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A method of graph reduction and its applications

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Abstract: The independent set problem for a given simple graph is to determine the size of a maximal set of its pairwise non-adjacent vertices. We propose a new way of graph reduction leading to a new proof of the NP-completeness of the independent set problem in the class of planar graphs and to the proof of NP-completeness of this problem in the class of planar graphs having only triangular internal facets of maximal vertex degree 18.

Keywords: independent sets, planar graph, planar triangulation, computational complexity

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1 Introduction

An *independent set* (briefly, i.s.) of a simple graph is a subset of its pairwise non-adjacent vertices. A *maximal independent set* (briefly, a maximal i.s.) of a graph G is an i.s. of the graph G with maximal number of vertices; its size $\alpha(G)$ is called the *independence number* of the graph G. The *independent set problem* (briefly, the IS problem) for a given graph G and a natural number k is to check whether the inequality $\alpha(G) \geq k$ holds or not.

Several algorithmic instruments for the graph reduction are known for solving the IS problem. For example, if in a graph G a vertex *a adjacently absorbs* a vertex b (that is, $ab \in E(G)$ and $N(a) \supseteq N(b) \setminus \{a\}$, where $N(v)$ is the neighbourhood of a vertex v), then $\alpha(G) = \alpha(G \setminus \{a\})$. This is the so-called the *adjacency absorption rule*. The adjacency absorption is a particular case of the so-called *compressions* [1] (a compression is a self-mapping of the vertex set of a graph which is not an automorphism and under which any two distinct non-adjacent vertices are mapped into distinct non-adjacent vertices). So, a compressions transforms a graph into its induced subgraph; it obviously preserves the independence number. Recall that a graph H is *induced by a subgraph* of a graph G if H is obtained by removing some vertices of the graph G.

In the present paper, we propose a new method of graph transformation. This method is based on "local surgery": if in a given graph G there is a fragment G_1 with a special entry in the graph G, then the fragment G_1 is replaced by the graph G_2; note that the difference between the independence numbers of the resulting graph and the graph G depends only on G_1 and G_2 and does not depend on G. Out method is a particular case of the so-called replacements schemes introduced in [3]. In [3] a fairly general class of transformations is considered under which the independence number is accurately preserved, moreover, it is noted that nothing principally new will arise if the independence number is allowed to change by some constant. However, in [3] too little attention was paid to concrete replacements schemes and their applications to the analysis of the computational complexity of the IS problem in various classes of graphs.

We shall employ our method to prove the NP-completeness of the IS problem in two classes of graphs. The first class is the class of planar graphs $\mathcal{P}$. The NP-completeness of the IS problem in the class $\mathcal{P}$ was first proved in [6]; the principal idea of this paper is the "planarization" of a given graph; that is, an introduc-

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tion of a special "shunt" into each intersection of edges of the graph. Unfortunately, in [6] no explanation is provided about how the corresponding "shunt" was obtained. Using the "local surgery" method we propose a different "shunt", whose construction is seemingly simpler. The second class is the set of planar almost-triangulations with maximal vertex degree 18. Recall that a *planar almost-triangulation* is a planar graph in which each face (except, possibly, an exterior face) is bounded by a 3-cycle. Our result improves to some extent the corresponding assertion from [4] on the NP-completeness of the IS problem in the class 4-connected almost-triangulations with vertex-logarithmic maximal degree of vertices. For some available cases of polynomial solvability of the IS problem in subsets of the class $\mathcal{P}$, see [5, 8–10].

2 Notation

Throughout we shall adopt the following notation:

$\triangleq$ means the equality by definition,

$\overline{a,b}$ denotes the set $\{a, a+1, \dots, b\}$, where $a < b$ are given integer numbers,

$N(x)$ is the neighbourhood of a vertex x, $\deg(x)$ is the degree of x,

if G is a graph and $V' \subseteq V(G)$, then $G[V']$ is the subgraph of the graph G induced by the vertex subset V', and $G \backslash V'$ is the result of removing from the graph G all elements of the set V' (together with their incident edges),

O_2 is the empty graph with two vertices, $K_{1,3}$ is the graph with four vertices, of which one vertex is adjacent to the remaining three vertices, which are in turn pairwise non-adjacent.

3 The replacement operation and its significance

Let H_1 and H_2 be graphs, $A \subseteq V(H_1) \cap V(H_2)$. We say that H_1 and H_2 are *α-similar with respect to A* if there exists a constant c such that the equality $\alpha(H_1 \setminus X) = \alpha(H_2 \setminus X) + c$ is satisfied for any $X \subseteq A$ (and in particular, for $X = \varnothing$). It is clear that $c = \alpha(H_1) - \alpha(H_2)$.

Let G be some graph and let H be some its induced subgraph. A subset $A \subseteq V(H)$ will be called *H-separating* if no vertex of the graph $H \setminus A$ is adjacent to any of the vertices of the graph $G \setminus V(H)$.

