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# Polynomial curves on trinomial hypersurfaces

by

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**1. Introduction.** It is well known that the Fermat equation $z_0^p + z_1^p + z_2^p = 0$, $p \geq 3$, has no non-trivial solution over the polynomial ring $\mathbb{C}[x]$. The reason is that the projective curve defined by the Fermat equation in $\mathbb{P}^2$ is not rational.

It is natural to consider more general equations

$$z_0^p + z_1^q + z_2^r = 0, \quad p, q, r \in \mathbb{Z}_{\geq 2}, \tag{1}$$

and to look for polynomial solutions. Geometrically such a solution corresponds to a polynomial curve $\tau\colon \mathbb{C} \to V_{p,q,r}$, where $V_{p,q,r} := V(z_0^p + z_1^q + z_2^r)$ is called the *Pham–Brieskorn surface* in $\mathbb{C}^3$. Here we have *trivial solutions*, namely,

$$z_0(x) = \alpha\phi(x)^{m/p}, \quad z_1(x) = \beta\phi(x)^{m/q}, \quad z_2(x) = \gamma\phi(x)^{m/r},$$

where $m = \operatorname{lcm}(p, q, r)$, $\phi(x) \in \mathbb{C}[x]$, and $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha^p + \beta^q + \gamma^r = 0$.

The following result is stated in [8, Theorem 0.1(a)] with references to [6], [7] and [15, Corollary of Lemma 8].

Theorem 1. *The Pham–Brieskorn surface $V_{p,q,r}$ admits a non-trivial polynomial curve if and only if one of the following conditions hold.*

(i) *At least one of the numbers $p, q, r$ is coprime to the others.*
(ii) $\gcd(p, q) = \gcd(p, r) = \gcd(q, r) = 2$.

Moreover, the conditions of Theorem 1 characterize rational Pham–Brieskorn surfaces (see [6, p. 117]).

Now we come to a special class of non-trivial polynomial curves on $V_{p,q,r}$. Let us recall that a triple $(p, q, r)$ of positive integers is called *platonic* if we

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have $1/p+1/q+1/r>1$. It is well known that up to renumbering the platonic triples are

$$(5,3,2),\quad (4,3,2),\quad (3,3,2),\quad (p,2,2),\quad (p,q,1),\quad p,q\in\mathbb{Z}_{>0}.$$

In 1873, Schwarz [20] found polynomial solutions of equation (1) in coprime polynomials $z_0(x), z_1(x), z_2(x)$ for every platonic triple $(p,q,r)$ with $p,q,r\geq 2$; see also [22] and [8] for explicit formulas.

In 1883, Halphen [9] proved that equation (1) has no solution in nonconstant coprime polynomials when $1/p+1/q+1/r\leq 1$. We refer to [16] for a historical account of the subject.

Following [8, Theorem 0.1(b)], we reformulate these results in terms of polynomial curves.

THEOREM 2. *The Pham–Brieskorn surface $V_{p,q,r}$ admits a polynomial curve not passing through the origin if and only if $(p,q,r)$ is a platonic triple.*

There are several ways to generalize the theory of Pham–Brieskorn surfaces to higher dimensions. One way is to consider Pham–Brieskorn hypersurfaces

$$V(z_0^{p_0}+z_1^{p_1}+\cdots+z_m^{p_m})\subseteq\mathbb{C}^{m+1};$$

see [8, Example 2.21] and references therein for related results.

In this paper we investigate the case of trinomial hypersurfaces of arbitrary dimension. Trinomial relations in many variables arise naturally in connection with torus actions of complexity 1, multigraded algebras and Cox rings of algebraic varieties [3, 10–14].

Following [13], in Section 2 we consider two types of trinomial affine hypersurfaces, discuss their geometric properties and define a torus action of complexity one for hypersurfaces of each type. Theorems 3 and 4 are generalizations of Theorem 1 to the case of trinomial hypersurfaces. It turns out that for hypersurfaces of Type 2 rationality is equivalent to existence of a non-trivial polynomial curve, while for Type 1 this is not the case.

In Section 5 we define Schwarz–Halphen curves on trinomial hypersurfaces and study their basic properties. An extension of Theorem 2 to the hypersurface case is given in Theorem 5. As one may expect, a significant role in our arguments is played by the Mason–Stothers abc-Theorem.

**2. Preliminaries.** In this section we consider two types of trinomials over the field $\mathbb{C}$ of complex numbers [13], [14].

