N_{t_{k+1}^N,T}(y)\big)\hat{\theta}^N_{t_k^N,T}(y)
\end{aligned}$$

We have:

$$\partial_x^4\tilde{p}_N\left(t_{k+1}^N,T,x+\lambda\Big(\sqrt{\gamma_{k+1}^N}z+F_N(t_k^N,\Theta)\Theta\gamma_{k+1}^N\Big),y\right)\le\frac{C}{(T-t_k^N)^2}\frac{(T-t_k^N)^{-\frac12}}{\left(1+\frac{|u-v|}{\sqrt{T-t_k^N}}\right)^{S-2}}.$$

Note that since F_N is bounded, $|v|\le C\Big(|z|\sqrt{\gamma_{k+1}^N}+\gamma_{k+1}^N|y|\Big)$. Besides, for all $u,v\in\mathbb{R}$ with $|v|\le\varepsilon$, we have:

$$\frac{1}{(1+|u-v|)^{S-2}}\le\frac{C(\varepsilon,S)}{(1+|u|)^{S-2}},\quad C(\varepsilon,S)=2^S(1+\varepsilon^S).\tag{3.36}$$

Using that estimate, we get:

$$\begin{aligned}
&\partial_x^4\tilde{p}_N\left(t_{k+1}^N,T,x+\lambda\Big(\sqrt{\gamma_{k+1}^N}z+F_N(t_k^N,\Theta)\Theta\gamma_{k+1}^N\Big),y\right)\\
&\le\frac{C}{(T-t_k^N)^2}\frac{(T-t_k^N)^{-\frac12}}{\left(1+\dfrac{|u|}{\sqrt{T-t_k^N}}\right)^{S-2}}\Big(1+(\gamma_{k+1}^N)^{\frac{S-2}{2}}|z|^{S-2}+(\gamma_{k+1}^N)^{S-2}|y|^{S-2}\Big)
\end{aligned}$$

Now, observe that $|u|\ge\frac12|x-\hat{\theta}^N_{t_{k+1}^N,T}(y)|$ for N sufficiently large. We get:

$$\begin{aligned}
&\partial_x^4\tilde{p}_N\left(t_{k+1}^N,T,x+\lambda\Big(\sqrt{\gamma_{k+1}^N}z+F_N(t_k^N,\Theta)\Theta\gamma_{k+1}^N\Big),y\right)\\
&\le\frac{C}{(T-t_k^N)^2}\frac{(T-t_k^N)^{-\frac12}}{\left(1+\dfrac{|x-\hat{\theta}^N_{t_{k+1}^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-2}}\Big(1+(\gamma_{k+1}^N)^{\frac{S-2}{2}}|z|^{S-2}+(\gamma_{k+1}^N)^{S-2}|y|^{S-2}\Big).
\end{aligned}$$

Now for the term multiplying the derivative, using estimate (3.23) and recalling F_N is bounded, we have:

$$\begin{aligned}
&\Big|\lambda\gamma_{k+1}^N\Big[\partial_u F_N\Big(t_k^N,\Theta\Big)\Theta+F_N\big(t_k^N,\Theta\big)\Big]\Big(z+F_N\Big(t_k^N,\Theta\Big)\sqrt{\gamma_{k+1}^N}\Theta\Big)^3\Big|\\
\le\ & C\gamma_{k+1}^N\Big(a_N\sqrt{\gamma_0^N}|\Theta|+1\Big)\times\Big(|z|+\sqrt{\gamma_{k+1}^N}|\Theta|\Big)^3\\
\le\ & C\gamma_{k+1}^N\Big(1+|z|^4+a_N\sqrt{\gamma_0^N}|y|^4+a_N\sqrt{\gamma_0^N}|x-\hat{\theta}^N_{t_{k+1}^N,T}(y)|^4\Big).
\end{aligned}$$

Note that we bounded $|\Theta|=|\mu x+(1-\mu)\hat{\theta}^N_{t_k,T}(y)|$ by $C_T(|y|+|x-\hat{\theta}^N_{t_k^N,T}(y)|)$. Finally, this gives us the estimate for the fourth derivative term:

$$\begin{aligned}
R_{\tilde{p}_N}^4(z,\lambda,\mu)\ \le\ &\frac{C}{(T-t_k^N)^2}\gamma_{k+1}^N\Big(1+|z|^4+a_N\sqrt{\gamma_0^N}|y|^4+a_N\sqrt{\gamma_0^N}|x-\hat{\theta}^N_{t_{k+1}^N,T}(y)|^4\Big)\\
&\times\Big(1+(\gamma_{k+1}^N)^{\frac{S-2}{2}}|z|^{S-2}+(\gamma_{k+1}^N)^{S-2}|y|^{S-2}\Big)\frac{(T-t_k^N)^{-\frac12}}{\left(1+\dfrac{|x-\hat{\theta}^N_{t_{k+1}^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-2}}.
\end{aligned}$$

From the additional γ_{k+1}^N in the numerator, we can cancel a singularity. Also, we expand and rearrange the terms to get the estimate:

$$R^4_{\tilde{p}_N}(z,\lambda,\mu) \le \frac{C}{T-t_k^N}\frac{(T-t_k^N)^{-\frac{1}{2}}}{\left(1+\frac{|x-\hat{\theta}^N_{t^N_{k+1},T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-6}}\Big(1+|z|^{S+2}+a_N\sqrt{\gamma_0^N}|y|^{S+2}\Big).$$

Similarly, we get for the third derivative term:

$$R^3_{\tilde{p}_N}(z,\lambda,\mu) \le \frac{C}{T-t_k^N}\frac{(T-t_k^N)^{-\frac{1}{2}}}{\left(1+\frac{|x-\hat{\theta}^N_{t^N_{k+1},T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-6}}\Big(1+|z|^{S+2}+a_N\sqrt{\gamma_0^N}|y|^{S+2}\Big).$$

To complete the estimation of the term I in the remainder, we need to integrate the last estimate against the density p_ξ. Notice that thanks to our upper bound, we get rid of the integrals in μ and λ. We have:

$$I \le \frac{C\sqrt{\gamma_{k+1}^N}}{T-t_k^N}\frac{(T-t_k^N)^{-\frac{1}{2}}}{\left(1+\frac{|x-\hat{\theta}^N_{t^N_{k+1},T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-6}}\int_{\mathbb{R}}\Big(1+|z|^{S+2}+a_N\sqrt{\gamma_0^N}|y|^{S+2}\Big)$$
$$\times|\hat{\theta}^N_{t^N_{k+1},T}(y)-x|p_\xi(z)dz.$$

Again, we can cancel a singularity by matching it with $|\hat{\theta}^N_{t^N_{k+1},T}(y)-x|$ to finally get the estimate:

$$I \le \frac{C\sqrt{\gamma_0^N}}{\sqrt{T-t_k^N}}\frac{(T-t_k^N)^{-\frac{1}{2}}}{\left(1+\frac{|x-\hat{\theta}^N_{t^N_{k+1},T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-7}}\Big(1+a_N\sqrt{\gamma_0^N}|y|^{S+2}\Big).$$

We can combine that last estimate with (3.35) to get to the estimate for I. Notice that since $a_N \to +\infty$, we have $\sqrt{\gamma_0^N} \le a_N\sqrt{\gamma_0^N}$, which allows us to put the same estimate for both I and M_N^1.