Let H_1 and H_2 be graphs, $A \subseteq V(H_1) \cap V(H_2)$, and let H_1 and H_2 be α-similar with respect to A. Assume that the graph G contains an induced subgraph H_1 with an H_1-separating set A. The *replacement of H_1 by H_2 in the graph G* consists in the formation of a graph G^* with the vertex set $(V(G) \setminus V(H_1)) \cup V(H_2)$ and the edge set $(E(G) \setminus E(H_1)) \cup E(H_2)$.

The following lemma holds.

Lemma 1. *If the graph G^* is the result of the replacement of H_1 by H_2 in the graph G, then the graphs G^* and G are α-similar with respect to A.*

Proof. Assume that $A = \{v_1, \dots, v_k\}$. Consider an arbitrary subset $I \subseteq \overline{1,k}$. Let $G_I \triangleq G \setminus \{v_i \mid i \in I\}$ and $G_I^* \triangleq G^* \setminus \{v_i \mid i \in I\}$. Let S_I be a maximal i.s. of the graph G_I, $M_I \triangleq S_I \setminus V(H_1)$ and $X_I \triangleq \bigcup\limits_{x \in M_I} (N(x) \cap V(H_1))$, where the neighbourhood of a vertex x is considered in the graph G_I. Since $X_I \subseteq A$ and A is an H_1-separating set, we have $\alpha(G_I) = |M_I| + \alpha(H_1 \setminus X_I)$. If the set M_I in the graph G_I^* is augmented with a maximal i.s. of the graph $H_2 \setminus X_I$, then we obtain an i.s. of cardinality $|M_I| + \alpha(H_2 \setminus X_I)$. Therefore, $\alpha(G_I^*) \geq |M_I| + \alpha(H_2 \setminus X_I) = \alpha(G_I) - \alpha(H_1 \setminus X_I) + \alpha(H_2 \setminus X_I) = \alpha(G_I) - \alpha(H_1) + \alpha(H_2)$. The converse inequality is proved in the same way. □

The proof of Lemma 1 shows that $\alpha(G^*) = \alpha(G) + \alpha(H_2) - \alpha(H_1)$.

Let H be some graph and $A \subseteq V(H)$. Consider the family $\mathfrak{M}(H, A)$ consisting of subsets $X \subseteq A$ such that $\alpha(H \setminus (A \setminus X)) > \alpha(H \setminus (A \setminus Y))$ for any $Y \subset X$. For example, for a simple path $H \triangleq (v_1, v_2, v_3)$ the set $\mathfrak{M}(H, \{v_1, v_2, v_3\})$ coincides with the set $\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_3\}\}$. The following result holds.

Lemma 2. *Let H_1 and H_2 be graphs and $A \subseteq V(H_1) \cap V(H_2)$. Then H_1 and H_2 are α-similar with respect to A if and only if $\mathfrak{M}(H_1, A) = \mathfrak{M}(H_2, A)$.*

Proof. It is clear that if H_1 and H_2 are α-similar with respect to A, then $\mathfrak{M}(H_1, A) = \mathfrak{M}(H_2, A)$.

Assume that $A = \{v_1, \dots, v_k\}$ and that $\mathfrak{M}(H_1, A) = \mathfrak{M}(H_2, A)$. For each $i \in \overline{1, 2}$, we define the function $f_{(H_i,A)}(\cdot)\colon \{0,1\}^k \longrightarrow \{0, \dots, k\}$ as follows. For a Boolean vector $(x_1, \dots, x_k)$ the value $f_{(H_i,A)}(\cdot)$ on this vector is defined to be $\alpha(H_i) - \alpha(H_i \setminus \{v_j | \, x_j = 1\})$. It is clear that both functions $f_{(H_1,A)}(\cdot)$ and $f_{(H_2,A)}(\cdot)$ are monotone, and besides $f_{(H_1,A)}(0, \dots, 0) = f_{(H_2,A)}(0, \dots, 0) = 0$. Hence, for each $i \in \overline{1, 2}$ the function $f_{(H_i,A)}(\cdot)$ is uniquely determined by the tuple $(M_1^{(i)}, M_2^{(i)}, \dots, M_k^{(i)})$, where $M_j^{(i)}$ is the set of all bottom arguments on which the function $f_{(H_i,A)}(\cdot)$ assumes the value j. For each $i \in \overline{1, 2}$ the set $\mathfrak{M}(H_i, A)$ consists exactly of all elements corresponding to the (bottom) arguments on which the value of the function $f_{(H_i,A)}(\cdot)$ is changed. Hence from $\mathfrak{M}(H_1, A) = \mathfrak{M}(H_2, A)$, $f_{(H_1,A)}(0, \dots, 0) = f_{(H_2,A)}(0, \dots, 0) = 0$ it follows that $M_j^{(1)} = M_j^{(2)}$ for any $j \in \overline{1, k}$. This proves that the functions $f_{(H_1,A)}(\cdot)$ and $f_{(H_2,A)}(\cdot)$ coincide. Hence H_1 and H_2 are α-similar with respect to A. □

