

TOPOLOGICAL CLASSIFICATION OF Ω -STABLE FLOWS ON SURFACES BY MEANS OF EFFECTIVELY DISTINGUISHABLE MULTIGRAPHS

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ABSTRACT. Structurally stable (rough) flows on surfaces have only finitely many singularities and finitely many closed orbits, all of which are hyperbolic, and they have no trajectories joining saddle points. The violation of the last property leads to Ω -stable flows on surfaces, which are not structurally stable. However, in the present paper we prove that a topological classification of such flows is also reduced to a combinatorial problem. Our complete topological invariant is a multigraph, and we present a polynomial-time algorithm for the distinction of such graphs up to an isomorphism. We also present a graph criterion for orientability of the ambient manifold and a graph-associated formula for its Euler characteristic. Additionally, we give polynomial-time algorithms for checking the orientability and calculating the characteristic.

1. Introduction. A traditional method of qualitative studying of a flows dynamics with a finite number of special trajectories on surfaces consists of a splitting the ambient manifold by regions with a predictable trajectories behavior known as *cells*. Such a view on continuous dynamical systems rises to the classical work by A. Andronov and L. Pontryagin [2] published in 1937. In that paper, they considered a system of differential equations

$$\dot{x} = v(x), \tag{1}$$

where $v(x)$ is a C^1 -vector field given on a disc bounded by a curve without a contact in the plane and found a roughness criterion for the system (1).

A more general class of flows on the 2-sphere was considered in works by E. Leontovich-Andronova and A. Mayer [12, 13], where a topological classification of such flows was also based on splitting by cells, whose types and relative positions

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(the *Leontovich-Mayer scheme*) completely define a qualitative decomposition of the phase space of the dynamical system into trajectories. The main difficulty in generalisations of this result to flows on arbitrary orientable surfaces is the possibility of new types of trajectories, namely *unclosed recurrent trajectories*. The absence of non-trivial recurrent trajectories for rough flows on the plane and on the sphere is an immediate corollary from the Poincaré-Bendixson theory for these surfaces, but this is not so trivial for orientable surfaces of genus $g > 0$. At first, it was proved by A. Mayer [14] in 1939 for rough flows with no singularities on the 2-torus¹ and later by M. Peixoto [20, 21] for structurally stable² flows on surfaces of any genus (see also [19]).

In 1971, M. Peixoto obtained a topological classification of structurally stable flows on arbitrary surfaces [22]. As before, he did it by studying all admissible cells and he introduced a combinatorial invariant called a *directed graph* generalizing the Leontovich-Mayer scheme. In 1976, D. Neumann and T. O'Brien [16] considered the so-called *regular flows* on arbitrary surfaces, such flows have no non-trivial periodic trajectories (i.e. periodic trajectories other than limit cycles) and include the flows above as a particular case. They introduced a complete topological invariant for the regular flows named *an orbit complex*, which is a space of flow orbits equipped with some additional information.

In 1998, A. Oshemkov and V. Sharko [17] introduced a new invariant for Morse-Smale flows on surfaces, namely a *three-colour graph*, and described an algorithm to distinct such graphs, which was not, however, polynomial, i.e. its working time is not limited by some polynomial on the length of input information. In the same work they obtained a complete topological classification of Morse-Smale flows on surfaces in terms of atoms and molecules introduced in the work of A. Fomenko [3].

Structurally stable (rough) flows on surfaces have only finitely many singularities and finitely many closed orbits, all of which are hyperbolic, they also have no trajectories joining saddle points. The violation of the last property leads to Ω -stable flows on surfaces, which are not structural stable. However, in the present paper we prove that a topological classification of such flows is also reduced to a combinatorial problem. The complete topological invariant is an equipped graph and we give a polynomial-time algorithm for the distinction of such graphs up to isomorphism. We also present a graph criterion for orientability of the ambient manifold and a graph-associated formula for its Euler characteristic. Additionally, we give polynomial-time algorithms for checking the orientability and calculating the euler characteristic.

2. The dynamics of an Ω -stable flow.

2.1. Global properties. Let ϕ^t be some Ω -stable flow on a closed surface S . Due to [18], the non-wandering set Ω_{ϕ^t} of the flow ϕ^t consists of a finite number of hyperbolic fixed points and hyperbolic closed trajectories (limit cycles), which are called *basic sets*, denote them $\Omega_1, \dots, \Omega_k$.

¹Actually in [14] A. Mayer found the conditions of roughness for cascades (discrete dynamical systems) on the circle and he also got the topological classification for these cascades.

²The term “rough system” introduced by A. Andronov and L. Pontryagin in [2] is slightly different from its English counter part “structurally stable system” introduced by M. Peixoto in [20, 21], but the sets of rough and structural stable systems coincide.

Proposition 1 (Corollary 5.3, [25]). *Every basic set Ω_i of the flow ϕ^t possesses the stable manifold $W_{\Omega_i}^s = \{x \in S \mid \phi^t(x) \rightarrow \Omega_i \text{ for } t \rightarrow +\infty\}$ and the unstable manifold $W_{\Omega_i}^u = \{x \in S \mid \phi^t(x) \rightarrow \Omega_i \text{ for } t \rightarrow -\infty\}$ and*

$$S = \bigcup_{i=1}^k W_{\Omega_i}^s = \bigcup_{i=1}^k W_{\Omega_i}^u.$$

2.2. Fixed points. The hyperbolicity of the fixed points of the flow ϕ^t is expressed by the following fact.

Proposition 2 ([19], Theorem 5.1 in Chapter 2 and [24], Theorem 7.1 in Chapter 4). *The flow ϕ^t in some neighbourhood of a fixed point $q \in \Omega_{\phi^t}$ is topologically equivalent to one of the following linear flows*

$$\begin{aligned} a^t(x, y) &= (2^{-t}x, 2^{-t}y), \\ b^t(x, y) &= (2^{-t}x, 2^t y), \\ c^t(x, y) &= (2^t x, 2^t y). \end{aligned}$$

In the cases a^t, b^t, c^t the fixed point q is called *sink, saddle, source* with 0,1,2-dimensional unstable manifold W_q^u accordingly. We will denote by $\Omega_{\phi^t}^0, \Omega_{\phi^t}^1, \Omega_{\phi^t}^2$ the set of all sinks, saddles, sources of ϕ^t accordingly.

It follows from the criterion of the Ω -stability in [23] that the saddle points do not organize *cycles*, i.e. collections of points

$$q_1, \dots, q_k, q_{k+1} = q_1$$

with a property

$$W_{q_i}^s \cap W_{q_{i+1}}^u \neq \emptyset, \quad i = 1, \dots, k.$$

2.3. Closed trajectories. Let \mathfrak{c} be a closed trajectory of ϕ^t and $p \in \mathfrak{c}$. Let Σ_p be a smooth cross-section passing through the point p transversal to trajectories of ϕ^t near p . Let $V_p \subset \Sigma_p$ be a neighbourhood of p such that for every point $x \in V_p$ there is a value $\tau_x \in \mathbb{R}^+$ with properties $\phi^{\tau_x}(x) \in V_p$ and $\phi^t(x) \notin V_p$ for any $0 < t < \tau_x$. Then Σ_p is called a *Poincaré cross-section* and the map $F_p: V_p \rightarrow \Sigma_p$ given by the formula $F_p(x) = \phi^{\tau_x}(x), x \in V_p$ is called a *Poincaré map*.

The hyperbolicity of the closed trajectory \mathfrak{c} is expressed by the following fact.

Proposition 3 ([19], Proposition 1.2 in Chapter 3 and Theorem 5.5 in Chapter 2). *Poincaré map $F_p: V_p \rightarrow F_p(V_p)$ is a diffeomorphism with the fixed point p in a neighbourhood of which F_p is topologically conjugate to one of the following linear diffeomorphisms*

$$\begin{aligned} a_+(x) &= \frac{x}{2}, \quad a_-(x) = -\frac{x}{2}, \\ c_+(x) &= 2x, \quad c_-(x) = -2x. \end{aligned}$$

In the cases a_{\pm}, c_{\pm} the closed trajectory \mathfrak{c} is called an *attractive, repelling limit cycle* accordingly. Denote by $\Omega_{\phi^t}^c$ the set of all limit cycles of ϕ^t .

In any case a limit cycle \mathfrak{c} has a neighbourhood $U_{\mathfrak{c}}$, which is disjoint with other limit cycles and fixed points of ϕ^t and whose boundary $R_{\mathfrak{c}}$ is transverse to the trajectories of ϕ^t . The neighbourhood $U_{\mathfrak{c}}$ is homeomorphic to an annulus or a Möbius band (see Fig. 1) in the cases a_+, c_+ or a_-, c_- accordingly and can be constructed in the following way.

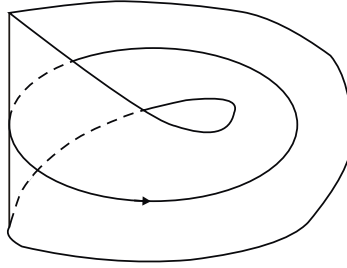


FIGURE 1. The case when U_c is homeomorphic to a Möbius band

For every points $a, b \in V_p$ let us denote by $m_{a,b}$ the segment of V_p bounded by the points a, b and by $\mu_{a,b}$ the length of this segment. In the cases a_+, c_+ let us choose two points $x_1^*, x_2^* \in (V_p \setminus \{p\})$ on different connected components of $V_p \setminus \{p\}$. Then R_c is the union of two circles

$$\left\{ \phi^{\frac{\mu_{x_1^*, F_p(x_1^*)}}{F_p(x_1^*)} \tau_x}(x) : x \in m_{x_1^*, F_p(x_1^*)} \right\} \text{ and } \left\{ \phi^{\frac{\mu_{x_2^*, F_p(x_2^*)}}{F_p(x_2^*)} \tau_x}(x) : x \in m_{x_2^*, F_p(x_2^*)} \right\}.$$

In the cases a_-, c_- let us choose a point $x^* \in (V_p \setminus \{p\})$. Then

$$R_c = \left\{ \phi^{\frac{\mu_{x^*, F_p^2(x^*)}}{F_p^2(x^*)} 2\tau_x}(x) : x \in m_{x^*, F_p^2(x^*)} \right\}.$$

A moving of Σ_p along the trajectories in the positive time gives a *consistent with c orientation on R_c* . Thus, in further we will assume that R_c is oriented consistently with c .

3. The directed graph for a flow $\phi^t \in G$. Denote by G a class of Ω -stable flows ϕ^t with at least one fixed saddle point or at least one limit cycle³ on a surface S . That is the flow class we consider in our work.

Recall that a *graph* Γ is an ordered pair (B, E) such that B is a finite non-empty set of *vertices*, E is a set of pairs of the vertices called *edges*. Besides, if E is a multiset then Γ is called a *multigraph*. Recall that a *multiset* is a set with the opportunity of multiple inclusion of its elements. For simplicity, we call a multigraph a *graph* everywhere below.