The same upper bound can be obtained quite similarly for $I\!I$. Besides, as we pointed out above, substituting $\tilde{p}_N$ in (3.34), we can compensate the singularity to obtain a similar upper bound. Now, to conclude to the upper bound for $R_{\tilde{p}_N}(t_k^N,T,x,y)$, we recall the identity:

$$R_{\tilde{p}_N}(t_k^N,T,x,y) = I + I\!I + \Big(\tilde{L}^N_{t^N_{k+1}} - \tilde{L}^N_{t^N_k}\Big)\tilde{p}_N(t_k^N,T,x,y).$$

Observe the announced estimate holds for each of the above contribution, and thus for the remainder.

As for M_N, the estimate follows from (3.32) and (3.35). □

3.2.2 Proof of Step 3

This step consists in replacing H_N with its approximation K_N+M_N. First, we split the sum:

$$\begin{aligned}
&\sum_{r=0}^N \tilde{q}_N \otimes_N H_N^{[r]}(t_k^N,T,x,y) - \sum_{r=0}^N \tilde{q}_N \otimes_N (K_N+M_N)^{[r]}(t_k^N,T,x,y) \\
=\ &\sum_{r=0}^N\Big(\tilde{q}_N \otimes_N H_N^{[r]} - \tilde{q}_N \otimes_N (H_N+M_N)^{[r]}\Big)(t_k^N,T,x,y) \\
&+\sum_{r=0}^N\Big(\tilde{q}_N \otimes_N (H_N+M_N)^{[r]} - \tilde{q}_N \otimes_N (K_N+M_N)^{[r]}\Big)(t_k^N,T,x,y) =: I + I\!I,
\end{aligned} \tag{3.37}$$

and to control I and $I\!I$, we proceed by induction. We prove the following result first.

Lemma 3.9. *For all $0 \le t_k^N \le T-\delta$, $\forall \delta > 0$, all $x \in K_x, y \in K_y$, where K_x, K_y are compact sets, for all $r \in \mathbb{N}$, there exists a constant $C > 0$ depending on the compacts K_x, K_y, such that:*

$$\begin{aligned}&\Big(\tilde{q}_N \otimes_N H_N^{[r]} - \tilde{q}_N \otimes_N (H_N + M_N)^{[r]}\Big)(t_k^N, T, x, y) \\ &\le CK^r (1 + (\sqrt{\gamma_0^N}|x|)^{S-7})(1 + a_N\sqrt{\gamma_0^N}|y|^{S+2})\sqrt{\gamma_0^N}(T-t_k^N)^{\frac{r}{2}} \\ &\quad \times \prod_{j=1}^{r+1} B\left(\frac{j}{2}, \frac{1}{2}\right) \frac{(T-t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \hat{\theta}_{t_k^N,T}^N(y)|}{\sqrt{T-t_k^N}}\right)^{S-7}},\end{aligned}$$

where

$$K = \left(1 + \max_{x \in K_x, y \in K_y, t_k^N} \frac{|x - \hat{\theta}_{t_k^N,T}^N(y)|}{\sqrt{T-t_k^N}}\right)^{\frac{S}{2}+1} = K(K_x, K_y, T, \delta). \tag{3.38}$$

Proof. We proceed by induction. First, for $r = 1$, we have:

$$\tilde{q}_N \otimes_N H_N^{[1]} - \tilde{q}_N \otimes_N (H_N + M_N)^{[1]} = \tilde{q}_N \otimes_N M_N.$$

Thus, using Lemma 3.8, we write:

$$\begin{aligned}|\tilde{q}_N \otimes_N M_N|(t_k^N, T, x, y) &\le \sum_{i=k}^{M(N)} \gamma_{i+1}^N \int_{\mathbb{R}} dz g_C(t_i^N - t_k^N, x - \theta_{t_k^N,t_i^N}^N(z)) \frac{C\sqrt{\gamma_0^N}}{\sqrt{T-t_i^N}} \\ &\quad \times \frac{(T-t_i^N)^{-\frac{1}{2}}}{\left(1 + \frac{|z - \hat{\theta}_{t_{i+1}^N,T}^N(y)|}{\sqrt{T-t_k^N}}\right)^{S-7}} (1 + a_N\sqrt{\gamma_0^N}|y|^{S+2}) \\ &\le \sum_{i=k}^{M(N)} \gamma_{i+1}^N \int_{\mathbb{R}} dz \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \theta_{t_k^N,t_i^N}^N(z)|}{\sqrt{t_i^N - t_k^N}}\right)^{S-7}} \frac{C\sqrt{\gamma_0^N}}{\sqrt{T-t_i^N}} \frac{(T-t_i^N)^{-\frac{1}{2}}(1 + a_N\sqrt{\gamma_0^N}|y|^{S+2})}{\left(1 + \frac{|z - \hat{\theta}_{t_{i+1}^N,T}^N(y)|}{\sqrt{T-t_k^N}}\right)^{S-7}}.\end{aligned}$$

We have to estimate:

$$I = \int_{\mathbb{R}} dz \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \theta_{t_k^N,t_i^N}^N(z)|}{\sqrt{t_i^N - t_k^N}}\right)^{S-7}} \frac{(T-t_i^N)^{-\frac{1}{2}}}{\left(1 + \frac{|z - \hat{\theta}_{t_{i+1}^N,T}^N(y)|}{\sqrt{T-t_k^N}}\right)^{S-7}}.$$

We write:

$$\theta_{t_k^N,t_i^N}^N(z) - x = \theta_{t_k^N,t_i^N}^N\Big(z - \theta_{t_i^N,t_k^N}^N(x)\Big).$$

Hence, we have for some constant $C = C_T > 1$:

$$C^{-1}|z - \theta_{t_i^N,t_k^N}^N(x)| \le |\theta_{t_k^N,t_i^N}^N(z) - x| \le C|z - \theta_{t_i^N,t_k^N}^N(x)|.$$