4 New proof of the NP-completeness of the IS problem in the class of planar graphs

The classical proof of the NP-completeness of the IS problem in the class $\mathcal{P}$, as proposed in [6], depends on the polynomial reduction of the IS problem to this problem for planar graphs and is based on the following idea. Let G be an arbitrary graph. Consider the planar "drawing" of the graph G in which the vertices of G are shown by points, and the edges of G by intervals. We shall assume that no three edges intersect at one point and no edge contains any vertex as its inner point. It is clear that such a drawing always exists. Moreover, each such embedding of the graph G may be constructed in $O(|V(G)|)$-time (see [7]). In the drawing, we consider each intersection of two edges. The number of intersections of edges is estimated from above by $\binom{|E(G)|}{2}$. For a point of intersection of any two intersecting edges (a, b) and (a', b'), we consider a small Euclidean neighbourhood of this point which does not contain any vertex or a part of an edge distinct from the two intersecting edges under consideration. We remove from the graph G the edges (a, b) and (a', b'), augment the resulting graph with the edges (a, v), (u, b), (a', v'), (u', b'), and identify the vertices v, u, v', u' with similarly-named vertices of the "shunt" H depicted in Fig. 1.

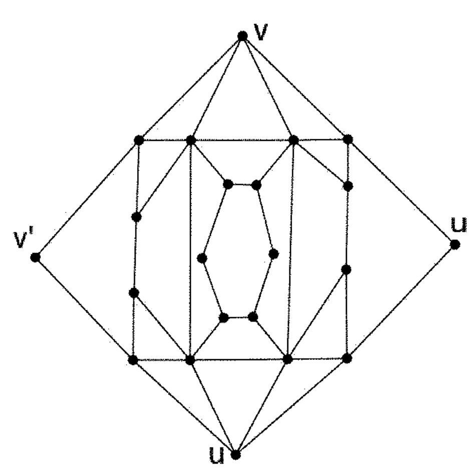
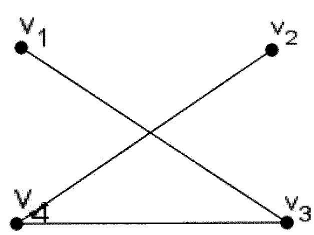


Fig. 1: The "shunt" H

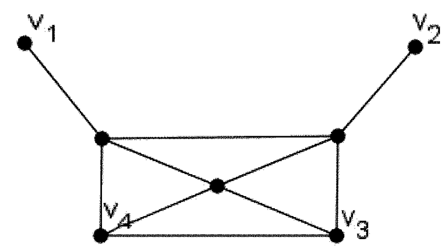
Clearly, the graph thus obtained is planar. To see this it suffices to place the “shunt” H in the neighbourhood of the point of intersection of the edges (a, b) and (a', b') and place the vertices v, u, v', u' on the boundary of this neighbourhood. According to [6], this “shunting” of each intersection point of edges increases the independence number of the graph by 9. Thus, the IS problem is polynomially reducible to the same problem in the class $\mathcal{P}$.

Unfortunately, in [6] no explanation is given how the graph H was obtained. Next, using the “surgery” from the previous section of the present paper, we construct a different “shunt” H^*. Even though the graph H^* contains more vertices than the graph H, the process of its construction is more clear, to our opinion, than that of the graph H.

Lemma 3. *The graphs G_1 and G_2 from Fig. 2 are α-similar with respect to $\{v_1, v_2, v_3, v_4\}$.*



The graph G_1



The graph G_2

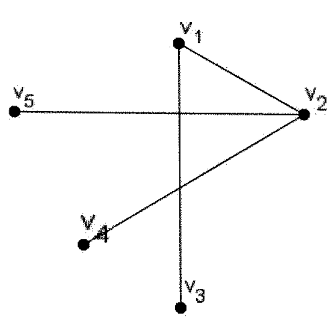
Fig. 2: The graphs G_1 and G_2

Proof. The equality

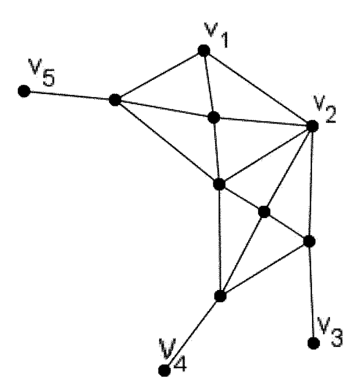
$$\mathfrak{M}(G_1, \{v_1, v_2, v_3, v_4\}) = \mathfrak{M}(G_2, \{v_1, v_2, v_3, v_4\}) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_1, v_2, v_3, v_4\}\}$$

may be verified directly. Hence by Lemma 2 the graphs G_1 and G_2 are α-similar with respect to $\{v_1, v_2, v_3, v_4\}$. □