If a graph includes an edge $e = (a, b)$, then both vertices a and b are called *incident* to the edge e . The vertices a and b are *connected by e* . A graph is called *directed* if every its edge is an ordered pair of vertices. A finite sequence

$$\tau = (b_0, (b_0, b_1), b_1, \dots, b_{i-1}, (b_{i-1}, b_i), b_i, \dots, b_{k-1}, (b_{k-1}, b_k), b_k)$$

of vertices and edges of a graph is called a *path*, the number k is called *the length* of the path and it is equal to the number of edges of the path. The path τ is called *simple* if it contains only pairwise disjoint edges. The simple path τ is called a *cycle* if $b_0 = b_k$. A graph is called *connected* if every two vertices can be connected by a path.

³If flow ϕ^t has neither fixed saddle points nor closed trajectories, then its non-wandering set consists of exactly two fixed points: a source and a sink, all such flows are topologically equivalent, that is the reason why we exclude such flows from the class G .

Let $\mathcal{R} = \bigcup_{c \in \Omega_{\phi^t}^c} R_c$. We call \mathcal{R} a *cutting set* and the connected components of \mathcal{R} *cutting circles*.

Let $\hat{S} = S \setminus \mathcal{R}$. We call an *elementary region* a connected component of the set \hat{S} . We have already showed the way of constructing some cutting circle in the previous paragraph. Notice, that from the way of designing there follows, that such circles or circle always exist near each limit cycle, because we may construct it as small as we want. As well we showed that each neighbourhood of limit cycle bounded by two or one cutting circle is homeomorphic to an annulus or a Möbius band respectively. So we may present all types of elementary regions. Due to Proposition 1, the elementary region can be one of the following four types:

- 1) a region of the type \mathcal{L} contains exactly one limit cycle;
- 2) a region of the type \mathcal{A} contains exactly one source or exactly one sink;
- 3) a region of the type \mathcal{M} contains at least one saddle point;
- 4) a region of the type \mathcal{E} does not contain elements of basic sets.

Definition 3.1. A directed graph Υ_{ϕ^t} is said to be a *graph of the flow* $\phi^t \in G$ (see Fig. 2) if

- (1) the vertices of Υ_{ϕ^t} bijectively correspond to the elementary regions of ϕ^t ;
- (2) every directed edge of Υ_{ϕ^t} , which connects a vertex a with a vertex b , corresponds to the cutting circle R , which is a common boundary of the regions A and B corresponding to a and b , such that any trajectory of ϕ^t passes R starting at A and ending in B by increasing the time.

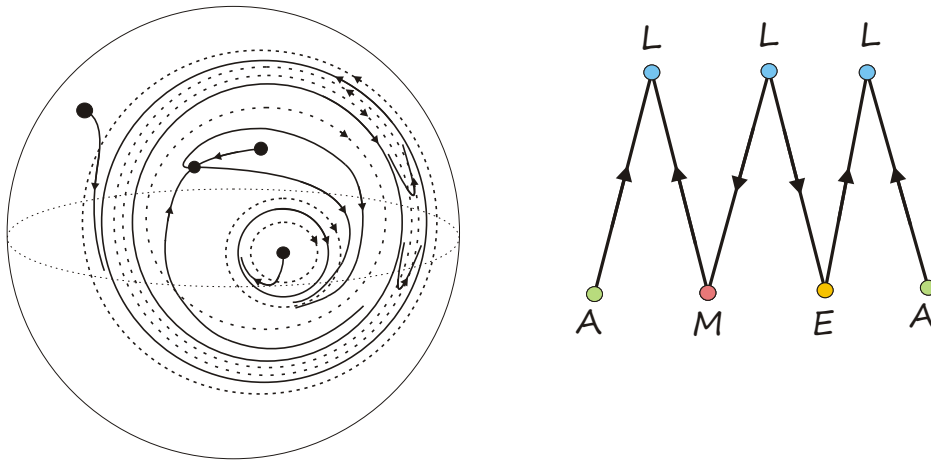


FIGURE 2. ϕ^t and Υ_{ϕ^t}

We will call an \mathcal{L} -, \mathcal{A} -, \mathcal{E} - or \mathcal{M} -*vertex* a vertex of Υ_{ϕ^t} , which corresponds to a \mathcal{L} -, \mathcal{A} -, \mathcal{E} - or \mathcal{M} -region accordingly.

The following Proposition immediately follows from the dynamics of the flow ϕ^t and the structure of the cutting set.

Proposition 4. *Let Υ_{ϕ^t} be the directed graph of a flow $\phi^t \in G$, then:*

- 1) every \mathcal{M} -vertex can be connected only with \mathcal{L} -vertices, furthermore, with every vertex by a single edge;
- 2) every \mathcal{E} -vertex can be incident only to two edges which connect this vertex with two different \mathcal{L} -vertices, and one of these edges enters to the \mathcal{E} -vertex, the other one exits;
- 3) every \mathcal{A} -vertex can be connected only with a \mathcal{L} -vertex, furthermore, by a single edge;
- 4) every \mathcal{L} -vertex has degree (the number of incident edges) 1 or 2, and if its degree is 2, then both edges either enter or exit the vertex.

Isomorphicity of the directed graphs is necessary condition for flows from G to be topological equivalent. To make the directed graph a complete topological invariant for the class G , below we equip the graph Υ_{ϕ^t} with additional information.

4. Equipment of the directed graph. In this section, we describe how to assign some additional information to vertices and edges of the directed graph of a flow from G .

4.1. \mathcal{A} -vertex. The flows in \mathcal{A} -regions can belong to only the two equivalence classes: a source pool and a sink pool, which we can distinguish by directions of edges incident to \mathcal{A} -vertices.

4.2. \mathcal{L} -vertex. The flows in \mathcal{L} -regions can belong to only the four equivalence classes: an annulus with a stable limit cycle, an annulus with an unstable one, a Möbius band with a stable one, a Möbius band with an unstable one, which we can distinguish by directions of edges and by quantities of edges incident to \mathcal{L} -vertices.

4.3. \mathcal{E} -vertex. The flows in \mathcal{E} -regions can belong to only the two equivalence classes corresponding to the consistent and the inconsistent orientation of connecting components of \mathcal{E} 's boundary (see Fig. 3). However, the structure of an \mathcal{E} -region cannot be determined by the directed graph, therefore, we will attribute the *weight* to the vertex corresponding to an \mathcal{E} -region. The weight is “+” in the consistent case and “-” in the inconsistent one.

4.4. \mathcal{M} -vertex. The flows in \mathcal{M} -regions cannot be determined by the directed graph. Then we will equip vertices corresponding to them by *four-colour graphs* for a description of the dynamics of the flow in the regions. In more details.

Let us consider some \mathcal{M} -region which is either a 2-manifold with a boundary or a closed surface. In the first case let us attach the union D of some disjoint 2-disks to the boundary to get a closed surface M , in the second case we also denote the closed surface by M and will suppose that $D = \emptyset$. Let us extend $\phi^t|_{\mathcal{M}}$ to an Ω -stable flow $f^t: M \rightarrow M$ such that f^t coincides with ϕ^t out of D and Ω_{f^t} has exactly one fixed point (a sink or a source) in each connected component of D .

Let $\Omega_{f^t}^0, \Omega_{f^t}^1, \Omega_{f^t}^2$ be the sets of all sources, saddle points and sinks of f^t accordingly. By the definition of the region \mathcal{M} the flow f^t has at least one saddle point. Let

$$\tilde{M} = M \setminus (\Omega_{f^t}^0 \cup W_{\Omega_{f^t}^1}^s \cup W_{\Omega_{f^t}^1}^u \cup \Omega_{f^t}^2).$$

A connected component of \tilde{M} is called a *cell*.

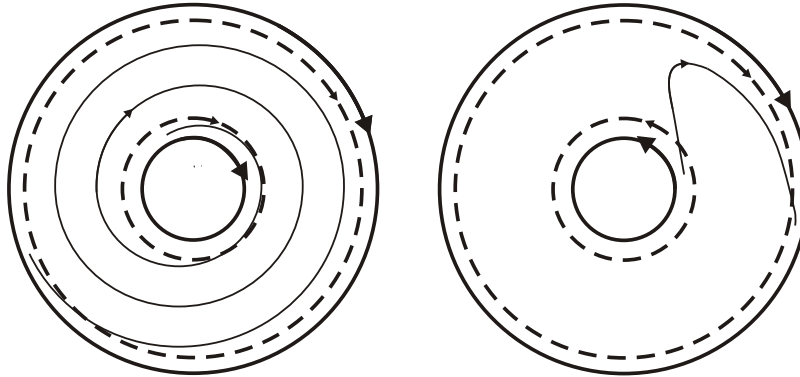


FIGURE 3. The cases of the consistent (leftward) and the inconsistent (rightward) orientation of boundary's connecting component of some \mathcal{E} -region.

Lemma 4.1 ([11], the main Theorem). *Every cell J of the flow f^t contains a single sink ω and a single source α in its boundary, and the whole cell is the union of trajectories going from α to ω .*

Let us choose a trajectory θ_J in the cell J , we will call it a t -curve. Let

$$\mathcal{T} = \bigcup_{J \subset \bar{S}} \theta_J, \quad \bar{M} = \tilde{M} \setminus \mathcal{T}.$$

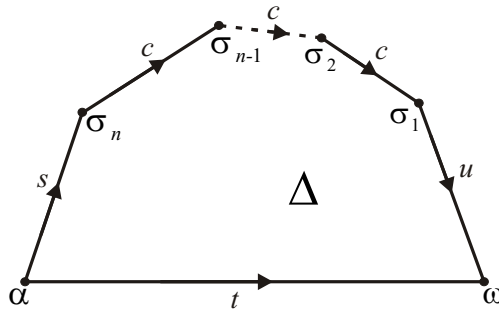


FIGURE 4. A polygonal region

Let us call a c -curve a separatrix connecting saddle points (from the word “connection”), a u -curve an unstable saddle separatrix with a sink in its closure, a s -curve a stable saddle separatrix with a source in its closure. We will call a *polygonal region* Δ a connecting component of \bar{M} .

Lemma 4.2 ([10], Lemma 3.4). *Every polygonal region Δ is homeomorphic to an open disk and its boundary consists of a unique t -curve, a unique u -curve, a unique s -curve, and a finite (may be empty) set of c -curves (see Fig. 4).*

Denote by Δ_{f^t} the set of all polygonal regions of f^t (see Fig. 5, where a flow f^t and all its polygonal regions are presented).