*Type 1.* We fix positive integers $n_1, n_2$ and let $n=n_1+n_2$. For each $i=1,2$, we take a tuple $l_i\in\mathbb{Z}_{>0}^{n_i}$ and define a monomial

$$T_i^{l_i}:=T_{i1}^{l_{i1}}\ldots T_{in_i}^{l_{in_i}}\in\mathbb{C}[T_{ij};\, i=1,2,\, j=1,\ldots,n_i].$$

By a *trinomial of Type 1* we mean a polynomial of the form $T_1^{l_1} + T_2^{l_2} + 1$. A *trinomial hypersurface of Type 1* is the zero set

$$X = V(T_1^{l_1} + T_2^{l_2} + 1) \subseteq \mathbb{C}^n.$$

It is easy to check that $X$ is an irreducible smooth affine variety of dimension $n - 1$.

*Type 2.* Fix positive integers $n_0, n_1, n_2$ and let $n = n_0 + n_1 + n_2$. For each $i = 0, 1, 2$, fix a tuple $l_i \in \mathbb{Z}_{>0}^{n_i}$ and define a monomial

$$T_i^{l_i} := T_{i1}^{l_{i1}} \dots T_{in_i}^{l_{in_i}} \in \mathbb{C}[T_{ij};\, i = 0, 1, 2,\, j = 1, \dots, n_i].$$

By *a trinomial of Type 2* we mean a polynomial of the form $T_0^{l_0} + T_1^{l_1} + T_2^{l_2}$. A *trinomial hypersurface of Type 2* is

$$X = V(T_0^{l_0} + T_1^{l_1} + T_2^{l_2}) \subseteq \mathbb{C}^n.$$

One can check that $X$ is an irreducible normal affine variety of dimension $n-1$. Clearly, every trinomial surface of Type 2 is either the Pham–Brieskorn surface $V_{p,q,r}$ or is isomorphic to the affine plane $\mathbb{C}^2$.

The following simple lemma describes the singular locus of $X$.

LEMMA 1. *A point $(t_{01}, \dots, t_{2n_2})$ on a trinomial hypersurface $X$ of Type 2 is singular if and only if for every $i = 0, 1, 2$ either there exist $1 \le j < k \le n_i$ with $t_{ij} = t_{ik} = 0$, or we have $t_{ij} = 0$ for some $1 \le j \le n_i$ with $l_{ij} \ge 2$.*

*Proof.* A point $x \in X$ is singular if and only if

$$\frac{\partial (T_0^{l_0} + T_1^{l_1} + T_2^{l_2})}{\partial T_{ij}}(x) = 0 \qquad \text{for all } i = 0, 1, 2 \text{ and all } 1 \le j \le n_i.$$

This implies the assertion. ∎

Recall that the *complexity* of an effective action $T \times X \to X$ of an algebraic torus $T$ on an irreducible algebraic variety $X$ is defined as $\dim X - \dim T$. Trinomial hypersurfaces of both types are equipped with a torus action of complexity 1. Namely, assume that every variable $T_{ij}$ is an eigenvector of a weight $w_{ij}$ with respect to a $T$-action. Then we have relations

$$\sum_{j=1}^{n_1} l_{1j} w_{1j} = \sum_{j=1}^{n_2} l_{2j} w_{2j} = 0$$

for Type 1 and relations

$$\sum_{j=1}^{n_0} l_{0j} w_{0j} = \sum_{j=1}^{n_1} l_{1j} w_{1j} = \sum_{j=1}^{n_2} l_{2j} w_{2j}$$

for Type 2. These relations define a subgroup in the torus of all invertible diagonal matrices on $\mathbb{C}^n$ whose connected component is a subtorus $T$ of codimension 2, and the restricted action $T \times X \to X$ is effective.

For Type 1, the monomials $T_1^{l_1}$ and $T_2^{l_2}$ are non-constant regular invariants of the $T$-action. On the other hand, for a trinomial hypersurface $X$ of Type 2, every $T$-orbit on $X$ contains the origin in its closure, and thus every regular $T$-invariant is a constant.

Example 1. On the hypersurface

$$X = V(T_{01}^2 T_{02}^4 + T_{11}^6 + T_{21}^8) \subseteq \mathbb{C}^4$$

we have the $(\mathbb{C}^\times)^2$-action given by

$$(t_1, t_2) \cdot (T_{01}, T_{02}, T_{11}, T_{21}) = (t_1^{12} t_2^{-2} T_{01}, t_2 T_{02}, t_1^4 T_{11}, t_1^3 T_{21}).$$

## 3. Horizontal curves on trinomial hypersurfaces of Type 2

Definition 1. A *polynomial curve* on an algebraic variety $X$ is a regular non-constant morphism $\tau\colon \mathbb{C} \to X$.

