REPRESENTATIONS OF FINITE-DIMENSIONAL QUOTIENT ALGEBRAS OF THE 3-STRING BRAID GROUP

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ABSTRACT. We consider quotients of the group algebra of the 3-string braid group B_3 by p-th order generic polynomial relations on the elementary braids. In cases $p = 2, 3, 4, 5$ these quotient algebras are finite dimensional. We give semisimplicity criteria for these algebras and present explicit formulas for all their irreducible representations.

Keywords: Braid group, irreducible representations, semisimplicity.

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Introduction

A classical theorem by H.S.M. Coxeter states that factorizing the n-strand braid group B_n by p-th order relation $\sigma^p = 1$ on its elementary braid generator σ results in a finite quotient if and only if

$$1/n + 1/p > 1/2 . \tag{0.1}$$

In case of B_3 such factorization gives finite quotient groups for $p = 2, 3, 4, 5$, of orders, respectively, 6, 24, 96, and 600 [C]. Generalizing this setting one can consider quotients of the group algebra $\mathbb{C}[B_n]$ obtained by imposing p-th order monic polynomial relation on the elementary braids. Under condition (0.1) the resulting quotient algebras are finite dimensional and, by the Tits deformation theorem (see [CR], §68, or [HR], section 5), in a generic situation they are isomorphic to the group algebras of the corresponding Coxeter's quotient groups and, hence, semisimple. As a next step it would be interesting to identify the semisimplicity conditions and to describe explicitly irreducible representations of the finite dimensional quotients.

A significant progress in this direction have been achieved by I. Tuba and H. Wenzl. In paper [TW] they have classified all the irreducible representations of B_3 in dimensions ≤ 5. Their classification scheme in dimensions ≤ 4 gives all the irreducible representations for the quotients in cases $n = 3$, $p = 2, 3, 4$, and describes their semisimplicity conditions. However the $\mathbb{C}[B_3]$ quotient algebras for $p = 5$ admit irreducible representations of dimensions up to 6 and the classification in [TW] does not cover them. In this note we construct all the 6-dimensional irreducible representations of these algebras and identify their semisimplicity conditions. We are working in the diagonal basis for the first elementary braid generator g_1, and we restrict our considerations to the case where all p roots of its minimal polynomial are distinct. For the sake of completeness we present formulas for representations from I. Tuba and H. Wenzl list in this basis too.

Our paper is organized as follows. In the next section we fix notations and derive preliminary results on possible values of a central element of B_3 in low dimensional irreducible representations ($d \leq 6$). Section 2 contains our main results: theorem 4 — criteria of semisimplicity of the $p = 2, 3, 4, 5$ quotients of $\mathbb{C}[B_3]$, and proposition 2 — explicit formulae for all their irreducible representations. Before going on with the considerations let us mention a number of related results and approaches.

In [W] B. Westbury suggested approach to representation theory of B_3 using representations of a particular quiver. It was subsequently used by L. Le Bruyn to construct Zariski dense rational parameterizations of the irreducible representations of B_3 of any dimension [B1, B2]. This approach has proved to be effective in treating a problem of braid reversion (see [B1]). However it does not provide representation's semisimplicity criteria. A 5-dimensional variety of the irreducible 6-dimensional representations of B_3 constructed below belongs to a 8-dimensional family of B_3-representations of type 6b (see Fig.1 in [B1]).

For a more general case of B_n, $n > 3$, series of irreducible representations related with the Iwahori-Hecke ($p = 2$ case) and Birman-Murakami-Wenzl algebras ($p = 3$ case with additional restrictions) are well investigated (for a review, see [LR]). Some other particular families of the B_n-representations have been found in [FLSV, AK].

1. Braid group B_3 and its quotients: spectrum of elementary braids

The three strings braid group B_3 is generated by a pair of *elementary braids* – g_1 and g_2 – satisfying the braid relation

$$g_1g_2g_1 = g_2g_1g_2. \tag{1.1}$$

Alternatively it can be given in terms of generators

$$a = g_1g_2, \quad b = g_1g_2g_1, \tag{1.2}$$

and relations

$$a^3 = b^2 = c\,, \tag{1.3}$$

where $c = (g_1g_2)^3 = (g_1g_2g_1)^2$ is a central element of B_3 which generates the center $\mathbb{Z}(B_3)$. Thus, the quotient group $B_3/\mathbb{Z}(B_3) = \langle a, b|\, a^3 = b^2 = 1\rangle$ is the free product of two cyclic groups $\mathbb{Z}_3 * \mathbb{Z}_2$ which is known to be isomorphic to $PSL(2, \mathbb{Z})$.

Let X be a set of pairwise different nonzero complex numbers:

$$X = \{x_1, x_2, \dots, x_n\}, \qquad x_i \in \mathbb{C} \setminus \{0\}, \;\; x_i \neq x_j \;\forall i \neq j. \tag{1.4}$$

In this note we consider finite dimensional quotient algebras of the group algebra $\mathbb{C}[B_3]$ obtained by imposing following polynomial conditions on the elementary braids:[1]

$$P_X(g) = \prod_{i=1}^{n=|X|} (g - x_i 1) = 0, \qquad \text{where } g \text{ is either } g_1, \text{ or } g_2. \tag{1.5}$$

As was already mentioned in the introduction the quotient algebras

$$Q_X = \mathbb{C}[B_3]/\langle P_X(g)\rangle. \tag{1.6}$$

are finite dimensional iff $|X| = n < 6$. With a particular choice of polynomials $P_X(g) = g^n - 1$ they are the group algebras of the quotient groups $B_3/\langle g^n\rangle$ and, by the Tits deformation argument, $Q_X \simeq \mathbb{C}[B_3/\langle g^n\rangle]$ for $n < 6$ and for generic choice of $x_i \in X$ and, therefore, in a generic situation Q_X is semisimple.

In the next section we will classify irreducible representations of these algebras. It turns out that their dimensions are less or equal to 6. In the rest of this section we will show that in these irreducible representations the spectra of the central element c (1.1) and of generators a and b (1.2) are, up to a discrete factor, defined by the eigenvalues x_i of the elementary braids.

Let V be a finite dimensional linear space, $\dim V = d$. Let $\rho_{X,V}$ be a family of irreducible representations $Q_X \to \mathrm{End}(V)$. We will assume that their characters are continuous functions of parameters $x_i \in X$.[2] Throughout this section we also assume that $d \geq n$ and that the minimal polynomials of operators $\rho_{X,V}(g_{1,2})$ coincide with P_X. The latter assumptions do not cause any loss of generality since **a)** all roots of the characteristic polynomials of $\rho_{X,V}(g_{1,2})$ belong to X, and **b)** given a family $\rho_{X',V}$ we can treat it as a family of representations of the quotient algebras Q_X of a minimal possible set $X \subset X'$ removing from X' all the elements which do not show up in the

[1] In the braid group elementary braids g_1 and g_2 are conjugate to each other and, hence, conditions on them are identical.