Lemma 4. *The graphs G_3 and G_4 from Fig. 3 are α-similar with respect to $\{v_1, v_2, v_3, v_4, v_5\}$.*



The graph G_3



The graph G_4

Fig. 3: The graphs G_3 and G_4

Proof. In the graph G_3 we consider the subgraph whose edges are shown in bold in Fig. 4. This subgraph is isomorphic to the graph G_1. We replace the subgraph by the graph G_2 (see Fig. 4). In the resulting graph we also consider the subgraph whose edges are shown in bold and which is isomorphic to G_1, and replace it by G_2 (see Fig. 4). As a result, we get the graph isomorphic to the graph G_4. Hence by Lemmas 1 and 3 the graphs G_3 and G_4 are α-similar with respect to $\{v_1, v_2, v_3, v_4, v_5\}$. □

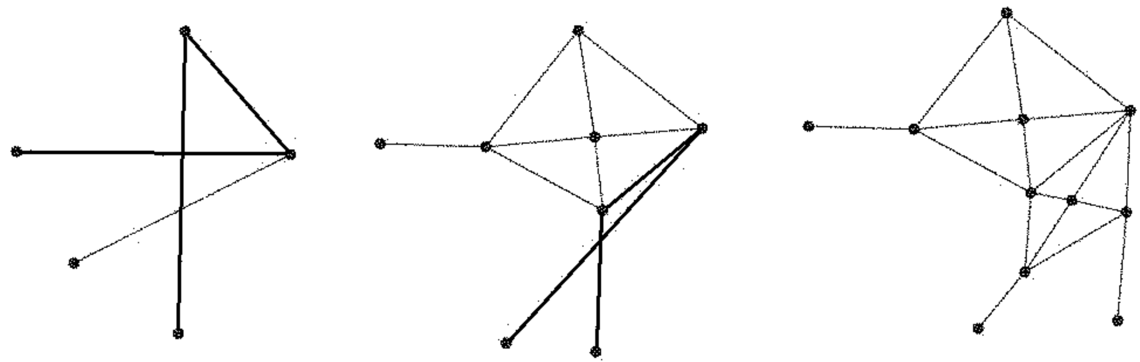
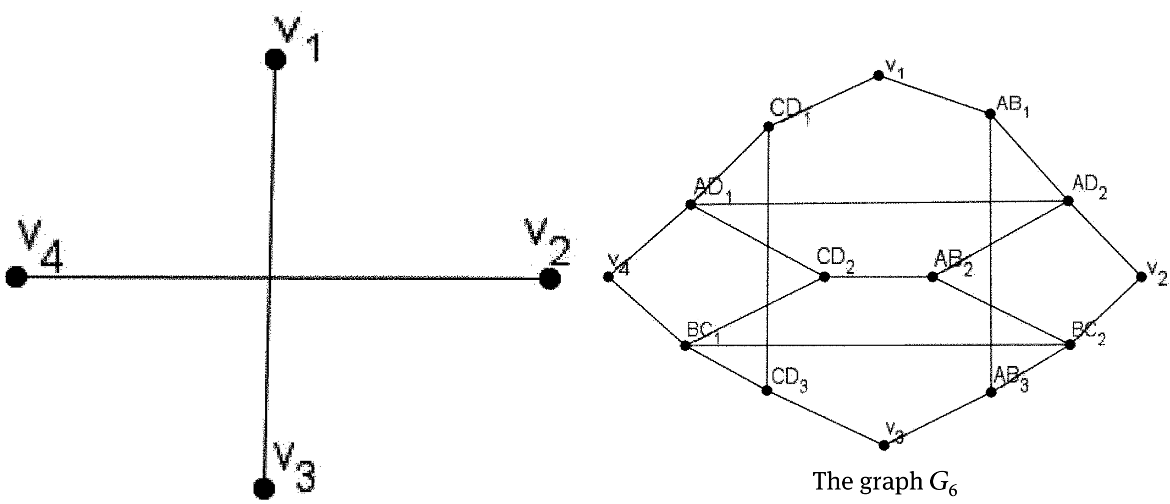


Fig. 4: Transformation of the graph G_3 into the graph G_4

Lemma 5. *The graphs G_5 and G_7 from Figs. 5 and 6 are α-similar with respect to $\{v_1, v_2, v_3, v_4\}$.*



The graph G_6

The graph G_5

Fig. 5: The graphs G_5 and G_6

Proof. It is easy to check that $\mathfrak{M}(G_5, \{v_1, v_2, v_3, v_4\}) = \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_1, v_2, v_3, v_4\}\}$. Consider the graph G_6 from Fig. 5. We claim that the set $\mathfrak{M}(G_6, \{v_1, v_2, v_3, v_4\})$ is the same.