Assume that a variety $X$ is affine and carries an action $T \times X \to X$ of an algebraic torus $T$. Every one-parameter subgroup $\gamma\colon \mathbb{C}^\times \to T$ and every point $x_1 \in X$ with a non-closed orbit $\gamma(\mathbb{C}^\times) \cdot x_1$ define a polynomial curve

$$\tau\colon \mathbb{C} \to X, \quad \tau(t) = \gamma(t) \cdot x_1 \quad \text{for all } t \neq 0 \quad \text{and} \quad \tau(0) = x_0,$$

where $x_0$ is the unique point in the boundary of the closure of the non-closed orbit $\gamma(\mathbb{C}^\times) \cdot x_1$. Our aim now is to define and study a class of polynomial curves which is in a sense complementary to this class of curves. The following definition is a special case of the standard notion of a quasisection [18, Section 2.5].

Definition 2. A polynomial curve $\tau\colon \mathbb{C} \to X$ on an irreducible $T$-variety $X$ of complexity one is called *horizontal* if there exists a $T$-invariant open subset $W$ in $X$ such that $\tau(\mathbb{C})$ intersects all $T$-orbits on $W$.

In the case of the Pham–Brieskorn surface $X = V_{p,q,r}$, every polynomial curve on $X$ is either horizontal or the closure of a $T$-orbit on $X$. Curves of the latter type correspond to trivial polynomial solutions mentioned in the introduction.

Lemma 2. *A polynomial curve $\tau\colon \mathbb{C} \to X$ on a trinomial hypersurface of Type 2 is horizontal if and only if the rational function $T_0^{l_0}/T_1^{l_1}$ is non-constant along the image $\tau(\mathbb{C})$.*

*Proof.* If $\lambda_0 T_0^{l_0} = \lambda_1 T_1^{l_1}$ on $\tau(\mathbb{C})$ for some $(\lambda_0, \lambda_1) \in \mathbb{C} \setminus (0,0)$, then $\tau(\mathbb{C})$ is contained in a proper closed $T$-invariant subset $V(\lambda_0 T_0^{l_0} - \lambda_1 T_1^{l_1})$, and the curve cannot be horizontal.

Conversely, assume that the function $T_0^{l_0}/T_1^{l_1}$ is non-constant along $\tau(\mathbb{C})$. Let us consider the open subset $W_0$ in $X$ consisting of all points where each coordinate $T_{ij}$ is non-zero. Since the stabilizer in $T$ of a point on $W_0$ is trivial, all $T$-orbits in $W_0$ form a one-parameter family of orbits of codimension 1 in $X$. The intersection of the curve $\tau(\mathbb{C})$ with $W_0$ is not contained in a $T$-orbit and thus it intersects generic $T$-orbits in $W_0$. This implies that the curve is horizontal. ∎

REMARK 1. One may obtain examples of horizontal polynomial curves on a trinomial hypersurface $X$ as generic orbits of a regular action $\mathbb{G}_a \times X \to X$, where $\mathbb{G}_a$ is the additive group of the ground field $\mathbb{C}$ and the action comes from a homogeneous locally nilpotent derivation of the algebra $\mathbb{C}[X]$ (see [1, Lemma 2]).

For a trinomial $T_0^{l_0} + T_1^{l_1} + T_2^{l_2}$ of Type 2, we let $d_i = \gcd(l_{i1}, \ldots, l_{in_i})$.

THEOREM 3. *Let $X$ be a trinomial hypersurface of Type 2. The following conditions are equivalent:*

(i) *The hypersurface $X$ is rational.*
(ii) *The hypersurface $X$ admits a horizontal polynomial curve.*
(iii) *Either at least one of the numbers $d_0, d_1, d_2$ is coprime to the others, or* $\gcd(d_0, d_1) = \gcd(d_0, d_2) = \gcd(d_1, d_2) = 2$.

*Proof.* Conditions (i) and (iii) are equivalent by [2, Proposition 5.5].

Let us prove implication (ii)⇒(i). Assume that the hypersurface $X$ admits a horizontal polynomial curve $\tau$. Consider the rational quotient $\pi\colon X \to Y$, i.e. a rational morphism to an algebraic variety $Y$ with $\mathbb{C}(Y) = \mathbb{C}(X)^T$ defined by the inclusion $\mathbb{C}(X)^T \subseteq \mathbb{C}(X)$ (see [18, Section 2.4] for more details). Then $Y$ is a curve and $\pi$ restricted to $\tau(\mathbb{C})$ gives rise to a dominant rational morphism from $\mathbb{C}$ to $Y$. This shows that the curve $Y$ is rational. On the other hand, the variety $X$ contains an open subset isomorphic to $T \times Y'$, where $Y'$ is a curve birational to $Y$. This proves that the variety $X$ is rational.