[2] All representations constructed in the next section satisfy the continuity condition.

characteristic polynomials of $\rho_{X',V}(g_{1,2})$. The characteristic polynomial of elementary braids $g_{1,2}$ in representation $\rho_{X,V}$ then has a form

$$\Pi_\rho(g) = \prod_{i=1}^{n=|X|} (g - x_i)^{m_i}, \qquad \text{where } \; m_i \in \mathbb{N}^+ : \; \sum_{i=1}^{n} m_i = d. \tag{1.7}$$

In particular, $\det \rho_{X,V}(g_{1,2}) = \prod_{i=1}^n x_i^{m_i}$.

Denote

$$A := \rho_{X,V}(a), \quad B := \rho_{X,V}(b), \quad \rho_{X,V}(c) := C_\rho \,\mathrm{Id}_V. \tag{1.8}$$

Here we have taken into account that, by Schur's lemma, central element c acts in the irreducible representation as a scalar operator. Calculating determinant of $\rho_{X,V}(c)$ one finds relation

$$\left(\prod_{i=1}^n x_i^{m_i}\right)^6 = (C_\rho)^d\,. \tag{1.9}$$

By (1.3) operators A and B satisfy equalities

$$A^3 = B^2 = C_\rho \,\mathrm{Id}_V. \tag{1.10}$$

Notice that A and B can not be scalar, otherwise $\rho_{X,V}(g_1)$ and $\rho_{X,V}(g_2)$ have common basis of eigenvectors and the representation $\rho_{X,V}$ is reducible. Thus, A and B should have at least two different eigenvalues taking values in sets

$$\mathrm{Spec}A \subset C_\rho^{1/3} \cdot \{1, \nu, \nu^{-1}\}, \;\; \nu := e^{2\pi \mathrm{i}/3}, \qquad \mathrm{Spec}B \subset C_\rho^{1/2} \cdot \{1, -1\}\,. \tag{1.11}$$

The following proposition describes explicitly the spectrum of operators A and B in low dimensional representations.

Proposition 1. *Let $\rho_{X,V} : Q_X \to \mathrm{End}(V)$ be a family of irreducible representations of algebras Q_X (1.6) such that*
a) their characters are continuous functions of parameters $x_i \in X$;
b) characteristic and minimal polynomials of the elementary braids $\rho_{X,V}(g_{1,2})$ are given, respectively, by Π_ρ (1.7) and P_X (1.5).

Let A, B, C_ρ be as defined in (1.8). Denote $\nu := e^{2\pi\mathrm{i}/3}$, and introduce notation $e_k(X)$ for k-th elementary symmetric polynomial in the set of variables $X = \{x_i\}_{i=1,\dots,n}$.

Then for $n = |X| \le 5$ and $d = \dim V \le 6$ coefficient C_ρ and eigenvalues of operators A and B can take following values.

$$d = n = 2: \;\; C_\rho = -e_2(X)^3,$$
$$\mathit{Spec}\,A = -e_2(X)\cdot\{\nu, \nu^{-1}\}, \quad \mathit{Spec}\,B = \mathrm{i}\, e_2(X)^{\frac{3}{2}}\cdot\{1, -1\}; \tag{1.12}$$

$$d = n = 3: \;\; C_\rho = e_3(X)^2,$$
$$\mathit{Spec}\,A = e_3(X)^{\frac{2}{3}}\cdot\{1, \nu, \nu^{-1}\}, \quad \mathit{Spec}\,B = e_3(X)\cdot\{1, -1^{\sharp 2}\}; \tag{1.13}$$

$$d = n = 4: \;\; \text{for any root } h(X) := \sqrt[2]{e_4(X)}: \quad C_\rho = h(X)^3,$$
$$\mathit{Spec}\,A = h(X)\cdot\{1^{\sharp 2}, \nu, \nu^{-1}\}, \quad \mathit{Spec}\,B = h(X)^{\frac{3}{2}}\cdot\{1^{\sharp 2}, -1^{\sharp 2}\}; \tag{1.14}$$

$$d = n = 5: \;\; \text{for any root } f(X) := \sqrt[5]{e_5(X)}: \quad C_\rho = f(X)^6,$$
$$\mathit{Spec}\,A = f(X)^2\cdot\{1, \nu^{\sharp 2}, (\nu^{-1})^{\sharp 2}\}, \quad \mathit{Spec}\,B = f(X)^3\cdot\{1^{\sharp 3}, -1^{\sharp 2}\}; \tag{1.15}$$

$$d=6,\ n=5,\ m_i=2,\ 1\le i\le 5: \qquad C_\rho=-x_ie_5(X),$$
$$Spec\,A=-\sqrt[3]{x_ie_5(X)}\cdot\{1^{\sharp 2},\nu^{\sharp 2},(\nu^{-1})^{\sharp 2}\},\quad Spec\,B=\mathrm{i}\sqrt[2]{x_ie_5(X)}\cdot\{1^{\sharp 3},-1^{\sharp 3}\}. \tag{1.16}$$

Proof. Denote Tr_V an operation of taking trace in representation $\rho_{X,V}$. To prove assertions of the proposition we analyze functions $\mathrm{Tr}_V(g_1^kg_2)$, for $k=2,\dots,5$.

Case $d=n=2$. Using minimal polynomial for g_1 and characteristic polynomial for g_2 we calculate

$$\mathrm{Tr}B\,=\,\mathrm{Tr}_V\,(g_1g_2g_1)\,=\,\mathrm{Tr}_V\,\left(g_1^2g_2\right)\,=\,e_1(X)\,(\mathrm{Tr}A-e_2(X))\,.$$

Noticing that spectral condition (1.11) for the non-scalar 2×2 matrix B assumes $\mathrm{Tr}B=0$ we conclude that $\mathrm{Tr}A=e_2(X)$. From (1.9) we have $C_\rho=\pm e_2(\mathrm{x})^3$, which together with spectral condition on A (1.11) leaves us the only possibility to fulfill relations for the traces of A and B, namely the one presented in (1.12).

Case $d=n=3$. We shall evaluate $\mathrm{Tr}_Vg_1^3g_2$ in two different ways. First, we use cyclic property of the trace and the braid relation (1.1):

$$\mathrm{Tr}_Vg_1^3g_2\,=\,\mathrm{Tr}_Vg_1^2g_2g_1\,=\,\mathrm{Tr}_V(g_1g_2)^2\,=\,\mathrm{Tr}A^2. \tag{1.17}$$

Second, we apply minimal polynomial for g_1 and characteristic polynomial for g_2:

$$\mathrm{Tr}_Vg_1^3g_2\,=\,e_1(X)\mathrm{Tr}B\,-\,e_2(X)\mathrm{Tr}A\,+\,e_3(X)e_1(X).$$

Comparing the results of these calculations and taking into account that, by (1.11) and (1.9), traces of powers of A and B can be expressed in terms of (roots of) $e_3(X)$ and, hence, are algebraically independent from $e_1(X)$ and $e_2(X)$ we find that $\mathrm{Tr}A=\mathrm{Tr}A^2=0$, $\mathrm{Tr}B=-e_3(X)$. On the other hand from (1.9) one finds $C_\rho=\sqrt[3]{1}\,e_3(X)^2$ which, together with the spectral conditions (1.11), gives (1.13) as the only possibility to satisfy the above relations for traces.