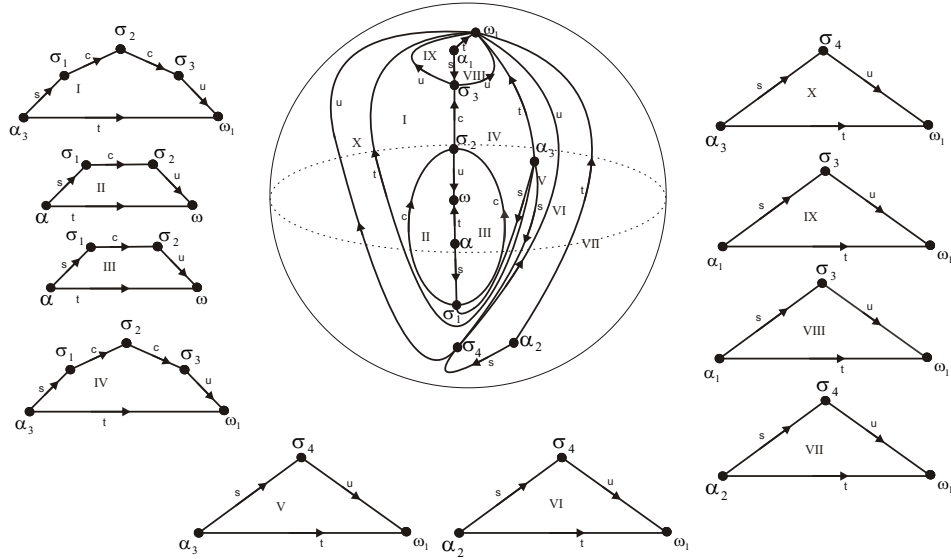


FIGURE 5. An example of the flow f^t together with the polygonal regions

Definition 4.3. A multigraph is called an n -colour graph if the set of its edges is the disjoint union of n subsets, each of which consists of edges of the same colour.

We say that a four-colour graph $\Gamma_{\mathcal{M}}$ with edges of colours u, s, c, t bijectively corresponds to f^t if:

- 1) the vertices of $\Gamma_{\mathcal{M}}$ bijectively correspond to the polygonal regions of Δ_{f^t} ;
- 2) two vertices of $\Gamma_{\mathcal{M}}$ are incident to an edge of colour s, t, u or c if the polygonal regions corresponding to these vertices has a common s -, t -, u - or c -curve; that establishes an one-to-one correspondence between the edges of $\Gamma_{\mathcal{M}}$ and the colour curves;
- 3) if some vertex b of $\Gamma_{\mathcal{M}}$ is incident to more than one c -edge (the number n_b of c -edges is more than 1), then we order the c -edges by

$$c_1^b, \dots, c_{n_b}^b$$

by a moving (according to the direction from the source to the sink on t -curve) along the boundary of the corresponding polygonal region (see, for example, Figure 6).

Definition 4.4. We say that the graph $\Gamma_{\mathcal{M}}$ is the four-colour graph of the flow f^t corresponding to $\phi^t|_{\mathcal{M}}$.

Definition 4.5. Two four-colour graphs $\Gamma_{\mathcal{M}}$ and $\Gamma_{\mathcal{M}'}$ corresponding to $\phi^t|_{\mathcal{M}}$ and $\phi^t|_{\mathcal{M}'}$ respectively are said to be *isomorphic* if there is an one-to-one correspondence ψ of the vertices and the edges of the first graph to the vertices and the edges of the second graph preserving the colours of all edges and the numbers of c -edges.

4.5. **(\mathcal{M}, \mathcal{L})- and (\mathcal{L}, \mathcal{M})-edge.** Let us denote by π_{f^t} the one-to-one correspondence described above between the polygonal regions and the vertices, also between the colour curves of f^t and the colour edges of $\Gamma_{\mathcal{M}}$ respectively.

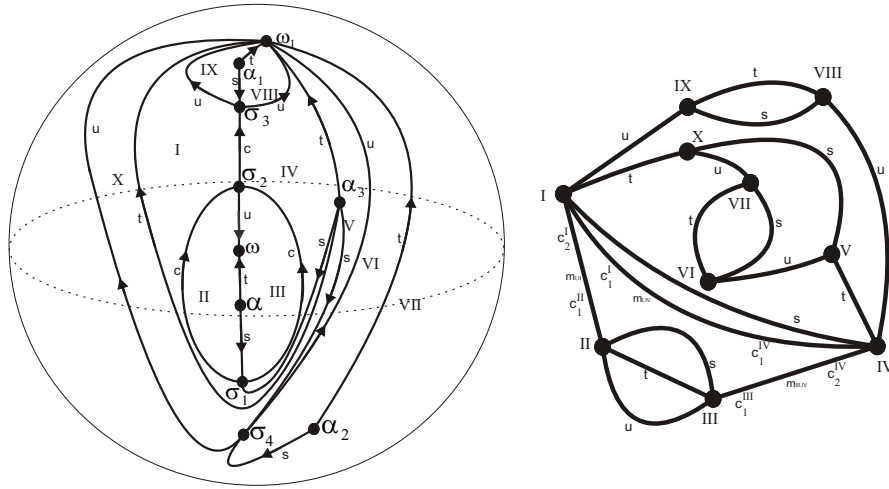


FIGURE 6. An example of f^t and its four-colour graph

Let us call a *st-cycle* (*tu-cycle*) a cycle of $\Gamma_{\mathcal{M}}$ consisting only of *s-* and *t-*edges (*t-* and *u-*edges). Let us call *u-* and *s-*edges exiting out a vertex b as *nominal c-*edges and assign the numbers 0 and $n_b + 1$ to them respectively. Let us call a *c*-cycle* a simple cycle

$$b_1, (b_1, b_2), b_2, \dots, b_{2k}, b_{2k+1}, b_{2k+1} = b_1,$$

if

$$(b_{2i-1}, b_{2i}) = c_m^{b_{2i}}, (b_{2i}, b_{2i+1}) = c_{m+1}^{b_{2i}} = c_l^{b_{2i+1}}, (b_{2i+1}, b_{2i+2}) = c_{l-1}^{b_{2i+1}}.$$

Proposition 5 ([10], Proposition 3). *The projection π_{ft} gives an one-to-one correspondence between the sets $\Omega_{ft}^0, \Omega_{ft}^1, \Omega_{ft}^2$ and the sets of *tu-*, *c*-*, and *st-*cycles respectively.*

By our construction $M = \mathcal{M} \cup D$, where D is either empty or each its connected component contains exactly one sink ω (source α) of the flow f^t , uniquely corresponding to a cutting circle R_c for a limit cycle c of the flow ϕ^t , which uniquely corresponds to an $(\mathcal{M}, \mathcal{L})$ -edge ($(\mathcal{L}, \mathcal{M})$ -edge) of the graph Υ_{ϕ^t} . Due to Proposition 5 the node ω (α) uniquely corresponds to a *tu-cycle* (an *st-cycle*), denote it by $\tau_{\mathcal{M}, \mathcal{L}}$ ($\tau_{\mathcal{L}, \mathcal{M}}$). Moreover, due to Proposition 5, we can embed the graph $\Gamma_{\mathcal{M}}$ such that the cycle $\tau_{\mathcal{M}, \mathcal{L}}$ ($\tau_{\mathcal{L}, \mathcal{M}}$) coincides with R_c . Thus we induce an orientation from R_c to the cycle and call the cycle $\tau_{\mathcal{M}, \mathcal{L}}$ ($\tau_{\mathcal{L}, \mathcal{M}}$) *oriented* one.

5. The formulation of the results.

Definition 5.1. Let Υ_{ϕ^t} be the directed graph of a flow $\phi^t \in G$. We will say that Υ_{ϕ^t} is the *equipped graph* of ϕ^t and denote it by $\Upsilon_{\phi^t}^*$ if:

- (1) every \mathcal{E} -vertex is equipped with the weight “+” or “-” in consistent and inconsistent case respectively;
- (2) every \mathcal{M} -vertex is equipped with a four-colour graph $\Gamma_{\mathcal{M}}$ corresponding to the flow f^t constructed in Subsection 4.4;
- (3) every edge $(\mathcal{M}, \mathcal{L})$ ($(\mathcal{L}, \mathcal{M})$) is equipped with an oriented *tu-cycle* (*st-cycle*) $\tau_{\mathcal{M}, \mathcal{L}}$ ($\tau_{\mathcal{L}, \mathcal{M}}$) of $\Gamma_{\mathcal{M}}$ corresponding to the limit cycle c of \mathcal{L} and oriented consistently with R_c (see Fig. 7).

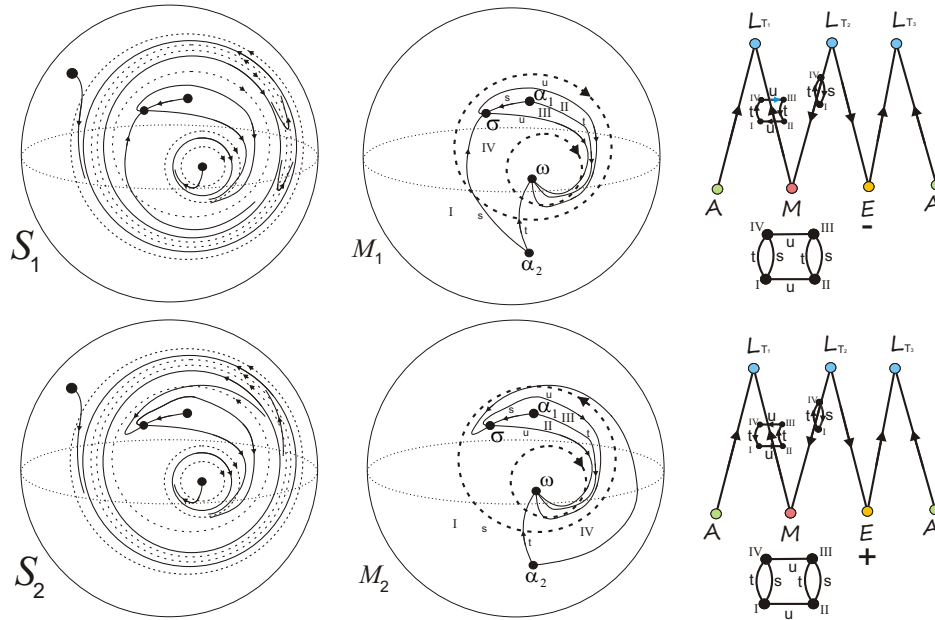


FIGURE 7. Two flows from G and their equipped graphs

On Fig. 8 you can see the two examples of flows from G whose difference might be recognized only by oriented cycles of four-colour graphs, and on Fig. 9 there are examples of flows on a torus whose difference might be recognized only by weight of \mathcal{E} -vertices.

Let us denote by $\pi_{\phi^t}^*$ the one-to-one correspondence described above between the elementary regions and the vertices, the cutting circles and the edges, the directions of the trajectories and the directions of the edges, the consistencies of the orientations of the boundary's connecting components of \mathcal{E} -regions and the weights of the \mathcal{E} -vertices, the \mathcal{M} -regions and the four-colour graphs, the stable limit cycles and the tu -cycles, the unstable limit cycles and the st -cycles, the orientations of the stable limit cycles and the orientations of the cycles $\tau_{\mathcal{M},\mathcal{L}}$, the orientations of the unstable limit cycles and the orientations of the cycles $\tau_{\mathcal{L},\mathcal{M}}$ accordingly.