We now turn to (iii)⇒(ii). Let us prove first that a rational Pham–Brieskorn surface $V_{p,q,r} = V(z_0^p + z_1^q + z_2^r)$ admits a horizontal polynomial curve. In this part we use a method proposed in [7] and fill a gap in the arguments given there.

Take $\epsilon \in \mathbb{C}$ with $\epsilon^q = -1$. We have

$$(x^r + 1)^p + (\epsilon(2x^r + 1))^q + x^r l(x) = 0$$

for some polynomial $l(x)$. Assume first that $\gcd(p, r) = \gcd(q, r) = 1$. Then there exist $u, v \in \mathbb{Z}_{>0}$ such that $vr - upq = 1$. Let us take

$$z_0 = l(x)^{uq}(x^r + 1), \qquad z_1 = \epsilon l(x)^{up}(2x^r + 1), \qquad z_2 = l(x)^v x.$$

This curve is horizontal because the polynomials $x^r + 1$ and $x$ are coprime.

Now assume that $\gcd(p, q) = \gcd(p, r) = \gcd(q, r) = 2$ and $p \geq q \geq r$. Then $p = 2p_1$, $q = 2q_1$, $r = 2r_1$ with pairwise coprime $p_1, q_1, r_1$.

Consider an equation

$$l_0(x)^2 w_0(x)^{2p_1} + l_1(x)^2 w_1(x)^{2q_1} + l_2(x)^2 w_2(x)^{2r_1} = 0. \tag{2}$$

Take positive integers $u_i, v_i$ such that

$$u_1p_1 - v_1q_1r_1 = 1, \quad u_2q_1 - v_2p_1r_1 = 1, \quad u_3r_1 - v_3p_1q_1 = 1.$$

The polynomials

$$\begin{aligned} z_0 &= l_0(x)^{u_1} l_1(x)^{v_2r_1} l_2(x)^{v_3q_1} w_0(x), \\ z_1 &= l_0(x)^{v_1r_1} l_1(x)^{u_2} l_2(x)^{v_3p_1} w_1(x), \\ z_2 &= l_0(x)^{v_1q_1} l_1(x)^{v_2p_1} l_2(x)^{u_3} w_2(x) \end{aligned} \tag{3}$$

satisfy the equation $z_0^p + z_1^q + z_2^r = 0$. Moreover, if $w_0(x)$ has a prime factor that does not appear in $w_1(x)$ and does not divide $l_0(x)l_1(x)l_2(x)$, then we obtain a horizontal curve. Hence it suffices to find a solution of equation (2) that meets the latter condition.

We define $s(x) = \alpha(x^{2r_1}+1)^{p_1}$ with some $\alpha \in \mathbb{C}$ and $m(x) = s(x) - (x^{2r_1}+2)^{q_1}$. Then

$$\begin{aligned} (x^{2r_1}+2)^{2q_1} + 4\alpha^2 m(x)^2 (x^{2r_1}+1)^{2p_1} &= (s(x)-m(x))^2 + (2s(x)m(x))^2 \\ &= (s(x)+m(x))^2. \end{aligned}$$

Note that $m(0) = \alpha - 2^{q_1}$. So the left hand side with $x = 0$ equals

$$2^{2q_1} + 4\alpha^2(\alpha - 2^{q_1})^2.$$

Let $\alpha_0$ be a root of this polynomial. Then we have

$$(2\alpha_0 m(x))^2 (x^{2r_1}+1)^{2p_1} + (x^{2r_1}+2)^{2q_1} + l_2(x)^2 x^{2r_1} = 0 \tag{4}$$

with some polynomial $l_2(x)$. Since $m(x)$ is coprime to both $x^{2r_1}+1$ and $x^{2r_1}+2$, the polynomial curve coming from (4) via (3) is horizontal. This completes the proof in the surface case.

