

ERGODIC PROPERTIES OF TAME DYNAMICAL SYSTEMS

A.V. ROMANOV

ABSTRACT. We study the problem on the weak-star decomposability of a topological \mathbb{N}_0 -dynamical system (Ω, φ) , where φ is an endomorphism of a metric compact set Ω , into ergodic components in terms of the associated enveloping semigroups. In the tame case (where the Ellis semigroup $E(\Omega, \varphi)$ consists of B_1 -transformations $\Omega \rightarrow \Omega$), we show that (i) the desired decomposition exists for an appropriate choice of the generalized sequential averaging method; (ii) every sequence of weighted ergodic means for the shift operator $x \rightarrow x \circ \varphi$, $x \in C(\Omega)$, contains a pointwise convergent subsequence. We also discuss the relationship between the statistical properties of (Ω, φ) and the mutual structure of minimal sets and ergodic measures.

1. INTRODUCTION

We are interested in topological \mathbb{N}_0 -dynamical systems, that is, semicascade (Ω, φ) generated by a continuous endomorphism φ of a metric compact set Ω . The aim of the present paper is to develop a common point of view on the following three aspects of the theory of such systems:

- (1) The weak-star convergence of various ergodic means (averages along the orbits of the system) for scalar test functions $x \in X \doteq C(\Omega)$ or Radon measures $\mu \in X^*$. (For the case of Cesàro means, this approach goes back to Kryloff–Bogoliouboff [16] and Oxtoby [18].)
- (2) Relations between minimal sets and ergodic measures.
- (3) The decomposability of the dynamical system (Ω, φ) into irreducible (ergodic) subsystems depending on the choice of the averaging method.

The main results are obtained for the class of tame systems introduced (under a different name) by Köhler [14] and studied in detail in the papers [5, 6, 8, 9, 13]. There are several equivalent definitions of tame dynamical systems; for example, one says that a system (Ω, φ) is tame, $(\Omega, \varphi) \in \mathcal{D}_{\text{tm}}$, if its Ellis semigroup consists of endomorphisms of Ω belonging to the first Baire class. The interest in such objects is due to the relatively simple topology of their enveloping semigroups often combined with a pretty complex phase dynamics. A number of assertions on the convergence of generalized ergodic means for $(\Omega, \varphi) \in \mathcal{D}_{\text{tm}}$ were established in [15, 20], where the paper [15] deals with the more general case in which arbitrary amenable operator semigroups act on X . There are reasons to believe that the tame–untame dichotomy is somehow related to the absence or existence of chaotic phase dynamics. In any case, every untame semicascade on $[0, 1]$ proves to be chaotic in the sense of Li–Yorke [14].

2010 *Mathematics Subject Classification*: Primary 37A30, 47A35; Secondary 20M20
Key words and phrases. ergodic means, tame dynamical system, enveloping semigroup.

We discuss the following properties of weak-star ergodic (see Sec. 2.1) operator nets (or sequences) $\mathcal{V} \subset \mathcal{L}(X^*)$ identified with the corresponding averaging methods.

- (a) The convergence of *all* nets (sequences) \mathcal{V} ; the convergence of *some* ergodic sequences \mathcal{V} .
- (b) The possibility of a statistical description of the behavior of orbits of (Ω, φ) with the use of ergodic measures.

Properties (a) of a discrete semiflow (Ω, φ) are considered in connection with the following dynamic characteristics:

- (i) The orbital subsystems are uniquely ergodic.
- (ii) The supports of ergodic measures are minimal.
- (iii) The minimal subsystems are uniquely ergodic.

Section 3 classifies and strengthens the corresponding results contained in the recent papers [15, 19, 20]. Theorem 3.3 establishes that if $(\Omega, \varphi) \in \mathcal{D}_{\text{tm}}$, then every ergodic sequence \mathcal{V} contains a convergent subsequence; in particular, there exists a convergent subsequence of Cesàro means.

