



On interrelations between divergence-free and Hamiltonian dynamics



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ABSTRACT

A mathematically correct description is presented on the interrelations between the dynamics of divergence free vector fields on an oriented 3-dimensional manifold M and the dynamics of Hamiltonian systems. It is shown that for a given divergence free vector field X with a global cross-section there exist some 4-dimensional symplectic manifold $\tilde{M} \supset M$ and a smooth Hamilton function $H : \tilde{M} \rightarrow \mathbb{R}$ such that for some $c \in \mathbb{R}$ one gets $M = \{H = c\}$ and the Hamiltonian vector field X_H restricted on this level coincides with X . For divergence free vector fields with singular points such an extension is impossible but the existence of local cross-section allows one to reduce the dynamics to the study of symplectic diffeomorphisms in some sub-domains of M . We also consider the case of a divergence free vector field X with a smooth integral having only finite number of critical levels. It is shown that such a noncritical level is always a 2-torus and restriction of X on it possesses a smooth invariant 2-form. The linearization of the flow on such a torus (i.e. the reduction to the constant vector field) is not always possible in contrast to the case of an integrable Hamiltonian system but in the analytic case (M and X are real analytic), due to the Kolmogorov theorem, such a linearization is possible for tori with Diophantine rotation numbers.

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1. Introduction

It is a rather frequent case when the Lagrangian description of liquid flows discovers structures characteristic for Hamiltonian dynamics [1,5,25,32]. Our aim in this note is to display the reason of this in the explicit form. In a sense, we proceed what was done by Arnold in [1,2]. To show the interrelations between these two types of dynamics we present some calculations which have to demonstrate how symplectic 2-dimensional maps arise in the Lagrangian description of liquid flows. One has to say that models, where divergence-free 3-dimensional vector fields are studied, appear not only at the Lagrangian description of liquid currents. Such equations arise also in the study of motions of thin liquid films flowing down along the inclined plane [5]. The same features are characteristic in other situations where the divergence-free flows arise [4]. This is observed in models of magnetic hydrodynamics, plasma confinement problems where magnetic line fields are studied [13,15,17,24]. An important application in plasma physics involves integration of magnetic field lines

$$\frac{dx}{dt_0} = B(x),$$

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where t_0 is an artificial time-like parameter parameterizing motion along the field line, and $\text{div}B$ is equal to zero [17]. In any case, the study of divergence-free vector fields is a very interesting problem itself, many features of their flow orbit structure deserve a detailed investigation, see, for instance [12,34].

The Lagrangian description of stationary 3-dimensional flows in the space \mathbb{R}^3 with coordinates (x, y, z) has the form

$$\dot{x} = A(x, y, z, t), \quad \dot{y} = B(x, y, z, t), \quad \dot{z} = C(x, y, z, t),$$

where vector field $V = (A, B, C)$ is the velocity field of liquid particles. If the liquid is incompressible, then $\text{div}V \equiv 0$ and the field is divergence-free. We consider here the autonomous case and suppose also that vector field V has some (possibly local) cross-section, that is, there is a 2-dimensional smooth submanifold N such that N is transverse to the vector field at points of N and for which orbits starting at points of some of its subdomain $N_1 \subset N$ return to N in finite times.

It is more convenient to carry out the calculations in the invariant (coordinate-free) form [4]. Consider a smooth (C^∞) 3-dimensional oriented manifold M with a smooth volume form Ω . Recall that for each smooth vector field X on M a smooth function is defined called *the divergence of the vector field*, $\text{div} X$. This function is given by the relation

$$L_X \Omega = d(\iota_X \Omega) = (\text{div} X)\Omega,$$

where $\iota_X \Omega$, called the interior product of Ω and X , is the 2-form such that for any $m \in M$, $\xi, \eta \in T_m M$ gives $\iota_X \Omega(\xi, \eta) = \Omega(X, \xi, \eta)$. If $\text{div} X \equiv 0$, then they say the vector field be divergence-free. For any given diffeomorphism $f : M \rightarrow M$ the pullback form $f^* \Omega$ is defined as $[f^* \Omega]_m(\xi, \eta, \zeta) = \Omega_{f(m)}(Df(\xi), Df(\eta), Df(\zeta))$. One says that f preserves the volume if the identity $f^* \Omega = \Omega$ holds true.

Let a vector field X be divergence-free. Then the flow for this vector field, i.e. one-parameter group of diffeomorphisms f^t (shifts along flow orbits) consists of divergence-free diffeomorphisms [4].

A nondegenerate volume form Ω on an oriented manifold (M, Ω) generates the correspondence between vector fields X on M and 2-forms ω . This is given by the relation

$$\omega_X = \iota_X \Omega, \tag{1}$$

for any given vector field X on M .

A standard example of a 3-dimensional divergence-free vector field (DFVF, for brevity) is a flow on a nonsingular level of a Hamilton function H for a Hamiltonian vector field X_H on some smooth symplectic 4-dimensional manifold (N, ω) with the Hamiltonian $H : N \rightarrow \mathbb{R}$ and symplectic 2-form ω . Suppose $dH \neq 0$ on some level $H = c$, then this level is a smooth 3-dimensional orientable submanifold M of N (nonsingular level). The flow in N generated by the vector field X_H preserves 2-form ω and hence, the iterated volume form $\Omega = \omega \wedge \omega$ as well. Let us endow N with some smooth Riemannian metric g . Then a vector field ∇H on M is correctly defined by the relation $\iota_{\nabla H} g = g(\nabla H, \cdot) = dH(\cdot)$. By construction, one gets $\nabla H \neq 0$ on M . Therefore, the function $\rho = 1/g(\nabla H, \nabla H)$ and a vector field $n = \rho \nabla H$ are defined and differentiable. Observe that

$$dH(n) = (\iota_{\nabla H} g)(n) = g(\nabla H, n) \equiv 1. \tag{2}$$

Let us set $\Omega_n = (\iota_n \Omega)|_M$. It turns out that Ω_n is nondegenerate, i.e. is a volume form and invariant w.r.t. the flow on M . The invariant proof of this assertion is given below in Section 3. In statistical physics the measure on M generated by Ω_n is usually called the *Liouville measure*.

