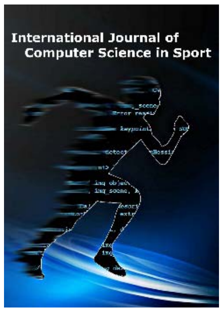
International Journal of Computer Science in Sport

Volume 17, Issue 2, 2018

Journal homepage: http://iacss.org/index.php?id=30

sciendo

DOI: 10.2478/ijcss-2018-0006



Generalized Knockout Tournament Seedings

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Abstract

Generalized knockout tournament seedings for an arbitrary number of participants in one match are designed. Several properties of knockout tournament seedings are investigated. Enumeration results for knockout tournament seedings with different properties are obtained. Several new generalized knockout tournaments seedings are proposed and justified by a set of properties.

KEYWORDS: ELIMINATION TOURNAMENT, COMBINATORIAL OPTIMIZATION, OR IN SPORTS, SEEDING.

Introduction

There are two main types of sports tournaments: knockout tournaments and round robin tournaments. In knockout tournaments, all losers in each round are eliminated, and all winners are promoted to the next round. In round robin tournaments, all participants play against each other. The main advantage of knockout tournaments is a significantly lower number of matches and rounds. As a result, spectator interest increases from round to round. For example, if the number of participants in a tournament equals $N = 2^n$, then the number of games (with two players in one match) in a knockout tournament equals $2^n - 1$, whereas that in a round robin tournament equals $2^{n-1}(2^n - 1)$. Similarly, the number of rounds in a knockout tournament equals n, and that in a round robin tournament equals $2^n - 1$. The lower requirements for time and facilities and the increasing spectator interest are the main reasons for the popularity of knockout tournaments.

There are many single-winner games with a higher number of players in one match (e.g., card games such as blackjack and poker). Football teams are typically divided into groups, with four teams in each group. A round robin subtournament within one group can thus be considered one match with four teams.

Running tracks, swimming pools, bowling lanes and other sports facilities have limited capacities. Thus, it is not possible to organize one race for all athletes. Because of these limited capacities, several rounds of races are usually organized, e.g., a qualification round, regular races, and the final. With a high number of participants, the knockout tournament structure of the competition is applied. In our setting, the lane position does not matter. Only the set of race (match) participants matters. Real sports tournaments have their own specific rules (e.g., both relative and absolute results matter), but in this paper, we develop a general theory of such tournaments that can be applied to all tournaments.

This paper generalizes a knockout tournament seedings model that was developed for a standard two participants in a one-match framework (Karpov, 2016) by considering tournaments with more than two participants in one match. There are many ways of scheduling knockout tournaments. Different knockout tournament schedules are called *seedings* (the assignment of players to tournament brackets, with information about the initial order of participants' strengths coming mainly from historical data). In computational social choice, knockout tournaments correspond to a voting tree or an agenda (Vassilevska Williams, 2016).

There are several theoretical approaches to seeding type justification, but the results are ambiguous. The related economic literature has considered the costs and benefits of different tournament designs (seedings) for heterogeneous contestants (Groh et. al 2012; Kräkel, 2012; Rosen, 1986; Stracke et al 2015; Wei et al. 2018). Prize structure, effort functions and some other assumptions matter in this context. Clear solutions are obtained mainly for four-participant tournaments.

A combinatorial optimization approach in knockout tournament seeding studies was applied in (Dagaev & Suzdaltsev 2018; Karpov 2016). Three different seedings are justified under different assumptions in these studies, and the main two are standard seeding and equal gap seeding.

This paper develops an axiomatic approach in a generalized knockout tournament framework (tournaments with more than two participants in one match). We define several desirable properties of seedings and enumerate seedings that satisfy these properties. Several new knockout tournament seedings, which generalize the standard seeding and the equal gap seeding, are proposed and justified by the set of properties. Sports tournament organizers can easily apply the proposed seedings to real competitions.

The paper contains enumeration and representation results. Because of the novelty of the combinatorial object, all enumeration formulas are new. The representation results are the main contribution of the paper. They are based on the enumeration results and follow those results.

The structure of the paper is as follows. Section 2 describes generalized knockout tournament seedings, its properties and the enumeration results. Section 3 presents representation theorems for different seedings. Section 4 concludes.

Framework

Let k be the number of participants in one match, n be the number of rounds and $X = \{1,2,\dots,k^n\}$ be the set of participants of the knockout tournament (henceforth, tournament). The indices of the participants represent the order of the participants' strengths, where participant 1 is the strongest and participant k^n is the weakest.

The knockout tournament seeding, or simply the seeding, is a hypergraph with k^n vertices labelled from 1 to k^n that are described by a following set system (nested set system). There are k^{n-1} disjoint sets of k vertices (each such set is one match), k^{n-2} disjoint sets of k^2 vertices such that each new set unites k sets of k vertices (each such set is a subtournament with two rounds), k^{n-3} disjoint sets of k^3 vertices such that each new set unites k sets of k^2 vertices (each such set is a subtournament with three rounds), etc.