The main results of the paper are gathered in Sec. 4. We show (Theorem 4.5) that a tame dynamical system (Ω, φ) has distinct (depending on the choice of a sequential averaging method) decompositions into ergodic components and describe all such decompositions in terms of some operator semigroup $\mathcal{K}_c \subseteq \mathcal{L}(X^*)$ related to (Ω, φ) . In one interpretation, the decomposability of $(\Omega, \varphi) \in \mathcal{D}_{\text{tm}}$ into ergodic components implies the existence of an ergodic sequence \mathcal{V} such that the asymptotic \mathcal{V} -distributions of all orbits are determined by ergodic measures. Thus, tame semicascades possess property (b).

Section 5 contains a short survey of some typical examples of tame and untame \mathbb{N}_0 -systems. In particular, we present the recent results [17] on the efficient tame–untame dichotomy for affine semicascades on the tori \mathbb{T}^d , $d \geq 1$.

Most of the results of the paper were presented by the author at the 5th Mini-workshop on Operator Theoretic Aspects of Ergodic Theory held at the Eberhardt Karls Universität Tübingen, November 17–18, 2017.

2. PRELIMINARIES

We deal with semicascades (Ω, φ) , where φ is a continuous endomorphism of a metric compact set Ω . Given Ω , we sometimes identify (Ω, φ) with φ when using terms like “minimal endomorphism.” Let $X = C(\Omega)$, let $Ux = x \circ \varphi$, $x \in X$, be the Koopman operator, and let $V = U^* \in \mathcal{L}(X^*)$. We have $\|U\|_{\mathcal{L}(X)} = \|V\|_{\mathcal{L}(X^*)} = 1$. By $\mathcal{P}(\Omega)$ we denote the convex set of Borel probability measures on Ω , which is compact in the w^* -topology of the space X^* , and by X_1 we denote the subspace of X^{**} formed by bounded functions of the first Baire class. Let us present necessary information on ergodic means, enveloping semigroups associated with (Ω, φ) , and tame dynamical systems.

2.1. Ergodic means. We slightly modify the classical definition [2] for the case of a cyclic semigroup $\{V^n\}$ of shift operators and say that a net $\{V_\alpha\} \subseteq \text{co}\{V^n, n \in \mathbb{N}_0\}$ in $\mathcal{L}(X^*)$ is ergodic if

$$(1) \quad (\text{Id} - V)V_\alpha \xrightarrow{w^*Q} 0 : \quad (x, (\text{Id} - V)V_\alpha \mu) \rightarrow 0, \quad x \in X, \quad \mu \in X^*.$$

Here $V_\alpha = U_\alpha^*$, $U_\alpha \in \mathcal{L}(X)$, and the net $\{U_\alpha\} \subseteq \text{co}\{U^n, n \in \mathbb{N}_0\}$ is said to be ergodic as well. If $V_\alpha \xrightarrow{W^*O} Q$, $Q \in \mathcal{L}(X^*)$, then $Q^2 = Q$. In view of the duality $(Ux, \mu) = (x, V\mu)$, $x \in X$, $\mu \in X^*$, the convergence $V_\alpha \xrightarrow{W^*O} Q$ in $\mathcal{L}(X^*)$ is equivalent to the convergence $U_\alpha x \xrightarrow{w^*} Q^*x$ in X^{**} , where $x \in X$ and $Q^* \in \mathcal{L}(X^{**})$. For ergodic sequences $\{U_n\} \subset \mathcal{L}(X)$, this convergence is equivalent to the pointwise convergence of functions, $U_n x \rightarrow \bar{x} \in X_1$. Note that ergodicity is inherited when passing to subnets and subsequences. When speaking of ergodic means in what follows, we most often mean operator nets (or sequences) in $\mathcal{L}(X^*)$.