5.1. The classification result.

Definition 5.2. Two equipped graphs $\Upsilon_{\phi^t}^*$ and $\Upsilon_{\phi^{t'}}^*$ are said to be *isomorphic* if there is a one-to one correspondence ξ between all edges and vertices of $\Upsilon_{\phi^t}^*$ and all edges and vertices of $\Upsilon_{\phi^{t'}}^*$ preserving their equipments in the following way:

- (1) the weights of vertices \mathcal{E} and $\xi(\mathcal{E})$ are equal;
- (2) for vertices \mathcal{M} and $\xi(\mathcal{M})$, there is an isomorphism $\psi_{\mathcal{M}}$ of the four-colour graphs $\Gamma_{\mathcal{M}}, \Gamma_{\xi(\mathcal{M})}$ such that $\psi_{\mathcal{M}}(\tau_{\mathcal{M},\mathcal{L}}) = \tau_{\xi(\mathcal{M}),\xi(\mathcal{L})}$ and the orientations of $\psi_{\mathcal{M}}(\tau_{\mathcal{M},\mathcal{L}})$ and $\tau_{\xi(\mathcal{M}),\xi(\mathcal{L})}$ coincide (similarly for $\tau_{\mathcal{L},\mathcal{M}}$).

Theorem 5.3. Flows $\phi^t, \phi^{t'} \in G$ are topologically equivalent if and only if the equipped graphs $\Upsilon_{\phi^t}^*$ and $\Upsilon_{\phi^{t'}}^*$ are isomorphic.

5.2. The realisation results. To solve the realization problem, we introduce the notions of an admissible four-colour graph and an equipped graph.

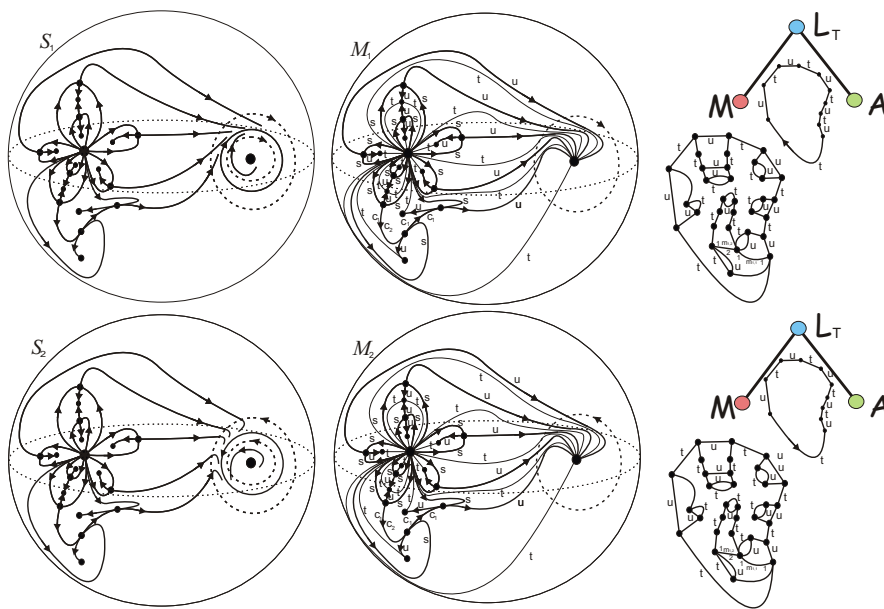


FIGURE 8. Two examples of flows from G differing only by orientation of the limit cycle between \mathcal{M} and \mathcal{A} and their equipped graphs

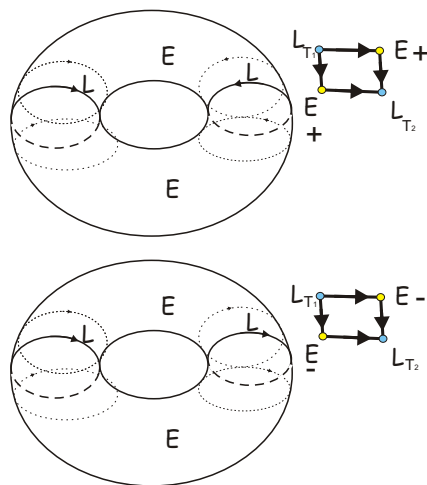


FIGURE 9. Two examples of flow from G without \mathcal{A} - and \mathcal{M} -regions differing only by orientation of the limit cycle and their equipped graphs

Let Γ be a four-colour graph with the properties:

- (1) every edge of the four-colour graph is coloured in one of the four colours: s, u, t, c ;

(2) every vertex of the four-colour graph is incident to exactly one edge of the colours s, u, t . Besides, the number n_b of c -edges incident to a vertex b can be any (may be null) and these edges $c_1^b, \dots, c_{n_b}^b$ are ordered if $n_b \geq 1$.

Definition 5.4. We say a four-colour graph Γ is *admissible* if it contains a c^* -cycle and every cycle has four vertices.

Lemma 5.5 ([10], Lemma 2.1). *The graph $\Gamma_{\mathcal{M}}$ is admissible.*

Lemma 5.6 ([10], Theorem 3). *Every admissible four-colour graph Γ corresponds to a closed surface M and an Ω -stable flow $f^t: M \rightarrow M$ from G without limit cycles, besides:*

(1) *The Euler characteristic of M can be obtained by the formula*

$$\chi(M) = \nu_0 - \nu_1 + \nu_2, \quad (2)$$

where ν_0, ν_1, ν_2 are the numbers of all tu -, c^* - and st -cycles of Γ respectively;

(2) *M is non-orientable if and only if Γ has at least one cycle with an odd length.*

Definition 5.7. We call Υ^* an *admissible equipped graph* if it is a connected directed graph Υ with \mathcal{A} -, \mathcal{L} -, \mathcal{E} - and \mathcal{M} -vertices satisfying the items (1)–(4) of Proposition 4, so that

- every \mathcal{M} -vertex is equipped with an admissible four-colour graph $\Gamma_{\mathcal{M}}$,
- every edge entering into (exiting out of) any \mathcal{M} -vertex is equipped with an oriented st -cycle (ut -cycle) of the four-colour graph,
- every \mathcal{E} -vertex is assigned with a weight “+” or “−”.

Lemma 5.8. *The graph $\Upsilon_{\phi^t}^*$ is admissible.*

For every \mathcal{M} -vertex of an admissible equipped graph Υ^* , let us denote by $X_{\mathcal{M}}$ the result of applying the formula (2) to the corresponding admissible four-colour graph $\Gamma_{\mathcal{M}}$. Denote by $Y_{\mathcal{M}}$ the quantity of edges, which are incident to \mathcal{M} and denote by $N_{\mathcal{A}}$ the quantity of \mathcal{A} -vertices of Υ^* .

Theorem 5.9. *Every admissible equipped graph Υ^* corresponds to an Ω -stable flow $\phi^t: S \rightarrow S$ from G on a closed surface S , besides:*

(1) *The Euler characteristic of S can be calculated by the formula*

$$\chi(S) = \sum_{\mathcal{M}} (X_{\mathcal{M}} - Y_{\mathcal{M}}) + N_{\mathcal{A}}; \quad (3)$$

(2) *S is orientable if and only if every four-colour graph equipping Υ^* has not cycles of an odd length and every \mathcal{L} -vertex is incident to exactly two edges.*

5.3. The algorithm results. An algorithm for solving the isomorphism problem is considered to be *efficient* if its working time is bounded by a polynomial on the length of the input data, in this case the input data is the number of the vertices and the edges of the graph. Algorithms of such kind are also called *polynomial-time* or simply *polynomial*. This commonly recognized definition of efficient solvability rises to A. Cobham [5]. A common standard of intractability is NP-completeness [6]. The complexity status of the isomorphism problem is still open, for the class of all graphs, neither its polynomial-time solvability nor its NP-completeness is proved. Fortunately, four-colour graphs and directed graphs of flows on ambient surfaces can be embedded into the carrying surfaces. Indeed, we have the following theorems.

Theorem 5.10. *Isomorphism of the equipped graphs $\Upsilon_{\phi^t}^*$, $\Upsilon_{\phi^{t'}}^*$ of flows $\phi^t, \phi^{t'} \in G$ can be recognized in polynomial time.*

Theorem 5.11. *The orientability of the ambient surface S for an Ω -stable flow ϕ^t can be tested in a linear time and the Euler characteristic of S can be determined in quadratic time by means of the equipped graph $\Upsilon_{\phi^t}^*$.*

6. The proof of the classification Theorem 5.3. In this section we consider Ω -stable flow $\phi^t \in G$ on closed surface S and prove that the isomorphic class of its equipped graph $\Upsilon_{\phi^t}^*$ is a complete topological invariant.

6.1. The necessary condition of Theorem 5.3. Let two Ω -stable flows $\phi^t, \phi'^t \in G$ given on a closed surface S be topological equivalent, i.e. there is a homeomorphism $h: S \rightarrow S$ mapping trajectories of ϕ^t to trajectories of ϕ'^t . Without loss of generality we assume that the cutting set \mathcal{R}' of ϕ'^t is created so that $\mathcal{R}' = h(\mathcal{R})$, where \mathcal{R} is the cutting set of ϕ^t . Also we can assume that the restriction \mathcal{T}' of the set of t -curves of ϕ'^t to the \mathcal{M} -regions of ϕ'^t is created so that $\mathcal{T}' = h(\mathcal{T})$, where \mathcal{T} is the restriction of the set of t -curves of ϕ^t to the \mathcal{M} -regions of ϕ^t . Then h maps the elementary and the polygonal regions of ϕ^t to the elementary and the polygonal regions of ϕ'^t respectively.

Recall that $\pi_{\phi^t}^*$ is the one-to-one correspondence between the elementary regions and the vertices, the cutting circles and the edges, the directions of the trajectories and the directions of the edges, the consistencies of the orientations of the limit circles for the \mathcal{E} -regions and the weights of the \mathcal{E} -vertices, the \mathcal{M} -regions and the four-colour graphs, the stable limit cycles and the tu -cycles, the unstable limit cycles and the st -cycles respectively. Let us define the isomorphism $\xi: \Upsilon_{\phi^t}^* \rightarrow \Upsilon_{\phi'^t}^*$ by the formula

$$\xi = \pi_{\phi'^t}^* h (\pi_{\phi^t}^*)^{-1}.$$

Notice that h carries out the topological equivalence of ϕ^t and ϕ'^t , then it preserves the types of elementary regions and, hence, ξ preserves the types of the vertices. Also notice that h preserves the orientations on the trajectories, then the weights of vertices \mathcal{E} and $\xi(\mathcal{E})$ are equal.

Let $\Gamma_{\mathcal{M}}$ be the four-colour graph for some vertex \mathcal{M} , $\Gamma_{\xi(\mathcal{M})}$ be the four-colour graph for the vertex $\mathcal{M}' = \xi(\mathcal{M})$. Recall that $\phi^t|_{\mathcal{M}} = f^t|_{\mathcal{M}}$ ($\phi'^t|_{\mathcal{M}'} = f'^t|_{\mathcal{M}'}$) and π_{f^t} ($\pi_{f'^t}$) is the one-to-one correspondence between the polygonal regions and the vertices, also between the colour curves of f^t (f'^t) and the colour edges of the four-colour graph $\Gamma_{\mathcal{M}}$ ($\Gamma_{\mathcal{M}'}$) respectively.