For example, a seeding of a tournament with $k = 2$ participants in each match and $n = 3$ rounds is described by the set system $\{1,4\},\{2,3\},\{5,8\},\{6,7\},\{1,2,3,4\},\{5,6,7,8\},\{1,2,3,4,5,6,7,8\}$, but it is more convenient to describe this seeding as a nested set system, $\Big\{\big\{\{1,4\},\{2,3\}\big\},\big\{\{5,8\},\{6,7\}\big\}\Big\}$. This is called the *nested set representation of seeding.* There are two subtournaments $\big\{\{1,4\},\{2,3\}\big\}$ and $\big\{\{5,8\},\{6,7\}\big\}$, each of which also contains two subtournaments. In each subsequent round, the winners of the subtournaments meet. The order of sets inside the subtournament does not matter. $\Big\{\big\{\{1,4\},\{2,3\}\big\},\big\{\{5,8\},\{6,7\}\big\}\Big\}$ and $\Big\{\big\{\{1,4\},\{3,2\}\big\},\big\{\{6,7\},\{8,5\}\big\}\Big\}$ represent the same seeding.

For $k = 2$, the most popular tournament seeding (called *standard seeding*) creates pairs in the first round that match the strongest participant with the weakest participant, the second strongest participant with the second weakest participant, etc. The pairs in subsequent rounds are determined in a way that prevents the first two participants from being in a head-to-head match before the final and delays the confrontations between other strong participants until later rounds. Strong participants are rewarded for their success through such a seeding. For $k = 2,\ n = 3$, it is $T_{2,3}^{\text{standard}} = \Big\{\big\{\{1,8\},\{4,5\}\big\},\big\{\{2,7\},\{3,6\}\big\}\Big\}$.

The terms tournament and seeding are used here interchangeably. By a tournament, we mean the specific seeding corresponding to a tournament.

Each tournament with $n \geq 2$ rounds is a set that consists of k subtournaments. Each subtournament with $n \geq 2$ rounds is a set that also consists of k subtournaments. $T_{k,n}^{m,i}$ is a subtournament *i* with *m* rounds. It is a part of a tournament with *n* rounds and *k* participants in each match. For notational convenience, let $T_{k,n}^{n} = T_{k,n}$ and $T_{k,n}^{0,i} = \{i\}$. Subtournaments $T_{k,n}^{m,i}, T_{k,n}^{m,j}$ are nonoverlapping if there is no participant that plays in both subtournaments. A tournament with n rounds is a set $T_{k,n} = \bigcup_{i=1}^{k} T_{k,n}^{n-1,i}$, where all subtournaments are nonoverlapping. A subtournament with m rounds is a set $T_{k,n}^{m,i} = \bigcup_{i=1}^{k} T_{k,n}^{m-1,i}$, where all subtournaments are nonoverlapping.

Let $\mathbb{T}_{k,n}$ be the set of all possible seedings with k participants in one match and n rounds. The cardinality of the set of all possible seedings is denoted as $\#\mathbb{T}_{k,n}$. Let $\mathbb{T}_{k,n}^{property}$ be the set of all possible seedings with k participants in one match and n rounds that satisfy a property; the cardinality of this set is denoted as $\#\mathbb{T}_{k,n}^{property}$.

Proposition 1. The number of seedings is equal to

$$\#\mathbb{T}_{k,n} = (k!)^{\frac{1-k^n}{k-1}}(k^n!) \tag{1}$$

Proof. There are $k^n!$ permutations of participants. Each permutation corresponds to the nested set representation of a seeding. There are $\sum_{i=0}^{n-1} k^i = \frac{1-k^n}{1-k}$ matches, subtournaments and tournaments. For each of them, there are $k!$ permutations of participants (subtournaments), but they do not change a tournament.

Considering the tournament to be the union of k subtournaments, we obtain the recursive representation of formula (1):

$$\#\mathbb{T}_{k,n} = \frac{k^n!}{k!\,(k^{n-1}!)^k}\left(\#\mathbb{T}_{k,n-1}\right)^k \tag{2}$$

All combinatorial formulas are new; some sequences in the case of $k = 2$ or $n = 2$ are mentioned in the On-line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, henceforth OEIS, and are added by the author.

For $k = 2$, it is the A067667 sequence in the OEIS. For $n = 2$, it is the A057599 sequence in the OEIS. Even for the small values of $k = 2$ and $n = 2$, these sequences are quickly growing functions. For higher values of k and n, the number of seedings becomes astronomically large, e.g., $\#\mathbb{T}_{3,3} = 833'712'928'048'000'000$.

A knockout tournament seeding is a purely combinatorial object, without any assumptions about participants' behaviour. To study the properties of seedings, we assume that a stronger participant always wins a match with a weaker participant. To find a particular seeding for practical use, we introduce several properties. Some of them have a close prototype in (Karpov, 2016), where the case of $k = 2$ is considered.

The first property makes a seeding invariant under the strength/weakness ranking transformation. There are no special rules for weak or strong participants. A seeding designed for strength-ordered participants is equal to a seeding designed for weakness-ordered participants.

Symmetry. A seeding is invariant under the point mapping $i \to k^n + 1 - i$.

$\{\{1,4\},\{2,3\}\},\{\{5,8\},\{6,7\}\}$ and $\{\{1,8\},\{2,7\}\},\{\{3,4\},\{5,6\}\}$ are examples of symmetric seedings.