Case $d=n=4$. Similarly to the case $d=n=3$ we calculate $\mathrm{Tr}_Vg_1^4g_2$ in two ways:

$$\mathrm{Tr}_Vg_1^4g_2\,=\,\mathrm{Tr}_V(g_1g_2)^2g_1\,=\,C_\rho\,\mathrm{Tr}_V(g_1g_2)^{-1}g_1\,=\,C_\rho\,e_3(X)/e_4(X), \tag{1.18}$$
$$\mathrm{Tr}_Vg_1^4g_2\,=\,e_1(X)\mathrm{Tr}A^2\,-\,e_2(X)\mathrm{Tr}B\,+\,e_3(X)\mathrm{Tr}A\,-\,e_4(X)e_1(X),$$

where in the last line we take additionally into account eq.(1.17). Hence, using an algebraic independence of C_ρ and thus of $\mathrm{Tr}A$, $\mathrm{Tr}A^2$ and $\mathrm{Tr}B$ from the elementary symmetric polynomials $e_i(X)$, $i=1,2,3$, one concludes: $\mathrm{Tr}A=C_\rho/e_4(X)$, $\mathrm{Tr}A^2=e_4(X)$, $\mathrm{Tr}B=0$. The latter conditions are only compatible with eqs.(1.9) and (1.11) in two cases given in (1.14).

Case $d=n=5$. Here we calculate $\mathrm{Tr}_Vg_1^5g_2$:

$$\begin{aligned}\mathrm{Tr}_Vg_1^5g_2\,&=\,C_\rho\,\mathrm{Tr}_V(g_1g_2)^{-1}g_1^2\,=\,C_\rho\,\mathrm{Tr}_Vg_1^{-1}g_2\\ &=\frac{C_\rho}{e_5(X)}\left(C_\rho\frac{e_4(X)}{e_5(X)}\,-\,e_1(X)\mathrm{Tr}A^2\,+\,e_2(X)\mathrm{Tr}B\,-\,e_3(X)\mathrm{Tr}A\,+\,e_4(X)e_1(X)\right),\end{aligned}$$

where passing to the second line we expressed g_1^{-1} in terms of positive powers of g_1 using its minimal polynomial and then used $d=5$ analogue of formula (1.18).

Calculating $\mathrm{Tr}_Vg_1^5g_2$ in another way we obtain

$$\mathrm{Tr}_Vg_1^5g_2\,=\,e_1(X)\left(C_\rho\frac{e_4(X)}{e_5(X)}\right)-e_2(X)\mathrm{Tr}A^2+e_3(X)\mathrm{Tr}B-e_4(X)\mathrm{Tr}A+e_5(X)e_1(X).$$

Now collecting coefficients in the independent polynomials $e_i(X))$, $i = 1,2,3,4$, and taking into account eq.(1.9) we find $C_\rho = e_5(X)^{6/5}$, $\mathrm{Tr}A = -e_5(X)^{2/5}$, $\mathrm{Tr}A^2 = -e_5(X)^{4/5}$, $\mathrm{Tr}B = e_5(X)^{3/5}$, which in combination with (1.11) finally leads to conditions (1.15).

Case $d = 6$, $n = 5$: We calculate $\mathrm{Tr}_V g_1^5 g_2$ in two ways similarly to the previous case, but using now different expressions $\mathrm{Tr}_V g_1 = e_1(X) + x_i$, $\mathrm{Tr}_V g_1^{-1} = e_4(X)/e_5(X) + x_i^{-1}$, following from the characteristic polynomial (1.7). Collecting then coefficients in independent polynomials we derive $C_\rho = -x_i e_5(X)$, $\mathrm{Tr}A = \mathrm{Tr}A^2 = \mathrm{Tr}B = 0$, which in combination with (1.11) assumes (1.16). □

2. Low dimensional representations of Q_X and semisimplicity

In this section we construct explicitly representations of algebras Q_X whose data coincide with those given in the proposition 1. Investigating reducibility conditions for these representations we obtain semisimplicity criteria for algebras Q_X and classify their irreducible representations. We derive formulas for the representations in the basis of eigenvectors of g_1.

Proposition 2. *Algebras Q_X in cases $|X| \le 5$ have following representations of dimensions $\dim V \le 6$.*

- $\underline{|X| = \dim V = 1 :}$

$$\rho_X^{(1)}(g_1) = \rho_X^{(1)}(g_2) = x_1. \tag{2.1}$$

- $\underline{|X| = \dim V = 2 :}$

$$\rho_X^{(2)}(g_1) = diag\{x_1, x_2\}, \qquad \rho_X^{(2)}(g_2) = \frac{1}{x_1 - x_2}\begin{pmatrix} -x_2^2 & -x_1x_2 \\ x_1^2 - x_1x_2 + x_2^2 & x_1^2 \end{pmatrix}. \tag{2.2}$$

- $\underline{|X| = \dim V = 3 :}$

$$\rho_X^{(3)}(g_1) = diag\{x_1, x_2, x_3\}, \quad \rho_X^{(3)}(g_2) = \begin{pmatrix} \frac{x_2x_3(x_2+x_3)}{\Delta_1(X)} & \frac{x_3(x_1^2+x_2x_3)}{\Delta_1(X)} & \frac{x_2(x_1^2+x_2x_3)}{\Delta_1(X)} \\ \frac{x_3(x_2^2+x_1x_3)}{\Delta_2(X)} & \frac{x_1x_3(x_1+x_3)}{\Delta_2(X)} & \frac{x_1(x_2^2+x_1x_3)}{\Delta_2(X)} \\ \frac{x_2(x_3^2+x_1x_2)}{\Delta_3(X)} & \frac{x_1(x_3^2+x_1x_2)}{\Delta_3(X)} & \frac{x_1x_2(x_1+x_2)}{\Delta_3(X)} \end{pmatrix}, \tag{2.3}$$

where we introduced notation

$$\Delta_i(X) := \prod_{j=1,\ j\neq i}^{|X|} (x_j - x_i). \tag{2.4}$$

- $\underline{|X| = \dim V = 4.}$ *There exist two inequivalent representations depending on a choice of the square root $h = \sqrt{e_4(X)}$:*