All vertices of the graph G_6 may be split into 3 cycles: (v_4, AD_1, CD_2, BC_1), (v_2, AD_2, AB_2, BC_2), $(v_1, AB_1, AB_3, v_3, CD_3, CD_1)$. Any i.s. of the graph G_6 contains at most two vertices of each of the first two cycles and at most three vertices of the third cycle. Assume that $\alpha(G_6) = 7$. Then any maximal i.s. of the graph G_6 contains exactly two vertices of the first two cycles and three vertices of the third cycle. In the first cycle, a maximal i.s. of the graph G_6 may contain either only the vertices AD_1 and BC_1 or only the vertices v_4 and CD_2; the second cycle may contain either AD_2 and BC_2 or AB_2 and v_2. The pairs of vertices (v_4, CD_2) and (v_2, AB_2) may not simultaneously belong to any i.s. of the graph G_6. Hence, for any maximal i.s. of the graph G_6 at least one of the pairs of vertices (AD_1, BC_1) and (AD_2, BC_2) belong to it. Consequently, for any maximal i.s. of the graph G_6 at least one of the two families of vertices (CD_1, CD_3) and (AB_1, AB_3) does not belong to it. Hence, no maximal i.s. of the graph G_6 may contain simultaneously three vertices of the cycle $(v_1, AB_1, AB_3, v_3, CD_3, CD_1)$, which leads to a contradiction. As a result, $\alpha(G_6) < 7$. Additionally, $\{v_1, v_4, AD_2, CD_2, BC_2, CD_3\}$ is an i.s. of the graph G_6, and hence, $\alpha(G_6) = 6$.

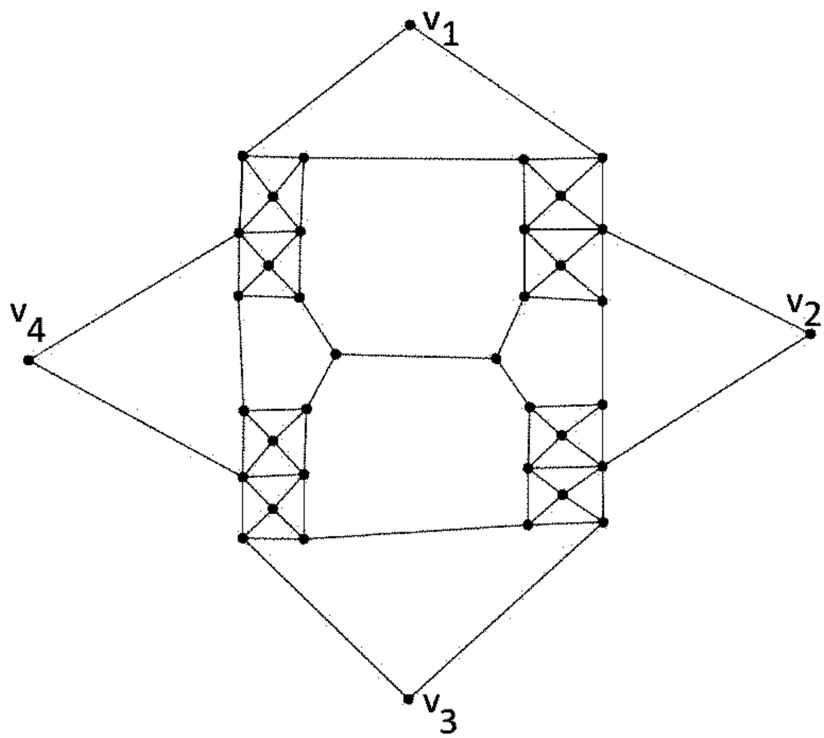


Fig. 6: The graph G_7

Consider in the graph G_6 the following subsets of its vertices $\{v_1, v_2, AB_2, AB_3, AD_1, BC_1\}$, $\{v_3, v_4, CD_1, CD_2, AD_2, BC_2\}$, $\{v_2, v_3, AB_1, AB_2, AD_1, BC_1\}$, $\{v_1, v_4, CD_2, CD_3, AD_2, BC_2\}$. Each such a subset is an i.s. of the graph G_6. Hence $\alpha(G \setminus \{v_i, v_{(i+1) \bmod 4}\}) = \alpha(G)$ for any $i \in \overline{1, 4}$.

After removing the vertices v_2 and v_4 from the graph G_6 all the remaining vertices belong to the cycle $(v_1, AB_1, AD_2, AB_2, BC_2, AB_3, v_3, CD_3, BC_1, CD_2, AD_1, CD_1)$ consisting of 12 vertices. If $\alpha(G_6 \setminus \{v_2, v_4\}) = 6$, then the vertices of any maximal i.s. of the graph $G_6 \setminus \{v_2, v_4\}$ should interlace in this cycle, but this is impossible. Besides, in the graph $G_6 \setminus \{v_2, v_4\}$ there exists an i.s. $\{v_1, AD_2, BC_2, v_3, BC_1\}$; that is, $\alpha(G_6 \setminus \{v_2, v_4\}) = 5$.