Proposition 2. *The number of seedings that satisfy the symmetry property is equal to*

for an odd k

$$\#\mathbb{T}_{k,n}^{S}=\frac{2^{\frac{k^{n}-nk+n-1}{2}}}{(k!)^{\frac{1}{2}\left(\frac{k^{n}-1}{k-1}-n\right)}}\frac{\left(\frac{k^{n}-1}{2}\right)!}{\left[\left(\frac{k-1}{2}\right)!\right]^{n}} \tag{3}$$

for an even k

$$\#\mathbb{T}_{k,n}^{S}=\left(\frac{k^{n}}{2}\right)!\sum_{i=0}^{k/2}\frac{\left(\#\mathbb{T}_{k,n-1}^{S}\right)^{2i}}{\left(\left(\frac{k^{n-1}}{2}\right)!\right)^{2i}i!\left(\frac{k}{2}-i\right)!}\left(\frac{2^{k^{n-1}-1}}{(k!)^{\frac{k^{n-1}-1}{k-1}}}\right)^{\frac{k}{2}-i} \tag{4}$$

Proof. A pair of sets $A,B\subseteq\{1,\dots,x\}$ are said to be symmetric if and only if $|A|=|B|=y$, $y<x$, and if $i\in A$, then $x+1-i\in B$. A set $A\subseteq\{1,\dots,x\}$ is said to be self-symmetric if and only if $|A|=y$, $y<x$, and if $i\in A$, then $x+1-i\in A$.

Odd k. For each tournament, there is only one self-symmetric set of the cardinality of k^{n-1}. There are $\#\mathbb{T}_{k,n-1}^{S}$ ways to define a symmetric subtournament generated by the self-symmetric set. There are $\frac{k-1}{2}$ symmetric pairs of sets of the cardinality of k^{n-1}. There are $2^{k^{n-1}-1}\#\mathbb{T}_{k,n-1}$ ways to define two subtournaments generated by the symmetric pair. Considering a tournament to be the union of k subtournaments, we obtain

$$\#\mathbb{T}_{k,n}^{S}=\frac{\left(\frac{k^{n}-1}{2}\right)!}{\left(\frac{k^{n-1}-1}{2}\right)!\,(k^{n-1}!)^{\frac{k-1}{2}}\left(\frac{k-1}{2}\right)!}\#\mathbb{T}_{k,n-1}^{S}\left(2^{k^{n-1}-1}\#\mathbb{T}_{k,n-1}\right)^{\frac{k-1}{2}} \tag{5}$$

With $\#\mathbb{T}_{k,1}^{S}=1$, we obtain

$$\#\mathbb{T}_{k,n}^{S}=\prod_{i=2}^{n}\frac{\left(\frac{k^{i}-1}{2}\right)!}{\left(\frac{k^{i-1}-1}{2}\right)!\left(\frac{k-1}{2}\right)!}\frac{2^{\frac{(k^{i-1}-1)(k-1)}{2}}}{(k!)^{\frac{k^{i-1}-1}{2}}} \tag{6}$$

Simplifying, we obtain the result.

Even k. We have an even number of self-symmetric sets of the cardinality of k^{n-1}. Thus, we have

$$\#\mathbb{T}_{k,n}^{S}=\sum_{i=0}^{k/2}\frac{\left(\frac{k^{n}}{2}\right)!}{\left(\left(\frac{k^{n-1}}{2}\right)!\right)^{2i}(k^{n-1}!)^{\frac{k}{2}-i}(2i)!\left(\frac{k}{2}-i\right)!}\left(2^{k^{n-1}-1}\#\mathbb{T}_{k,n-1}\right)^{\frac{k}{2}-i}\left(\#\mathbb{T}_{k,n-1}^{S}\right)^{2i} \tag{7}$$

Substituting $\#\mathbb{T}_{k,n-1}$, we obtain the result. ■

For $k=2$, it is A261187 sequence in (OEIS). In this case, formula (4) has a simpler representation

$$\#\mathbb{T}_{2,n}^{S}=2^{\mathrm{n}-1}!\,y_{n} \tag{8}$$

where $y_n = 0.5(y_{n-1})^2 + 1$, with $y_1 = 1$.

For an odd k, there is another representation of the recurrence (5). For each tournament, there is only one self-symmetric set of the cardinality k (one match set). There are $\frac{k^{n-1}-1}{2}$ symmetric pairs of sets of the cardinality of k. There are 2^{k-1} ways to define a symmetric pair of sets from a self-symmetric set of the cardinality of $2k$. Considering the tournament to be the union of k^{n-1}matches, we obtain

$$\#\mathbb{T}_{k,n}^{S} = \frac{\left(\frac{k^n-1}{2}\right)!}{\left(\frac{k-1}{2}\right)!\,(k!)^{\frac{k^{n-1}-1}{2}}\left(\frac{k^{n-1}-1}{2}\right)!}\left(2^{k-1}\right)^{\frac{k^{n-1}-1}{2}}\#\mathbb{T}_{k,n-1}^{S} \tag{9}$$

With $\#\mathbb{T}_{k,1}^{S} = 1$, we obtain formula (6). These two representations of the tournament (tournament as the union of k subtournaments (formula (5)) or of k^{n-1} matches (formula (9))) are applied to all derivations of subsequent combinatorial formulas.

Following Wright (2014), competitive intensity is a key property for sports competition design. The closer in strength the participants are, the higher the competitive intensity is. The two strongest participants of a match are the main rivals. From round to round, the two strongest participants of each match become stronger, and the strengths of the participants become closer. The intensity of competition increases, which supports spectator interest. In the final match, the two strongest participants play against each other.