$$\begin{aligned} \rho_{h,X}^{(4)}(g_1) &= diag\{x_1, x_2, x_3, x_4\}, \\ \rho_{h,X}^{(4)}(g_2) &= \begin{pmatrix} \frac{\alpha_1}{\Delta_1(X)} & \frac{\beta_1\gamma_3\gamma_4}{\Delta_1(X)} & \frac{\beta_1\gamma_2\gamma_4}{\Delta_1(X)} & \frac{\beta_1\gamma_2\gamma_3}{\Delta_1(X)} \\ \frac{\beta_2}{\Delta_2(X)} & \frac{\alpha_2}{\Delta_2(X)} & \frac{\beta_2\gamma_2}{\Delta_2(X)} & \frac{\beta_2\gamma_2}{\Delta_2(X)} \\ \frac{\beta_3}{\Delta_3(X)} & \frac{\beta_3\gamma_3}{\Delta_3(X)} & \frac{\alpha_3}{\Delta_3(X)} & \frac{\beta_3\gamma_3}{\Delta_3(X)} \\ \frac{\beta_4}{\Delta_4(X)} & \frac{\beta_4\gamma_4}{\Delta_4(X)} & \frac{\beta_4\gamma_4}{\Delta_4(X)} & \frac{\alpha_4}{\Delta_4(X)} \end{pmatrix}. \end{aligned} \tag{2.5}$$

$$
\begin{array}{rcl}
\textit{Here} \quad \alpha_i(h,X) & := & e_3(X^{\setminus i})\, e_1(X^{\setminus i}) \,-\, h\, e_2(X^{\setminus i}), \quad X^{\setminus i} := X \setminus \{x_i\}, \\
\beta_i(h,X) & := & e_4(X)/x_i^2 \,-\, h, \quad i = 1,2,3,4, \\
\gamma_a(h,X) & := & x_1 x_a \,+\, x_b x_c \,-\, h, \quad a,b,c \in \{x_2,x_3,x_4\} \textit{ are pairwise distinct.}
\end{array} \tag{2.6}
$$

• $\underline{|X| = \dim V = 5.}$ *There exist five inequivalent representations corresponding to different values of the root* $f(X) := \sqrt[5]{e_5(X)}$*:*

$$\rho^{(5)}_{f,X}(g_1) \;=\; diag\{x_1,x_2,x_3,x_4,x_5\}, \quad \rho^{(5)}_{f,X}(g_2) \,=\, ||m_{ij}||_{i,j=1}^{5}, \tag{2.7}$$

$$m_{ii}(f,X) \;:=\; \frac{e_4(X^{\setminus i})\, e_1(X^{\setminus i}) + f\, x_i\, e_3(X^{\setminus i}) + f \prod_{k=1,\, k\neq i}^{5} (f + x_k)}{\Delta_i(X)}, \tag{2.8}$$

$$m_{ij}(f,X) \;:=\; \frac{(x_i^2 + f\, x_i + f^2) \prod_{k=1,\, k\neq i,j}^{5} (f^2 + x_i x_k)}{f\, x_i\, x_j\, \Delta_i(X)}, \quad \forall i \neq j. \tag{2.9}$$

• $\underline{|X| = 5, \dim V = 6.}$ *There exist five inequivalent representations* $\rho^{(6)}_{i,X}$, $i = 1, \dots, 5$, *corresponding to all admissible values* $C_\rho = -x_i e_5(X)$ *of the central element* c. *Formulas for* $\rho^{(6)}_{5,X}$ *are given in table 1. Formulas for the other representations can be obtained by the transposition of the eigenvalues* x_5 *and* x_i*:* $\rho^{(6)}_{i,X} = \sigma_{i5} \circ \rho^{(6)}_{5,X}$, $i = 1 \dots 4$.

Remark 1. As it is noticed in section 1 a representation of Q_X stays also a representation of $Q_{X'}$ if $X \subset X'$.

Remark 2. Irreducible representations of B_3 of dimensions $d \leq 5$ were classified by Imre Tuba and Hans Wenzl in [TW]. We reproduce their table of representations in the basis where g_1 takes a diagonal form. In their approach I.Tuba and H.Wenzl have used different basis in which matrices of the braids g_1 and g_2 assume a special 'ordered' triangular from. This allows them analyzing also algebras whose minimal polynomials P_X have multiple roots and, hence, matrices of the braids $g_{1,2}$ are not diagonalizable. These cases are missed in our approach. Instead, our method is suitable for construction of the 6-dimensional representations for algebras Q_X, $|X| = 5$ and, thus, allows us classifying irreducible representations for these algebras and studying their semisimplicity.

Proof. By our initial assumptions matrices of braids $g_{1,2}$ in any representation are diagonalizable. We choose a basis where $\rho_{X,V}(g_1) := D_g$ is diagonal. By (1.7) the diagonal components of D_g are x_i taken with multiplicities m_i.

Keeping in mind that in an irreducible representation matrices A and B of braids a and b are also diagonalizable (see eq.(1.10)) we use for them parameterization

$$A = U^{-1} D_a U, \quad B = V D_b V^{-1}. \tag{2.10}$$

Here D_a and D_b are diagonal matrices whose diagonal components are elements of $Spec\, A$ and $Spec\, B$. For irreducible representations of dimensions ≤ 6 they were defined in proposition 1. Due to relation $g_1 = a^{-1} b$ matrices U and V have to satisfy condition

$$U\, D_g\, V \;=\; D_a^{-1}\, U\, V\, D_b. \tag{2.11}$$

Table 1. 6-dimensional representation of Q_X, $|X| = 5$.