As $\Gamma_{\mathcal{M}}$ is the four-colour graph of the region \mathcal{M} , then $\Gamma_{\mathcal{M}} = \pi_{\phi^t}^*(\mathcal{M})$. Let $\Gamma_{\mathcal{M}'} = \Gamma_{\xi(\mathcal{M})} = \pi_{\phi'^t}^*(h(\mathcal{M}))$. As h maps the polygonal regions of f^t to the polygonal regions of f'^t , then there exists an isomorphism $\psi: \Gamma_{\mathcal{M}} \rightarrow \Gamma_{\mathcal{M}'}$ defined by the formula

$$\psi_{\mathcal{M}} = \pi_{f'^t} h \pi_{f^t}^{-1}.$$

As $\mathcal{R}' = h(\mathcal{R})$, then $\psi(\tau_{\mathcal{M}, \mathcal{L}}) = \tau_{\xi(\mathcal{M}), \xi(\mathcal{L})}$ and the orientations of $\psi_{\mathcal{M}}(\tau_{\mathcal{M}, \mathcal{L}})$ and $\tau_{\xi(\mathcal{M}), \xi(\mathcal{L})}$ coincide (similarly for $\tau_{\mathcal{L}, \mathcal{M}}$). Thus ξ is the required isomorphism.

So, one topological equivalence class of flows defines one isomorphisity class of graphs.

6.2. The sufficient condition of Theorem 5.3. Assume that two graphs $\Upsilon_{\phi^t}^*$ and $\Upsilon_{\phi'^t}^*$ are isomorphic by ξ . To prove the topological equivalence of the flows we need to construct homeomorphisms between elementary regions mapping the trajectories of ϕ^t to the trajectories of ϕ'^t so that for two elementary regions the homeomorphisms on their common boundaries coincide.

I. \mathcal{M} -region. Let us consider some \mathcal{M} -region of the flow ϕ^t . Consider the region

$$\mathcal{M}' = (\pi_{\phi^t}^*)^{-1} \xi \pi_{\phi^t}^*(\mathcal{M})$$

of the flow ϕ^t . Their four-colour graphs $\Gamma_{\mathcal{M}}$ and $\Gamma_{\mathcal{M}'}$ are isomorphic by means of ψ . Let $f^t: M \rightarrow M$ ($f'^t: M' \rightarrow M'$) be the flow corresponding to $\Gamma_{\mathcal{M}}$ ($\Gamma_{\mathcal{M}'}$). Recall that in Subsection 4.4 we defined the flow f^t (f'^t) on surface M (M') such that $M \cap S = \mathcal{M}$ ($M' \cap S = \mathcal{M}'$) and $\phi^t|_{\mathcal{M}} = f^t|_{\mathcal{M}}$ ($\phi^t|_{\mathcal{M}'} = f'^t|_{\mathcal{M}'}$).

The fact that f^t and f'^t are topologically equivalent if and only if $\Gamma_{\mathcal{M}}$ and $\Gamma_{\mathcal{M}'}$ are isomorphic is proved in [10], Theorem 1. So there exists the homeomorphism $h_M: M \rightarrow M'$ mapping trajectories of f^t to trajectories of f'^t . Suppose without loss of generality that cutting circles of \mathcal{M} and \mathcal{M}' are homeomorphic by means of h_M (in [10] we really constructed the homeomorphism so that these circles become homeomorphic). As $M \cap S = \mathcal{M}$ and $M' \cap S = \mathcal{M}'$, then we define the homeomorphism $h_{\mathcal{M}}: cl(\mathcal{M}) \rightarrow cl(\mathcal{M}')$ by the formula

$$h_{\mathcal{M}} = h_M|_{cl(\mathcal{M})}.$$

Thus we have the homeomorphism

$$h_{\mathcal{M}}: cl(\mathcal{M}) \rightarrow cl(\mathcal{M}')$$

for every \mathcal{M} -region of the flow ϕ^t .

II. \mathcal{E} -region. Let us consider some \mathcal{E} -region of the flow ϕ^t . Consider the \mathcal{E}' -region of the flow ϕ^t such that

$$\mathcal{E}' = (\pi_{\phi^t}^*)^{-1} \xi \pi_{\phi^t}^*(\mathcal{E}).$$

These two regions are of the same type because of the weight of the vertices corresponding to them.

Let E_1 and E_2 be the connected components of $\partial\mathcal{E}$. Then they are cutting circles and, hence, $E'_i = (\pi_{\phi^t}^*)^{-1} \xi \pi_{\phi^t}^*(E_i)$, $i = 1, 2$ are cutting circles which are the connected components of $\partial\mathcal{E}'$.

Let $h_{E_1}: E_1 \rightarrow E'_1$ be an arbitrary homeomorphism preserving orientations of E_1 and E'_1 . Let $x_0 \in E_1$ and $\{x_1\} = \mathcal{O}_{x_0} \cap E_2$. Let $x'_0 = h_{E_1}(x_0)$ and $\{x'_1\} = \mathcal{O}_{x'_0} \cap E'_2$. Let us construct the homeomorphism $h_{\mathcal{E}}: cl(\mathcal{E}) \rightarrow cl(\mathcal{E}')$ so that $h_{\mathcal{E}}|_{l_{x_0, x_1}} = h_{l_{x_0, x_1}}: l_{x_0, x_1} \rightarrow l_{x'_0, x'_1}$.

Thus we have the homeomorphism

$$h_{\mathcal{E}}: cl(\mathcal{E}) \rightarrow cl(\mathcal{E}')$$

for every \mathcal{E} -region of the flow ϕ^t .

III. \mathcal{A} -region. Let us consider some \mathcal{A} -region of the flow ϕ^t with a source α . Consider the region

$$(\pi_{\phi^t}^*)^{-1} \xi \pi_{\phi^t}^*(\mathcal{A})$$

of the flow ϕ^t . We perfectly know that it is the \mathcal{A}' -region with a source because of the directions the of edges.

\mathcal{A} (\mathcal{A}') is surely surrounded by some \mathcal{L} , ($\mathcal{L}' = (\pi_{\phi^t}^*)^{-1} \xi \pi_{\phi^t}^*(\mathcal{L})$).

Recall that

$$u = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

Due to Proposition 2 the source α (α') has a neighbourhood u_{α} ($u_{\alpha'}$) and the homeomorphism $h_{\alpha}: u_{\alpha} \rightarrow u$ ($h_{\alpha'}: u_{\alpha'} \rightarrow u$) such that $\phi^t|_{u_{\alpha}}$ ($\phi^t|_{u_{\alpha'}}$) is conjugate to $c^t|_u$. Also recall that $S_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r\}$ for $r \in (0, 1]$ and $S_r^{\alpha} = h_{\alpha}^{-1}(S_r)$ ($S_r^{\alpha'} = h_{\alpha'}^{-1}(S_r)$). Notice that $S_1^{\alpha} = \partial u_{\alpha}$ ($S_1^{\alpha'} = \partial u_{\alpha'}$)

We know that $\partial\mathcal{A}$ ($\partial\mathcal{A}'$) is oriented. Then now we orient $\partial u_\alpha = S_1^\alpha$ ($\partial u_{\alpha'} = S_1^{\alpha'}$) consistently with $\partial\mathcal{A}$ ($\partial\mathcal{A}'$). Let $h_{S_1^\alpha}: S_1^\alpha \rightarrow S_1^{\alpha'}$ be the arbitrary homeomorphism preserving orientations of S_1^α and $S_1^{\alpha'}$. Let $x \in S_1^\alpha$ ($x' \in S_1^{\alpha'}$) and \mathcal{O}_x ($\mathcal{O}_{x'}$) be the trajectory of x (x'). Let $x^\alpha \in (cl(u_\alpha) \setminus \{\omega\})$, then $x^\alpha = S_r^\alpha \cap \mathcal{O}_x$ for some $r \in (0, 1]$ and $x \in S_1^\alpha$. Let us define the homeomorphism $h_{u_\alpha}: cl(u_\alpha) \rightarrow cl(u_{\alpha'})$ so that $h_{u_\alpha}(\alpha) = \alpha'$ and $h_{u_\alpha}(x^\alpha) = x'^{\alpha'}$, where $x'^{\alpha'} = S_r^{\alpha'} \cap \mathcal{O}_{h_{S_1^\alpha}(x)}$.

Let $x_0 \in \partial u_\alpha$, $x'_0 \in \partial u_{\alpha'}$ and $x'_0 = h_{u_\alpha}(x_0)$. Let \mathcal{O}_{x_0} ($\mathcal{O}_{x'_0}$) be the trajectory of x_0 (x'_0) and $\{x_1\} = \mathcal{O}_{x_0} \cap \partial\mathcal{A}$ ($\{x'_1\} = \mathcal{O}_{x'_0} \cap \partial\mathcal{A}'$). Let us define the homeomorphism $h_{cl(\mathcal{A}) \setminus u_\alpha}: cl(\mathcal{A}) \setminus u_\alpha \rightarrow cl(\mathcal{A}') \setminus u_{\alpha'}$ so that $h_{cl(\mathcal{A}) \setminus u_\alpha}|_{l_{x_0, x_1}} = h_{l_{x_0, x_1}}: l_{x_0, x_1} \rightarrow l_{x'_0, x'_1}$ for any $x_0 \in \partial u_\alpha$.

So we define the homeomorphism $h_{\mathcal{A}}: cl(\mathcal{A}) \rightarrow cl(\mathcal{A}')$ by the formula

$$h_{\mathcal{A}}(x) = \begin{cases} h_{u_\alpha}(x) & \text{if } x \in u_\alpha, \\ h_{cl(\mathcal{A}) \setminus u_\alpha}(x) & \text{if } x \in cl(\mathcal{A}) \setminus u_\alpha. \end{cases}$$

The homeomorphism for \mathcal{A} -region with a sink can be constructed similarly. Thus we have a homeomorphism

$$h_{\mathcal{A}}: cl(\mathcal{A}) \rightarrow cl(\mathcal{A}')$$

for every \mathcal{A} -region of the flow ϕ^t .

IV. \mathcal{L} -region. Here we will follow the construction in [8]. Let us consider some \mathcal{L} -region of the flow ϕ^t with an unstable (for definiteness) limit cycle \mathbf{c} inside. Consider a region

$$(\pi_{\phi^t}^*)^{-1} \xi \pi_{\phi^t}^*(\mathcal{L})$$

of the flow ϕ^t . We perfectly know that it is an \mathcal{L}' -region of the flow ϕ^t with an unstable limit cycle \mathbf{c}' which is the same type to \mathcal{L} because of directions of edges and their numbers. We also know that the orientations of limit cycles and cutting circles of \mathcal{L} and \mathcal{L}' are oriented consistently because the orientations of $\psi(\tau_{\mathcal{L}, \mathcal{M}})$ and $\tau_{\xi(\mathcal{L}), \xi(\mathcal{M})}$ are equivalent.