Now we return to the manifold (M, Ω) with a divergence-free vector field X and assume X to have a smooth cross-section N . This means that $X(m) \notin T_m N$ for any $m \in N$. Suppose also for some subdomain $N_1 \subset N$ orbits of X return to N in finite time. Then the Poincaré map $P : N_1 \rightarrow N$ is defined. This map is a diffeomorphism from N_1 onto the image $P(N_1)$. As is known, Poincaré map P on N_1 generated by the flow f^t is defined as follows. For a point $x \in N_1$ its image $P(x)$ is defined as $P(x) = f^{T(x)}(x)$, where $T(x)$ is the first return time to N for the orbit through x . Due to transversality of N to the vector field at the points of N and smoothness of X , function $T(x)$ depends smoothly on x , hence P is a smooth map $P : N_1 \rightarrow N$. The following assertion is used in many papers, we present the proof for the reader convenience.

Proposition 1. *P is symplectic w.r.t. 2-form ω_X on N , in particular, ω_X is nondegenerate.*

Proof. To prove this fact, let us remark first the 2-form on ω_X as a function of two vectors $\omega_X(\xi, \eta)$ is bilinear and skew-symmetric. This follows from the properties of Ω . It is also nondegenerate. Indeed, let for a fixed $\xi \in T_x N$ one has $\omega_X(\xi, \eta) = 0$ for any $\eta \in T_x N$. We need to prove that $\xi = 0$. If $\xi \neq 0$, consider the plane L_ξ in $T_x M$ spanned by two independent vectors ξ and $X(x)$. If $\eta \in L_\xi$, then $\Omega(X(x), \xi, \eta) = 0$, but if $\eta \notin L_\xi$, then vectors $X(x), \xi, \eta$ are not coplanar and $\Omega(X(x), \xi, \eta) \neq 0$ since Ω is the volume form. But intersection of two transverse planes $T_x N$ and L_ξ is the straight-line $l_\xi \subset T_x N$ spanned by ξ . So, $\omega_X(\xi, \eta) = 0$ implies $\eta \in l_\xi$. But η is, by assumption, an arbitrary vector in $T_x N$, so $\xi = 0$. Also, on 2-dimensional manifold N any nondegenerate skew-symmetric 2-form is closed.

Let us shift the triple $(X(x), \xi, \eta)$ to $T_y M$ by the differential of the flow, where $y = f^{T(x)}(x) \in N$. We get $(Df^T X(x), Df^T \xi, Df^T \eta)$, here $T(x)$ is the transition time for point $x \in N_1$. Then the equality holds

$$\Omega(X(x), \xi, \eta) = (f^T)^* \Omega(X(x), \xi, \eta) = \Omega(Df^T(X(x)), Df^T(\xi), Df^T(\eta)),$$

since flow maps preserve Ω . Denote ω the restriction of the form ω_X on N .

To verify the symplecticity of the Poincaré map P we calculate the differential DP at x . Let us take on N a smooth curve $c(s)$ through x , $c(0) = x$, $c'(0) = \xi$, and transform it by the map $P(c(s)) = f^{T(c(s))}(c(s)) \in N$. Differentiating the transformed curve and setting $s = 0$ gives

$$DP(\xi) = Df^{T(x)}(\xi) + X(P(x))dT_x(\xi).$$

Thus the difference between $DP(\xi)$ and $Df^{T(x)}(\xi)$ is proportional to the vector $X(P(x))$ with the proportionality factor $dT_x(\xi)$. In addition, in accordance with the definition of the orbit, one has

$$\frac{d}{d\tau}(f^{t+\tau}(x))|_{\tau=0} = \frac{d}{d\tau}(f^\tau \circ f^t(x))|_{\tau=0} = X(f^t(x)).$$

Calculating this in the reverse order and using the group properties of the flow diffeomorphisms we come to

$$\frac{d}{d\tau}(f^{t+\tau}(x))|_{\tau=0} = \frac{d}{d\tau}(f^t \circ f^\tau(x))|_{\tau=0} = Df^t(X(x)).$$

Thus we have the identity for the flow

$$Df^t(X(x)) \equiv X(f^t(x)). \quad (3)$$

Now we come to the equality

$$\begin{aligned} \omega_x(\xi, \eta) &= \Omega(X(x), \xi, \eta) = (f^T)^* \Omega(X(x), \xi, \eta) = \Omega(Df^T X, Df^T \xi, Df^T \eta) \\ &= \Omega(X(P(x)), DP(\xi) - dT_x(\xi)X(P(x)), DP(\eta) - dT_x(\eta)X(P(x))) \\ &= \Omega(X(P(x)), DP(\xi), DP(\eta)) = \omega_{P(x)}(DP(\xi), DP(\eta)). \quad \blacksquare \end{aligned}$$

A similar construction being applied to the case of divergence-free vector fields on a smooth n -dimensional oriented manifold (M, Ω) with a nondegenerate n -form Ω and a smooth cross-section N leads to the study of volume preserving mappings on N w.r.t. the nondegenerate $(n - 1)$ -form ω . In the case of nonautonomous T -periodic in t divergence-free vector fields on an oriented 3-dimensional M this gives for its Poincaré map in period T the volume-preserving diffeomorphism of M .

These calculations show that if a divergence-free flow has a cross-section, then the related Poincaré map is symplectic and all known results on such maps are applicable (see, for instance, [21,30]). In particular, if a divergence free vector field X_0 has a smooth integral (integrable, see Section 3) and possesses a domain in the phase manifold filled with invariant tori, then its perturbation obeys the conclusions of KAM theory (see, for instance, [9]). Such type results were elaborated when studying divergence-free vector fields near singularities [8]. Also, all structures related with the stochastic properties of these maps reflect themselves in the chaotic behavior of the trajectories of liquid particles. But, of course, it is not obligatory, when the flow has a global cross-section. For instance, the well known ABC flow [4,5] on 3-torus most likely does not have global cross-sections. On the other hand, majority of such flows have periodic orbits, hence local cross-sections exist. In any case, this is the main reason why structures characteristic for the Hamiltonian flows are observed in liquid currents [1,5,19,24,25,31,32].