$\rho^{(6)}_{5,X}(g_1) \;=\; diag\{x_1, x_2, x_3, x_4, x_5, x_5\}\,, \qquad \rho^{(6)}_{5,X}(g_2) \;=\; \Vert g_{ij}\Vert_{i,j=1}^{\;\;6},$	
$G := \Vert g_{ij}\Vert_{i,j=1}^{\;\;4}:$	$g_{ii} = \frac{e_4(X^{\setminus i})e_1(X^{\setminus i}) - x_i x_5 e_3(X^{\setminus i})}{\Delta_i(X)},\ X^{\setminus i} := X \setminus \{x_i\},\ {\scriptstyle i=1,\ldots,4};$ $g_{1a} \;=\; \frac{p_a\, q_b\, q_c}{x_1^2 \Delta_a(X)}, \quad g_{a1} \;=\; \frac{p_1}{x_a^2 \Delta_1(X)}, \quad g_{ab} \;=\; \frac{q_a\, p_b}{x_a^2 \Delta_b(X)},$ *where indices* $a, b, c \in \{2, 3, 4\}$ *are pairwise distinct, and* $q_a(X) := x_1 x_a + x_b x_c\,, \quad p_i(X) := e_5(X) - x_i^3 x_5^2\,;$
$G_{31} := \begin{pmatrix} g_{51} & g_{52} \\ g_{61} & g_{62} \end{pmatrix}:$	$diag\{\frac{1}{\Delta_1(X)}, \frac{1}{\Delta_2(X)}\};$
$G_{32} := \begin{pmatrix} g_{53} & g_{54} \\ g_{63} & g_{64} \end{pmatrix}:$	$\begin{pmatrix} q_4\, r & q_3\,(\sigma_{34} \circ r) \\ (\sigma_{12} \circ r) & (\sigma_{12}\sigma_{34} \circ r) \end{pmatrix},$ *where* $r(X) := \frac{x_3}{x_1(x_2 - x_1)\Delta_3(X^{\setminus 2})},$ *and* $\forall f(X):\ \sigma_{ij} \circ f(\ldots x_i \ldots x_j \ldots) := f(\ldots x_j \ldots x_i \ldots);$
$G_{33} := \begin{pmatrix} g_{55} & g_{56} \\ g_{65} & g_{66} \end{pmatrix}:$	$\begin{pmatrix} u & q_3\, q_4\, v \\ (\sigma_{12} \circ v) & (\sigma_{12} \circ u) \end{pmatrix},$ *where* $v(X) := \frac{p_2(X)}{x_1 x_5 (x_2 - x_1)\Delta_5(X^{\setminus 2})},$ *and* $u(X) := \frac{x_1 x_2 (x_3 + x_4)(x_3 x_4 - x_1 x_5) + x_3 x_4 (x_2 - x_1)(x_1^2 + x_2 x_5)}{(x_2 - x_1)\Delta_5(X^{\setminus 2})};$
$G_{23} := \begin{pmatrix} g_{35} & g_{36} \\ g_{45} & g_{46} \end{pmatrix}:$	$\frac{1}{x_5 \Delta_5(X)} \begin{pmatrix} \frac{w}{x_3^2} & \frac{q_3\,(\sigma_{12} \circ w)}{x_3^2} \\ \frac{(\sigma_{34} \circ w)}{x_4^2} & \frac{q_4\,(\sigma_{12}\sigma_{34} \circ w)}{x_4^2} \end{pmatrix},$ $w(X) := p_1(X)\Big(x_1 x_2 x_3 x_4 \{x_1 x_3 + x_5(x_2 + x_4)\} - x_5^3 \{x_1 x_3 (x_2 + x_4) + x_5 x_2 x_4\}\Big);$
$G_{13} := \begin{pmatrix} g_{15} & g_{16} \\ g_{25} & g_{26} \end{pmatrix}:$	$\frac{1}{\Delta_5(X)} \begin{pmatrix} \frac{z}{x_1} & \frac{q_3\, q_4\,(\sigma_{12}\sigma_{23} \circ w)}{x_1^2 x_5} \\ \frac{(\sigma_{23} \circ w)}{x_2^2 x_5} & \frac{(\sigma_{12} \circ z)}{x_2} \end{pmatrix},$ $z(X) := (e_1 e_3 - x_1^2 e_2)(x_1 e_1 e_3 - e_2 x_5^3) x_1 x_5 +$ $e_3(x_1 - x_5)\Big(x_1^2 (e_1 - x_1)\{e_3(x_1 - x_5) - e_1 x_5^3\} + (x_1 e_2 - e_3)\{x_1 e_2 + (x_1 - x_5) x_5^2\} x_5\Big),$ *where* e_i *are elementary symmetric polynomials in variables* x_2, x_3, x_4.

We solve this matrix equality for U and V in cases where diagonal matrices D_g, D_a and D_b are as described in proposition 1. Formulae for representations given in proposition 2 follow then, e.g., from relation $g_2 = g_1^{-1} a$: $\rho_{X,V}(g_2) = D_g^{-1} A$.

Solving (2.11) is straightforward but rather tedious computation. For an interested reader we give few details of it in cases $d = 2, 3, 4$.

Case $d = 2$. We choose

$$D_g = diag\{x_1, x_2\}, \quad D_a = -e_2(X)\, diag\{\nu, \nu^{-1}\}, \quad D_b = \mathrm{i} e_2(X)^{\frac{3}{2}}\, diag\{1, -1\}.$$

Noticing that matrices U/V are defined up to left/right multiplication by a diagonal matrix we use for them following ansatzes

$$U = \begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix},$$

where stars stay for unknown components. With this settings eq.(2.11) defines U and V up to conjugation by a diagonal matrix. We choose a solution which gives nice expression (2.2) for $\rho_X^{(2)}(g_2)$:

$$U = \begin{pmatrix} 1 & -\frac{x_1}{\nu^{-1}x_1+\nu x_2} \\ -\frac{\nu x_1+\nu^{-1}x_2}{x_1} & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -\frac{\mathrm{i}\sqrt{e_2}}{x_1-x_2+\mathrm{i}\sqrt{e_2}} \\ \frac{x_1-x_2-\mathrm{i}\sqrt{e_2}}{\mathrm{i}\sqrt{e_2}} & 1 \end{pmatrix},$$

Note that, unlike U and V, resulting expression for $\rho_X^{(2)}(g_2)$ is defined with the only restriction $x_1 \neq x_2$ and does not depend on a choice of root $\sqrt{e_2}$.

Case $d = 3$. We choose

$$D_g = diag\{x_1, x_2, x_3\}, \quad D_a = e_3(X)^{\frac{2}{3}}\, diag\{1, \nu^{-1}, \nu\}, \quad D_b = e_3(X)\, diag\{1, -1, -1\},$$

and use ansatzes

$$U = \begin{pmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & * & * \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}.$$

Solution of eq.(2.11) which gives formula (2.3) for $\rho_X^{(3)}(g_2)$ reads

$$U = \begin{pmatrix} 1 & \frac{x_1+h}{x_2+h} & \frac{x_1+h}{x_3+h} \\ \frac{x_2+\nu h}{x_1+\nu h} & 1 & \frac{x_2+\nu h}{x_3+\nu h} \\ \frac{x_3+\nu^{-1}h}{x_1+\nu^{-1}h} & \frac{x_3+\nu^{-1}h}{x_2+\nu^{-1}h} & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -1 & -1 \\ -\frac{(x_1-x_3)(x_2^2+x_1x_3)}{(x_2-x_3)(x_1^2+x_2x_3)} & 1 & 0 \\ -\frac{(x_1-x_2)(x_3^2+x_1x_2)}{(x_3-x_2)(x_1^2+x_2x_3)} & 0 & 1 \end{pmatrix}.$$