Increasing competitive intensity. *In each subsequent round, a winner faces a rival that is stronger than the strongest rival in the previous round.*

Proposition 3. *The number of seedings that satisfy the increasing competitive intensity property is equal to*

$$\#\mathbb{T}_{k,n}^{ICI} = \left((k-2)!\right)^{\frac{1-k^{n-1}}{k-1}}\prod_{i=2}^{n}\left[\frac{(k^i-2)!}{\left((k^{i-1}-1)!\right)^2\left((k^{i-1})!\right)^{k-2}}\right]^{k^{n-i}} \tag{10}$$

Proof. The strongest and second strongest participants should be in different subtournaments. Thus, we have

$$\#\mathbb{T}_{k,n}^{ICI} = \frac{1}{(k-2)!}\frac{(k^n-2)!}{\left((k^{n-1}-1)!\right)^2\left((k^{n-1})!\right)^{k-2}}\left(\#\mathbb{T}_{k,n-1}^{ICI}\right)^k \tag{11}$$

With $\#\mathbb{T}_{k,1}^{ICI} = 1$, we obtain the result. ■

For $k = 2$, formula (10) is also the number of binary heaps (the sequence A056972 in (OEIS)). Increasing competitive intensity is a very weak condition, with $\lim_{k\to\infty}\frac{\#\mathbb{T}_{k,2}^{ICI}}{\#\mathbb{T}_{k,2}} = 1$ and $\lim_{k\to\infty}\frac{\#\mathbb{T}_{k,3}^{ICI}}{\#\mathbb{T}_{k,3}} = e^{-1}$. The next property strengthens the increasing competitive intensity property, thus guaranteeing the strongest final match, the strongest semifinal, etc.

Delayed confrontation (Schwenk 2000). Participants rated among the top k^j participants shall never meet until the number of participants has been reduced to k^j or fewer.

It is a core property for tournament design. This property is aimed to support spectator interest. Matches with and between the strongest participants draw the interest of spectators. These participants should not be dropped at the beginning of the tournament. This property allocates strong participants equally between subtournaments. Thus, there is no subtournament with only weak or only strong participants.

Proposition 4. *The number of seedings that satisfy the delayed confrontation property is equal to*

$$\#\mathbb{T}_{k,n}^{DC} = \big((k-1)!\big)^{\frac{k-k^n}{k-1}} \prod_{i=2}^{n} \big(k^i - k^{i-1}\big)! \tag{12}$$

Proof. From the delayed confrontation property, participants $\{k^{n-1}+1, \ldots, k^n\}$ should lose in round 1, participants $\{k^{n-2}+1, \ldots, k^{n-1}\}$ should lose in round 2, etc. Thus, we have

$$\#\mathbb{T}_{k,n}^{DC} = \frac{(k^n - k^{n-1})!}{((k-1)!)^{k^{n-1}}} \#\mathbb{T}_{k,n-1}^{DC} \tag{13}$$

With $\#\mathbb{T}_{k,1}^{DC} = 1$, we obtain the result. ■

For $k = 2$, it is A261125 sequence in (OEIS). The delayed confrontation property does not require an assumption about the deterministic result of each match. Strong participants are divided between different subtournaments and do not play against each other. We introduce several refinements of the delayed confrontation property: sincerity rewarded, equal differences, equal sums, balance and equal partition of losers.

The sincerity rewarded property goes back to Schwenk (2000). We should encourage strong participants; otherwise, they will have incentives to lose in pretournament games and get a weaker rival (a model with such incentives is developed in (Dagaev, Sonin 2017)).

Sincerity rewarded. *In addition to the delayed confrontation property, in each round r, the absolute value of the difference between the strongest participants' ranks in a match among the top k^{n-r} participants strictly increases with the strength of the top participant.*

The standard seeding satisfies this property. The strongest participant plays against the weakest participant, thus guaranteeing the highest absolute value of the difference between participants' ranks.

The weakest violation of the sincerity rewarded property leads to the equal differences property. It implements an idea from the favouritism minimize property from Schwenk (2000). We generalize the competitive intensity measure of Dagaev, Suzdaltsev (2018) for a k higher than 2. The competitive intensity is an absolute value of the difference between the strongest participant's rank and the second strongest participant's rank in the match. By equalizing the competitive intensities of all matches of the round, we obtain the equal differences property.

Equal differences. *In addition to the delayed confrontation property, all matches of one round should have an equal absolute value of the difference between the strongest participant rank and the second strongest participant rank in the match.*

Proposition 5. *The number of seedings that satisfy the equal differences property is equal to*

$$\#\mathbb{T}_{k,n}^{ED} = \big((k-2)!\big)^{\frac{k-k^n}{k-1}} \prod\nolimits_{\mathrm{i}=2}^{\mathrm{n}} \big(k^i - 2k^{i-1}\big)! \tag{14}$$

Proof. From the equal differences property, participants $\{1, \ldots, k^{n-1}\}$ should be matched with participants $\{k^{n-1}+1, \ldots, 2k^{n-1}\}$. Thus, we have

$$\#\mathbb{T}_{k,n}^{ED} = \frac{[(k^n - 2k^{n-1})!]}{((k-2)!)^{k^{n-1}}} \#\mathbb{T}_{k,n-1}^{ED} \tag{15}$$

With $\#\mathbb{T}_{k,1}^{ED} = 1$, we obtain the result. ■

The subsequent property equates the qualities of matches (Dagaev, Suzdaltsev 2018) and supports spectator interest in all matches. By the quality of a match, we mean the sum of the ranks of the match's participants.

Equal sums. *In addition to the delayed confrontation property, all matches of one round should have equal sum of ranks of match's participants.*

The subsequent property simplifies the symmetry property in the presence of the delayed confrontation property.