Thus, we have:

$$\int_{\mathbb{R}} dz \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \theta^N_{t_k^N, t_i^N}(z)|}{\sqrt{t_i^N - t_k^N}}\right)^{S-7}} \frac{(T - t_i^N)^{-\frac{1}{2}}}{\left(1 + \frac{|z - \hat{\theta}^N_{t_{i+1}^N, T}(y)|}{\sqrt{T - t_i^N}}\right)^{S-7}}$$

$$\leq \int_{\mathbb{R}} dz \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|z - \theta^N_{t_i^N, t_k^N}(x)|}{\sqrt{t_i^N - t_k^N}}\right)^{S-7}} \frac{(T - t_i^N)^{-\frac{1}{2}}}{\left(1 + \frac{|z - \hat{\theta}^N_{t_{i+1}^N, T}(y)|}{\sqrt{T - t_i^N}}\right)^{S-7}}$$

$$\leq C \frac{(T - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|\hat{\theta}^N_{t_{i+1}^N, T}(y) - \theta^N_{t_i^N, t_k^N}(x)|}{\sqrt{T - t_k^N}}\right)^{S-7}}.$$

Now, we can rewrite

$$\frac{|\hat{\theta}^N_{t_{i+1}^N, T}(y) - \theta^N_{t_i^N, t_k^N}(x)|}{\sqrt{T - t_k^N}} = \left| \frac{\hat{\theta}^N_{t_{i+1}^N, t_k^N}\left(\hat{\theta}^N_{t_k^N, T}(y) - x\right)}{\sqrt{T - t_k^N}} + \frac{\hat{\theta}^N_{t_{i+1}^N, t_k^N}\left(\theta^N_{t_k^N, t_{i+1}^N} \theta^N_{t_{i+1}^N, t_k^N}(x) - \hat{\theta}^N_{t_k^N, t_{i+1}^N} \theta^N_{t_i^N, t_k^N}(x)\right)}{\sqrt{T - t_k^N}} \right|.$$

Observe now that:

$$\begin{aligned} & \frac{\left|\hat{\theta}^N_{t_{i+1}^N, t_k^N}\left(\theta^N_{t_k^N, t_{i+1}^N} \theta^N_{t_{i+1}^N, t_k^N}(x) - \hat{\theta}^N_{t_k^N, t_{i+1}^N} \theta^N_{t_i^N, t_k^N}(x)\right)\right|}{\sqrt{T - t_k^N}} \\ \leq \; & C_T \frac{\left|\theta^N_{t_k^N, t_{i+1}^N} \theta^N_{t_{i+1}^N, t_k^N}(x) - \hat{\theta}^N_{t_k^N, t_{i+1}^N} \theta^N_{t_i^N, t_k^N}(x)\right|}{\sqrt{T - t_k^N}} \\ \leq \; & C_T \gamma_0^N \frac{\left|\theta^N_{t_{i+1}^N, t_k^N}(x)\right|}{\sqrt{T - t_k^N}} \leq C_T \sqrt{\gamma_0^N} |x|. \end{aligned}$$

Thus, using equation (3.36), we can write:

$$\frac{(T - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|\hat{\theta}^N_{t_{i+1}^N, T}(y) - \theta^N_{t_i^N, t_k^N}(x)|}{\sqrt{T - t_k^N}}\right)^{S-7}} \leq \frac{\left(1 + (\sqrt{\gamma_0^N}|x|)^{S-7}\right)(T - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|\hat{\theta}^N_{t_{i+1}^N, t_k^N}\left(\hat{\theta}^N_{t_k^N, T}(y) - x\right)|}{\sqrt{T - t_k^N}}\right)^{S-7}}.$$

Now, since we have for some $C > 1$:

$$C^{-1} \frac{\left|\hat{\theta}^N_{t_k^N, T}(y) - x\right|}{\sqrt{T - t_k^N}} \leq \frac{\left|\hat{\theta}^N_{t_{i+1}^N, t_k^N}\left(\hat{\theta}^N_{t_k^N, T}(y) - x\right)\right|}{\sqrt{T - t_k^N}} \leq C \frac{\left|\hat{\theta}^N_{t_k^N, T}(y) - x\right|}{\sqrt{T - t_k^N}},$$

we obtain the following estimate:

$$\int_{\mathbb{R}} dz \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \theta^N_{t_k^N, t_i^N}(z)|}{\sqrt{t_i^N - t_k^N}}\right)^{S-7}} \frac{(T - t_i^N)^{-\frac{1}{2}}}{\left(1 + \frac{|z - \hat{\theta}^N_{t_{i+1}^N, T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}}$$

$$\leq C\Big(1 + (\sqrt{\gamma_0^N}|x|)^{S-7}\Big) \frac{(T - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|\hat{\theta}^N_{t_k^N, T}(y) - x|}{\sqrt{T - t_k^N}}\right)^{S-7}}.$$

Observe that:

$$\sum_{i=k}^{M(N)} \gamma_{i+1}^N \frac{C\sqrt{\gamma_0^N}}{\sqrt{T - t_i^N}} \leq C\sqrt{\gamma_0^N} \int_{t_k^N}^{T} \frac{1}{\sqrt{T-u}} du \leq C\sqrt{\gamma_0^N}\sqrt{T - t_k^N}.$$

Thus, we have the estimate

$$\begin{aligned} &\left|\tilde{q}_N \otimes_N H_N^{[1]} - \tilde{q}_N \otimes_N (H_N + M_N)^{[1]}\right|(t_k^N, T, x, y) \\ \leq\ & C\sqrt{\gamma_0^N}\sqrt{T - t_k^N} \frac{(T - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}} (1 + a_N\sqrt{\gamma_0^N}|y|^{S+2})(1 + \sqrt{\gamma_0^N}|x|)^{S-7}, \end{aligned}$$

which is the required estimate for $r = 1$. Assume now that the estimate holds for some $r \geq 1$. We write:

$$\begin{aligned} &\Big(\tilde{q}_N \otimes_N H_N^{[r+1]} - \tilde{q}_N \otimes_N (H_N + M_N)^{[r+1]}\Big) \\ =\ & \Big(\tilde{q}_N \otimes_N H_N^{[r]} - \tilde{q}_N \otimes_N (H_N + M_N)^{[r]}\Big) \otimes_N (H_N + M_N) - (\tilde{q}_N \otimes_N H_N^{[r]}) \otimes_N M_N. \end{aligned}$$