Similar considerations in Lagrangian periodic incompressible liquid flows lead to the study of iterations of volume preserving diffeomorphisms. There is a vast literature devoted to this topic. Many similarities with the structures observed in Hamiltonian dynamics can be found in these investigations. But there are many differences since the property of volume preservation gives much more freedom for the behavior of orbits [29,37].

2. Divergence-free vector fields and Hamiltonian systems

Now we suppose a divergence-free vector field X on an oriented smooth 3-manifold (M, Ω) to have a global cross-section. Due to a discussion above, this means that for some smooth closed 2-dimensional submanifold $N \subset M$ all orbits of X intersect N transversely and each orbit starting on N returns on N in a finite time. Then the Poincaré map $P : N \rightarrow N$ is globally defined on N and M itself is a bundle over S^1 with a leaf N . In other words, M is diffeomorphic to the suspension over a diffeomorphism $P : N \rightarrow N$ and a roof function $F : N \rightarrow \mathbb{R}$ being the return time $F(x)$ for the orbit through $x \in N$.

The construction above implies that there is a symplectic 2-form ω on N such that P is a symplectic diffeomorphism w.r.t. $\omega : P^* \omega = \omega$. In this section we want to show that there is a 4-dimensional smooth symplectic manifold (\tilde{M}, Λ) and a smooth Hamilton function H on \tilde{M} such that the Hamiltonian vector field X_H on some of its level $H = c$ coincides with the vector field X . Namely, the following theorem holds

Theorem 1. *There is a smooth manifold \tilde{M} of dimension four and a symplectic 2-form Λ on \tilde{M} such that*

- M is a smooth submanifold of \tilde{M} ,
- vector field X is extended till a Hamiltonian vector field \tilde{X} on (\tilde{M}, Λ) with a smooth Hamiltonian $H : \tilde{M} \rightarrow \mathbb{R}$ and M is the level $H = c$ for some c .

Proof. It is sufficient to consider M and the flow of the vector field X as given by the suspension over a diffeomorphism $P : N \rightarrow N$. To construct the manifold \tilde{M} and the vector field \tilde{X} we increase by one the dimension of the suspension construction.

Let us define the action $(x, r, s) \cdot n$ of the group \mathbb{Z} on the manifold $N \times \mathbb{R}^2$ setting $(P^n(x), r - n, s)$ for any $x \in N$ and all $n \in \mathbb{Z}$. This action is free and its set of orbits $\tilde{M} = (N \times \mathbb{R}^2)/\mathbb{Z}$ is a smooth manifold of dimension four and the quotient mapping $q : N \times \mathbb{R}^2 \rightarrow \tilde{M}$ is a regular covering.

If $\mathbb{R}^2 \times \mathbb{Z} \rightarrow \mathbb{R}^2$ is the auxiliary action defined as $(r, s) \cdot n = (r - n, s)$, then the factor-manifold $C = \mathbb{R}^2/\mathbb{Z}$ is diffeomorphic to the cylinder and the quotient mapping $q_0 : \mathbb{R}^2 \rightarrow C$ is the regular covering as well.

The formula $p(q(x, r, s)) = q_0(r, s)$ defines a submersion $p : \tilde{M} \rightarrow C$. Hence, a smooth bundle $\xi = (\tilde{M}, p, C)$ is defined.

Let $p_0 : N \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a natural projection. According to the construction, the following diagram is commutative

$$\begin{array}{ccc}
 N \times \mathbb{R}^2 & \xrightarrow{p_0} & \mathbb{R}^2 \\
 q \downarrow & & \downarrow q_0 \\
 \tilde{M} & \xrightarrow{p} & C
 \end{array} \tag{4}$$

Consider a flow \hat{h}^t on $N \times \mathbb{R}^2$ being defined by the formula $\hat{h}^t(x, r, s) = (x, r + t, s)$. This flow generates a vector field $\hat{X} = \partial/\partial r$. Since the flow \hat{h}^t commutes with the action of the group \mathbb{Z} , it induces the flow \tilde{h}^t on \tilde{M} . Denote its vector field as \tilde{X} .

A usual suspension over the diffeomorphism $P : N \rightarrow N$ is given, if everywhere in these constructions we set $s = 0$. Then $M = q(N \times \mathbb{R} \times 0)$ is a smooth 3-dimensional submanifold in \tilde{M} . The flow h^t corresponding to the vector field X on M is generated by the restriction of the flow \hat{h}^t on $N \times \mathbb{R} \times 0$. Therefore, one has $\tilde{X}|_M = X$, i.e. \tilde{X} is the extension of X from the submanifold $M \subset \tilde{M}$ to the whole \tilde{M} . Finally, the restriction of the bundle ξ over $S^1 = q_0(\mathbb{R} \times 0) \subset C$ gives the bundle $\xi_1 = (M, p_1, S^1)$.

The standard symplectic form $\lambda_0 = dr \wedge ds$ on \mathbb{R}^2 is invariant with respect to shifts $(r, s) \rightarrow (r - t, s)$, $t \in \mathbb{R}$. In particular, this is true for $t = n \in \mathbb{Z}$. Thus λ_0 generates a symplectic form λ on C . To further purposes, one needs to calculate the Lie derivative of the form $p^*\lambda$ in the direction of the vector field \tilde{X} .

Lemma 1. *The equality $L_{\tilde{X}}(p^*\lambda) = 0$ holds.*

Proof. By construction we get $\lambda_0 = q_0^*\lambda$. Due to (4) one has

$$q^*(p^*\lambda) = p_0^*\lambda_0. \tag{5}$$

The invariance of λ_0 with respect to the shifts implies that the form $p_0^*\lambda_0$ is preserved by the flow \hat{h}^t on $N \times \mathbb{R}^2$. In virtue of $\hat{h}^t \circ q = q \circ \hat{h}^t$ we have

$$q^*((\hat{h}^t)^*(p^*\lambda)) = (\hat{h}^t)^*(q^*(p^*\lambda)) = (\hat{h}^t)^*(p_0^*\lambda_0) = p_0^*\lambda_0 = q^*(p^*\lambda).$$

The mapping q is a covering, therefore the equalities obtained imply that $(\tilde{h}^t)^*(p^*\lambda) = p^*\lambda$. It remains to remember that the flow \tilde{h}^t is generated by the field \tilde{X} . ■

A natural projection $\mu_0 : N \times \mathbb{R}^2 \rightarrow N \times \mathbb{R} \times 0$ induces the map $\mu : \tilde{M} \rightarrow M$ via the formula $\mu(q(x, r, s)) = q(x, r, 0) = q(\mu_0(x, r, s))$. Here for any points $\hat{u} \in N \times \mathbb{R}^2$, $\tilde{u} = q(\hat{u})$ and $u = \mu(\tilde{u})$ we have

$$D\mu(\tilde{X}(\tilde{u})) = D\mu(Dq(\hat{X}(\hat{u}))) = Dq(D\mu_0(\hat{X}(\hat{u}))) = X(u).$$

Let us set $\tilde{\omega} = \mu^*\omega$ and $\tilde{\omega} = \iota_{\tilde{X}}\tilde{\omega}$. Then the preceding equalities imply $\tilde{\omega} = \mu^*\omega_X$.