One can obtain various ergodic sequences $\mathcal{V} = \{V_n\} \subset \mathcal{L}(X^*)$ based on summation methods for numerical sequences with an infinite numerical matrix $S = \{s_{n,k}\}$ satisfying the following conditions:

- (1) $s_{n,k} \geq 0$ and $\sum_{k=0}^{\infty} s_{n,k} = 1$ for each $n \geq 0$.
- (2) Each row of S contains finitely many entries $s_{n,k} > 0$.
- (3) $\lim_{n \rightarrow \infty} (s_{n,0} + \sum_{k=1}^{\infty} |s_{n,k} - s_{n,k-1}|) = 0$.

The sequence of operators $V_n = \sum_{k=0}^{\infty} s_{n,k} V^k$ proves to be ergodic, because $\|(\text{Id} - V)V_n\|_{\mathcal{L}(X^*)} \rightarrow 0$ as $n \rightarrow \infty$. For example, the weights corresponding to appropriate Riesz means have the form

$$s_{n,k} = \frac{p_k}{p_0 + p_1 + \dots + p_n} \quad (0 \leq k \leq n), \quad s_{n,k} = 0 \quad (k > n),$$

where $p_n \geq p_{n+1} > 0$ and $\sum_{n=0}^{\infty} p_n = \infty$. (One has $p_n \equiv 1$ for the Cesàro means.)

2.2. Enveloping semigroups. The Ellis semigroup $E(\Omega, \varphi)$ of a semicascade (Ω, φ) is the closure of the set $\{\varphi^n, n \in \mathbb{N}_0\}$ of transformations in the topology of the direct product Ω^Ω [4]. The Köhler semigroup $\mathcal{K}(\Omega, \varphi)$ is the closure of the set $\mathcal{K}^0 = \{V^n, n \in \mathbb{N}_0\}$ of operators in the W^*O -topology of the space $\mathcal{L}(X^*)$ [14]. Finally, the semigroup $\mathcal{K}_c(\Omega, \varphi)$ is defined as the W^*O -closure of the convex hull $\text{co}\mathcal{K}^0$ [19]. The right-topological semigroups $E(\Omega, \varphi)$, $\mathcal{K}(\Omega, \varphi)$, and $\mathcal{K}_c(\Omega, \varphi)$ are compact. Actually, $\mathcal{K}_c(\Omega, \varphi)$ is the enveloping semigroup of the action $\mathcal{P} \times W \xrightarrow{V} \mathcal{P}$ on $\mathcal{P} = \mathcal{P}(\Omega)$ of the abelian semigroup of polynomials $W = \text{co}\{t^n, n \geq 0\}$ with the usual multiplication.

Let us present some useful properties of the semigroup $\mathcal{K}_c = \mathcal{K}_c(\Omega, \varphi)$ (see [19, Sec. 1]). The nonempty kernel $\text{Ker } \mathcal{K}_c$ (the intersection of two-sided ideals) of the semigroup \mathcal{K}_c consists precisely of unit norm projections $Q \in \mathcal{K}_c$ such that $VQ = Q$, or, equivalently, $QX^* = \text{fix}(V) \doteq \{\mu \in X^* : V\mu = \mu\}$. A net $V_\alpha \subset \text{co}\mathcal{K}^0$ such that $V_\alpha \xrightarrow{W^*O} T \in \mathcal{K}_c$ is ergodic if and only if $T \in \text{Ker } \mathcal{K}_c$. Every element $Q \in \text{Ker } \mathcal{K}_c$ is the limit of some ergodic net of operators; i.e., there exist W^*O -convergent ergodic nets for any $\varphi \in C(\Omega, \Omega)$.

Remark 2.1. According to [19, Theorem 3.2], all ergodic nets (1) converge if and only if $\text{Ker } \mathcal{K}_c$ consists of a single element, which is necessarily the zero element of the semigroup \mathcal{K}_c . The paper [15] uses a slightly different (wider and more traditional) definition of ergodic nets; namely, it is assumed that $V_\alpha \in \overline{\text{co}}\mathcal{K}^0 = \mathcal{K}_c$ in (1). Nevertheless, the condition $\text{card } \text{Ker } \mathcal{K}_c = 1$ also implies the convergence of all nets of this kind [15, Theorem 4.3].