Now we turn to the case of a trinomial hypersurface of arbitrary dimension. It is well known that for all sufficiently large positive integers $c_i$ there exist positive integers $b_{i1}, \ldots, b_{in_i}$ such that

$$b_{i1}l_{i1} + \cdots + b_{in_i}l_{in_i} = c_id_i.$$

We take sufficiently large pairwise coprime $c_0, c_1, c_2$ that are coprime to $d_0, d_1, d_2$, find the corresponding $b_{ij}$, substitute $T_{ij} = z_i^{b_{ij}}$, and obtain

$$z_0^{c_0d_0} + z_1^{c_1d_1} + z_2^{c_2d_2} = 0. \tag{5}$$

If the hypersurface $X$ is rational, surface (5) is rational as well. We take a horizontal polynomial curve on this surface:

$$z_0 = \phi_0(x), \quad z_1 = \phi_1(x), \quad z_2 = \phi_2(x).$$

With $T_{ij} = \phi_i(x)^{b_{ij}}$ we obtain a polynomial curve on $X$. Let us check that this curve is horizontal. The rational invariants $T_i^{l_i}/T_j^{l_j}$ on this curve are equal to

$$\frac{\phi_i(x)^{c_i d_i}}{\phi_j(x)^{c_j d_j}}.$$

This fraction is non-constant for some $i, j$ just because the curve on surface (5) is horizontal. This completes the proof of Theorem 3. ∎

Remark 2. By [10, Theorem 1.1(ii)], a trinomial hypersurface of Type 2 is a factorial affine variety if and only if the numbers $d_0, d_1, d_2$ are pairwise coprime. In particular, every factorial trinomial hypersurface of Type 2 satisfies the conditions of Theorem 3.

Remark 3. In [4] we show that every irreducible simply connected curve on a toric affine surface $X$ is an orbit closure of an action $\mathbb{G}_m \times X \to X$ of the multiplication group $\mathbb{G}_m$ of the ground field. The results of this paper characterize existence of certain polynomial curves on affine hypersurfaces with a torus action of complexity one

Problem 1. *Let $X$ be a normal rational affine variety without non-constant invertible functions equipped with a torus action $T \times X \to X$ of complexity one such that $\mathbb{C}[X]^T = \mathbb{C}$. Does $X$ admit a horizontal polynomial curve?*

One possible approach to this problem is to use Cox rings and total coordinate spaces (see [3, Section 1.6] for details). Namely, under our assumptions the variety $X$ has a finitely generated divisor class group $\mathrm{Cl}(X)$ and a finitely generated Cox ring $R(X)$. Moreover, the ring $R(X)$ is the quotient of a polynomial ring by an ideal generated by trinomials [13, Theorem 1.8], and the total coordinate space $\overline{X} = \mathrm{Spec}(R(X))$ carries a torus action of complexity 1. So one may try to construct a horizontal polynomial curve on $\overline{X}$ and then to project it to a horizontal polynomial curve on $X$ via the quotient morphism $\overline{X} \to X$. The difficulty with this approach is that the total coordinate space need not be rational: see [2, Example 5.12] and the following example.

Example 2. Consider the surface $V_{3,3,3} = V(z_0^3 + z_1^3 + z_2^3)$ in $\mathbb{C}^3$. This surface is not rational and does not admit a horizontal polynomial curve. On the other hand, the quotient $X$ of $V_{3,3,3}$ by the group $H = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ acting as

$$(z_0, z_1, z_2) \mapsto (\epsilon_0 z_0, \epsilon_1 z_1, \epsilon_0\epsilon_1 z_2), \qquad \epsilon_0^3 = \epsilon_1^3 = 1,$$

is a rational $\mathbb{G}_m$-surface [13, Theorem 1.7]. One can check that the algebra $\mathbb{C}[X]$ is generated by the functions

$$a = z_0^3, \qquad b = z_1^3, \qquad c = z_2^3, \qquad d = z_0 z_1 z_2^2, \qquad e = z_0^2 z_1^2 z_2,$$

and the formulas

$$a = -(x+1)x^2, \quad b = -(x+1)x^3, \quad c = (x+1)^2x^2,$$
$$d = (x+1)^2x^3, \quad e = (x+1)^2x^4$$

define a horizontal polynomial curve on $X$.

**4. Horizontal curves on trinomial hypersurfaces of Type 1.** In this section we study existence of horizontal polynomial curves on trinomial hypersurfaces of Type 1. For this we need the following important result [21], [17], [19, Theorem 1.8]. Given a polynomial $p(x) \in \mathbb{C}[x]$, denote by $\omega(p(x))$ the number of its distinct roots (without counting multiplicities).

The Mason–Stothers abc-Theorem. *Let $a(x)$, $b(x)$, $c(x)$ be three coprime polynomials, not all three constant. Assume that $a(x) + b(x) + c(x) = 0$. Then*

$$\max\{\deg a(x), \deg b(x), \deg c(x)\} \leq \omega(a(x)b(x)c(x)) - 1.$$

Let us turn to a characterization of existence of horizontal polynomial curves.