1. Consider the case of the annulus.

Step 1. Let L^* and L^{**} be the two connecting components of $\partial\mathcal{L}$ and let $L'^* = (\pi_{\phi^t}^*)^{-1} \xi \pi_{\phi^t}^*(L^*)$, $L'^{**} = (\pi_{\phi^t}^*)^{-1} \xi \pi_{\phi^t}^*(L^{**})$. Let $h^*: L^* \rightarrow L'^*$ and $h^{**}: L^{**} \rightarrow L'^{**}$ be the contractions of the homeomorphisms constructed before on the closures of the elementary regions adjoined to \mathcal{L} (\mathcal{L}') with L^* and L^{**} as their common boundary accordingly.

Step 2. Recall that Σ_p ($\Sigma_{p'}$) is the Poincaré cross-section of \mathbf{c} (\mathbf{c}'), F_p ($F_{p'}$) is the Poincaré map and $\{p\} = \Sigma_p \cap \mathbf{c}$ ($\{p'\} = \Sigma_{p'} \cap \mathbf{c}'$). By Proposition 3 $F_p \in Diff^1(\Sigma_p)$. The point p is a source of F_p . Let $m_{a,b}$, $a, b \in \Sigma_p$ ($m_{a',b'}$, $a', b' \in \Sigma_{p'}$) be the segment in Σ_p ($\Sigma_{p'}$) with boundary $\{a, b\}$ ($\{a', b'\}$) and $\mu_{a,b}$ ($\mu_{a',b'}$) be its length.

Let $\{x^*\} = \Sigma_p \cap L^*$ and $\{x^{**}\} = \Sigma_p \cap L^{**}$. Let $x'^* \in L'^*$ and $x'^{**} \in L'^{**}$ be such that $x'^* = h^*(x^*)$ and $x'^{**} = h^{**}(x^{**})$. Let $\{x'_*\} = \Sigma_{p'} \cap L'^*$ and $\{x'_{**}\} = \Sigma_{p'} \cap L'^{**}$. Let $t^* \geq 0$ and $t^{**} \geq 0$ be the least non negative numbers such that $x'^* = \phi^{t^*}(x'_*)$ and $x'^{**} = \phi^{t^{**}}(x'_{**})$. Let

$$p'^* = \phi^{(\frac{\mu_{x'^*, p'}}{\mu_{x'^*, x'_*}}(t^* - t^{**}) + t^{**})}(p'),$$

$$\Sigma_{p'^*} = \left\{ \phi^{(\frac{\mu_{x'^*, x'}}{\mu_{x'^*, x'_*}}(t^* - t^{**}) + t^{**})}(x') : x' \in \Sigma_{p'} \right\}.$$

Step 3. Let us construct a homeomorphism $h_\Sigma: \Sigma_p \rightarrow \Sigma_{p'^*}$ by the next way. For $x \in m_{x^*, F_p^{-1}(x^*)}$ let $t_x^* \geq 0$ be such that $\phi^{t_x^*}(x) \in L^*$ and $t_{x'}^* \geq 0$ such that $\phi'^{(-t_{x'}^*)}(h^*(\phi^{t_x^*}(x))) \in m_{x'^*, F_{p'^*}^{-1}(x'^*)}$. Then

$$h_\Sigma(x) = \phi'^{(-t_{x'}^*)}(h^*(\phi^{t_x^*}(x)));$$

Similarly for $x \in m_{x^{**}, F_p^{-1}(x^{**})}$ let $t_x^{**} \geq 0$ be such that $\phi^{t_x^{**}}(x) \in L^{**}$ and $t_{x'}^{**} \geq 0$ such that $\phi'^{(-t_{x'}^{**})}(h^{**}(\phi^{t_x^{**}}(x))) \in m_{x'^{**}, F_{p'^*}^{-1}(x'^{**})}$. Then

$$h_\Sigma(x) = \phi'^{(-t_{x'}^{**})}(h^{**}(\phi^{t_x^{**}}(x)));$$

For $x \in m_{F_p^{-k}(x^*), F_p^{-k+1}(x^*)}$, where $k \in \mathbb{N}$ let

$$h_\Sigma(x) = F_{p'^*}^{-k}(x) \circ h_\Sigma \circ F_p^k(x);$$

For $x \in m_{F_p^{-l}(x^{**}), F_p^{-l+1}(x^{**})}$, where $l \in \mathbb{N}$ let

$$h_\Sigma(x) = F_{p'^*}^{-l}(x) \circ h_\Sigma \circ F_p^l(x).$$

Step 4. Let us define a homeomorphism $h_{\mathcal{L}}: cl(\mathcal{L}) \rightarrow cl(\mathcal{L}')$ by the next formulas.

$$\text{For } x \in \Sigma_p \setminus (m_{F_p^{-1}(x^*), x^*} \cup m_{F_p^{-1}(x^{**}), x^{**}})$$

$$\text{let } h_{\mathcal{L}}|_{l_{x, F_p(x)}} = h_{l_{x, F_p(x)}}: l_{x, F_p(x)} \rightarrow l_{h_\Sigma(x), h_\Sigma(F_p(x))}.$$

$$\text{For } x \in m_{F_p^{-1}(x^*), x^*}$$

$$\text{let } h_{\mathcal{L}}|_{l_{x, \phi^{t_x^*}(x)}} = h_{l_{x, \phi^{t_x^*}(x)}}: l_{x, \phi^{t_x^*}(x)} \rightarrow l_{h_\Sigma(x), h^*(\phi^{t_x^*}(x))}.$$

$$\text{For } x \in m_{F_p^{-1}(x^{**}), x^{**}}$$

$$\text{let } h_{\mathcal{L}}|_{l_{x, \phi^{t_x^{**}}(x)}} = h_{l_{x, \phi^{t_x^{**}}(x)}}: l_{x, \phi^{t_x^{**}}(x)} \rightarrow l_{h_\Sigma(x), h^{**}(\phi^{t_x^{**}}(x))}.$$

2. Consider the case of the Möbius band. In general the construction is similar to the case of the annulus but it has the few important differences.

Step 1. The boundary $\partial\mathcal{L}$ has only one connected component, and Σ_p crosses it in two points x^* and x^{**} . Denote $h^*: \partial\mathcal{L} \rightarrow \partial\mathcal{L}$ the homeomorphism constructed before on $\partial\mathcal{L}$. Let x'_* be one of the two points in which $\Sigma_{p'}$ crosses $\partial\mathcal{L}'$. Let $x'^* = h^*(x^*)$. Let $t^* \geq 0$ be the least non negative number such that $x'^* = \phi^{t^*}(x'_*)$. Let

$$p'^* = \phi^{t^*}(p') \quad \text{and} \quad \Sigma_{p'^*} = \left\{ \phi'^{t^*}(x') : x' \in \Sigma_{p'} \right\}.$$

Denote by x'^{**} the second point in which $\Sigma_{p'^*}$ crosses $\partial\mathcal{L}'$ (i.e. $x'^{**} \neq x'^*$).

Step 2. Let us construct a homeomorphism

$$h_\Sigma: (\Sigma_p \setminus m_{x^*, F_p^{-1}(x^*)}) \rightarrow (\Sigma_{p'^*} \setminus m_{x'^{**}, F_{p'^*}^{-1}(x'^*)})$$

by the next way: For $x \in m_{x^*, F_p^{-2}(x^*)}$ let $t_x^* \geq 0$ be such that $\phi^{t_x^*}(x) \in \partial\mathcal{L}$ and $t_{x'}^* \geq 0$ such that $\phi'^{(-t_{x'}^*)}(h^*(\phi^{t_x^*}(x))) \in m_{x'^*, F_{p'^*}^{-2}(x'^*)}$. Then

$$h_\Sigma(x) = \phi'^{(-t_{x'}^*)}(h^*(\phi^{t_x^*}(x)));$$

For $x \in m_{F_p^{-k}(x^*), F_p^{-k-2}(x^*)}$, where $k \in \mathbb{N}$, let

$$h_\Sigma(x) = F_{p'^*}^{-k}(x) \circ h_\Sigma \circ F_p^k(x);$$

Step 3. Let us define the homeomorphism $h_{\mathcal{L}}: cl(\mathcal{L}) \rightarrow cl(\mathcal{L}')$ by the next formulas

$$\begin{aligned} &\text{For } x \in \Sigma_p \setminus (m_{F_p^{-2}(x^*), x^*} \cup m_{x^{**}, F_p(x^*)}) \\ &\text{let } h_{\mathcal{L}}|_{l_{x, F_p(x)}} = h_{l_{x, F_p(x)}}: l_{x, F_p(x)} \rightarrow l_{h_{\Sigma}(x), h_{\Sigma}(F_p(x))}. \\ &\text{For } x \in m_{F_p^{-2}(x^*), x^*} \\ &\text{let } h_{\mathcal{L}}|_{l_{x, \phi^{t^*}(x)}} = h_{l_{x, \phi^{t^*}(x)}}: l_{x, \phi^{t^*}(x)} \rightarrow l_{h_{\Sigma}(x), h^*(\phi^{t^*}(x))}. \end{aligned}$$

The homeomorphism for \mathcal{L} -region with a stable limit cycle can be constructed similarly. Thus we have a homeomorphism

$$h_{\mathcal{L}}: cl(\mathcal{L}) \rightarrow cl(\mathcal{L}')$$

for every \mathcal{L} -region of the flow ϕ^t .

The final homeomorphism. We have built the homeomorphism for each elementary region. Thus, the final homeomorphism $h: S \rightarrow S$ we defined by the formula

$$h(x) = \begin{cases} h_{\mathcal{A}}(x) & \text{if } x \in cl(\mathcal{A}), \\ h_{\mathcal{E}}(x) & \text{if } x \in cl(\mathcal{E}), \\ h_{\mathcal{M}}(x) & \text{if } x \in cl(\mathcal{M}), \\ h_{\mathcal{L}}(x) & \text{if } x \in cl(\mathcal{L}). \end{cases}$$

So, Theorem 5.3 is proved. □

7. Realisation of an admissible equipped graph Υ^* by the Ω -stable flow ϕ^t on a surface S . Let Υ^* be some admissible equipped graph.

I. Let us construct an Ω -stable flow ϕ^t corresponding to Υ^* 's isomorphic class by creation the surface S and the continuous vector field.

Step 1. Let B be the set of Υ^* 's vertices and E be the set of its edges. Let us construct for every $b \in B$ a surface S_b with a boundary and a vector field \vec{V}_b on it, transversal to the boundary. The required Ω -stable flow on S will be glued from these pieces of dynamics by means annuli which correspond to the edges from E according to incidence.

\mathcal{A} -vertex. Let b be an \mathcal{A} -vertex. Then $S_b = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ and the vector field on the disk S_b we define by the vector-function $\vec{V}_b(x, y) = \{-x, -y\}$ ($\vec{V}_b(x, y) = \{x, y\}$), if the edges incident to b are directed to b (out of b).

\mathcal{E} -vertex. Let b be an \mathcal{E} -vertex. Let $W = [0, 1] \times [0, 1]$. Define the minimal equivalence relation $\sim_{\mathcal{E}}$ on W such that $(x, 0) \sim_{\mathcal{E}} (x, 1)$ for $x \in [0, 1]$. Let $S_b = W / \sim_{\mathcal{E}}$ and $q_b: W \rightarrow S_b$ be the natural projection. Define on the annulus S_b the vector field by the formula $\vec{V}_b(x, y) = q_b(\{\frac{1}{2}, 1\})$ ($\vec{V}_b(x, y) = q_b(\{\sin \frac{2\pi}{3}(x + \frac{1}{4}), \cos \frac{2\pi}{3}(x + \frac{1}{4})\})$), if the weight of \mathcal{E} is “+” (“-”).