We set $\Lambda = \tilde{\omega} + \varepsilon^{-1}p^*\lambda$ for some $\varepsilon \neq 0$.

Lemma 2. *2-form Λ on \tilde{M} is symplectic.*

Proof. First we prove that the form $\tilde{\omega}$ is nondegenerate on the leaves of the bundle $\xi = (\tilde{M}, p, C)$.

Each leaf F_1 of the bundle $\xi_1 = (M, p_1, S^1)$ is a cross-section for the flow generated by the vector field X . By Proposition 1 this means that the form ω_X is nondegenerate on F_1 .

Let $c = q_0(r, s) \in C$ and $c_1 = q_0(r, 0)$. We set $F = p^{-1}(c)$ and $F_1 = p_1^{-1}(c_1)$. Then one has $F = q(N \times r \times s)$, $F_1 = q(N \times r \times 0)$ and therefore the restriction $\mu|_F : F \rightarrow F_1$ is a diffeomorphism. But then the form $\tilde{\omega} = \mu^*\omega_X$ is nondegenerate on F .

The nondegeneracy of the form $\tilde{\omega}$ on bundle leaves of ξ and lemma 1 from [28] imply that Λ is symplectic on \tilde{M} . ■

Thus we have constructed a symplectic manifold (\tilde{M}, Λ) for which the first assertion of the theorem holds and the vector field \tilde{X} on \tilde{M} that is the extension of the field X . It remains to prove that \tilde{X} is Hamiltonian.

Lemma 3. *The form $\iota_{\tilde{X}}\Lambda$ is closed.*

Proof. For any vector field \tilde{Y} on manifold \tilde{M} one has

$$(\iota_{\tilde{X}}\Lambda)(\tilde{Y}) = \Lambda(\tilde{X}, \tilde{Y}) = \tilde{\omega}(\tilde{X}, \tilde{Y}) + \frac{1}{\varepsilon}(p^*\lambda)(\tilde{X}, \tilde{Y}).$$

But the following equalities hold $\tilde{\omega}(\tilde{X}, \tilde{Y}) = (i_{\tilde{X}}\tilde{\Omega})(\tilde{X}, \tilde{Y}) = \tilde{\Omega}(\tilde{X}, \tilde{X}, \tilde{Y}) = 0$, hence

$$i_{\tilde{X}}\Lambda = \frac{1}{\varepsilon}i_{\tilde{X}}(p^*\lambda). \tag{6}$$

Also we have $d(i_{\tilde{X}}(p^*\lambda)) = L_{\tilde{X}}(p^*\lambda) - i_{\tilde{X}}(d(p^*\lambda))$. Since $d(p^*\lambda) = p^*(d\lambda) = 0$, the preceding formula and Lemma 1 imply $d(i_{\tilde{X}}(p^*\lambda)) = 0$. In virtue of (6) lemma has been proved. ■

Lemma 4. *Periods of the form $i_{\tilde{X}}\Lambda$ along any 1-cycle in the manifold \tilde{M} are equal to zero.*

Proof. The inclusion $M \rightarrow \tilde{M}$ is a homotopic equivalence. Then it is sufficient to calculate periods along cycles lying in M . It is also evident that as a singular 1-cycle one can understand a piece-wise smooth loop $\gamma : I \rightarrow M, I = [0, 1]$, with the initial point $\gamma(0) = \gamma(1) = q(x_0, 0, 0)$.

Suppose $\hat{\gamma} : I \rightarrow N \times \mathbb{R} \times 0$ be a path with an initial point $\hat{\gamma}(0) = (x_0, 0, 0)$, for which $q \circ \hat{\gamma} = \gamma$. For any $t \in I$ as N_t and R_t we denote the leaves of trivial bundles $N \times \mathbb{R} \times 0 \rightarrow \mathbb{R}$ and $N \times \mathbb{R} \times 0 \rightarrow N$, through the point $\hat{\gamma}(t)$. Then we have $d\hat{\gamma}/dt = Y_2 + Y_1$, where $Y_2 \in T_{\hat{\gamma}(t)}N_t$ and $Y_1 \in T_{\hat{\gamma}(t)}R_t$. Here we have $Dp_0(Y_2) = 0$ and vectors Y_1 and $\hat{X}(\hat{\gamma}(t))$ are collinear. This implies $i_{\tilde{X}}(p_0^*\lambda_0)(Y_k) = (p_0^*\lambda_0)(\hat{X}(\hat{\gamma}(t)), Y_k) = 0$ for $k = 1, 2$. These equalities imply $i_{\tilde{X}}(p_0^*\lambda_0)(d\hat{\gamma}/dt) = 0$ and then

$$\int_{\hat{\gamma}} i_{\tilde{X}}(p_0^*\lambda_0) = \int_0^1 i_{\tilde{X}}(p_0^*\lambda_0)(d\hat{\gamma}/dt) = 0. \tag{7}$$

Due to (5) and (6) we come to

$$\int_{\gamma} i_{\tilde{X}}\Lambda = \frac{1}{\varepsilon} \int_{\gamma} i_{\tilde{X}}(p^*\lambda) = \frac{1}{\varepsilon} \int_{\hat{\gamma}} i_{\tilde{X}}(q^*(p^*\lambda)) = \frac{1}{\varepsilon} \int_{\hat{\gamma}} i_{\tilde{X}}(p_0^*\lambda_0). \tag{8}$$

From (7) and (8) the proof follows. ■

Lemmata 3 and 4 imply, in accordance with de Rham theorem [38, chapter 5], that the form $i_{\tilde{X}}\Lambda$ is exact. Therefore there is a smooth function $H : \tilde{X} \rightarrow \mathbb{R}$ for which the identity $dH = i_{\tilde{X}}\Lambda$ holds. But this means that the vector field \tilde{X} is globally Hamiltonian.