Theorem 4. *Let $X$ be a trinomial hypersurface of Type 1. The following conditions are equivalent:*

(i) *The hypersurface $X$ admits a horizontal polynomial curve.*
(ii) *$l_{ij} = 1$ for some $i = 1, 2$ and some $j = 1, \ldots, n_i$.*

*Proof.* (ii)⇒(i). Renumbering, we may assume that $l_{11} = 1$. Then we let

$$T_{11} = -x^{l_{21}} - 1, \quad T_{12} = \cdots = T_{1n_1} = 1, \quad T_{21} = x, \quad T_{22} = \cdots = T_{2n_2} = 1.$$

This gives a horizontal polynomial curve on $X$.

(i)⇒(ii). Let $T_{ij}(x)$ be a horizontal polynomial curve on $X$. We let

$$a(x) = T_{11}^{l_{11}}(x) \ldots T_{1n_1}^{l_{1n_1}}(x), \quad b(x) = T_{21}^{l_{21}}(x) \ldots T_{2n_2}^{l_{2n_2}}(x), \quad c(x) = 1.$$

Denote by $m_{ij}$ the number of distinct roots of the polynomial $T_{ij}(x)$. By the Mason–Stothers abc-Theorem, we have

$$\begin{aligned} m_{11}l_{11} + \cdots + m_{1n_1}l_{1n_1} &\leq \deg a(x) \leq \omega(a(x)b(x)) - 1 \\ &\leq m_{11} + \cdots + m_{1n_1} + m_{21} + \cdots + m_{2n_2} - 1, \end{aligned}$$

and similarly

$$m_{21}l_{21} + \cdots + m_{2n_2}l_{2n_2} \leq m_{11} + \cdots + m_{1n_1} + m_{21} + \cdots + m_{2n_2} - 1.$$

Summing up these two inequalities, we obtain

$$m_{11}(l_{11} - 2) + \cdots + m_{1n_1}(l_{1n_1} - 2) + m_{21}(l_{21} - 2) + \cdots + m_{2n_2}(l_{2n_2} - 2) \leq -2.$$

If all $l_{ij}$ are $\geq 2$, this is a contradiction. ∎

REMARK 4. Consider a trinomial hypersurface $X$ of Type 1 and let again $d_i = \gcd(l_{i1}, \dots, l_{in_i})$. By [14, Corollary 3.5], the hypersurface $X$ is rational if and only if either at least one of $d_1, d_2$ equals 1, or $d_1 = d_2 = 2$. Theorem 4 shows that not every rational trinomial hypersurface of Type 1 admits a horizontal polynomial curve. Moreover, by [13, Proposition 2.8], a trinomial hypersurface of Type 1 is factorial if and only if either $n_i l_{i1} = 1$ for some $i = 1, 2$, or $d_1 = d_2 = 1$. This shows that not every factorial trinomial hypersurface of Type 1 admits a horizontal polynomial curve.

**5. Schwarz–Halphen curves and platonic triples.** We keep the notation of the previous sections. For a polynomial curve

$$\tau\colon \mathbb{C} \to X, \qquad \tau(x) = (T_{ij}(x)),$$

we let

$$T_i^{l_i}(x) := T_{i1}(x)^{l_{i1}} \dots T_{in_i}(x)^{l_{in_i}}.$$

DEFINITION 3. A polynomial curve $\tau\colon \mathbb{C} \to X$ on a trinomial hypersurface of Type 2 is called a *Schwarz–Halphen curve* (an *SH-curve* for short) if the polynomials $T_0^{l_0}(x), T_1^{l_1}(x), T_2^{l_2}(x)$ are coprime.

In the case of a polynomial curve on the Pham–Brieskorn surface $V_{p,q,r}$, this condition means that the curve does not pass through the origin.

LEMMA 3. *Any SH-curve on a trinomial hypersurface $X$ of Type 2 is horizontal.*

*Proof.* By Lemma 2, it suffices to show that the rational function $T_0^{l_0}/T_1^{l_1}$ is non-constant along any SH-curve. If this is not the case, the polynomials $T_0^{l_0}(x)$ and $T_1^{l_1}(x)$ are proportional. Being coprime, they are constant. Then $T_2^{l_2}(x)$ is constant as well, so the curve is constant, a contradiction. ∎

Lemma 1 shows that the image $\tau(\mathbb{C})$ of an SH-curve $\tau\colon \mathbb{C} \to X$ is contained in the smooth locus $X^{\mathrm{reg}}$. The following example shows that the converse statement does not hold.