\mathcal{L} -vertex. Let b be an \mathcal{L} -vertex. Let

$$\mathcal{W} = \{(x, y) \in \mathbb{R}^2 : |x| \leq \frac{3 - \cos \pi y}{2}, 0 \leq y \leq 1\}.$$

Then \mathcal{W} is a curvilinear trapezium with the vertices $A(-1; 0), B(-2; 1), C(2; 1), D(1; 0)$. Define on \mathcal{W} the minimal equivalence relation $\sim_{\mathcal{L}}$ such that $(x, 0) \sim_{\mathcal{L}} (2x, 1)$ ($(x, 0) \sim_{\mathcal{L}} (-2x, 1)$) for $x \in AD$, if the vertex b is incident to two edges (one edge). Let $S_b = \mathcal{W} / \sim_{\mathcal{L}}$ and let $q_b: \mathcal{W} \rightarrow S_b$ be its natural projection. Then S_b is the annulus (the Möbius band). Define on S_b the vector field by the formula $\vec{V}_b(x, y) = q_b(\{0, 1\})$ ($\vec{V}_b(x, y) = q_b(\{0, -1\})$) and orient the boundary of S_b in the

direction of motion along the coordinate y from 0 to 1 (from 1 to 0), if the edges incident to b are directed to b (out of b).

\mathcal{M} -vertex. Let b be a \mathcal{M} -vertex. Then b is equipped with the four colour graph $\Gamma_{\mathcal{M}}$, corresponding to the surface M with the vector field \vec{V}_M , constructed in the proof of Lemma 5.6 in [10]. Let ω (α) be a sink (source) of \vec{V}_M such that $\omega = \pi_{V_M}^{-1}(\tau_{b,\mathcal{L}})$ ($\alpha = \pi_{V_M}^{-1}(\tau_{\mathcal{L},b})$), where π_{V_M} is the one-to-one correspondence between the elements of the field \vec{V}_M and the elements of the four colour graph $\Gamma_{\mathcal{M}}$. Let u_ω (u_α) is some neighbourhood of ω (of α) without other elements of the basic set inside and with the boundary transversal to the trajectories of \vec{V}_M . Let us orient ∂u_ω (∂u_α) consistently with the orientation of the cycle $\tau_{b,\mathcal{L}}$ ($\tau_{\mathcal{L},b}$). Then $S_b = M \setminus \bigcup_{\omega=\pi_{V_M}^{-1}(\tau_{b,\mathcal{L}})} \text{int } u_\omega \cup \bigcup_{\alpha=\pi_{V_M}^{-1}(\tau_{\mathcal{L},b})} \text{int } u_\alpha$ with the field $\vec{V}_b = \vec{V}_M|_{S_b}$. We

will suppose that each connected component of ∂S_b has an orientation due to the oriented cycle the orientation.

Step 2. Let $A = \mathbb{S}^1 \times [-1, 1]$ and we have two vector fields $\vec{V}^- = \{v^-(s), s \in \mathbb{S}^1\}$, $\vec{V}^+ = \{v^+(s), s \in \mathbb{S}^1\}$ on $\mathbb{S}^1 \times \{-1\}$, $\mathbb{S}^1 \times \{1\}$, accordingly, such that they are transversal to ∂A , \vec{V}^- has a direction to A , \vec{V}^+ has a direction out of A . Let

$$\vec{V}_A = \{\vec{v}(s, t) = \frac{1}{2} \left((1-t)v^-(s) + (1+t)v^+(s) \right), s \in \mathbb{S}^1, t \in [-1, 1]\}.$$

We will called the vector field \vec{V}_A by *an average of the boundaries*.

For every edge $e \in E$ denote by A_e a copy of the annulus A . Let us notice that the sets $\partial \left(\bigsqcup_{b \in B} S_b \right)$ and $\partial \left(\bigsqcup_{e \in E} A_e \right)$ consist of the same number of circles.

Let $h_{\Upsilon^*} : \partial \left(\bigsqcup_{b \in B} S_b \right) \rightarrow \partial \left(\bigsqcup_{e \in E} A_e \right)$ be a diffeomorphism such that if $h_{\Upsilon^*}(x) = y$ for $x \in S_b$, $y \in A_e$ then b, e are incident, moreover, h_{Υ^*} induces a concordant orientation on the connected components of ∂A_e for the edge e which is incident to \mathcal{M} -vertex and \mathcal{L} -vertex.

Let $\mathcal{S} = \bigsqcup_{b \in B} S_b \sqcup \bigsqcup_{e \in E} A_e$. Let us introduce on \mathcal{S} the minimal equivalent relation \sim_{Υ^*} such that $x \sim_{\Upsilon^*} h_{\Upsilon^*}(x)$. Then $\mathcal{S} / \sim_{\Upsilon^*}$ is a closed surface, denote it by S and by $q_S : \mathcal{S} \rightarrow S$ the natural projection. Then the required vector field \vec{V}_S on S coincides with $q_S(\vec{V}_{S_b})$ for every $b \in B$ and is the average of the boundaries on $q_S(A_e)$ for every $e \in E$.

II. Let us prove that the Euler characteristic of S can be calculated by the formula (3) $\chi(S) = \sum_{\mathcal{M}} (X_{\mathcal{M}} - Y_{\mathcal{M}}) + N_{\mathcal{A}}$, where $X_{\mathcal{M}}$ is the result of applying the formula (2) to the four-colour graph $\Gamma_{\mathcal{M}}$ corresponding to the vertex \mathcal{M} , $Y_{\mathcal{M}}$ is the quantity of the edges which are incident to \mathcal{M} , $N_{\mathcal{A}}$ is the quantity of \mathcal{A} -vertex of Υ^* .

It is well-known (see, for example, [4]) that $\chi(\Pi_p) = \chi(\Pi) - p$, where Π_p is the surface Π with p holes and if Π is a result of an identifying of the boundaries of Π_p^1 and Π_p^2 then $\chi(\Pi) = \chi(\Pi_p^1) + \chi(\Pi_p^2)$. As S is a result of the identifying of the boundaries of $\bigsqcup_{b \in B} S_b$ and $\bigsqcup_{e \in E} A_e$ and $\chi(A_e) = 0$ then to calculate $\chi(S)$ we need to calculate the characteristic of its elementary regions and to summarize them.

As $\chi(S_b) = 1$ for b being \mathcal{A} -vertex, $\chi(S_b) = 0$ for b being \mathcal{E} - or \mathcal{L} -vertex and $\chi(S_b) = X_{\mathcal{M}} - Y_{\mathcal{M}}$ for b being \mathcal{M} -vertex then we get the result.

III. Let us prove that S is orientable if and only if every four-colour graph equipping Υ^* has not odd length cycles and each \mathcal{L} -vertex is incident to exactly two edges.

Notice that S is orientable if and only if all its parts are orientable, i.e. all its elementary regions are orientable, that equivalently the condition that all \mathcal{L} -regions are the annuli and all four colour graphs equipping Υ^* do not have odd length cycles (see item (2) of Lemma 5.6). \square

8. Efficient algorithms to solve the isomorphism problem in the classes of four-colour and equipped graphs, to calculate the Euler characteristic and to determine orientability of the ambient surface. In this section, we consider the distinction (isomorphism) problem for four-colour and equipped graphs and the problems of calculation of the Euler characteristic of the ambient surface and determining its orientability. We present polynomial-time algorithms for their solution.

8.1. The isomorphism problem, a proof of Theorem 5.10. For two given four-colour (or equipped) graphs, the problem is to decide whether these graphs are isomorphic or not. Recall that four-colour graphs and directed graphs of flows can be embedded into the ambient surface. In other words, these graphs can be depicted on the ambient surface such that their vertices are points and their edges are Jordan curves on the surface, and no two edges are crossing in an internal point. This observation is useful for our purposes, as there exists a polynomial-time algorithm for the isomorphism problem of simple graphs embeddable into a fixed surface.

Definition 8.1. An unlabeled graph without loops, directed and multiple edges is called *simple*.

Proposition 6. [15] *The isomorphism problem for n -vertex simple graphs each embeddable into a surface of genus g can be solved in $O(n^{O(g)})$ time.*

First, let us consider only the case of four-colour graphs. We cannot directly apply Proposition 6 for distinction of four-colour graphs, as they are not simple. Nevertheless, it is possible to reduce the problem for four-colour graphs to the same problem for simple graphs with a small complexity of the reduction. To this end, we need the following operations with graphs.

Definition 8.2. The operation of k -subdivision of an edge (a, b) is to delete the edge from a graph, add vertices c_1, \dots, c_k and edges $(a, c_1), (c_1, c_2), \dots, (c_k, b)$.

Definition 8.3. The operation of (k_1, k_2) -subdivision of an edge (a, b) is to delete it from a graph, add vertices $c_1, c_2, \dots, c_{k_1}, v, u, w, d_1, d_2, \dots, d_{k_2}$ and edges $(a, c_1), (c_1, c_2), \dots, (c_{k_1}, v), (v, u), (u, w), (v, w), (v, d_1), (d_1, d_2), \dots, (d_{k_2}, b)$.

For the four-colour graph $\Gamma_{\mathcal{M}}$ of a given flow, we construct a simple graph $\Gamma_{\mathcal{M}}^*$ as follows. In the graph $\Gamma_{\mathcal{M}}$ we perform 1-subdivision of each s -edge, 2-subdivision of each t -edge, 3-subdivision of each u -edge. Let $e = (a, b)$ be an arbitrary c -edge of $\Gamma_{\mathcal{M}}$, $num_a(e)$ and $num_b(e)$ be the numbers of e in the sets of c -edges incident to a and b , correspondingly. We perform $(num_a(e) + 5, num_b(e) + 5)$ -subdivision of e . A similar operation is performed for all c -edges of the graph $\Gamma_{\mathcal{M}}$. The resultant graph $\Gamma_{\mathcal{M}}^*$ is simple (see Fig 10).