Case $d = 4$. We choose $D_g = diag\{x_1, x_2, x_3, x_4\}$,

$$D_a = h(X)\, diag\{1, 1, \nu, \nu^{-1}\}, \quad D_b = h(X)^{\frac{3}{2}}\, diag\{1, 1, -1, -1\},$$

and ansatzes for U, V:

$$U = \begin{pmatrix} I & \Psi^+ \\ \Psi^- & \Phi \end{pmatrix}, \quad V = \begin{pmatrix} I & \Lambda^+ \\ \Lambda^- & I \end{pmatrix},$$

where I is 2×2 unit matrix, $\Phi^\pm$ and $\Lambda^\pm$ are arbitrary 2×2 matrices, and 2×2 matrix Φ has unit diagonal components. Particular solution of eq.(2.11) which gives expression (2.5) for $\rho_{h,X}^{(4)}(g_2)$ reads

$$\Psi^+ = \begin{pmatrix} \frac{x_1(x_3-x_2)\beta_1\gamma_4}{x_3(x_1-x_2)\beta_3} & \frac{x_1(x_4-x_2)\beta_1\gamma_3}{x_4(x_1-x_2)\beta_4} \\ \frac{x_2(x_3-x_1)\beta_2}{x_3(x_2-x_1)\beta_3} & \frac{x_2(x_4-x_1)\beta_2}{x_4(x_2-x_1)\beta_4} \end{pmatrix}, \quad \Psi^- = \begin{pmatrix} \frac{x_1x_2}{(x_1x_2+\nu^{-1}h)(x_2x_3+\nu h)} & \frac{x_2x_4+\nu h}{x_3x_4+\nu h} \\ \frac{x_1x_2}{(x_1x_2+\nu h)(x_2x_4+\nu^{-1}h)} & \frac{x_2x_3+\nu^{-1}h}{x_3x_4+\nu^{-1}h} \end{pmatrix},$$

$$\Phi = \begin{pmatrix} 1 & \frac{x_2x_4+\nu h}{x_2x_3+\nu h} \\ \frac{x_2x_3+\nu^{-1}h}{x_2x_4+\nu^{-1}h} & 1 \end{pmatrix},$$

$$\Lambda^+ = -\begin{pmatrix} \frac{x_3(x_3-x_2)(x_1-\sqrt{h})\gamma_4}{x_1(x_1-x_2)(x_3-\sqrt{h})} & \frac{x_4(x_4-x_2)(x_1-\sqrt{h})\gamma_3}{x_1(x_1-x_2)(x_4-\sqrt{h})} \\ \frac{x_3(x_3-x_1)(x_2-\sqrt{h})}{x_2(x_2-x_1)(x_3-\sqrt{h})} & \frac{x_4(x_4-x_1)(x_2-\sqrt{h})}{x_2(x_2-x_1)(x_4-\sqrt{h})} \end{pmatrix},$$

$$\Lambda^- = -\frac{1}{\gamma_2}\begin{pmatrix} \frac{x_1(x_4-x_1)(x_3+\sqrt{h})}{x_3(x_4-x_3)(x_1+\sqrt{h})} & \frac{x_2(x_4-x_2)(x_3+\sqrt{h})\gamma_3}{x_3(x_4-x_3)(x_2+\sqrt{h})} \\ \frac{x_1(x_3-x_1)(x_4+\sqrt{h})}{x_4(x_3-x_4)(x_1+\sqrt{h})} & \frac{x_2(x_3-x_2)(x_4+\sqrt{h})\gamma_4}{x_4(x_3-x_4)(x_2+\sqrt{h})} \end{pmatrix}.$$

To get it we exclude consecutively matrices $\Lambda^{\pm}$, Ψ^-, Φ from equations (2.11) expressing them finally in terms of Ψ^+. The only condition imposed by eq.(2.11) on the components of Ψ^+ is

$$\frac{(\Psi^+)_{11}(\Psi^+)_{22}}{(\Psi^+)_{12}(\Psi^+)_{21}} = \frac{(x_3-x_2)(x_4-x_1)\gamma_4}{(x_4-x_2)(x_3-x_1)\gamma_3}.$$

Remaining three degrees of freedom are due to arbitrariness in conjugation of U and V by a diagonal matrix. We fix it to get the expression for $\rho_X^{(4)}(g_2)$ in the most suitable form.

Solving eq.(2.11) in cases $d = 5$, $\dim V = 5, 6$, is more lengthy. We skip it presenting final results of the calculations in eqs.(2.7)-(2.9) and in table 1. For them the braid relation (1.1) can be checked directly. □

Proposition 3. *For algebras Q_X (1.6) defined by a set of data X (1.4) representations $\rho^{(d)}_{\dots}$, $d \le 5$, described in proposition 2 are irreducible if and only if following conditions on their parameters are satisfied*

$$|X| = 2,\ \ \rho_X^{(2)}: \qquad I_{ij}^{(2)} := x_i^2 - x_ix_j + x_j^2 \neq 0, \tag{2.12}$$
where indices $i,j \in \{1,2\}$ are distinct;

$$|X| = 3,\ \ \rho_X^{(3)}: \qquad I_{ijk}^{(3)} := x_i^2 + x_jx_k \neq 0, \tag{2.13}$$
where $i,j,k \in \{1,2,3\}$ are pairwise distinct;

$$|X| = 4,\ \rho_{h,X}^{(4)}: \qquad I_{h,i}^{(4)} := x_i^2 - h \neq 0, \quad J_{h,ijkl}^{(4)} := x_ix_j + x_kx_l - h \neq 0, \tag{2.14}$$
where $i,j,k,l \in \{1,2,3,4\}$ are pairwise distinct;

$$|X| = 5,\ \rho_{f,X}^{(5)}: \qquad I_{f,i}^{(5)} := x_i^2 + x_if + f^2 \neq 0, \quad J_{f,ij}^{(5)} := x_ix_j + f^2 \neq 0, \tag{2.15}$$
where $i,j \in \{1,2,3,4,5\}$ are pairwise distinct;

Otherwise, they are reducible but indecomposable.

For representations $\rho_{s,X}^{(6)}$, $s = 1, \dots, 5$, also given in proposition 2 we present less detailed statement, which describes conditions under which all of them are irreducible:

$$|X| = 5,\ \rho_{s,X}^{(6)},\ _{1\le s\le 5}: \qquad I_i^{(6)} := e_5(X) + x_i^5 \neq 0, \quad J_{ij}^{(6)} := e_5(X) - x_i^3x_j^2 \neq 0,$$
$$K_{i,jklm}^{(6)} := x_jx_k + x_lx_m \neq 0, \tag{2.16}$$
where $i,j,k,l,m \in \{1,2,3,4,5\}$ are pairwise distinct.

Otherwise, among them there are reducible but indecomposable representations.

Proof. We will search for invariant subspaces in representations $\rho^{(d)}_{\dots}$ of proposition 2. Note that for any $y \in Q_X$ such that $Spec\,\rho_{X,V}(y)$ is multiplicity free an invariant subspace in V should be a linear span of some subset of a basis of eigenvectors of $\rho_{X,V}(y)$.