Consider the graph $G_6 \setminus \{v_1, v_3\}$. Assume that $\alpha(G_6 \setminus \{v_1, v_3\}) = 6$. Consider two subgraphs of the graph $G_6 \setminus \{v_1, v_3\}$ induced by the sets of vertices $\{v_4, AD_1, BC_1, CD_1, CD_2, CD_3\}$ and $\{v_2, AD_2, BC_2, AB_1, AB_2, AB_3\}$. Clearly, none of these subgraphs may contain more than three vertices of any i.s. of the graph $G_6 \setminus \{v_1, v_3\}$. The first of these subgraphs contains only two three–vertex i.s. (these are $\{v_4, CD_2, CD_1\}$ and $\{v_4, CD_2, CD_3\}$); the second subgraph also contains only two three-vertex i.s. (these are $\{v_2, AB_2, AB_1\}$ and $\{v_2, AB_2, AB_3\}$). Since the vertices CD_2 and AB_2 may not simultaneously belong to any i.s. of the graph $G_6 \setminus \{v_1, v_3\}$, we obtain a contradiction with the equality $\alpha(G_6 \setminus \{v_1, v_3\}) = 6$. Additionally, in the graph $G_6 \setminus \{v_1, v_3\}$ there exists the i.s. $\{v_4, CD_2, CD_1, v_2, AB_1\}$; that is, $\alpha(G_6 \setminus \{v_2, v_4\}) = 5$.

After removing from the graph G_6 any three vertices lying in the set $\{v_1, v_2, v_3, v_4\}$, one of the following i.s. is formed in the resulting subgraph: $\{v_1, AD_1, AB_2, AB_3, CD_3\}$, $\{v_2, AB_1, AB_2, CD_3, AD_1\}$, $\{v_3, BC_1, AB_2, AB_1, CD_1\}$, $\{v_4, CD_2, CD_1, BC_2, AB_1\}$. Hence, using the arguments from the previous two paragraphs, we see that the independence number of the resulting subgraph is 5.

After removing the vertices v_1, v_2, v_3, v_4 from the graph G_6, the vertex set of the resulting graph may be covered by two cycles $(AD_1, CD_1, CD_3, BC_1, CD_2)$ and $(AD_2, AB_1, AB_3, BC_2, AB_2)$. Each of these cycles contains at most two vertices of any i.s. of the graph $G_6 \setminus \{v_1, v_2, v_3, v_4\}$, and hence, $\alpha(G_6 \setminus \{v_1, v_2, v_3, v_4\}) < 5$. Besides, the graph $G_6 \setminus \{v_1, v_2, v_3, v_4\}$ contains the i.s. $\{CD_1, BC_1, AB_1, BC_2\}$. It follows that $\alpha(G_6 \setminus \{v_1, v_2, v_3, v_4\}) = 4$.

Thus, $\mathfrak{M}(G_5, \{v_1, v_2, v_3, v_4\}) = \mathfrak{M}(G_6, \{v_1, v_2, v_3, v_4\})$. Hence, by Lemma 2 the graphs G_5 and G_6 are α-similar with respect to $\{v_1, v_2, v_3, v_4\}$.

The process of generation of the graph G_7 from the graph G_6 is depicted in Fig. 7. In each step of this process one considers a subgraph induced by the boldface edges. This subgraph, which is isomorphic to the graph G_3, is replaced by the graph G_4.

Hence by Lemmas 1 and 4 the graphs G_6 and G_7 are α-similar with respect to $\{v_1, v_2, v_3, v_4\}$. This proves that the graphs G_5 and G_7 are α-similar with respect to $\{v_1, v_2, v_3, v_4\}$. □

The "shunt" H^*, mentioned at the beginning of the present section coincides with the graph G_7.