To complete the proof we need to show that M is a level of the Hamiltonian H . Fix some point $x_0 \in N$ and set $\tilde{u}_0 = q(x_0, 0, 0)$. Then the value of Hamiltonian $H : \tilde{M} \rightarrow \mathbb{R}$ at an arbitrary point $\tilde{u} = q(x, r, s)$ can be calculated by means of the formula $H(\tilde{u}) = \int_{\gamma} i_{\tilde{X}}\Lambda$, where $\gamma : [0, 1] \rightarrow \tilde{M}$ is some piece-wise smooth path with the end points $\gamma(0) = \tilde{u}_0$ and $\gamma(1) = \tilde{u}$.

We put also $\tilde{u}_s = q(x_0, 0, s)$ for all $s \in \mathbb{R}$. The integral $\int_{\gamma} i_{\tilde{X}}\Lambda$ depends only on the endpoint of the path γ . This allows one to regard that $\gamma = \alpha_s \gamma_{rs}$ where $\alpha_s(t) = q(x_0, 0, ts)$ and the path $\gamma_{rs} : [0, 1] \rightarrow \tilde{M}$ lies in the submanifold $M_s = q(N \times \mathbb{R} \times s)$, it joins points \tilde{u}_s and \tilde{u} . But for such a path γ_{rs} the equalities hold: (7) and (8). Hence we get $\int_{\gamma_{rs}} i_{\tilde{X}}\Lambda = 0$ and

$$H(\tilde{u}) = \int_{\alpha_s} i_{\tilde{X}}\Lambda + \int_{\gamma_{rs}} i_{\tilde{X}}\Lambda = \int_{\alpha_s} i_{\tilde{X}}\Lambda.$$

Thus, the value $H(\tilde{u})$ does not depend on the number r . Therefore, submanifolds $M_s \subset \tilde{M}$ are the level sets of the Hamiltonian H . In particular, there is a number $c \in \mathbb{R}$ such that $M = M_0 = \{\tilde{u} \in \tilde{M} | H(\tilde{u}) = c\}$. ■

It is evident that the extension of a DFVF given on smooth 3-dimensional oriented manifold M till a Hamiltonian vector field on some symplectic 4-dimensional \tilde{M} is impossible if X has singular points (equilibria) [15,26]. Indeed, as is known, equilibria of a Hamiltonian vector field coincide with critical points of the Hamiltonian. But usually a critical level of a Hamiltonian is not a smooth manifold since it has singularities at critical points (near them the level is not a smooth manifold).

Nevertheless, the reduction to 2-dimensional symplectic diffeomorphisms here is exploited also, for instance, when studying homoclinic dynamics. This can be seen, in particular, in [5]. In these cases the related Poincaré maps are discontinuous but symplectic since they are constructed on cross-sections to homoclinic orbits to an equilibrium where the discontinuity takes place on the trace of stable (unstable) manifold to the related saddle or saddle-focus equilibrium. Other equilibria are also possible but they are degenerate (have either zeroth or pure imaginary eigenvalues) because the condition $divX = 0$ holds at the equilibrium.

There is an interesting case where a close construction is exploited in other circumstances [11]. This concerns the case when a map $f : M \rightarrow M$ acts on a smooth manifold and it is assumed in addition that there is a symmetry for f , that a smooth vector field v on M exists such that $f^*v = v$ where $[f^*v](x) = Df^{-1}v(f(x))$. Hence f transforms v to itself and then orbits of the flow are transformed to the orbits of the same flow. Suppose that the flow of v is complete (all orbits are extendable onto whole \mathbb{R}) and has a global cross-section Σ . In this case there is a covering space $\Sigma \times \mathbb{R} \rightarrow M$ generated by the Poincaré map on Σ . It appears there exists a lift F of the map f to $\Sigma \times \mathbb{R}$ which has a form of a skew product map with the base Σ . Moreover, if f preserves a volume form Ω on M and v is divergence-free, then the map on the base Σ preserves the induced form on Σ . This shows connections with our results.

3. On integrable divergence-free vector fields

In this section we single out a class of DFVF with a simple structure. They can be taken as initial systems for applying perturbation methods.

Suppose on a smooth 3-dimensional oriented manifold (M, Ω) a smooth DFVF X be given. We shall call such a vector field to be *integrable* if it possesses a smooth integral F , i.e. a smooth function $F : M \rightarrow \mathbb{R}$ which satisfies the identity $dF(X) \equiv 0$. A particular case of this situation was studied, for instance, in [32]. There is also another case of integrability similar to that for a general 3-dimensional systems, where by the integrability one understands the existence two integrals independent almost everywhere. This case for a DFVF we call *super-integrable*.

If a vector field X is integrable, then the manifold M is foliated into levels of this function $F = c$. A first natural question in this case arises: do some restrictions exist on the topology of levels of function F and flows generated by X on the invariant subset $F = c$? Recall that usually the integrability of 3-dimensional vector fields requires to have two independent (almost everywhere) integrals (what we called super-integrability above). We shall show that for many goals it is enough to have only one smooth integral to investigate the orbit structure on the majority of levels of the integral. This also shows the close relation with integrable Hamiltonian vector fields where the (Liouville) integrability of a 2-degrees-of-freedom Hamiltonian vector field X_H follows from the existence of one additional smooth integral independent of H almost everywhere (due to the Liouville–Arnold theorem [3]). But as we shall see, the orbit behavior on levels $F = c$ can be a bit more complicated than in the integrable Hamiltonian case: the flow on such a level is not always linear, this depends on interrelations between arithmetics of rotation numbers and smoothness of the flow as in the Kolmogorov theorem [22]. The details on the orbit structure of the integrable Hamiltonian systems with two degrees of freedom see in [6,27].