Now, from the controls of Section 3 (specifically the proof of Lemma 3.5), and Lemma 3.8, we can estimate directly:

$$\begin{aligned} &\Big|(\tilde{q}_N \otimes_N H_N^{[r]}) \otimes_N M_N(t_k^N, T, x, y)\Big| \\ \leq\ & \sum_{i=k}^{M(N)} \gamma_{i+1}^N \int_{\mathbb{R}} |\tilde{q}_N \otimes_N H_N^{[r]}|(t_k^N, t_i^N, x, z)|M_N|(t_i^N, T, z, y) dz \\ \leq\ & C^r \sum_{i=k}^{M(N)} \gamma_{i+1}^N \int_{\mathbb{R}} dz (t_i^N - t_k^N)^{\frac{r}{2}} \prod_{j=1}^{r+1} B\left(\frac{j}{2}, \frac{1}{2}\right) g_C(t_i^N - t_k^N, \theta^N_{t_k^N, t_i^N}(z) - x) \\ & \times \frac{C\sqrt{\gamma_0^N}}{\sqrt{T - t_i^N}} \frac{(T - t_i^N)^{-\frac{1}{2}}}{\left(1 + \frac{|z - \hat{\theta}^N_{t_{i+1}^N, T}(y)|}{\sqrt{T - t_i^N}}\right)^{S-7}} (1 + a_N\sqrt{\gamma_0^N}|y|^{S+2}). \end{aligned}$$

To estimate the last integral, we bound the Gaussian by a polynomial estimate, which admittedly is not the sharpest way to proceed:

$$\begin{aligned} &\Big|(\tilde{q}_N \otimes_N H_N^{[r]}) \otimes_N M_N(t_k^N, T, x, y)\Big| \\ \leq\ & C^r \sum_{i=k}^{M(N)} \gamma_{i+1}^N (t_i^N - t_k^N)^{\frac{r}{2}} \prod_{j=1}^{r+1} B\left(\frac{j}{2}, \frac{1}{2}\right) \\ & \times \int_{\mathbb{R}} dz \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|\theta^N_{t_k^N, t_i^N}(z) - x|}{\sqrt{t_i^N - t_k^N}}\right)^{S-7}} \frac{C\sqrt{\gamma_0^N}}{\sqrt{T - t_i^N}} \frac{(T - t_i^N)^{-\frac{1}{2}}(1 + a_N\sqrt{\gamma_0^N}|y|^{S+2})}{\left(1 + \frac{|z - \hat{\theta}^N_{t_{i+1}^N, T}(y)|}{\sqrt{T - t_i^N}}\right)^{S-7}}. \end{aligned}$$

Observe that we can bound the integral over z as the convolution of two heavy tailed estimates. This yields the bound:

$$\begin{aligned}&\left|(\tilde{q}_N \otimes_N H_N^{[r]}) \otimes_N M_N(t_k^N, T, x, y)\right| \\ \leq\ & C^r \sum_{i=k}^{M(N)} \gamma_{i+1}^N \frac{C\sqrt{\gamma_0^N}}{\sqrt{T-t_i^N}} a_N \sqrt{\gamma_0^N}(t_i^N - t_k^N)^{\frac{r}{2}} \prod_{j=1}^{r+1} B\left(\frac{j}{2}, \frac{1}{2}\right) \frac{(T-t_k^N)^{-\frac{1}{2}}}{\left(1+\frac{|x-\hat{\theta}^N_{t_k^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-7}} \\ &\times(1+a_N\sqrt{\gamma_0^N}|y|^{S+2})(1+\sqrt{\gamma_0^N}|x|^{S-7}).\end{aligned}$$

Comparing the sum to an integral yields:

$$\sum_{i=k}^{M(N)} \gamma_{i+1}^N \frac{C\sqrt{\gamma_0^N}}{\sqrt{T-t_i^N}}(t_i^N - t_k^N)^{\frac{r}{2}} \prod_{j=1}^{r+1} B\left(\frac{r}{2}, \frac{1}{2}\right) \leq C\sqrt{\gamma_0^N}(T-t_k^N)^{\frac{r+1}{2}} \prod_{j=1}^{r+2} B\left(\frac{j}{2}, \frac{1}{2}\right).$$

Thus, we get the following estimate:

$$\begin{aligned}&\left|(\tilde{q}_N \otimes_N H_N^{[r]}) \otimes_N M_N(t_k^N, T, x, y)\right| \\ \leq\ & C^{r+1}\sqrt{\gamma_0^N}(T-t_k^N)^{\frac{r+1}{2}} \prod_{j=1}^{r+2} B\left(\frac{j}{2}, \frac{1}{2}\right) \frac{(T-t_k^N)^{-\frac{1}{2}}(1+a_N\sqrt{\gamma_0^N}|y|^{S+2})(1+\sqrt{\gamma_0^N}|x|^{S-7})}{\left(1+\frac{|x-\hat{\theta}^N_{t_k^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-7}}.\end{aligned}$$

Now, for $\left(\tilde{q}_N \otimes_N H_N^{[r]} - \tilde{q}_N \otimes_N (H_N+M_N)^{[r]}\right) \otimes_N (H_N+M_N)$, we use the induction hypothesis and Lemma 3.8:

$$\begin{aligned}&\left|\left(\tilde{q}_N \otimes_N H_N^{[r]} - \tilde{q}_N \otimes_N (H_N+M_N)^{[r]}\right) \otimes_N (H_N+M_N)(t_k^N, T, x, y)\right| \\ \leq\ & \sum_{i=k}^{M(N)} \gamma_{i+1}^N \int_{\mathbb{R}} \left|\tilde{q}_N \otimes_N H_N^{[r]} - \tilde{q}_N \otimes (H_N+M_N)^{[r]}\right|(t_k^N, t_i^N, x, z)|H_N+M_N|(t_i^N, T, z, y) \\ \leq\ & C^r(1+\sqrt{\gamma_0^N}|x|^{S-7}) \sum_{i=k}^{M(N)} \gamma_{i+1}^N \int_{\mathbb{R}} (1+a_N\sqrt{\gamma_0^N}|z|^{S+2})\sqrt{\gamma_0^N}(t_i^N - t_k^N)^{\frac{r+1}{2}} \prod_{j=1}^{r+2} B\left(\frac{j}{2}, \frac{1}{2}\right) \\ &\times \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left(1+\frac{|x-\hat{\theta}^N_{t_k^N,t_i^N}(z)|}{\sqrt{t_i^N - t_k^N}}\right)^{S-7}} C\frac{1+a_N\sqrt{\gamma_0^N}|y|^{S+2}}{\sqrt{T-t_i^N}} \frac{(T-t_i^N)^{-1/2}}{\left(1+\frac{|z-\hat{\theta}^N_{t_{i+1}^N,T}(y)|}{\sqrt{T-t_i^N}}\right)^{S-7}} dz.\end{aligned}$$