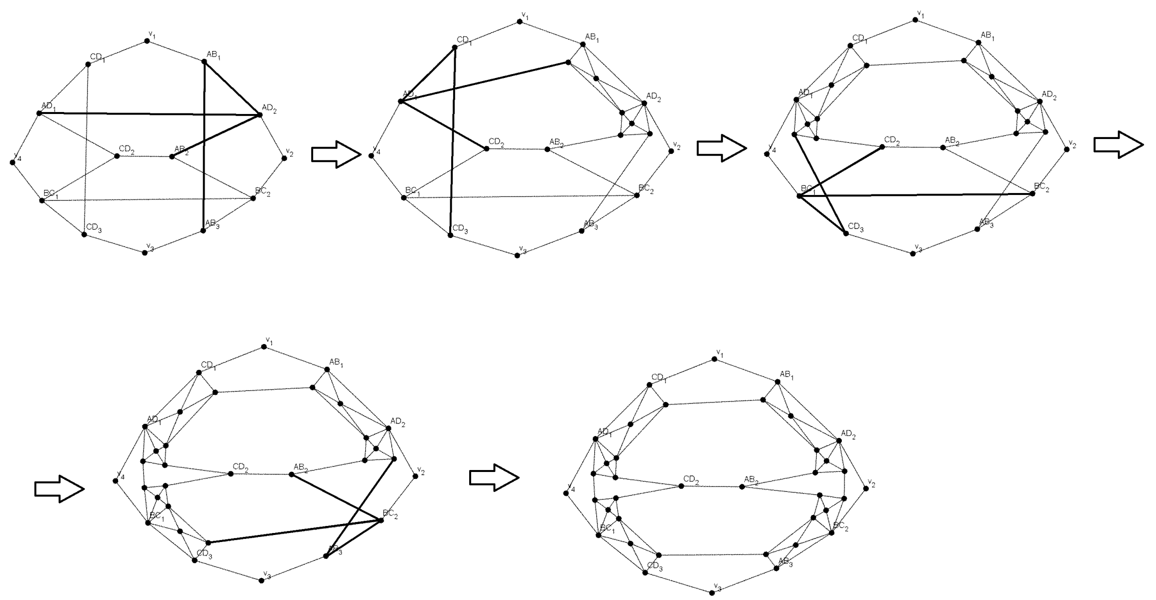


Fig. 7: The process of generation of the graph G_7 from the graph G_6

5 NP-completeness of the IS problem in one class of planar almost-triangulations with maximal vertex degree 18

The main result of the present section is the following statement.

Theorem 1. *The IS problem is NP-complete in the class of planar almost-triangulations with maximal vertex degree 18.*

It is well known that the IS problem is NP-complete in the class $\mathcal{P}(3)$ of planar graphs with vertex degree at most 3. This follows, for example, from the results of [6] and the machinery of reduction of graph vertices of [2]. We claim that the IS problem in the class $\mathcal{P}(3)$ is polynomially reducible to the same problem for planar almost-triangulations with maximal vertex degree 18. The result of Theorem 1 then follows.

If v is a pendant vertex of the graph G and if u is the neighbour of v, then $\alpha(G) = \alpha(G \setminus \{u\})$ by the adjacency absorption rule. If $G_1, \ldots, G_p$ are all connected components of the graph G, then $\alpha(G) = \sum_{i=1}^{p} \alpha(G_i)$. Thus, the IS problem in the class $\mathcal{P}(3)$ is polynomially reducible to the same problem for connected graphs from $\mathcal{P}(3)$ without pendant vertices.

Lemma 6. *Let the graph O_2 be formed by the vertices v and u, and let the graph $K_{1,3}$ be formed by the vertex x and its neighbours y, v, u, Then they are α-similar with respect to $\{v, u\}$.*

Proof. The equality $\mathfrak{M}(O_2, \{v, u\}) = \mathfrak{M}(K_{1,3}, \{v, u\})$ may be verified directly. Hence the graphs O_2 and $K_{1,3}$ are α-similar with respect to $\{v, u\}$ by Lemma 2. □

We note that if in the embedding of a planar graph some edge lies inside a face, then when counting the edges bounding this edge we count this edge two times. So, for example, in Fig. 8, the faces are bounded by seven and eight edges, respectively.

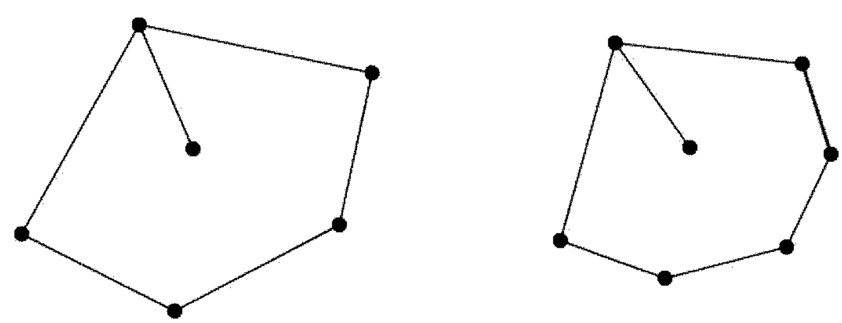


Fig. 8: The faces bounded by 7 and 8 edges

Below we will implement a polynomial reduction of the IS problem in the set of connected $\mathcal{P}(3)$-graphs without pendant vertices to the IS problem in the class of planar almost-triangulations with maximal vertex degree 18. This implementation will be represented as a solid text without separation into lemmas and the theorem.

Consider an arbitrary connected graph $G \in \mathcal{P}(3)$ without pendant vertices and take its planar embedding. All its faces are bounded by cycles. For each inner face of the embedding of the graph G we execute the following operation consisting of two suboperations.

Suboperation 1. This operation applies to the inner face of the graph G bounded by $n \geq 6$ edges. Such a face is split by k smaller faces, each having five or six cycle edges (see Fig. 9).