Let us consider the case when $dF \neq 0$ on a level $F = c$ (more precisely, its connected component) assuming this component be compact. We call such a level nonsingular. Thus a smooth closed connected 2-dimensional submanifold $\Sigma = \{F = c\}$ is an invariant set w.r.t. the flow generated by DFVF X .

Proposition 2. *A closed nonsingular invariant 2-dimensional level Σ is orientable. If X has not zeros on Σ , then it is an invariant torus. Moreover, all close to Σ levels are also smooth invariant tori with flows without zeros.*

Proof. Consider a tubular neighborhood B of Σ . Since $dF \neq 0$ on it, Σ separates B into parts with different signs of the function $F - c$. On the other hand, if Σ would be nonorientable, then it does not separate B [7]. Thus, Σ is an orientable closed invariant 2-dimensional manifold. If X has not zeros on Σ , then it cannot be the Klein bottle and it is a torus. In this case, all close levels of F are diffeomorphic to Σ by the Morse theory [33] and carry flows without zeros, that is, they are smooth invariant tori. ■

Now we consider the level $\Sigma = \{F = c_0\}$ without equilibria on Σ . In principle, a flow without equilibria on a smooth torus can have different structures. If the flow has a global cross-section, then its structure depends mainly on its Poincaré rotation number and the smoothness of the flow [16]. But the flow on a torus without a global cross-section may have a Reeb component [18]. We want to show that the divergence-free property implies the strong restrictions on the flow behavior. Namely, we shall prove the existence of smooth invariant measure for the flow.

In particular, 2-form $\omega_X(\xi, \eta)$ is completely degenerate on Σ . Indeed, if vector $\xi \in T_x\Sigma$ is not collinear to $X(x)$, then the plane L_ξ in T_xM spanned by two independent vectors ξ and $X(x)$ coincides with the tangent plane $T_x\Sigma$. Thus for any vector $\eta \in T_x\Sigma$ one has $\omega_X(\xi, \eta) = 0$. This is an analog for Σ to be a Lagrangian torus in the symplectic setting.

So, in order to get an invariant measure on Σ , we need to go in another way. The natural way is the following. Take a thin layer $F = c, |c - c_0| < \varepsilon$, where ε is small enough and positive. Then these levels of F are also smooth tori with vector fields without equilibria. The flow in this layer preserves the volume. Let us introduce some smooth Riemannian metrics in this layer. Since Σ is two-sidedly imbedded, then on Σ a smooth field of normal vectors exists. Choose such a field and denote $n(x) \in T_xM$ its normal vector at the point $x \in \Sigma$. Then a 2-form ω_n on $\Sigma, \omega_n(\cdot, \cdot) = \Omega(n(x), \cdot, \cdot)$ is defined.

Theorem 2. *A vector field of normals can be chosen in such a way that: (i) 2-form ω_n on Σ is nondegenerate; (ii) the restriction of X on the level $F = c_0$ defines the flow φ_Σ^t that preserves the form $\omega_n, (\varphi_\Sigma^t)^*\omega_n = \omega_n$.*

Proof. Choose $\varepsilon \in \mathbb{R}, \varepsilon > 0$, small enough such that on submanifold $M_\varepsilon = \{x \in M \mid |F(x) - c_0| < \varepsilon\}$ of M the inequalities $dF \neq 0$ and $X \neq 0$ stay valid.

Let g be some smooth Riemannian metrics on M_ε . Then a vector field ∇F on M_ε is correctly defined by the relation $i_{\nabla F}g = g(\nabla F, \cdot) = dF(\cdot)$. By construction, one gets $\nabla F \neq 0$ on M_ε . Therefore, the function $\rho = 1/g(\nabla F, \nabla F)$ and a vector field $n = \rho \nabla F$ are defined and differentiable. Observe that

$$dF(n) = (i_{\nabla F}g)(n) = g(\nabla F, n) \equiv 1. \tag{9}$$

Let us set as above $\omega_n = (i_n\Omega)|_\Sigma$.

The vector field X is tangent to Σ , therefore we get

$$L_X\omega_n = (L_X(i_n\Omega))|_\Sigma. \tag{10}$$

By definition $\iota_n \Omega = c(n \otimes \Omega)$, where c denote the contraction of the tensor $n \otimes \Omega$. In accordance to proposition 3.2 in [35] we have

$$L_X(c(n \otimes \Omega)) = c(L_X(n \otimes \Omega)) = c(L_X n \otimes \Omega) + c(n \otimes L_X \Omega),$$

where $L_X n = [X, n]$ is the Lie bracket of vector fields. As a result we come to the relation

$$L_X(\iota_n \Omega) = \iota_{[X, n]} \Omega + \iota_n(L_X \Omega). \tag{11}$$

Lemma 5. *The vector field $[X, n]$ is tangent to Σ .*

Proof. The vector $n(x)$ is orthogonal to the surface Σ , thus the representation $[X, n](x) = hn(x) + Z$ holds, where $h \in \mathbb{R}$ and $Z \in T_x \Sigma$. But then, due to (9) we get

$$[X, n](x)F = hn(x)F + ZF = h dF(n(x)) + dF(Z) = h.$$

From the other hand, one has $nF = dF(n) \equiv 1$ and $XF = dF(X) \equiv 0$. Therefore, the equality holds

$$[X, n]F = X(nF) - n(XF) \equiv 0.$$

The equalities obtained lead to $h = 0$ and hence $[X, n](x) = Z \in T_x \Sigma$. ■

Returning to the proof of the theorem, consider arbitrary $x \in \Sigma$ and $Y, Z \in T_x \Sigma$. It follows from the equality $\dim \Sigma = 2$ and Lemma 5 that three vectors Y, Z and $[X, n](x)$ are coplanar. Therefore we get $(\iota_{[X, n]} \Omega)(Y, Z) = \Omega([X, n](x), Y, Z) = 0$ and hence

$$(\iota_{[X, n]} \Omega)|_\Sigma = 0. \tag{12}$$

Since $L_X \Omega = 0$, then from (11) and (12) the equality $(L_X(\iota_n \Omega))|_\Sigma = 0$ follows and hence $L_X \omega_n = 0$, in virtue of (10).