Consider representations $\rho^{(d)}_{\dots}$ of dimension $d = \dim V \leq 5$. Here the spectrum of $\rho^{(d)}_{\dots}(g_1)$ is simple. Choose a basis of eigenvectors of $\rho^{(d)}_{\dots}(g_1)$: $\{v_k := \delta_{ki},\, 1 \leq i \leq d\}_{k=1,\dots d}$. Denote

$$V_Y := Span\{v_k : k \in Y\}, \text{ where } Y \subset \{1,2,3,4,5\}. \tag{2.17}$$

Obviously, any invariant subspace in the representation space V, if exists, should be of the form V_Y. Furthermore, if the representation is decomposable then the decomposition is

$$V = V_Y \oplus V_{\bar{Y}}, \text{ where } \bar{Y} := \{1,2,3,4,5\} \setminus Y. \tag{2.18}$$

Correspondingly, matrix $\rho^{(d)}_{\dots}(g_2)$ have to be block-triangular (resp., block-diagonal) with blocks labelled by indices from subsets Y and $\bar{Y}$, iff the representation is reducible (resp., decomposable). Let us analyze the block structure of $\rho^{(d)}_{\dots}(g_2)$ in cases $d = 3, 4, 5$ (case $d = 2$ is trivial).

Case $d = 3$. Representation $\rho^{(3)}_X$ (2.3) has 2-dimensional invariant subspace $V_{\{1,2\}}$ iff $I^{(3)}_{312} = 0$. Its complementary 1-dimensional subspace $V_{\{3\}}$ exists under conditions $I^{(3)}_{123} = I^{(3)}_{231} = 0$. Altogether conditions $I^{(3)}_{312} = I^{(3)}_{123} = I^{(3)}_{231} = 0$ lead to $x_1 = x_2 = x_3 = 0$ and, hence, they are incompatible. Invariance conditions in two other cases — $V_{\{2,3\}}$, $V_{\{1\}}$, and $V_{\{1,3\}}$, $V_{\{2\}}$ — differ from the above by a cyclic permutation of the subscript indices. It follows that $\rho^{(3)}_X$ is irreducible iff inequalities (2.13) are fulfilled, and otherwise it is indecomposable.

Case $d = 4$. Conditions for existence of invariant subspaces in $\rho^{(4)}_{h,X}$ are

$$V_{\{1,2,3\}}:\ I^{(4)}_{h,4} = 0; \qquad V_{\{4\}}:\ I^{(4)}_{h,3} = J^{(4)}_{h,1234} = 0, \text{ or } I^{(4)}_{h,2} = J^{(4)}_{h,1324} = 0; \tag{2.19}$$

$$V_{\{1,2\}}:\ I^{(4)}_{h,3} = I^{(4)}_{h,4} = 0; \qquad V_{\{3,4\}}:\ J^{(4)}_{h,1234} = 0, \text{ or } I^{(4)}_{h,1} = I^{(4)}_{h,2} = 0. \tag{2.20}$$

For the rest of invariant subspaces their existence conditions can be obtained by a cyclic permutations of subscripts $1, 2, 3, 4$ in (2.19) [3], or of subscripts $2, 3, 4$ in (2.20). Altogether these conditions justify irreducibility criterium (2.14). Decomposability, e.g., like $V = V_{\{1,2,3\}} \oplus V_{\{4\}}$, or like $V = V_{\{1,2\}} \oplus V_{\{3,4\}}$, demands

$$I^{(4)}_{h,1} = I^{(4)}_{h,2} = I^{(4)}_{h,3} = I^{(4)}_{h,4} = 0, \quad \text{or} \quad I^{(4)}_{h,3} = I^{(4)}_{h,4} = J^{(4)}_{h,1234} = 0,$$

or similar sets of relations with permuted subscripts $2, 3, 4$. One can check that these conditions are incompatible with initial settings for X (1.4).

Case $d = 5$. Invariant subspaces in $\rho^{(5)}_{f,X}$ exist under conditions:

$$V_{\{1,2,3,4\}}:\ I^{(5)}_{f,5} = 0; \qquad V_{\{5\}}:\ J^{(5)}_{f,12} = J^{(5)}_{f,34} = 0, \text{ or } \forall \text{ permutation of sbs } {\scriptstyle 2,3,4}, \text{ or} \tag{2.21}$$
$$J^{(5)}_{f,12} = I^{(5)}_{f,3} = I^{(5)}_{f,4} = 0, \text{ or } \forall \text{ permutation of subscripts } {\scriptstyle 1,2,3,4};$$

[3]The only exception is subspace $V_{\{1\}}$ which can not be invariant in this representation.

$$V_{\{1,2,3\}}:\ J^{(5)}_{f,45}=0,\ \text{or}\ I^{(5)}_{f,4}=I^{(5)}_{f,5}=0;\qquad V_{\{4,5\}}:\ I^{(5)}_{f,3}=J^{(5)}_{f,12}=0, \tag{2.22}$$

or $\forall$ permutation of subscripts $1{,}2{,}3$.

For the rest of invariant subspaces the existence conditions can be obtained by permutation of indices in formulas above. Taken together these conditions prove irreducibility criterium (2.15). On the other hand, an attempt to find decomposition into invariant subspaces, like $V = V_{\{1,2,3,4\}} \oplus V_{\{5\}}$, or like $V = V_{\{1,2,3\}} \oplus V_{\{4,5\}}$, results in a set of conditions

$$I^{(5)}_{f,1}=J^{(5)}_{f,23}=J^{(5)}_{f,45}=0,\ \text{or}\ I^{(5)}_{f,1}=I^{(5)}_{f,2}=I^{(5)}_{f,3}=J^{(5)}_{f,45}=0,\ \text{or}\ \forall\ \text{permutation of sbs}\ 1,2,3,4,5,$$

which are incompatible with (1.4). Thus, representations $\rho^{(5)}_{f,X}$ are always indecomposable.

Case $d = 6$ is more sophisticated. We carry out considerations for representation $\rho^{(6)}_{5,X}$ (see table 1). For the other 6-dimensional representations results follow then by transpositions of arguments x_i.

Take a basis of eigenvectors of $\rho^{(6)}_{5,X}(g_1)$: $\{v_k := \delta_{ki},\ 1 \le i \le 6\}_{k=1,\dots 6}$. Assume there exists an invariant subspace $V_{inv} \subsetneq V$ and consider its subspace

$$W := V_{inv} \cup V_{\{1,2,3,4\}}.$$

Spectrum of $\rho^{(6)}_{5,X}(g_1)$ in this subspace is simple and so, W has a form $W = V_Y$ (2.17) for some subset $Y \subset \{1, 2, 3, 4\}$. We consider separately cases with different Y.

Case $W = V_{\{1,2,3,4\}}$. Consider action of matrix $\rho^{(6)}_{5,X}(g_2)$ on W. Since components g_{51} and g_{62} of this matrix are always nonzero we conclude that vectors v_5 and v_6 belong to V_{inv} and hence, $V_{inv} = V$, which is a contradiction.

Case $W = V_{\{1\}}$. Considering action of $\rho^{(6)}_{5,X}(g_2)$ on $v_1 \in W \subset V_{inv}$ we obtain $v_5 \in V_{inv}$. Now let's assume that $V_{inv} = V_{\{1,5\}}$. Then the matrix $\rho^{(6)}_{5,X}(g_2)$ should take block-diagonal form with vanishing components $g_{21} = g_{31} = g_{41} = g_{61} = g_{25} = g_{35} = g_{45} = g_{65} = 0$. This happens iff $p_1(X) \equiv J^{(6)}_{15} = 0$. Thus, we conclude that representation $\rho^{(6)}_{5,X}$ under condition $J^{(6)}_{15} = 0$ has the invariant subspace $V_{\{1,5\}}$. This subspace is not further reducible.