By Lemmas 1 and 6 a subdivision of an inner face of the graph G increases its independence number by $k - 1$.

Suboperation 1 is applied to each inner face of the graph G. As a result, we get a planar graph G' in which every vertex has degree at most 6 (because each vertex of the graph G is contained in at most three its faces). The graph G' contains inner faces of only the following types: those bounded by 3-, 4-, 5-cycles and those bounded by seven and eight edges, as shown in Fig. 9.

Suboperation 2. An inner face of the graph G' is subdivided into triangles as shown in Fig. 10 (the edges of the faces of the graph G' are shown in bold).

Applying Suboperation 2 to each inner face of the graph G' we get the graph G''. Consider in each of the subdivisions from Fig. 10 the vertex labeled by the letter A. Such a vertex will be called an *A-vertex*. The neighbourhood of each of the A-vertices is strictly contained in the union of neighbourhoods of any of its neighbours and of the neighbour itself. Hence, by the adjacency absorption rule, the set of all A-vertices lies in some maximal i.s. of the graph G''. Hence, the removing of any A-vertex together with its neighbourhood from the graph G'' decreases its independence number exactly by 1. So, after removal of all A-vertices and their neighbourhoods, we get the graph isomorphic to the graph G'.

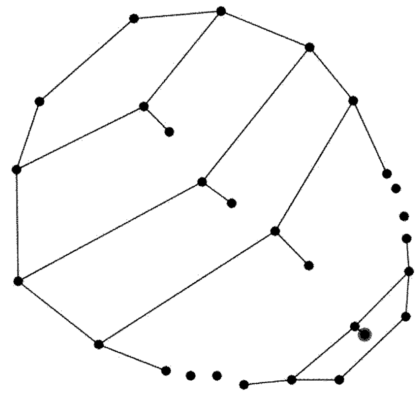


Fig. 9: Subdivision of a face of the graph, Suboperation 1

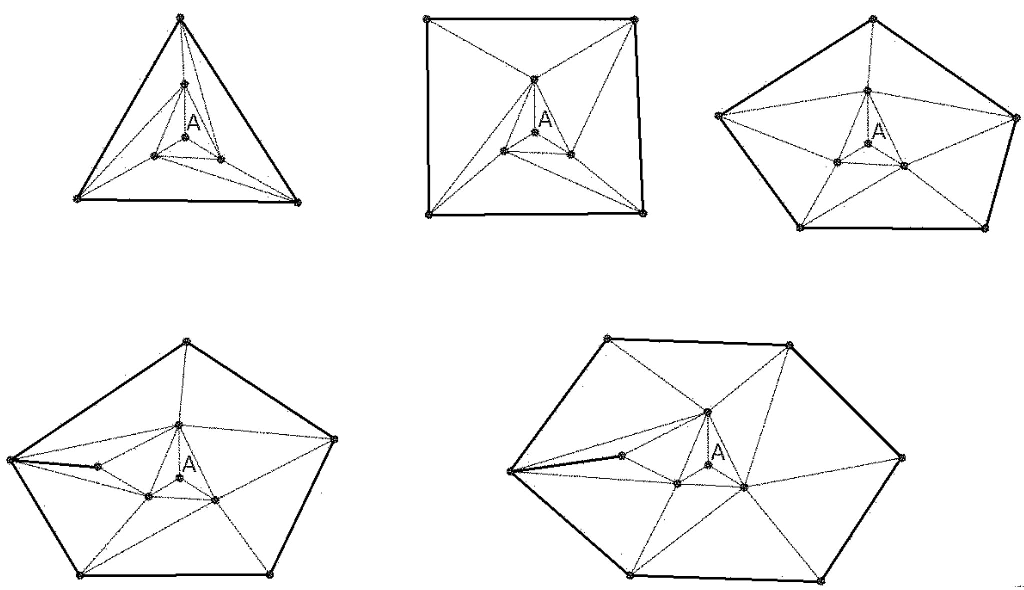


Fig. 10: Subdivision of a face of the graph, Suboperation 2

By the construction of the graph G'', we see that if x is a vertex of maximal degree and $\deg(x) \geq 8$, then x is not a pendant vertex of the graph G'. If $x \in V(G)$, then this vertex is contained in at most three faces of the graph G, and hence, in at most six faces of the graph G'. Hence, in the graph G'', in each face of the graph G' at most two new edges are incident with x. Hence, $\deg(x) \leq 18$. If x is a vertex of the graph G' incident with a pendant vertex of this graph, then x lies precisely in two edges of the graph G'. By the construction of the graph G'', one of these faces contains at most five edges incident with x, and the other face, at most four such edges. As a result, $\deg(x) \leq 7$. Consequently, the degree of any vertex of the graph G'' is at most 18.

Thus, the IS problem on the set $\mathcal{P}(3)$ is polynomially reducible to the same problem in the class of planar almost-triangulations with maximal vertex degree 18. The result of Theorem 1 then follows.

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