2.3. Tame dynamical systems. Tame \mathbb{N}_0 -systems can be defined as follows in function-theoretic terms (see [14]).

Definition 2.2. One says that a semicascade (Ω, φ) is tame $((\Omega, \varphi) \in \mathcal{D}_{\text{tm}})$ if, for any $x \in X$ and any subsequence $\{n(k)\} \subseteq \mathbb{N}_0$,

$$\inf_a \left\| \sum_{k=0}^{\infty} a_k x_{n(k)} \right\|_X = 0,$$

where $x_{n(k)} = x \circ \varphi^{n(k)}$, the sequences $a \in l^1$ have finitely many nonzero terms, and $\sum_{k=0}^{\infty} |a_k| = 1$.

Essentially, this condition is related to the problem on the isomorphic embeddability of l^1 in Banach spaces, which goes back to Rosenthal [21]. Let Π_b and Π_1 , respectively, be the sets of Borel endomorphisms and first Baire class endomorphisms of Ω . Each of the following properties is equivalent to Definition 2.2:

- (a) $E(\Omega, \varphi)$ is a Fréchet–Urysohn compact set.
- (b) $\text{card } E(\Omega, \varphi) \leq \mathfrak{c}$.
- (c) $\mathcal{K}_c(\Omega, \varphi)$ is a Fréchet–Urysohn compact set.
- (d) $E(\Omega, \varphi) \subset \Pi_1$.
- (e) $E(\Omega, \varphi) \subset \Pi_b$.

Properties (c) and (e) as equivalent definitions of a tame dynamical system arose in [15, Proposition 3.11] and [20, Theorem 3.4], respectively; the remaining properties can be found in [5, 6]. Essentially, the semigroups $E(\Omega, \varphi)$ and $\mathcal{K}_c(\Omega, \varphi)$ in conditions (a) and (c) are sequentially compact. According to (a), a dynamical system is tame if its Ellis semigroup is metrizable. Compact subsystems and direct products of tame systems prove to be tame themselves [5].

3. CONVERGENCE OF ERGODIC MEANS

A criterion for the weak-star convergence of Cesáro means

$$U_n = \frac{1}{n+1}(I + U + \dots + U^n), \quad V_n = \frac{1}{n+1}(I + V + \dots + V^n)$$

was obtained in [10, Theorem 1] and extended to arbitrary ergodic nets $\{U_\alpha\} \subset \mathcal{L}(X)$ and $\{V_\alpha\} \subset \mathcal{L}(X^*)$ in [19, Theorem 1.5]. Namely, the following theorem holds.

Theorem 3.1 (the separation principle). *Let $X_0 = \{x \in X : U_\alpha x \xrightarrow{w^*} \bar{x} \in X^{**}\}$. One has $X_0 = X$ if and only if the limit elements \bar{x} separate $\text{fix}(V)$.*

The latter condition means that for each invariant measure $\mu = V\mu$, $\mu \in X^*$, there exist continuous functions $x_1, x_2 \in X_0$ such that $(\bar{x}_1, \mu) \neq (\bar{x}_2, \mu)$. Further, X_0 is a nonempty closed U -invariant linear subspace of X , $\bar{x} = Tx$, $T \in \mathcal{L}(X_0, X^{**})$, and $\|T\| = 1$. In the case of ergodic sequences, one has $\bar{x} \in X_1$.