Case $W = V_{\{2,3\}}$. From the action of $\rho^{(6)}_{5,X}(g_2)$ on $v_2 \in V_{inv}$ we get $v_6 \in V_{inv}$, as $g_{26} \ne 0$. Assuming then $V_{inv} = V_{\{2,3,6\}}$ and checking block-triangularity of $\rho^{(6)}_{5,X}(g_2)$: $g_{12} = g_{13} = g_{16} = g_{42} = g_{43} = g_{46} = g_{52} = g_{53} = g_{56} = 0$, we find that this case is realized under condition $q_4(X) \equiv K^{(6)}_{5,1423} = 0$. Thus, $V_{\{2,3,6\}}$ is a minimal invariant subspace containing $W = V_{\{2,3\}}$.

Two cases considered above illustrate appearance of conditions like $J^{(6)}_{\dots} \ne 0$ and $K^{(6)}_{\dots} \ne 0$ in formulation of the proposition. Permuting arguments x_i, that is, considering all representations $\rho^{(6)}_{\dots}$ one can obtain all polynomials $J^{(6)}_{\dots}$, $K^{(6)}_{\dots}$ in the conditions of their reducibility. Consideration of the other cases with $W \ne \emptyset$ is similar. It does not result in any other independent reducibility conditions. In particular, for representation $\rho^{(6)}_{5,X}$ one obtains:

- in case $W = V_{\{2,3,4\}}$ minimal possible invariant subspace $V_{inv} = V_{\{2,3,4,5,6\}}$;
- in case $W = V_{\{1,4\}}$ minimal possible invariant subspace $V_{inv} = V_{\{1,4,5,6\}}$.

In searching for a decomposition of $\rho^{(6)}_{5,X}$ into a direct sum these invariant subspaces could be complements, respectively, for the subspaces $V_{inv} = V_{\{1,2\}}$ (case $W = V_{\{1\}}$) and $V_{inv} = V_{\{2,3,6\}}$ (case $W = V_{\{2,3\}}$). As we see, this does not happen. In all other reducible regimes with $W \neq \emptyset$ representations $\rho^{(6)}_{\dots}$ turn to be indecomposable.

It lasts considering case $W = \emptyset$. Assuming that V_{inv} is 2-dimensional, i.e. $V_{inv} = V_{\{5,6\}}$, we get a contradiction since block-triangularity conditions for $\rho^{(6)}_{5,X}$: $G_{13} = G_{23} = 0$ do not have any solution.

Still, there is a possibility to find 1-dimensional space V_{inv}. This happens if 2×2 matrices G_{13}, G_{23} and G_{33} for certain values of parameters x_i have common eigenspace V_{inv}, which is a null space for G_{13} and G_{23}. Calculating determinants of G_{13} and G_{23}:

$$\det G_{13} \sim K^{(6)}_{5,1234} J^{(6)}_{35} J^{(6)}_{45} (e_5(X) + x_5^5), \quad \det G_{23} \sim J^{(6)}_{15} J^{(6)}_{25} (e_5(X) + x_5^5),$$

we see that the only new possible regime where one observes nontrivial invariant subspace is given by condition $I^{(6)}_5 = 0$. Indeed, in this case one finds common eigenvector

$$\{(x_5^2 + x_2 x_3)(x_5^2 - x_1 x_3)(x_2^2 - x_2 x_5 + x_5^2),\ x_1 x_3 (x_1^2 - x_1 x_5 + x_5^2)\},$$

with eigenvalues 0, 0 and x_5, respectively, for G_{13}, G_{23} and G_{33}. The invariant subspace generated by this vector does not have an invariant direct summand, as there is no invariant subspaces containing $V_{\{1,2,3,4\}}$.

□

Our main result follows as a direct consequence of propositions 2 and 3:

Theorem 4. *For $|X| \leq 5$ algebra Q_X (1.6) defined by a set of data X (1.4) is semisimple iff:*

$$|X| = 2: \qquad I^{(2)}_{12} \neq 0; \tag{2.23}$$

$$|X| = 3: \qquad \{I^{(2)}_{ij}, I^{(3)}_{ijk}\} \cap \{0\} = \emptyset \ \textit{ for all pairwise distinct indices } {\scriptstyle i,j,k \in \{1,2,3\}}; \tag{2.24}$$

$$|X| = 4: \qquad \{I^{(2)}_{ij}, I^{(3)}_{ijk}, I^{(4)}_{h,i}, J^{(4)}_{h,ijkl}\} \cap \{0\} = \emptyset \tag{2.25}$$
$$\forall h : h^2 = e_4(X), \textit{ and for all pairwise distinct indices } {\scriptstyle i,j,k,l \in \{1,2,3,4\}};$$

$$|X| = 5: \qquad \{I^{(2)}_{ij}, I^{(3)}_{ijk}, I^{(4)}_{h,i}, J^{(4)}_{h,ijkl}, I^{(5)}_{f,i}, J^{(5)}_{f,ij}, I^{(6)}_{i}, J^{(6)}_{ij}, K^{(6)}_{i,jklm}\} \cap \{0\} = \emptyset \tag{2.26}$$
$$\forall f : f^5 = e_5(X), \quad \forall h : h^2 = e_4(X^{\backslash i}),$$
$$\textit{and for all pairwise distinct indices } {\scriptstyle i,j,k,l,m \in \{1,2,3,4,5\}}.$$

In the semisimple case all irreducible representations of these algebras are described in proposition 2.

Proof. Existence of reducible but indecomposable representations assumes nonsemisimplicity of an algebra. All the algebras Q_X which the theorem states to be nonsemisimple obey such representations according to proposition 3.

On the other hand, as follows from Artin-Wedderburn theorem an algebra over an algebraically closed field is semisimple if and only if sum of squares of dimensions of its inequivalent irreducible representations equals dimension of the algebra. Propositions 2 and 3 provide such sets of representations for algebras Q_X in semisimple regimes.

For instance, as is known $\dim Q_X = 600$ if $|X| = 5$. This algebra under conditions (2.26) has following inequivalent irreducible representations (see proposition 2 and remark 1): $\binom{5}{1} = 5$ times 1-dimensional, $\binom{5}{2} = 10$ times 2-dimensional, $\binom{5}{3} = 10$ times 3-dimensional, $2 \times \binom{5}{4} = 10$ times 4-dimensional, 5 times 5-dimensional, and 5 times 6-dimensional. Altogether: $5 * 1^2 + 10 * 2^2 + 10 * 3^2 + 10 * 4^2 + 5 * 5^2 + 5 * 6^2 = 600$ that fits the dimension of the algebra and proves its semisimplicity. $\square$

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