Balance. *In addition to the delayed confrontation property, all matches of one round should be invariant under the point mapping* $\mathrm{i} \to \mathrm{k}^{\mathrm{n-r+1}} + 1 - \mathrm{i}$, *where r is the number of the round.*

Proposition 6. *The number of seedings that satisfy the balance property is equal to*

for an odd k

$$\#\mathbb{T}_{k,n}^{B} = 0 \tag{16}$$

for an even k

$$\#\mathbb{T}_{k,n}^{B} = \left(\left(\frac{k}{2} - 1\right)!\right)^{\frac{k-k^n}{k-1}} \prod_{i=2}^{n} \left(\frac{k^i - 2k^{i-1}}{2}\right)! \tag{17}$$

Proof. Odd k. Only one match can be invariant under the point mapping $i \to k^{n-r+1} + 1 - i$.

Even k. The strongest k^{n-1} participants play in different matches against the weakest k^{n-1} participants. There are $\left(\frac{k^n - 2k^{n-1}}{2}\right)! \, ((0.5k-1)!)^{-k^{n-1}}$ ways to assign all other participants to k^{n-1} matches consistent with the balance property. Thus, we have

$$\#\mathbb{T}_{k,n}^{B} = \frac{\left(\frac{k^n - 2k^{n-1}}{2}\right)!}{((0.5k-1)!)^{k^{n-1}}} \#\mathbb{T}_{k,n-1}^{B} \tag{18}$$

With $\#\mathbb{T}_{\mathrm{k},1}^{\mathrm{B}} = 1$ we obtain the result.

The balance property implies the equal sums property. For $k = 2$, the balance property coincides with the delayed confrontation property.

The sincerely rewarded, equal differences, equal sums, and balance properties are quite strong, with

$$\#\mathbb{T}_{2,n}^{SR} = \#\mathbb{T}_{2,n}^{ED} = \#\mathbb{T}_{2,n}^{ES} = \#\mathbb{T}_{2,n}^{B} = 1 \tag{19}$$

The next property equates matches by the presence of the weakest participants. We eliminate the advantages of having many weak competitors.

Equal partition of losers. *In addition to the delayed confrontation property, in all matches of one round, there should be only one participant from the set of participants* $\{k^{n-r+1} - k^{n-r} + 1, \dots, k^{n-r+1}\}$, *where r is the number of the round.*

Proposition 7. *The number of seedings that satisfy the equal partition of losers property is equal to*

$$\#\mathbb{T}_{k,n}^{EPL} = \big((k-2)!\big)^{\frac{k-k^n}{k-1}} \prod_{i=2}^{n} k^{i-1}!\,\big(k^i - 2k^{i-1}\big)! \tag{20}$$

Proof. From the equal differences property, participants $\{1, \dots, k^{n-1}\}$ should be matched with participants $\{k^n - k^{n-1} + 1, \dots, k^n\}$. Thus, we have

$$\#\mathbb{T}_{k,n}^{EPL} = k^{n-1}!\,\frac{[(k^n - 2k^{n-1})!]}{((k-2)!)^{k^{n-1}}}\,\#\mathbb{T}_{k,n-1}^{EPL} \tag{21}$$

With $\#\mathbb{T}_{k,1}^{EPL} = 1$, we obtain the result.

The balance property implies the equal partition of losers. For $k = 2$, the equal partition of losers coincides with the delayed confrontation property.

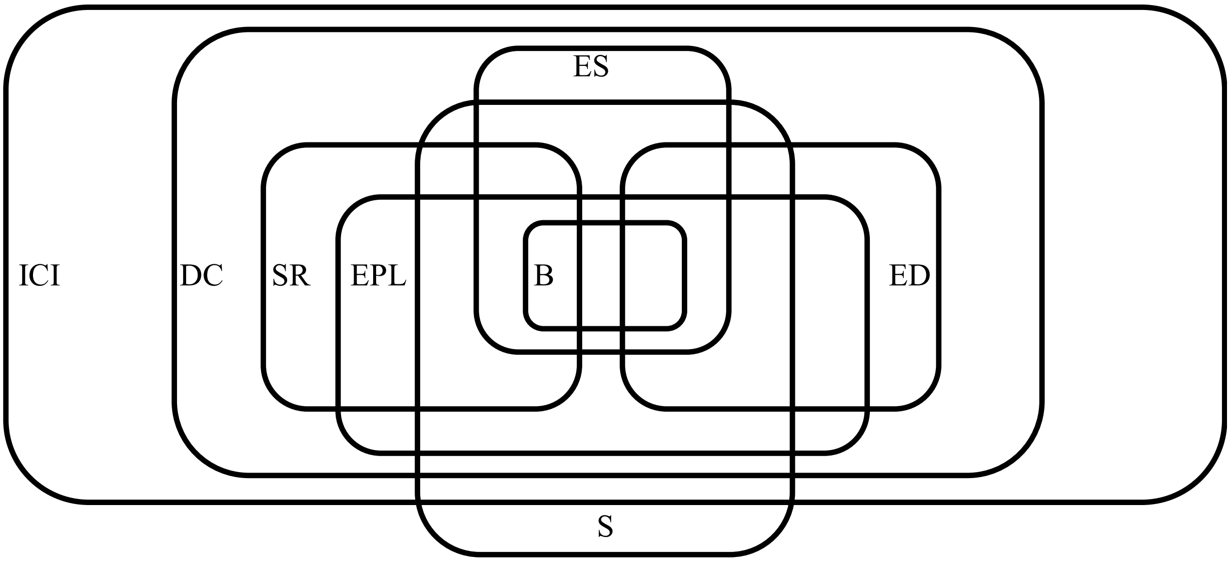

Figure 1. Venn diagram for seeding properties: ICI – increasing competitive intensity; DC – delayed confrontation; SR – sincerity rewarded; EPL – equal partition of losers; B – balance; ED – equal differences; ES – equal sums; S – symmetry. All labels are inside areas and close to the borders.