We use the following notation for an \mathbb{N}_0 -dynamical system (Ω, φ) : $m \subseteq \Omega$ is a minimal set; $\mu_e \in \mathcal{P}(\Omega)$ is an ergodic measure; $\bar{o}(\omega)$ is the closure of the orbit $o(\omega) = \{\varphi^n \omega, n \geq 0\}$ of an element $\omega \in \Omega$. We are interested in the following dynamic properties of (Ω, φ) and ergodic operator nets $\mathcal{V} \subset \mathcal{L}(X^*)$:

- (single m in \bar{o}):** Every $\bar{o}(\omega)$ contains a single m .
- (supp $\mu_e = m$):** The supports of μ_e are minimal.
- (single μ_e on m):** The minimal subsystems (m, φ) are uniquely ergodic.
- UE(\bar{o}):** The orbital subsystems $(\bar{o}(\omega), \varphi)$ are uniquely ergodic.
- (AEN):** All sets \mathcal{V} converge.

(AES): All sequences \mathcal{V} converge.

(SES): Some sequence \mathcal{V} converges.

Let us single out some general relations between these properties.

Lemma 3.2. *The following implications hold for an arbitrary semicascade (Ω, φ) :*

- (i) (AEN) \Rightarrow UE($\bar{\sigma}$) \Rightarrow (AES).
- (ii) (SES) \Rightarrow (single μ_e on m).
- (iii) UE($\bar{\sigma}$) \Leftrightarrow (single m in $\bar{\sigma}$) + (supp $\mu_e = m$) + (single μ_e on m).

Proof. The implication (AEN) \Rightarrow UE($\bar{\sigma}$) follows from [15, Lemma 5.9] with regard to Remark 2.1. The implication UE($\bar{\sigma}$) \Rightarrow (AES) was established in [19, Theorem 3.2]. Claim (ii) slightly generalizes Theorem 5.4 in [18]. If there exists a convergent ergodic operator sequence $V_n = U_n^*$ and the set $m \subseteq \Omega$ is minimal, then $(U_n x)(\omega) \rightarrow \bar{x}(\omega)$ for any $x \in X$ and $\omega \in m$, and $\bar{x}(\varphi\omega) \equiv \bar{x}(\omega)$ on m . Since the all orbits $o(\omega) \subseteq m$ are dense, it follows that the restriction $\bar{x}|_m$ is either constant or everywhere discontinuous. The latter is impossible for a function of the first Baire class, and the dynamical system (m, φ) is uniquely ergodic according to, say, the separation principle in Theorem 3.1. Claim (iii) is trivial. \square

We see that if some minimal set supports more than one ergodic measure, then there exist no convergent ergodic sequences (even though there always exist convergent ergodic nets). This effect holds for some minimal analytic diffeomorphisms of the torus \mathbb{T}^2 which have uncountably many ergodic measures [11, Corollary 12.6.4].

In the tame case, one can say much more about the convergence of ergodic means.

Theorem 3.3. *The following assertions hold for a tame \mathbb{N}_0 -system (Ω, φ) .*

- (i) *Every ergodic operator net $\{V_\alpha\} \subset \mathcal{L}(X^*)$ contains a convergent ergodic sequence $V_{\alpha(n)}$.*
- (ii) *Every ergodic operator sequence $\{V_n\} \subset \mathcal{L}(X^*)$ contains a convergent ergodic subsequence. In particular, the Cesáro means contain a convergent subsequence.*

Proof. Since $\mathcal{K}_c = \mathcal{K}_c(\Omega, \varphi)$ is compact, we assume without loss of generality that $V_\alpha \xrightarrow{W^*Q} Q$, where $Q \in \text{Ker } \mathcal{K}_c$. For the tame semicascade (Ω, φ) , the topological space \mathcal{K}_c is a Fréchet–Urysohn compact set, and hence the net $\{V_\alpha\}$ contains a sequence $V_{\alpha(n)} \xrightarrow{W^*Q} Q$, and this sequence is ergodic, because $Q \in \text{Ker } \mathcal{K}_c$. Claim (ii) follows from the sequential compactness of \mathcal{K}_c and from the preservation of ergodicity when passing to subsequences. \square

The ergodic sequence $\{V_{\alpha(n)}\}$ in Theorem 3.3 (i) is not a subsequence of the net $\{V_\alpha\}$ in general. Now let us find out how the ergodic and dynamic properties of tame systems are related.