Notice that the term $1+a_N\sqrt{\gamma_0^N}|y|^S$, coming from the unbounded feature of the drift, does not depends on z, and thus can be put in front of the integral. The corresponding term $1+a_N\sqrt{\gamma_0^N}|z|^{S+2}$ is harder to deal with. Observe that we have the following:

$$1+a_N\sqrt{\gamma_0^N}|z|^{S+2} \leq C\max(1, T^{\frac{S}{2}+1})\Big(\frac{|x-\hat{\theta}^N_{t_k^N,t_i^N}(z)|}{\sqrt{t_i^N - t_k^N}}+1\Big)^{\frac{S}{2}+1}\Big(\frac{|z-\hat{\theta}^N_{t_i^N,T}(y)|}{\sqrt{T-t_i^N}}+1\Big)^{\frac{S}{2}+1}.$$

where the constant $C > 0$ depends on the compacts K_x, K_y. Thus, plugging this estimate we get:

$$
\begin{aligned}
&\left|\left(\tilde{q}_N \otimes_N H_N^{[r]} - \tilde{q}_N \otimes_N (H_N + M_N)^{[r]}\right) \otimes_N (H_N + M_N)(t_k^N, T, x, y)\right| \\
\leq\ & C(1 + \sqrt{\gamma_0^N}|x|^{S-7})\sqrt{\gamma_0^N}(1 + a_N\sqrt{\gamma_0^N}|y|^{S+2}) \sum_{i=k}^{M(N)} \gamma_{i+1}^N (t_i^N - t_k^N)^{\frac{r+1}{2}} \\
&\times \frac{1}{\sqrt{T - t_i^N}} \prod_{j=1}^{r+2} B\left(\frac{j}{2}, \frac{1}{2}\right) \int_{\mathbb{R}} \Big(\frac{|x - \hat{\theta}^N_{t_k^N, t_i^N}(z)|}{t_i^N - t_k^N} + 1\Big)^{\frac{S}{2}+1} \Big(\frac{|z - \hat{\theta}^N_{t_{i+1}^N, T}(y)|}{T - t_i^N} + 1\Big)^{\frac{S}{2}+1} \\
&\times \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, t_i^N}(z)|}{\sqrt{t_i^N - t_k^N}}\right)^{S-7}} \frac{(T - t_i^N)^{-1/2}}{\left(1 + \frac{|z - \hat{\theta}^N_{t_{i+1}^N, T}(y)|}{\sqrt{T - t_i^N}}\right)^{S-7}} dz \\
\leq\ & C(1 + \sqrt{\gamma_0^N}|x|^{S-7})\sqrt{\gamma_0^N}(1 + a_N\sqrt{\gamma_0^N}|y|^{S+2})(T - t_k^N)^{\frac{r+2}{2}} \prod_{j=1}^{r+3} B\left(\frac{j}{2}, \frac{1}{2}\right) \\
&\times \int_{\mathbb{R}} \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, t_i^N}(z)|}{\sqrt{t_i^N - t_k^N}}\right)^{\frac{S}{2}-8}} \frac{(T - t_i^N)^{-1/2}}{\left(1 + \frac{|z - \hat{\theta}^N_{t_{i+1}^N, T}(y)|}{\sqrt{T - t_i^N}}\right)^{\frac{S}{2}-8}} dz
\end{aligned}
$$

By the convolution property of the polynomial densities, we have:

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, t_i^N}(z)|}{\sqrt{t_i^N - t_k^N}}\right)^{\frac{S}{2}-8}} \frac{(T - t_i^N)^{-1/2}}{\left(1 + \frac{|z - \hat{\theta}^N_{t_i^N, T}(y)|}{\sqrt{T - t_i^N}}\right)^{\frac{S}{2}-8}} dz \\
\leq\ & C \frac{(T - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, T}(y)|}{\sqrt{T - t_k^N}}\right)^{\frac{S}{2}-8}} \times \frac{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, T}(y)|}{\sqrt{T - t_k^N}}\right)^{\frac{S}{2}+1}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, T}(y)|}{\sqrt{T - t_k^N}}\right)^{\frac{S}{2}+1}} \\
\leq\ & CK \frac{(T - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}}.
\end{aligned}
$$

Now, estimating the sum over i by an integral yields the announced estimate. □

Now, to estimate $I\!I$ in (3.37), we proceed again by induction. We prove the following Lemma:

Lemma 3.10. *For all $0 \leq t_k^N \leq T - \delta$, $\delta > 0$, all $x \in K_x, y \in K_y$, for all $r \in \mathbb{N}$, there exists a constant $C > 0$ such that:*

$$
\begin{aligned}
&\left(\tilde{q}_N \otimes_N (H_N + M_N)^{[r]} - \tilde{q}_N \otimes_N (K_N + M_N)^{[r]}\right)(t_k^N, T, x, y) \\
&\leq C^r K^r (1 + a_N\sqrt{\gamma_0^N}|y|^S)\sqrt{\gamma_0^N}(T - t_k^N)^{\frac{r}{2}} \prod_{j=1}^{r} B\left(\frac{j}{2}, \frac{1}{2}\right) \frac{(T - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}},
\end{aligned}
$$

where K is defined in (3.38)

Proof. The proof uses similar techniques. For $r = 1$, notice that we have:

$$\tilde{q}_N \otimes_N (H_N + M_N)^{[1]} - \tilde{q}_N \otimes_N (K_N + M_N)^{[1]} = \tilde{q}_N \otimes_N (H_N - K_N).$$

We thus need a control of $H_N - K_N$, which follows (by definition of H_N and K_N) from a control of $\tilde{q}_N - \tilde{p}_N$. Specifically, in Step 4, we obtain an estimate on $\tilde{q}_N - \tilde{p}_N$, based on Edgeworth expansions. We then need to "take the derivative of the expansion". Since is done with the same arguments than those exposed in [3], we only give the estimate here. We have:

$$|H_N - K_N|(t_k^N, T, x, y) \le C\sqrt{\gamma_0^N} \frac{(T - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N,T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-3}}. \tag{3.39}$$