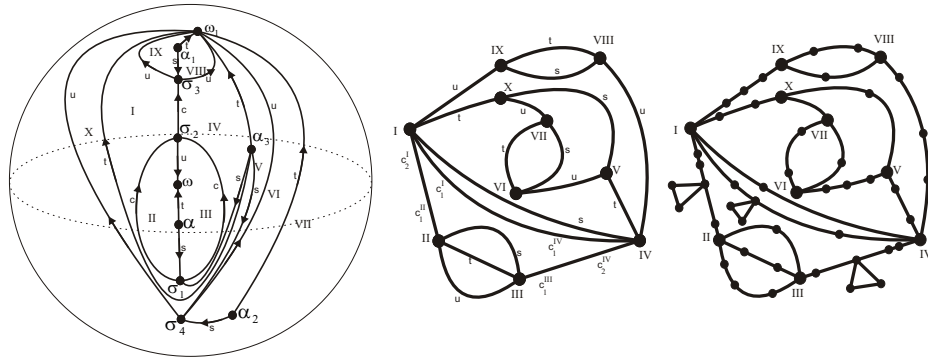


FIGURE 10. f^t , $\Gamma_{\mathcal{M}}$ and $\Gamma_{\mathcal{M}}^*$

Lemma 8.4. *Graphs $\Gamma_{\mathcal{M}}$ and $\Gamma_{\mathcal{M}'}$ are isomorphic iff graphs $\Gamma_{\mathcal{M}}^*$ and $\Gamma_{\mathcal{M}'}^*$ are isomorphic.*

Proof. Obviously, the graphs $\Gamma_{\mathcal{M}}^*$ and $\Gamma_{\mathcal{M}'}^*$ can be uniquely constructed with the graphs $\Gamma_{\mathcal{M}}$ and $\Gamma_{\mathcal{M}'}$. Let us show that the opposite fact is also true. It will follow the lemma.

Each polygonal region of Δ_{f^t} has at least three sides and, therefore, every vertex of $\Gamma_{\mathcal{M}}$ has at least three neighbours in this graph. Clearly, in the graph $\Gamma_{\mathcal{M}}^*$ none of the vertices of the graph $\Gamma_{\mathcal{M}}$ belongs to a triangle. Hence, the set of vertices of $\Gamma_{\mathcal{M}}$ consists of those and only those vertices of $\Gamma_{\mathcal{M}}^*$ that have at least three neighbours and do not belong to triangles. Deleted all vertices of $\Gamma_{\mathcal{M}}$ from $\Gamma_{\mathcal{M}}^*$, we obtain the disjunctive union of connected subgraphs, each of which is a path or a path with a triangle joined to an internal vertex of the path. These connected subgraphs are indicators of the existence of edges between the corresponding vertices of $\Gamma_{\mathcal{M}}$. If a subgraph is a path, then its length determines a colour in the set $\{s, t, u\}$ of the corresponding edge of $\Gamma_{\mathcal{M}}$. If a subgraph is a path with a joined triangle, then it corresponds to some c -edge $e = (a, b)$ of $\Gamma_{\mathcal{M}}$. Deleted all vertices of the triangle in the subgraph, we obtain two paths, whose lengths show the numbers of e in the sets of c -edges incident to the vertices a and b , respectively. Thus, knowing the graph $\Gamma_{\mathcal{M}}^*$, one can uniquely restore the graph $\Gamma_{\mathcal{M}}$. \square

Let us estimate the number of vertices of $\Gamma_{\mathcal{M}}^*$, assuming that $\Gamma_{\mathcal{M}}$ has n vertices and m edges. Clearly, any of m edges of the graph $\Gamma_{\mathcal{M}}$ corresponds to some subgraph of the graph $\Gamma_{\mathcal{M}}^*$ that has at most $2n + 18$ vertices. Therefore, the graph $\Gamma_{\mathcal{M}}^*$ has at most $(2n + 18) \cdot m$ vertices and it can be computed in polynomial time with the graph $\Gamma_{\mathcal{M}}$. Notice that $\Gamma_{\mathcal{M}}^*$ can be embedded into the ambient surface. By this fact and Lemma 8.4, the isomorphism problem for four-colour graphs can be reduced in polynomial time to the same problem for simple graphs, embedded into a fixed surface. Hence, the following result is true.

Lemma 8.5. *Isomorphism of four-colour graphs can be recognized in polynomial time.*

Next, we consider the isomorphism problem for the class of equipped graphs. Let $\Upsilon_{\phi^t}^*$ be an equipped graph. We will modify it as follows. We delete all $(\mathcal{M}, \mathcal{L})$ -edges and all $(\mathcal{L}, \mathcal{M})$ -edges (also forget about their associated tu -cycles and st -cycles) and

replace each \mathcal{M} -vertex by the corresponding graph $\Gamma_{\mathcal{M}}$. We also connect every \mathcal{L} -vertex with all vertices of the associated tu -cycle (st -cycle) in the corresponding graph $\Gamma_{\mathcal{M}}$ by edges oriented as $(\mathcal{M}, \mathcal{L})$ (resp. $(\mathcal{L}, \mathcal{M})$), arrange orientation of the cycle in $\Gamma_{\mathcal{M}}$ (preserving colors of its edges) as it was in $(\mathcal{M}, \mathcal{L})$ (resp. $(\mathcal{L}, \mathcal{M})$). The resultant graph Γ_t can be embedded into the ambient surface, as this is true for $\Upsilon_{\phi^t}^*$ and $\Gamma_{\mathcal{M}}$, for any \mathcal{M} -vertex, and by the fact that polygonal regions corresponding to \mathcal{L} -vertices and to their neighbours in tu -cycles and st -cycles have a common border.

We add two degree one neighbours to each \mathcal{A} -vertex, three degree one neighbours to each \mathcal{L} -vertex, four degree one neighbours to each \mathcal{E} -vertex with the “-” weight, and five degree one neighbours to each \mathcal{E} -vertex with the “+” weight. Additionally, in the graph Γ_t , we perform (2,1)-subdivision of any non-coloured oriented edge, (3,1)-subdivision of any oriented s -edge, (4,1)-subdivision of any oriented t -edge, (5,1)-subdivision of any oriented u -edge. Finally, for any \mathcal{M} , we apply subdivisions of all non-oriented edges in $\Gamma_{\mathcal{M}}$ as it was described earlier in the definition of $\Gamma_{\mathcal{M}}^*$. Clearly, the resultant graph Γ_t^* is simple, embeddable into the ambient surface, and it can be computed in polynomial time.

Lemma 8.6. *Equipped graphs $\Upsilon_{\phi^t}^*$ and $\Upsilon_{\phi^{t'}}^*$ are isomorphic if and only if Γ_t^* and $\Gamma_{t'}^*$ are isomorphic.*

Proof. Obviously, the graphs Γ_t^* and $\Gamma_{t'}^*$ can be uniquely constructed by the graphs $\Upsilon_{\phi^t}^*$ and $\Upsilon_{\phi^{t'}}^*$. Let us show that the opposite fact is also true. It will follow the Lemma.

Notice that any vertex of Γ_t^* not belonging to $\mathcal{A} \cup \mathcal{L} \cup \mathcal{E}$ has at most one degree one neighbour. Hence, a vertex of Γ_t^* is an \mathcal{A} -vertex of $\Upsilon_{\phi^t}^*$ iff it has exactly two degree one neighbours; a vertex of Γ_t^* is a \mathcal{L} -vertex of $\Upsilon_{\phi^t}^*$ iff it has exactly three degree one neighbours; a vertex of Γ_t^* is an \mathcal{E} -vertex of the weight “-” of $\Upsilon_{\phi^t}^*$ iff it has exactly four degree one neighbours; a vertex of Γ_t^* is an \mathcal{E} -vertex of the weight “+” of $\Upsilon_{\phi^t}^*$ iff it has exactly five degree one neighbours.

Therefore, one can determine all \mathcal{A} -, \mathcal{L} -, \mathcal{E} -vertices of $\Upsilon_{\phi^t}^*$ in the graph Γ_t^* . Hence, one can determine all $(\mathcal{A}, \mathcal{L})$ -, $(\mathcal{L}, \mathcal{A})$ -, $(\mathcal{L}, \mathcal{E})$ -, and $(\mathcal{E}, \mathcal{L})$ -edges of $\Upsilon_{\phi^t}^*$, knowing their ends and subgraphs of Γ_t^* between them. Considering a ball of radius five centering at a \mathcal{L} -vertex, one can determine orientation of the corresponding $(\mathcal{L}, \mathcal{M})$ -edge or $(\mathcal{M}, \mathcal{L})$ -edge, all vertices of its associated tu -cycle or st -cycle in the graph $\Upsilon_{\phi^t}^*$. Deleted all radius four balls centering at vertices in $\mathcal{A} \cup \mathcal{L} \cup \mathcal{E}$, we obtain the disjunctive union of subgraphs, which are analogues of the graphs of the form $\Gamma_{\mathcal{M}}^*$. By any such a subgraph, one can determine the corresponding graph $\Gamma_{\mathcal{M}}$, associated tu -cycles and st -cycles and their orientation. Thus, knowing the graph Γ_t^* , it is possible to uniquely restore the graph $\Upsilon_{\phi^t}^*$. \square

Recall that the graph Γ_t^* is simple, and it can be computed in polynomial time. By this fact and Lemma 8.6, the isomorphism problem for equipped graphs can be reduced in polynomial time to the same problem for simple graphs embedded into a fixed surface. Hence, Theorem 5.10 is true.

8.2. The Euler characteristic and the surface orientability, a proof of

Theorem 5.11. Now, we consider the problems of calculation of the Euler characteristic of the ambient surface and determining its orientability. For this purpose, we need the notion of bipartite graph.

Definition 8.7. A simple graph is called *bipartite* if the set of its vertices can be partitioned into two parts such that there is not an edge incident to two vertices in the same part.

By König theorem, a simple graph is bipartite if and only if it does not contain any odd cycles [9]. For any simple graph with n vertices and m edges, its bipartiteness can be recognized in $O(n + m)$ time by breath-first search [1]. Hence, by the second part of Theorem 5.9, to check orientability of the ambient surface, we forget about colours of edges of four-colour graphs and apply 2-subdivision to each their edge, to make them simple. Clearly, all of the new graphs are bipartite if and only if the ambient surface is orientable. Thus, orientability of the ambient surface can be tested in linear time on the length of a description of equipped graphs.

By Lemma 5.6, the Euler characteristic of a surface M is equal to $\nu_0 - \nu_1 + \nu_2$, where ν_0, ν_1, ν_2 are the numbers of all tu -, c^* -, and st -cycles of the four-colour graph $\Gamma_{\mathcal{M}}$ of a flow without limit cycles on M , respectively. Deleted all c -edges and all s -edges from $\Gamma_{\mathcal{M}}$, we obtain the disjunctive sum of tu -cycles. Similarly, deleting all c -edges and all u -edges, we obtain the disjunctive sum of st -cycles. Therefore, ν_0 and ν_2 can be computed in time proportional to the sum of the numbers of vertices and edges of $\Gamma_{\mathcal{M}}$. If an edge $e = (a, b)$ of $\Gamma_{\mathcal{M}}$ belongs to some its c^* -cycle C , then the vertex a has an odd or even number in C . Hence, assuming that this number of a is odd (or even) in C , by the number of e in the set of edges incident to b , one can determine an edge in C following the edge e . Hence, each edge of $\Gamma_{\mathcal{M}}$ is contained in at most two c^* -cycles and they can be found in time proportional to the number of edges of $\Gamma_{\mathcal{M}}$. Found all these cycles, one can remove e from $\Gamma_{\mathcal{M}}$ and similarly proceed our search of c^* -cycles in the resultant graph. Clearly, the found cycles will not be met one more time in the future searches of c^* -cycles. Therefore, ν_1 can be computed in time proportional to the square of the number of edges of $\Gamma_{\mathcal{M}}$. Thus, by the first part of Theorem 5.9, the statement of Theorem 5.11 holds.

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