Theorem 3.4. *A tame \mathbb{N}_0 -system (Ω, φ) possesses property (SES), and one has the equivalences*

$$(\text{AEN}) \Leftrightarrow \text{UE}(\bar{\sigma}) \Leftrightarrow (\text{AES}) \Leftrightarrow (\text{single } m \text{ in } \bar{\sigma}).$$

Proof. The existence of convergent operator ergodic sequences for a tame semicascade was established in Theorem 3.3. Assume that all such sequences converge and there exist two distinct elements $Q_1, Q_2 \in \text{Ker } \mathcal{K}_c(\Omega, \varphi)$. By Theorem 3.3 (i), there exist ergodic sequences $V_n^{(1)} \xrightarrow{W^*Q} Q_1$ and $V_n^{(2)} \xrightarrow{W^*Q} Q_2$. Then the mixed sequence

$V_{2n-1} = V_n^{(1)}$, $V_{2n} = V_n^{(2)}$ is ergodic but divergent. Thus, property (AES) implies the relation $\text{card Ker } \mathcal{K}_c = 1$, which is equivalent to property (AEN) by [19, Theorem 3.2], and for tame systems we have (AEN) \Leftrightarrow UE($\bar{\sigma}$) \Leftrightarrow (AES) by Lemma 3.2 (i). Finally, Theorem 4.6 in [20] provides the implication (single m in $\bar{\sigma}$) \Rightarrow (AES), and it remains to note that UE($\bar{\sigma}$) \Rightarrow (single m in $\bar{\sigma}$). \square

Independently, the equivalence (AEN) \Leftrightarrow (single m in $\bar{\sigma}$) for $(\Omega, \varphi) \in \mathcal{D}_{\text{tm}}$ was established in [15, Theorem 5.10]. Theorem 3.4, in particular, ensures the uniquely ergodicity of minimal tame semicascades, a fact obtained for a wider class of tame systems as early as in [9, 13]. This assertion was strengthened in [15, Lemma 5.12]: the uniqueness of a minimal set $m \subseteq \Omega$ implies the uniquely ergodicity of $(\Omega, \varphi) \in \mathcal{D}_{\text{tm}}$. In this connection, it is useful to state the following remark.

Remark 3.5. *The supports of ergodic measures of tame \mathbb{N}_0 -systems are either minimal or contain more than one minimal set.*

On the other hand, it follows from [12, Theorem 3.1] that if an arbitrary semicascade (Ω, φ) has a unique minimal set and the Cesáro means are weakly-star convergent, then there exists either one ergodic measure of uncountably many such measures. The second possibility can indeed be realized [12, Sec. 4].

Remark 3.6. *Even for tame systems, the convergence of one ergodic sequence does not imply the convergence of all other ergodic sequences; i.e., (SES) $\not\Rightarrow$ (AES). Namely, a tame Bernoulli subshift for which the Cesáro means are convergent but the property (single m in $\bar{\sigma}$) is not satisfied was constructed in [15, Example 5.14]. By Theorem 3.4, this semicascade does not satisfy condition (AES) either.*

4. ASYMPTOTIC DISTRIBUTIONS OF ORBITS

Here we transfer some constructions in [16, 18] related to the pointwise convergence on Ω of the Cesáro means $U_n x$ for continuous test functions $x \in X = C(\Omega)$ to arbitrary ergodic sequences. Instead of the individual ergodic theorem (which fails for general averaging methods), we use a priori information on the pointwise convergence of some generalized ergodic means. Our main task is to establish the possibility of decomposition of a tame dynamical system into irreducible (ergodic) components. By $\mathcal{P}_{\text{in}}(\Omega)$ and $\mathcal{P}_e(\Omega)$ we denote the subsets of φ -invariant and φ -ergodic measures, respectively, in $\mathcal{P}(\Omega)$, and by X_1 we denote the set of bounded scalar functions of the first Baire class on Ω . A set $\Theta \subseteq \Omega$ is *bi-invariant* if $\varphi^{-1}\Theta = \Theta$. Further, let $D(\Omega)$ be the set of Dirac measures δ_ω on Ω , and let $\mathcal{K}_c = \mathcal{K}_c(\Omega, \varphi) \subseteq \mathcal{L}(X^*)$ be the operator semigroup defined in Sec. 2.2.