Figure 1 presents a Venn diagram for all the seeding properties. If area X belongs to area Y, then all seedings that satisfy property X will also satisfy property Y. An empty intersection means that it is impossible to satisfy both properties. A nonempty intersection means that for some values of *n* and *k*, there exists a seeding that satisfies both properties. This diagram represents the general case. For particular values of parameters *n* and *k*, the diagram may be simpler; e.g., some areas may be equal, and some intersections may be empty.

The number of seedings that satisfies each particular property is quite large, but the cardinality of intersections may be small or even empty for particular parameters. The following section presents representation results based on the abovementioned properties.

Representation theorems

In this section, we generalize two main seedings: standard seeding and equal gap seeding. For $k = 2$, the representation results for these seedings are presented in Karpov (2016).

Standard seeding

For $k = 2$, the most popular seeding is the *standard seeding.* It is defined recursively. For any *m* from 1 to n, we have

$$T_{2,n}^{m,i} = \left\{T_{2,n}^{m-1,i}, T_{2,n}^{m-1,2^{n-m+1}-i+1}\right\}, i = \overline{1, 2^{n-m}}$$

Thus, for $n = 3$, we have

$$T_{2,3}^{standard} = \left\{\{\{1,8\},\{4,5\}\},\{\{2,7\},\{3,6\}\}\right\}$$

There are several justifications of the standard seeding.

Proposition 8. (Karpov 2016) *For $k = 2$, the standard seeding is a unique seeding that satisfies the equal rank sums property.*

Proposition 9. *For $k = 2$, the standard seeding is a unique seeding that satisfies the sincerely rewarded property.*

Proof. Participant $2^{n-1} - 1$ has a weaker rival than participant 2^{n-1}, etc. Because participant $2^{n-1} + 1$ should have a rival, we should have a match $\{2^{n-1}, 2^{n-1}\}$. The standard seeding is the only way to pair all other participants. ■

Proposition 10. *For $k = 2$, the standard seeding is a unique seeding that satisfies the balance property.*

Proposition 10 follows from proposition 6. The standard seeding also satisfies the equal partition of losers property. There is no direct generalization of the standard seeding for an arbitrary k. For $k = 3$ and $n = 2$, $\left\{\{1,6,8\},\{2,4,9\},\{3,5,7\}\right\}$ and $\left\{\{1,5,9\},\{2,6,7\},\{3,4,8\}\right\}$ satisfy the symmetry and equal rank sums properties but not the sincerely rewarded property, $\left\{\{1,6,7\},\{2,5,8\},\{3,4,9\}\right\}$ satisfies the symmetry and sincerity rewarded properties, but not the equal rank sums property. We develop two seedings, for $k = 3$ and $k = 4$, that satisfy the core properties of the standard seeding and give a corresponding justification.

For $k = 3$, the *modified standard seeding* is defined recursively. For any *m* from 1 to *n*, we have

$$T_{3,n}^{m,i} = \left\{T_{3,n}^{m-1,i}, T_{3,n}^{m-1,2\cdot 3^{n-m}-i+1}, T_{3,n}^{m-1,2\cdot 3^{n-m}+i}\right\}, i = \overline{1, 3^{n-m}}$$

Thus, for $k = 3$ and $n = 3$, we have

$$T_{3,3}^{MS} = \left\{\{\{1,18,19\},\{6,13,24\},\{7,12,25\}\},\{\{2,17,20\},\{5,14,23\},\{8,11,26\}\},\{\{3,16,21\},\{4,15,22\},\{9,10,27\}\}\right\}.$$

Proposition 11. *For $k = 3$, the modified standard seeding is a unique seeding that satisfies the sincerity rewarded and symmetry properties.*

Proof. It is true for $n = 1$. Suppose it is true for $\mathrm{n} - 1$. Let us prove for n.

Because the sincerity rewarded property leads to delayed confrontation, it is sufficient to define only first-round matches. By the sincerity rewarded property, the strongest 3^{n-1} participants play in different matches. By the symmetry property, the weakest 3^{n-1}

participants play in different matches. By the sincerity rewarded property, the strongest participant among participants $\{1,\dots,3^{n-1}\}$ plays against the weakest participant among participants $\{3^{n-1}+1,\dots,2\cdot 3^{n-1}\}$, the second strongest plays the second weakest, etc., and we have the following matches $\{i, 2\cdot 3^{n-1}-i+1, x\}$, where participant x is weaker than participant $2\cdot 3^{n-1}-i+1$. By the symmetry property, the participant $2\cdot 3^{n-1}-i+1$ corresponds to the participant $3^n-2\cdot 3^{n-1}+i-1+1=2\cdot 3^{n-1}-3^{n-1}+i$. It is a second weakest participant of a match. Thus, there is only one way to assign the third participant of the match (it is an image of the participant $i'=3^{n-1}-i+1$ of the symmetric match, $3^{n-1}-i+1\to 2\cdot 3^{n-1}+i$). We design a unique seeding for a tournament with n rounds.