The orthogonality of n to the surface Σ implies also that the form ω_n is nondegenerate and consequently is the area form on Σ . ■

Thus, for any nondegenerate level $\Sigma_c = \{F = c\}$ we have the vector field X_c being the restriction of X onto the invariant submanifold Σ_c . This vector field is nonsingular and preserves the area form ω_n . Let us choose some angle variables (φ, ψ) on the torus Σ . Then 2-form ω_n takes the form $a(\varphi, \psi)d\varphi \wedge d\psi$ with the smooth doubly periodic positive function a and the vector field has the form

$$\dot{\varphi} = A(\varphi, \psi), \quad \dot{\psi} = B(\varphi, \psi),$$

where $A^2 + B^2 \neq 0$ and both smooth functions A, B are doubly periodic. The measure preservation means here

$$\frac{\partial}{\partial \varphi}(aA) + \frac{\partial}{\partial \psi}(aB) = 0.$$

Denote

$$\lambda_1 = \int_\Sigma A a d\varphi \wedge d\psi, \quad \lambda_2 = \int_\Sigma B a d\varphi \wedge d\psi.$$

The number $\lambda = \lambda_1/\lambda_2$ called the Poincaré rotation number plays the main role in the orbit dynamics on the torus Σ . As is known, if λ is rational or one of λ_i is equal to zero, then all orbits of the flow are periodic (this is because of the existence of a smooth invariant measure). But if λ is irrational and the flow is of smoothness C^2 then all orbits on the torus are transitive. More subtle ergodic properties of the flow depend on the interrelations between the arithmetic type of λ and a smoothness of functions A, B [22]. For instance, for the case of C^5 -smooth r.h.s. the flow can have a continuous spectrum [20] (also see details in [23]).

One can think that the integrability of X imply the flow structure like in the integrable Hamiltonian system. In fact, the existence of an integral in M does not imply that this additional integral can be extended onto the 4-dimensional symplectic manifold constructed above. This is not the case even for the case of a Hamiltonian system with two degrees of freedom. Such a system can be nonintegrable in the whole phase space but be integrable on some separate level of Hamiltonian. To present such an example, let us consider some smooth symplectic four-dimensional manifold (M, Λ) with a symplectic form Λ and a smooth Hamiltonian H_0 . We assume the related Hamiltonian vector field X_{H_0} be integrable, that is there is an additional smooth integral $K, \{H_0, K\} \equiv 0$. Consider a perturbation of this vector field $H = H_0 + \varepsilon H_1$. Let us fix c . One can choose the function H_1 in the form $H_1 = (H_0 - c)F$ such that on the level $H_0 = c$ the integrable system has some integrable structure, and a function F can be taken arbitrarily. Let $J_X : T_X M \rightarrow T_X^* M$ be the isomorphism between 1-forms and vector fields on M defined by the symplectic form Λ . Then JdH is the Hamiltonian vector field generated by function H . Thus we get the Hamiltonian vector field

$$J(dH_0 + \varepsilon F dH_0 + \varepsilon(H_0 - c)dF).$$

On the set $H = c$ we have $(H_0 - c)(1 + \varepsilon F) = 0$, thus it coincides with the level of the function H_0 and therefore is the invariant submanifold where the dynamics is integrable. Indeed, $X_H = J(1 + \varepsilon F)dH_0$ on this level, hence the vector field is

obtained by the change time from the integrable vector field X_{H_0} on the level $H_0 = c$. It is evident that function F can be chosen in such a way that the complete dynamics would be nonintegrable. For instance, it can be reached, if on the level $H_0 = c$ the orbit structure will contain a saddle periodic orbit with its merged stable and unstable manifolds. Function F can be chosen in such a way that on close levels stable and unstable manifolds of saddle periodic orbits existing when changing $H = c$ would be split transversely. This shows that the integral K generally cannot be extended onto the whole phase space.

Remark 1. All constructions above where they appeal to the Riemannian metrics use a possibility to construct a smooth Riemannian metrics on a smooth manifold M , applying, for instance, a partition of unity [35]. If the manifold M is real analytic this construction does not work and one needs to use other tools. For example, to find a Riemannian metrics in this case one can exploit the Morrey–Grauert theorem about an analytic embedding of M into the Euclidean space (see, for instance, [36]) and then to restrict the Euclidean metrics to this embedded manifold. Thus all constructions can be done analytic as well. In particular, this concerns the case of an integrable DFVF and the flow on its nonsingular level without zeros. The flow in this case is real analytic and all conclusions of the Kolmogorov theorem hold.

3.1. On global structure of integrable divergence free vector fields

For the case of integrable DFVFs one can develop a global theory of such vector fields similar to the case of integrable Hamiltonian vector fields on smooth symplectic four-dimensional manifolds [6,27]. Recall that if H be a smooth Hamilton function and F its additional integral, one can restrict the Hamiltonian vector field X_H on some nondegenerate level $V_c = \{H = c\}$ where $dH \neq 0$. Suppose the restriction F_c of this additional integral be a smooth Bottian function [6], that is it has finitely many critical values and the related critical sets of F_c are organized into the finitely many critical closed smooth curves such that the restriction of F_c on the transverse disk D to the critical curve l generates a Morse function on D with a nondegenerate critical point at the trace of l on D . For the restriction of the flow X_H on V_c these critical curves are usually closed periodic orbits for X_H and for Bottian integral they can be only of two types: elliptic and hyperbolic ones in dependence of what type of a nondegenerate critical point on D is. If the level V_c is a closed manifold (compact without a boundary), then almost all levels of F are invariant Lagrangian tori (by the Liouville–Arnold theorem). This foliation can be described via the invariant introduced by Fomenko (see details in [6]).

A similar theory can be also elaborated for the case of integrable DFVFs. Suppose a smooth closed oriented 3-dimensional manifold (M, Ω) be given and X be a DFVF which we assume to be integrable and without equilibria (since M is oriented and closed, its Euler characteristic is zero). Let F be the related smooth integral, i.e. $dF(X) \equiv 0$, such that it has a finite number of critical levels. We assume that each critical set of F consists of finitely many smooth disjoint closed curves l_1, \dots, l_m such that on a cross-section to such curve l_i the restriction of F is a smooth function that has a Morse critical point at the trace of l_i . A simplest example of such situation is an integrable Hamiltonian vector field on a symplectic manifold restricted on the non-degenerate level of its additional integral when this restriction is a Bottian function. There are many such examples in mechanics (see, for instance, [6,10]).