EXAMPLE 3. Consider the hypersurface $X$ given by

$$T_{01}^3 T_{02} + T_{11}^3 T_{12} + T_{21}^2 T_{22} = 0$$

and the curve $\tau\colon \mathbb{C} \to X$ defined by

$$T_{01} = x+1, \quad T_{02} = x, \quad T_{11} = x-1, \quad T_{12} = x, \quad T_{21} = x, \quad T_{22} = -2(x^2+3).$$

This curve is not an SH-curve, but all of its points are smooth on $X$.

The following result generalizes Theorem 2 to higher dimensions. In the proof we use the idea of the proof of [19, Theorem 18.4].

THEOREM 5. *Let $X$ be a trinomial hypersurface of Type 2. Assume that $l_{i1} \le \dots \le l_{in_i}$ for $i = 0, 1, 2$. Then the following conditions are equivalent:*

(i) *The hypersurface $X$ admits an SH-curve.*
(ii) *The hypersurface $X$ admits a polynomial curve $\tau\colon \mathbb{C} \to X^{\mathrm{reg}}$.*
(iii) *The triple $(l_{01}, l_{11}, l_{21})$ is platonic.*

*Proof.* The implication (i)⇒(ii) has been observed above. For (iii)⇒(i), assume that $(l_{01}, l_{11}, l_{21})$ is a platonic triple and let $T_{ij}(x) = 1$ for all $i = 0, 1, 2$ and all $1 < j \le n_i$. By Theorem 2, the surface $V(T_{01}^{l_{01}} + T_{11}^{l_{11}} + T_{21}^{l_{21}})$ admits an SH-curve.

We turn to (i)⇒(iii). Let $\tau\colon \mathbb{C} \to X$ be an SH-curve. Without loss of generality we assume that $l_{01} \ge l_{11} \ge l_{21} \ge 2$. Let

$$a(x) = T_0^{l_0}(x), \qquad b(x) = T_1^{l_1}(x), \qquad c(x) = T_2^{l_2}(x).$$

Denote by $m_{ij}$ the number of pairwise distinct roots of the polynomial $T_{ij}(x)$. Then the Mason–Stothers abc-Theorem implies

$$\sum_j l_{0j} m_{0j} \le \sum_j m_{0j} + \sum_j m_{1j} + \sum_j m_{2j} - 1, \tag{6}$$

$$\sum_j l_{1j} m_{1j} \le \sum_j m_{0j} + \sum_j m_{1j} + \sum_j m_{2j} - 1, \tag{7}$$

$$\sum_j l_{2j} m_{2j} \le \sum_j m_{0j} + \sum_j m_{1j} + \sum_j m_{2j} - 1. \tag{8}$$

Summing (6)–(8), we obtain

$$\begin{aligned}\sum_j l_{0j} m_{0j} + \sum_j l_{1j} m_{1j} + \sum_j l_{2j} m_{2j} \\ \le 3\Big(\sum_j m_{0j} + \sum_j m_{1j} + \sum_j m_{2j}\Big) - 3.\end{aligned} \tag{9}$$

Thus we have $l_{21} = 2$. If $l_{11} = 2$ then the triple $(l_{01}, l_{11}, l_{21})$ is platonic.

Assume that $l_{11} \ge 3$. Let $l_{21} = \cdots = l_{2s_2} = 2$ and $l_{2j} \ge 3$ with $j > s_2$. We denote $l_{2j}$ and $m_{2j}$ with $j > s_2$ by $l''_{2j}$ and $m''_{2j}$ respectively, and $m_{21}, \ldots, m_{2s_2}$ by $m'_{2j}$.

One obtains from (8) the inequality

$$\sum_j m'_{2j} + \sum_j (l''_{2j} - 1) m''_{2j} \le \sum_j m_{0j} + \sum_j m_{1j} - 1. \tag{10}$$

Hence

$$\sum_j m'_{2j} \le \sum_j m_{0j} + \sum_j m_{1j} - 1. \tag{11}$$

It follows from (9) and (11) that

$$\begin{aligned}\sum_j l_{0j}m_{0j}+\sum_j l_{1j}m_{1j}+\sum_j l''_{2j}m''_{2j} &\le 3\Big(\sum_j m_{0j}+\sum_j m_{1j}+\sum_j m''_{2j}\Big)\\ &\quad+\sum_j m'_{2j}-3\\ &\le 4\Big(\sum_j m_{0j}+\sum_j m_{1j}\Big)+3\sum_j m''_{2j}-4.\end{aligned}$$

Thus

$$\sum_j l_{0j}m_{0j}+\sum_j l_{1j}m_{1j}\le 4\Big(\sum_j m_{0j}+\sum_j m_{1j}\Big)-4. \tag{12}$$

This proves that $l_{11}=3$. Let $l_{11}=\cdots=l_{1s_1}=3$ and $l_{1j}\ge 4$ with $j>s_1$. We denote $l_{1j}$ and $m_{1j}$ with $j>s_1$ by $l''_{1j}$ and $m''_{1j}$ respectively, and $m_{11},\dots,m_{1s_1}$ by $m'_{1j}$.