Using this estimate, we can control the case $r = 1$ with the same arguments as those used above. Next, for $r \ge 1$, we write:

$$\begin{aligned} & \tilde{q}_N \otimes_N (H_N + M_N)^{[r+1]} - \tilde{q}_N \otimes_N (K_N + M_N)^{[r+1]} \\ = & \left(\tilde{q}_N \otimes_N (H_N + M_N)^{[r]} - \tilde{q}_N \otimes_N (K_N + M_N)^{[r]}\right) \otimes_N (K_N + M_N) \\ & + \tilde{q}_N \otimes_N (H_N + M_N)^{[r]} \otimes_N (H_N - K_N). \end{aligned}$$

For the first contribution, we can use Lemma 3.8 to estimate $K_N + M_N$ and the induction hypothesis for $\tilde{q}_N \otimes_N (H_N + M_N)^{[r]} - \tilde{q}_N \otimes_N (K_N + M_N)^{[r]}$. For the second contribution, we can first obtain similarly to the control of $\tilde{q}_N \otimes_N H_N$ a control of $\tilde{q}_N \otimes_N (H_N + M_N)$, using Lemma 3.8. We then take the convolution of that estimate with the estimate 3.39, to get the announced estimate.

□

We can now conclude as for Step 3. Piecing together the estimates form Lemmas 3.9 and 3.10, we get the estimate:

Proposition 3.11. *Let $k \ge 1$. For all $0 \le t_k^N \le T - \delta$, $\delta > 0$, all $x \in K_x, y \in K_y$ compact sets, there exists a constant $C > 0$, such that:*

$$\begin{aligned} & \left|\sum_{r=0}^{N} \tilde{q}_N \otimes_N H_N^{[r]} - \tilde{q}_N \otimes_N (K_N + M_N)^{[r]}\right| (t_k^N, T, x, y) \\ \le & C_T(1 + \sqrt{\gamma_0^N}|x|^{S-7})(1 + a_N\sqrt{\gamma_0^N}|y|^S)\sqrt{\gamma_0^N} \frac{(T - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N,T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}}. \end{aligned}$$

3.2.3 Proof of Step 4

For this step, we need to replace $\tilde{q}_N$ with $\tilde{p}_N$. Observe carefully now that for the frozen densities, we have:

$$\begin{aligned} \big(\tilde{q}_N - \tilde{p}_N\big)(t, T, x, y) = & g_\sigma\left(T - t, \theta^N_{t,T}(y) - x\right) - g_\sigma\left(T - t, \hat{\theta}^N_{t,T}(y) - x\right) \\ & + g_\sigma\left(T - t, \hat{\theta}^N_{t,T}(y) - x\right) - p_{S_N}\left(\hat{\theta}^N_{t,T}(y) - x\right). \end{aligned} \tag{3.40}$$

The first contribution can be handled with similar arguments than those developed in Section 3.1 (see Lemma (3.3) above). We have the following estimate:

Lemma 3.12. *There exists a constant $C, C' > 0$, for all $t \le T$, for all $(x, y) \in \mathbb{R} \times \mathbb{R}$,*

$$\begin{aligned} & \left|g_\sigma\left(T - t, \theta^N_{t,T}(y) - x\right) - g_\sigma\left(T - t, \hat{\theta}^N_{t,T}(y) - x\right)\right| \\ \le & C\sqrt{(T - t)}a_N\sqrt{\gamma_0^N}|y|e^{C(T-t)^2 a_N^2 \gamma_0^N |y|^2} g_{C'}(T - t, \theta_{t,T}(y) - x). \end{aligned}$$

Again, observe that we chose to put the transport of the differential equation $\dot{x}_t = (-am'(\bar{\theta}_t) + \frac{1}{2})x_t$ at the price of the additional exponential factor that goes to 1 when N goes to infinity. This is in order to get a homogeneous upper bound for the tails.

The second contribution in (3.40) can be controlled by Bhattacharya and Rao [2], and consists in controlling the convergence of the sum *i.i.d.* variables $\sum_{i=k}^{M(N)-1} \sqrt{\gamma_{i+1}^N}\xi_{i+1}$ to its Gaussian limit σW_{T-t}. The only difference is that the argument of the density is $\hat{\theta}_{t,T}^N(y) - x$ instead of the usual $y - x$. Up to minor modification in [3], we get the following Lemma:

Lemma 3.13. *There exists $S > 4$, there exists a constant $C > 0$, for all $t \leq T$, for all $(x, y) \in \mathbb{R} \times \mathbb{R}$,*

$$\left| g_\sigma\left(T - t, \hat{\theta}_{t,T}^N(y) - x\right) - p_{S_N}\left(\hat{\theta}_{t,T}^N(y) - x\right)\right| \leq C\sqrt{\frac{\gamma_0^N}{T-t}} \frac{1}{\left(1 + \frac{|\hat{\theta}_{t,T}^N(y)-x|}{\sqrt{T-t}}\right)^{S-2}}.$$

Combining Lemmas 3.12 and 3.13, we get the estimate:

$$\begin{aligned}(\tilde{q}_N - \tilde{p}_N)(t, T, x, y) \leq C\frac{\gamma_0^N}{\sqrt{T-t}}|y|e^{C_T^2(\gamma_0^N|y|)^2} g_{C'}(T - t, \hat{\theta}_{t,T}^N(y) - x)\\ +\sqrt{\frac{\gamma_0^N}{T-t}}\frac{1}{\left(1 + \frac{|\hat{\theta}_{t,T}^N(y)-x|}{\sqrt{T-t}}\right)^{S-2}}.\end{aligned}$$

Besides, we can notice that:

$$\begin{aligned}\sqrt{(T-t)}a_N\sqrt{\gamma_0^N}|y|e^{C(T-t)^2a_N^2\gamma_0^N|y|^2} &\leq \sqrt{(T-t)}a_N\sqrt{\gamma_0^N}|y|e^{C(T-t)^2|y|^2}\\ &\leq a_N\sqrt{\gamma_0^N}e^{C(T-t)^2|y|^2}.\end{aligned}$$

To simplify the convolutions, notice also that we can use Lemma 3.4 to put the same argument in the Gaussian and in the polynomial estimate. Thus, we get the following estimate for the difference of the frozen densities:

$$\begin{aligned}(\tilde{q}_N - \tilde{p}_N)(t_k^N, T, x, y) \leq C\Big(a_N\sqrt{\gamma_0^N}e^{C(T-t_k^N)^2|y|^2}g_C(T - t_k^N, \theta_{t_k^N,T}(y) - x)\\ +\sqrt{\frac{\gamma_0^N}{T-t_k^N}}\frac{1}{\left(1 + \frac{|\hat{\theta}_{t_k^N,T}^N(y)-x|}{\sqrt{T-t_k^N}}\right)^{S-2}}\Big).\end{aligned} \tag{3.41}$$