We assume the existence of a convergent ergodic operator sequence $\mathcal{V} = \{V_n\} \subset \mathcal{L}(X^*)$, $V_n \xrightarrow{W^*Q} Q \in \text{Ker } \mathcal{K}_c$, and write this convergence briefly as $\mathcal{V} \rightarrow Q$. In this case, $(U_n x)(\omega) \rightarrow \bar{x}(\omega)$ for the dual ergodic sequence $\{U_n\} \subset \mathcal{L}(X)$, $U_n^* = V_n$, and all $\omega \in \Omega$ and $x \in X$; furthermore, the function $\bar{x} \in X_1$ is invariant ($\bar{x} \circ \varphi = \bar{x}$), and to each point $\omega \in \Omega$ there corresponds a measure $\mu_\omega = Q\delta_\omega \in \mathcal{P}_{\text{in}}(\Omega)$, $\bar{x}(\omega) = (x, \mu_\omega)$, determining the asymptotic \mathcal{V} -distribution of the orbit $o(\omega)$. Essentially, this means that $V_n \delta_\omega \xrightarrow{W^*} \mu_\omega$. A linear projection Q in X^* induces a mapping $\Psi_{\mathcal{V}} : \Omega \rightarrow \mathcal{P}_{\text{in}}(\Omega)$ of the first Baire class. The notation $\Psi_{\mathcal{V}}$ is convenient, even though this mapping is completely determined by the limit element Q of the sequence \mathcal{V} .

Lemma 4.1. *If an ergodic sequence $\mathcal{V} = \{V_n\}$ is convergent, then $\Psi_{\mathcal{V}}\Omega \supseteq \mathcal{P}_e(\Omega)$.*

In other words, for any convergent ergodic sequence \mathcal{V} , every ergodic measure determines the asymptotic \mathcal{V} -distribution of some orbit.

Proof. Let $\mu \in \mathcal{P}_e(\Omega)$, $x \in X$, $U_n^* = V_n$ and $c(x) = (x, \mu)$. Under the assumptions of the lemma, $(U_n x, \mu) = (x, V_n \mu) = (x, \mu)$, the dual sequence $(U_n x)(\omega)$ converges to $\bar{x}(\omega)$ for all $\omega \in \Omega$, $\bar{x} = \bar{x} \circ \varphi$, and $(\bar{x}, \mu) = c(x)$ by the Lebesgue theorem. An argument like [3, Proposition 7.15, (i) \Rightarrow (iv)] shows that, by virtue of the ergodicity of μ , the bounded invariant function $\bar{x} \in X_1$ is identically equal to a constant $c(x)$ on some Borel set $\Theta_{x,\mu} \subseteq \Omega$ of full μ -measure. Consequently, for the points $\omega \in \Theta_{x,\mu}$ we have

$$(2) \quad (x, V_n \delta_\omega) = (U_n x, \delta_\omega) \rightarrow (\bar{x}, \delta_\omega) = (\bar{x}, \mu) = (x, \mu).$$

Now we take x from an arbitrary countable set Y everywhere dense in X and obtain relations (2) for $x \in Y$ and $\omega \in \Theta_\mu$, where $\Theta_\mu = \bigcap_{x \in Y} \Theta_{x,\mu}$ and $\mu(\Theta_\mu) = 1$. Since $\|V_n\| \leq 1$ for all $n \in \mathbb{N}_0$, it follows that the same is true for any $x \in X$ and $\omega \in \Theta_\mu$.