Then (7) can be rewritten as

$$2\sum_j m'_{1j}+\sum_j (l''_{1j}-1)m''_{1j}\le \sum_j m_{0j}+\sum_j m_{2j}-1. \tag{13}$$

Summing (10) and (13), we obtain

$$\sum_j m'_{1j}\le 2\sum_j m_{0j}-2+\sum_j(2-l''_{1j})m''_{1j}+\sum_j(2-l''_{2j})m''_{2j}. \tag{14}$$

From (8) and (14) we get

$$\sum_j m'_{2j}\le 3\sum_j m_{0j}-3+\sum_j(3-l''_{1j})m''_{1j}+\sum_j(3-2l''_{2j})m''_{2j}. \tag{15}$$

Using (6), (14), (15), we obtain

$$\begin{aligned}\sum_j l_{0j}m_{0j} &\le \sum_j m_{0j}+\sum_j m'_{1j}+\sum_j m''_{1j}+\sum_j m'_{2j}+\sum_j m''_{2j}-1\\ &\le \sum_j m_{0j}+2\sum_j m_{0j}-2+\sum_j(6-2l''_{1j})m''_{1j}\\ &\quad+3\sum_j m_{0j}-3+\sum_j(6-3l''_{2j})m''_{2j}-1\\ &\le 6\sum_j m_{0j}-6.\end{aligned}$$

This proves that $l_{01}\le 5$ and thus the triple $(l_{01},l_{11},l_{21})$ is platonic.

Finally, let us prove (ii)⇒(i). Consider a curve $\tau\colon \mathbb{C}\to X^{\mathrm{reg}}$ and assume that the polynomials $T_0^{l_0}(x)$, $T_1^{l_1}(x)$, $T_2^{l_2}(x)$ are not coprime. Let $L(x)$ be a linear form that divides all these three polynomials. There exist indices

$1 \le j_s \le n_s$, $s = 0, 1, 2$, such that $L(x)$ divides the polynomials $T_{sj_s}(x)$, $s = 0, 1, 2$.

If at least one of the exponents $l_{sj_s}$ equals 1, then the triple $(l_{01}, l_{11}, l_{21})$ is platonic and we use the implication (iii)⇒(i).

If all the exponents $l_{si_s}$ are greater than 1, we consider the root $x = \alpha$ of the linear form $L(x)$. By Lemma 1, the point $\tau(\alpha)$ is a singular point on $X$, a contradiction.

This completes the proof of Theorem 5. ■

Remark 5. By [15, Section 2], an algebraic variety $X$ is said to be $\mathbb{A}^1$-*poor* if there exists a subvariety $Y$ of $X$ of codimension at least 2 such that every polynomial curve on $X$ meets $Y$. Theorem 5 implies that every trinomial hypersurface $X$ of Type 2 such that the triple $(l_{01}, l_{11}, l_{21})$ is not platonic is $\mathbb{A}^1$-poor. Indeed, any polynomial curve on $X$ meets the singular locus $Y$ of $X$. In particular, such hypersurfaces are *rigid* in the sense that $X$ admits no non-trivial $\mathbb{G}_a$-action, or equivalently the algebra $\mathbb{C}[X]$ admits no non-zero locally nilpotent derivation. Rigid factorial trinomial hypersurfaces of Type 2 are characterized in [1, Theorem 1]. Moreover, an explicit description of the automorphism group of a rigid trinomial hypersurface can be found in [5, Theorem 5.5].

Remark 6. If $\tau\colon \mathbb{C} \to X$ is a polynomial curve on a trinomial hypersurface $X$ of Type 1, then the polynomials $T_1^{l_1}(x)$ and $T_2^{l_2}(x)$ are coprime automatically. Thus every polynomial curve on $X$ is an SH-curve.

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**Abstract** (will appear on the journal's web site only)

We prove that every rational trinomial affine hypersurface admits a horizontal polynomial curve. This result provides an explicit non-trivial polynomial solution to a trinomial equation. Also we show that a trinomial affine hypersurface admits a Schwarz–Halphen curve if and only if the trinomial comes from a platonic triple. It is a generalization of Schwarz–Halphen's Theorem for Pham–Brieskorn surfaces.