This takes care of replacing $\tilde{q}_N$ with $\tilde{p}_N$. To complete step 4, we need to point out that the estimate for $\tilde{q}_N - \tilde{p}_N$ can be convoluted with $(K_N + M_N)^{[r]}$ Specifically, we have the following lemma:

Lemma 3.14. *For all $0 \leq t_k^N \leq T - \delta$, $\delta > 0$, for all $(x, y) \in K_x \times K_y$, there exists a constant $C > 0$ depending on K_x, K_y, T and δ such that for all N:*

$$\left|\sum_{r=0}^{N}(\tilde{q}_N - \tilde{p}_N) \otimes_N (K_N + M_N)^{[r]}\right| \leq C\sqrt{\gamma_0^N}\frac{(T - t_k^N)^{-\frac{1}{2}}}{\left(1 + \frac{|x-\hat{\theta}_{t_k^N,T}^N(y)|}{\sqrt{T-t_k^N}}\right)^{S-7}}.$$

Proof. We prove this estimate by induction. For $k = 1$, we have, define:

$$\begin{aligned}I := C\gamma_0^N\sum_{i=k}^{M(N)}\frac{\gamma_{i+1}^N}{\sqrt{t_i^N - t_k^N}}\int_{\mathbb{R}}dz|z|e^{C^2(\gamma_0^N|z|)^2}g_C(t_i^N - t_j^N, \hat{\theta}_{t_k^N,t_i^N}^N(z) - x)\\ \times\frac{1 + a_N\sqrt{\gamma_0^N}|y|^{S+2}}{\sqrt{T-t_i^N}}\frac{\sqrt{T-t_i^N}^{-1}}{\left(1 + \frac{z-\hat{\theta}_{t_i^N,T}^N(y)}{\sqrt{T-t_i^N}}\right)}.\end{aligned}$$

We use the inequalities:

$$(\gamma_0^N|z|)^2 \le C\left[(\gamma_0^N)^2|x-\hat\theta^N_{t_k^N,t_i^N}(z)|^2 + (\gamma_0^N|x|)^2\right].$$

Now, for x, y such that $\max(|x|,|y|) \le \frac{1}{(a_N\sqrt{\gamma_0^N})^{\frac{1}{S+2}}}$, we obtain:

$$e^{C^2(\gamma_0^N|z|)^2} \le e^{(\gamma_0^N)^2|x-\hat\theta^N_{t_k^N,t_i^N}(z)|^2} \times e^{(\gamma_0^N|x|)^2} \le Ce^{(\gamma_0^N)^2|x-\hat\theta^N_{t_k^N,t_i^N}(z)|^2},$$

and

$$\begin{aligned}\sqrt{\gamma_0^N}|z| \le C(\sqrt{\gamma_0^N}|x-\hat\theta^N_{t_k^N,t_i^N}(z)| + \sqrt{\gamma_0^N}|x|) &\le C(1+|x-\hat\theta^N_{t_k^N,t_i^N}(z)|)\\ &\le C\max(1,\sqrt{T})\left(1+\frac{|x-\hat\theta^N_{t_k^N,t_i^N}(z)|}{\sqrt{t_i^N-t_k^N}}\right).\end{aligned}$$

It follows from the above inequalities that

$$\begin{aligned}&\sqrt{\gamma_0^N}|z|e^{C(\gamma_0^N|z|)^2}g_C(t_i^N-t_k^N,\hat\theta^N_{t_k^N,t_i^N}(z)-x)\\ \le\;& Ce^{C(\gamma_0^N)^2\frac{|x-\hat\theta^N_{t_k^N,t_i^N}(z)|^2}{\sqrt{t_i^N-t_k^N}}}\left(1+\frac{|x-\hat\theta_{t_k^N,t_i^N}(z)|}{\sqrt{t_i^N-t_k^N}}\right)g_C(t_i^N-t_k^N,\hat\theta^N_{t_k^N,t_i^N}(z)-x)\\ \le\;& Cg_{C'}(t_i^N-t_k^N,\hat\theta^N_{t_k^N,t_i^N}(z)-x).\end{aligned}$$

Now, we have:

$$\begin{aligned}I \;\le\;& C\sqrt{\gamma_0^N}\sum_{i=k}^{M(N)}\frac{\gamma_{i+1}^N}{\sqrt{t_i^N-t_k^N}}\int_{\mathbb{R}}dz\,g_{C'}(t_i^N-t_k^N,\hat\theta^N_{t_k^N,t_i^N}(z)-x)\\ &\times\frac{1+a_N\sqrt{\gamma_0^N}|y|^{S+2}}{T-t_i^N}\frac{\sqrt{T-t_i^N}^{-1}}{\left(1+\frac{|z-\hat\theta^N_{t_i^N,T}(y)|}{\sqrt{T-t_i^N}}\right)^{S-7}}\\ \le\;& C\sqrt{\gamma_0^N}(1+a_N\sqrt{\gamma_0^N}|y|^{S+2})\sum_{i=k}^{M(N)}\frac{\gamma_{i+1}^N}{\sqrt{t_i^N-t_k^N}}\frac{1}{\sqrt{T-t_i^N}}\\ &\times\int_{\mathbb{R}}dz\frac{\sqrt{t_i^N-t_k^N}^{-1}}{\left(1+\frac{|\hat\theta^N_{t_k^N,t_i^N}(z)-x|}{\sqrt{t_i^N-t_k^N}}\right)^{S-7}}\frac{\sqrt{T-t_i^N}^{-1}}{\left(1+\frac{|z-\hat\theta^N_{t_i^N,T}(y)|}{\sqrt{T-t_i^N}}\right)^{S-7}}\\ \le\;& C\sqrt{\gamma_0^N}(1+a_N\sqrt{\gamma_0^N}|y|^{S+2})\Big(1+(\sqrt{\gamma_0^N}|x|)^{S-7}\Big)B(\frac{1}{2},\frac{1}{2})\frac{\sqrt{T-t_k^N}^{-1}}{\left(1+\frac{|x-\hat\theta^N_{t_k^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-7}}.\end{aligned}$$

We obtain:

$$\begin{aligned}&\Big|(\tilde q_N-\tilde p_N)\otimes_N(K_N+M_N)(t_k,T,x,y)\Big|\\ \le\;& C\sqrt{\gamma_0^N}(1+a_N\sqrt{\gamma_0^N}|y|^{S+2})\Big(1+(\sqrt{\gamma_0^N}|x|)^{S-7}\Big)B(\frac{1}{2},\frac{1}{2})\frac{\sqrt{T-t_k^N}^{-1}}{\left(1+\frac{|x-\hat\theta^N_{t_k^N,T}(y)|}{\sqrt{T-t_k^N}}\right)^{S-7}}.\end{aligned}$$