For $k=4$, the *modified standard seeding* is defined recursively. For any m from 1 to n, we have

$$T_{4,n}^{m,i}=\left\{T_{4,n}^{m-1,i}, T_{4,n}^{m-1,\frac{4^{n-m+1}}{2}-i+1}, T_{4,n}^{m-1,\frac{4^{n-m+1}}{2}+i}, T_{4,n}^{m-1,4^{n-m+1}-i+1}\right\}, i=\overline{1,4^{n-m}}$$

Thus, for $k=4$ and $n=2$, we have

$$T_{4,2}^{MS}=\{\{1,8,9,16\},\{2,7,10,15\},\{3,6,11,14\},\{4,5,12,13\}\}$$

Proposition 12. For $k=4$, *the modified standard seeding is a unique seeding that satisfies the sincerely rewarded and balance properties.*

Proof. By the balance property, for any m from 1 to n, we have $T_{4,n}^{m,i}=\left\{T_{4,n}^{m-1,i}, T_{4,n}^{m-1,x}, T_{4,n}^{m-1,y}, T_{4,n}^{m-1,4^{n-m+1}-i+1}\right\}, i=\overline{1,4^{n-m}}$, where $i<x\le 4^{n-m+1}-i$ and $i<y\le 4^{n-m+1}-i$. By the sincerely rewarded property, we have $T_{4,n}^{m,i}=\left\{T_{4,n}^{m-1,i}, T_{4,n}^{m-1,\frac{4^{n-m+1}}{2}-i+1}, T_{4,n}^{m-1,\frac{4^{n-m+1}}{2}+i}, T_{4,n}^{m-1,4^{n-m+1}-i+1}\right\}, i=\overline{1,4^{n-m}}$.

For the higher *k*, it is still not possible to satisfy all properties of the standard seeding.

Proposition 13. *For* $k=5$, *there is no seeding that satisfies the sincerity rewarded, symmetry, and equal sums properties.*

Proof. It is sufficient to consider the case of $n=2$ to prove the impossibility result. It is the last two rounds of any tournament. Let us consider the set of the middle participants (there exist two participants that are weaker than the middle participant, and there exist two participants that are stronger than the middle participant) of each first-round match. By the symmetry property, the set of the middle participants is self-symmetric (a set $A\subseteq\{1,\dots,x\}$ is said to be self-symmetric if and only if $|A|=y$, $y<x$, and if $i\in A$, then $x+1-i\in A$). Participant 13 belongs to the set of middle participants. There are two middle participants stronger than participant 13 and two participants weaker than participant 13. The two strongest participants in each match are stronger than participant 13. The two weakest participants in each match are weaker than participant 13.

The sum of the participants' ranks in one match equals 65. There are three cases.

1. The sum of the ranks of the two strongest participants in each match is equal to 11. Then, the sum of the ranks of the two weakest participants in each match is equal to 41. The rank of the middle participant in each first-round match should be equal to 13, but that is impossible.

2. The sum of the ranks of the two strongest participants in each match is equal to 11 or 12. Then, the sum of ranks of the two weakest participants in each match is equal to 41 or 40. The rank of middle participants can be equal to 12,13 or 14, but that is impossible.
3. The sum of the ranks of the two strongest participants in each match is equal to 11, 12 or 13, and the strongest participant has participant 12 in the pair. Then, the sum of the ranks of the two weakest participants in each match is equal to 41, 40 or 39. The rank of the middle participants can be equal to 11, 12, 13, 14 or 15, but that is impossible.

Equal gap seeding

For $k = 2$, the equal gap seeding is investigated in (Karpov 2016). Here, we generalize it. The *equal gap seeding* is defined recursively. For any *m* from 1 to n, we have

$$T_{k,n}^{m,i} = \bigcup_{j=0}^{k-1} T_{k,n}^{1,i+jk^{n-m}}, i = \overline{1, \mathrm{k}^{n-m}}$$

Thus, for $k = 2$ and $n = 4$,we have

$$T_{2,3}^{EG} = \Big\{\big\{\{\{1,9\},\{5,13\}\},\{\{3,12\},\{7,15\}\}\big\},\big\{\{\{2,10\},\{6,14\}\},\{\{4,13\},\{8,16\}\}\big\}\Big\}$$

for $k = 3$ and $n = 3$, we have

$$T_{3,3}^{EG} = \big\{\{\{1,10,19\},\{4,13,22\},\{7,16,25\}\},\{\{2,11,20\},\{5,14,23\},\{8,17,26\}\},\{\{3,12,21\},\{6,15,24\},\{9,18,27\}\}\big\}$$

for $k = 4$ and $n = 2$,we have

$$T_{4,2}^{EG} = \{\{1,5,9,13\},\{2,6,10,14\},\{3,7,11,15\},\{4,8,12,16\}\}$$

There are several justifications of the equal gap seeding.

Proposition 14 (Karpov 2016). *For $k = 2$, the equal gap seeding is a unique seeding that satisfies the equal differences property.*

For $k = 2$, the equal gap seeding also satisfies the symmetry property.

Proposition 15. *For $k = 3$, the equal gap seeding is a unique seeding that satisfies the equal differences and symmetry properties.*

Proof. It is true for $n = 1$. Suppose it is true for $\mathrm{n} - 1$. Let us prove for n.