Suppose now γ be a periodic orbit of X and $\{F = F(\gamma) = f\}$ be a connected set containing γ . If this level is not critical, then its connected component containing γ is an invariant smooth torus with a rational rotation number since it contains a periodic orbit. Due to the existence of smooth invariant measure on this torus all orbits on the torus are periodic and γ cannot be isolated. Thus, if γ is an isolated periodic orbit of X , then the connected set of $F = f$ either coincides with γ and γ is an elliptic periodic orbit for X or γ is a saddle periodic orbit and the connected set of $F = f$ coincides with the merged stable and unstable manifolds of γ (recall that F is preserved along any orbit of X).

In the first case γ is a closed curve of maximums (minimums) for F and for the second case γ is the curve of saddle critical points for F . This is easily derived from the Bott property like in [14].

By analogy with the Hamiltonian case we shall call such an integral to be Bottian. As we know, all noncritical levels of F are disjoint invariant tori. We assume, in addition, that almost all of these tori carry flows with irrational rotation numbers. This assumption prevents the super-integrability – the existence of the second independent almost everywhere integral for X .

First of all, we observe that an orbit of X through a critical point of F , since this orbit is not an equilibrium of X , consists of critical points of F . Indeed, by definition, one has $(D\Phi^t)^*(dF) = dF(D\Phi^t)$. If $dF(\xi) = 0$ for any $\xi \in T_x M$ (the point x is critical), then $dF(\eta) = 0$ for any $\eta = D\Phi^t(\xi)$ at the point $y = \Phi^t(x)$. Since Φ^t is a diffeomorphism, then $D\Phi^t$ is the isomorphism of tangent spaces. Hence, each connected critical set of F being a smooth closed curve is a periodic orbit of X . Denote such orbit γ . Choose a local transverse disk D to γ at some its point $m \in \gamma$ and consider a smooth function f being the restriction of F on D . Point m is a nondegenerate (Morse) critical point for f (the Bott property). Hence, m can be of two types, a center or a saddle. Since F is the integral, then for the case of center γ is enclosed by the family of invariant tori for X . If m is a saddle, then through m two smooth segments pass composing together the intersection of D with the local level of the set $F = F(m)$. Generically, for the saddle case the related periodic orbit of X through m is a saddle periodic orbit, hence one segment generates a local stable manifold of the orbit γ and another segment does a local unstable manifold of γ .

As is known, in the whole phase space a saddle periodic orbit can be of two types: orientable and nonorientable. For a periodic orbit of X the flow generates on the cross-section D the Poincaré map $P : D \rightarrow D$ being a symplectic mapping w.r.t. the restriction of Ω on D (see above). This symplectic map has a fixed point at m . The multipliers of $DF|_m$ can be generically either two complex conjugate numbers on the unit circle (elliptic fixed point) or two different nonzero real numbers μ, μ^{-1} (saddle fixed point). An orientable periodic orbit corresponds to the positive multipliers μ, μ^{-1} of $DF|_m$ and nonorientable

periodic orbit corresponds to the negative multipliers of $DF|_m$. Local stable and unstable 2-dimensional manifolds $W^s(\gamma)$, $W^u(\gamma)$ for the orientable periodic orbits are both cylinders and for the nonorientable periodic orbit they are both Möbius strips. Their continuation by the flow of X gives global stable and unstable manifolds which we also denote as $W^s(\gamma)$, $W^u(\gamma)$. Consider the ω -limit set of an orbit from $W^u(\gamma)$. Since M is closed, this set is not empty, invariant and closed.

To ease the exposition, we assume that all critical sets of F are either elliptic periodic orbits (critical sets of maximums or minimums) or saddle periodic orbits for X . Then for the case of Bottian integral all orbits of X and their limit sets in M are known. They are either closed periodic orbits composed by critical sets of F or they belong to stable (unstable) manifolds of saddle periodic orbits (saddle critical curves) or they belong to invariant 2-tori. In the latter case their ω - (α -) limit sets belong to the same torus as well. In the case under consideration, all limit set for an orbit in $W^u(\gamma)$ is one of saddle periodic orbits. More precisely, the following assertion is valid

Lemma 6. Consider a critical level $\{F = F(\gamma)\}$ being a compact set in M and let $\gamma_1, \dots, \gamma_s$ be all saddle periodic orbits of X in this set. Then any connected component of the set $\{F = F(\gamma)\} \setminus \{\gamma_1, \dots, \gamma_s\}$ consists of homo- or heteroclinic orbits of one or two different saddle periodic orbits from the collection $\{\gamma_1, \dots, \gamma_s\}$.

This lemma means that all stable and unstable manifolds of the same or different saddle periodic orbits coalesce.

Now one may consider, as in [14], the evolution of tori and their reconstructions when a natural parameter of the system, the value of integral F , varies. Since M is closed, F takes its maximal f_+ and minimal f_- critical values. For a Bottian integral F we have the related critical sets – a finite set of elliptic periodic orbits. Each maximal (and minimal) periodic orbit γ_i^+ (γ_j^-) gives rise to the one-parameter family of tori starting from γ_i^+ (if f_+ corresponds to several maximal critical curves, then we get several related families of tori). Tori of different families (or even from the same family), as F varies, can collide when they approach to the critical levels of saddle critical curves, reconstruct and continue forming other families. The complete reconstructions can be described by the invariant like “moleculas” (see details in [6]).

Remark 2. If a DFVF in question has equilibria, then its integrability with a smooth integral is in question. For instance, if this integral would have nondegenerate (of the Morse type) critical points, then the local structure of levels for this integral is determined by the Morse lemma [33]. The singular level containing the critical point is nonsmooth at the critical point (it has a cone-like singularity). But this level is an invariant set of the flow and usually this point is simultaneously a singular point of the vector field. Therefore, either this singular point has to be degenerate or the integral should have more degenerate critical points. Examples of (super-)integrable vector field in [34] show this. Another possibility is a nondegenerate (hyperbolic) singular point of a DFVF and its smooth integral with lines of critical points being not Bottian.

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