Suppose now that we have:

$$\begin{aligned}
&\Big|(\tilde{q}_N - \tilde{p}_N) \otimes_N (K_N + M_N)^{[r]}(t_k, T, x, y)\Big| \\
\leq\; & C^r K^r \sqrt{\gamma_0^N}(1 + a_N\sqrt{\gamma_0^N}|y|^{S+2})\Big(1 + (\sqrt{\gamma_0^N}|x|)^{S-7}\Big) \\
&\times (T - t_k^N)^{\frac{r-1}{2}} \prod_{j=1}^{r} B(\frac{j}{2}, \frac{1}{2}) \frac{\sqrt{T - t_k^N}^{-1}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\Big|(\tilde{q}_N - \tilde{p}_N) \otimes_N (K_N + M_N)^{[r+1]}(t_k, T, x, y)\Big| \\
\leq\; & C^r K^r \sqrt{\gamma_0^N}(1 + a_N\sqrt{\gamma_0^N}|y|^{S+2})\Big(1 + (\sqrt{\gamma_0^N}|x|)^{S-7}\Big) \prod_{j=1}^{r} B(\frac{j}{2}, \frac{1}{2}) \\
&\times \sum_{i=k}^{M(N)} \gamma_{k+1}^N \int_{\mathbb{R}} dz (1 + a_N|z|^{S+2}) \frac{(t_i^N - t_k^N)^{\frac{r-1}{2}} \sqrt{t_i^N - t_k^N}^{-1}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, T_i}(z)|}{\sqrt{t_i^N - t_k^N}}\right)^{S-7}} \frac{\sqrt{T - t_i^N}^{-1}}{\left(1 + \frac{|z - \hat{\theta}^N_{t_i^N, T}(y)|}{\sqrt{T - t_i^N}}\right)^{S-7}}.
\end{aligned}$$

Now, we use the inequality

$$1 + a_N\sqrt{\gamma_0^N}|z|^{S+2} \leq C\max(1, T^{\frac{S}{2}+1}) \left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, t_i^N}(z)|}{\sqrt{t_i^N - t_k^N}}\right)^{\frac{S}{2}+1} \left(1 + \frac{|z - \hat{\theta}^N_{t_i^N, T}(y)|}{\sqrt{T - t_i^N}}\right)^{\frac{S}{2}+1}.$$

Substituting this estimate and convoluting the polynomial functions, we have:

$$\int_{\mathbb{R}} \frac{\sqrt{t_i^N - t_k^N}^{-1}}{\left(1 + \frac{|\hat{\theta}^N_{t_k^N, t_i^N}(z) - x|}{\sqrt{t_i^N - t_k^N}}\right)^{\frac{S}{2}-8}} \frac{\sqrt{T - t_i^N}^{-1}}{\left(1 + \frac{|\hat{\theta}^N_{t_i^N, T}(y) - z|}{\sqrt{T - t_i^N}}\right)^{\frac{S}{2}-8}} dz \leq \frac{\sqrt{T - t_k^N}^{-1}}{\left(1 + \frac{|\hat{\theta}^N_{t_k^N, T}(y) - x|}{\sqrt{T - t_k^N}}\right)^{\frac{S}{2}-8}}.$$

We can now recover the right power, writing:

$$\begin{aligned}
\frac{\sqrt{T - t_k^N}^{-1}}{\left(1 + \frac{|\hat{\theta}^N_{t_k^N, T}(y) - x|}{\sqrt{T - t_k^N}}\right)^{\frac{S}{2}-8}} &= \frac{\sqrt{T - t_k^N}^{-1}}{\left(1 + \frac{|\hat{\theta}^N_{t_k^N, T}(y) - x|}{\sqrt{T - t_k^N}}\right)^{\frac{S}{2}-8}} \times \frac{\left(1 + \frac{|\hat{\theta}^N_{t_k^N, T}(y) - x|}{\sqrt{T - t_k^N}}\right)^{\frac{S}{2}+1}}{\left(1 + \frac{|\hat{\theta}^N_{t_k^N, T}(y) - x|}{\sqrt{T - t_k^N}}\right)^{\frac{S}{2}+1}} \\
&\leq CK \frac{\sqrt{T - t_k^N}^{-1}}{\left(1 + \frac{|\hat{\theta}^N_{t_k^N, T}(y) - x|}{\sqrt{T - t_k^N}}\right)^{\frac{S}{2}-7}},
\end{aligned}$$

recalling that

$$K = \left(1 + \max_{x \in K_x, y \in K_y, \{t_k^N\}} \frac{|\hat{\theta}^N_{t_k^N, T}(y) - x|}{\sqrt{T - t_k^N}}\right)^{\frac{S}{2}+1}.$$

Besides, comparing the sum to an integral, we get:

$$\sum_{i=k}^{M(N)} \gamma_{k+1}^N (t_i^N - t_k^N)^{\frac{r-1}{2}} (T - t_i^N)^{-\frac{1}{2}} \leq (T - t_k^N)^{\frac{r}{2}} B(\frac{r+1}{2}, \frac{1}{2}).$$

Finally, we obtain the desired expression:

$$\begin{aligned}
&\left|(\tilde{q}_N - \tilde{p}_N) \otimes_N (K_N + M_N)^{[r+1]}(t_k, T, x, y)\right| \\
\leq \quad & C^{r+1}K^{r+1}\sqrt{\gamma_0^N}(1 + a_N\sqrt{\gamma_0^N}|y|^{S+2})\Big(1 + (\sqrt{\gamma_0^N}|x|)^{S-7}\Big) \\
&\times (T - t_k^N)^{\frac{r}{2}} \prod_{j=1}^{r+1} B(\frac{j}{2}, \frac{1}{2}) \frac{\sqrt{T - t_k^N}^{-1}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}}.
\end{aligned}$$

Summing up we obtain

$$\sum_{r=0}^{N} \left|(\tilde{q}_N - \tilde{p}_N) \otimes_N (K_N + M_N)^{[r]}(t_k, T, x, y)\right| \leq C\sqrt{\gamma_0^N} \frac{\sqrt{T - t_k^N}^{-1}}{\left(1 + \frac{|x - \hat{\theta}^N_{t_k^N, T}(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}}.$$

where the constant C depends on the compacts sets K_x, K_y, and T but does not depends on N.

□

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