Because the equal differences property leads to the delayed confrontation, it is sufficient to define only first-round matches. By the equal differences property, the strongest 3^{n-1} participants play in different matches against participants $\{3^{n-1} + 1, \dots, 2 \cdot 3^{n-1}\}$. The absolute difference between the ranks of the strongest and the second strongest participant in the match equals 3^{n-1}. Because of the symmetry property, the absolute difference between the ranks of the strongest and the second strongest participants in the match also equals 3^{n-1}. Thus, we have

$$T_{3,n}^{1,i} = \bigcup_{j=0}^{2} \{i + j3^{n-1}\}, i = \overline{1, 3^{n-1}}$$

The *modified equal gap seeding* is defined recursively. For any *m* from 1 to *n*, we have

$$T_{4,n}^{m,i} = \left\{T_{4,n}^{m-1,i}, T_{4,n}^{m-1,i+4^{n-m}}, T_{4,n}^{m-1,4^n-i+1-4^{n-m}}, T_{4,n}^{m-1,4^n-i+1}\right\}, i = \overline{1, 4^{n-m}}$$

Thus, for $k = 4$ and $n = 2$,we have

$$T_{4,2}^{modified\ equal\ gap} = \{\{1,5,12,16\},\{2,6,11,15\},\{3,7,10,14\},\{4,8,9,13\}\}$$

The modified equal gap seeding satisfies the equal sums and equal differences properties, uniting the properties of the standard seeding and the equal gap seeding.

Proposition 16. *For $k = 4$, the modified equal gap seeding is a unique seeding that satisfies the equal differences and balance properties.*

Proof. By the equal differences property, in round m, the strongest 4^{n-m} participants play in 4^{n-m} matches against participants $\{4^{n-m}+1, \ldots, 2\cdot 4^{n-m}\}$. Because of the symmetry property, the absolute difference between the ranks of the weakest and second weakest participants in the match also equals 4^{n-m}. By the balance property, the sum of the ranks in each match equals $\frac{4^{n-m+1}(4^{n-m+1}+1)}{8}$. All of the strong and weak pairs considered above have different sums of ranks. There is only one way to define a tournament. For any m from 2 to n, we have $T_{4,n}^{m,i} = \left\{T_{4,n}^{m-1,i}, T_{4,n}^{m-1,i+4^{n-m}}, T_{4,n}^{m-1,4^n-i+1-4^{n-m}}, T_{4,n}^{m-1,4^n-i+1}\right\}, i = \overline{1, 4^{n-m}}$. ■

For a higher k, there is no good generalization of the equal gap seeding.

Proposition 17. *For $k = 5$, there is no seeding that satisfies the equal differences, symmetry, and equal sums properties.*

Proof. It is sufficient to consider the case of $n = 2$ to prove the impossibility result. It is the last two round of any tournament. By the equal differences property, the strongest 5 participants play in 5 matches against participants $\{6, \ldots, 10\}$. Because of the symmetry property, the absolute difference between the ranks of the weakest and second weakest participants in the match also equals 5. The sum of the ranks of these four participants is even. The sum of participants' ranks in one match equals 65. The rank of the middle participant should be odd in all first-round matches, but that is impossible. ■

For $k = 7$, there exists a seeding that satisfies the equal differences, symmetry, equal sums and equal partition of losers properties:

$$T_{7,2} = \left\{\begin{matrix}\{1,8,23,29,35,36,43\},\{2,9,18,31,34,37,44\},\{3,10,20,26,33,38,45\},\{4,11,22,24,28,39,46\}, \\ \{5,14,17,24,30,40,47\},\{6,13,16,19,32,41,48\},\{7,14,15,29,35,42,49\}\end{matrix}\right\}$$

The fourth match is self-symmetric. The first and seventh matches, the second and sixth matches, and the third and fifth matches generate symmetric pairs of matches. This example is not unique.

For even $k \geq 6$, there are many seedings that satisfy the equal differences and balance properties.

Proposition 18. For even $k \geq 6$, *the number of tournaments that satisfy the equal differences and balance properties is equal to*

$$\#\mathbb{T}_{k,n}^{ED,B} = \left(\left(\frac{k}{2}-2\right)!\right)^{\frac{k-k^n}{k-1}} \prod_{i=2}^{n} \left(\frac{k^i - 4k^{i-1}}{2}\right)! \quad (22)$$

Proof. By the equal differences and balance properties, the strongest k^{n-1} participants play in different matches with the weakest k^{n-1} participants, the second strongest k^{n-1} participants and the second weakest k^{n-1} participants. There are $\left(\frac{k^n-4k^{n-1}}{2}\right)!\,((0.5k-2)!)^{-k^{n-1}}$ ways to assign all other participants to k^{n-1} matches, consistent with the balance property. Thus, we have

$$\#\mathbb{T}_{k,n}^{ED,B} = \frac{\left(\frac{k^n-4k^{n-1}}{2}\right)!}{((0.5k-2)!)^{k^{n-1}}}\#\mathbb{T}_{k,n-1}^{ED,B} \quad (23)$$

With $\#\mathbb{T}_{k,1}^{ED,B} = 1$, we obtain the result.

Conclusion

For the cases of three and four participants in one match, we investigate generalizations of standard seeding and equal gap seeding. These cases are the most important from a practical point of view. For a higher number of participants in one match, we need additional or/and different properties, which can be obtained from the specific requirements of a particular tournament.

Acknowledgements

The author would like to thank Dmitry Dagaev, Constantine Sorokin and Fuad Aleskerov for their valuable comments. The paper was prepared within the framework of the Basic Research Program at the National Research University Higher School of Economics (HSE) and supported within the framework of a subsidy by the Russian Academic Excellence Project '5–100'. The work was conducted by the International Laboratory of Decision Choice and Analysis (DeCAn Lab) of the National Research University Higher School of Economics.

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