Thus, $V_n \delta_\omega \xrightarrow{w^*} \mu$ for $\omega \in \Theta_\mu$. \square

For a convergent ergodic sequence $\mathcal{V} = \{V_n\}$, we set

$$\Omega_{\mathcal{V}} = \{\omega \in \Omega : \mu_\omega \in \mathcal{P}_e(\Omega)\},$$

where $V_n \delta_\omega \xrightarrow{w^*} \mu_\omega$; then, for *ergodic* measures μ , the subsets $\Omega_{\mu,\mathcal{V}} = \Psi_{\mathcal{V}}^{-1} \mu$ form partitions of $\Omega_{\mathcal{V}}$ into \mathcal{V} -quasi-ergodic components. The sets $\Omega_{\mathcal{V}}$ and $\Omega_{\mu,\mathcal{V}}$ are bi-invariant. They are Borel sets, which follows by a purely topological reasoning [18, pp. 119–120] not related in any way to the specific features of Cesàro averaging. Since $\Omega_{\mu,\mathcal{V}} \supseteq \Theta_\mu$, where Θ_μ is the set in the proof of Lemma 4.1, we obtain the following assertion.

Corollary 4.2. *If an ergodic sequence \mathcal{V} is convergent, then to each ergodic measure μ there corresponds a Borel \mathcal{V} -quasi-ergodic set $\Omega_{\mu,\mathcal{V}}$ of full μ -measure.*

Now let us discuss the main topic of the present paper.

Definition 4.3. We say that an \mathbb{N}_0 -system (Ω, φ) is *ergodically decomposable* if there exists a convergent operator ergodic sequence \mathcal{V} such that $\Omega_{\mathcal{V}} = \Omega$, or, equivalently, $\Psi_{\mathcal{V}} \Omega = \mathcal{P}_e(\Omega)$.

In this case, the *topological dynamical system* (Ω, φ) essentially admits a decomposition into ergodic subsystems $(\Omega_{\mu,\mathcal{V}}, \varphi)$, $\mu \in \mathcal{P}_e(\Omega)$. Here $\mathcal{V} \rightarrow Q \in \text{Ker } \mathcal{K}_c$, and to each continuous function $x \in X$ there corresponds a function $\bar{x} = Q^* x \in X_1$ taking a constant value $(\bar{x}, \mu) = (x, \mu)$ on each quasi-ergodic set $\Omega_{\mu,\mathcal{V}}$. Thus, for each measure $\mu \in \mathcal{P}_e(\Omega)$, the *measure-preserving dynamical system* $(\Omega_{\mu,\mathcal{V}}, \varphi)$ is ergodic with respect to μ in the standard sense [3, Definition 6.18]. In the interpretation given in [11, Sec. 4.1], the ergodic decomposability of the semicascade (Ω, φ) means that the asymptotic \mathcal{V} -distributions of all orbits are determined by ergodic measures. Note also that the mapping $\Psi_{\mathcal{V}} : \Omega \rightarrow \mathcal{P}_e(\Omega)$ inducing the decomposition of (Ω, φ) is a sequential pointwise limit of continuous mappings and hence belongs to the first Baire class, so that its points of continuity form a dense G_δ -set in Ω .

It turns out that the ergodic decomposability on an \mathbb{N}_0 -dynamical system is related to the existence of operator ergodic sequences converging to the extreme points of the kernel of the semigroup $\mathcal{K}_c = \mathcal{K}_c(\Omega, \varphi)$.

Proposition 4.4. *If an ergodic sequence \mathcal{V} converges to $Q \in \text{ex Ker } \mathcal{K}_c$, then the dynamical system (Ω, φ) is ergodically decomposable.*

Proof. Let $\mathcal{V} = \{V_n\}$. By [20, Proposition 2.10], one has $Q : D(\Omega) \rightarrow \mathcal{P}_e(\Omega)$, and since $V_n \xrightarrow{W^*O} Q$, it follows that $V_n \delta_\omega \xrightarrow{W^*} \mu \in \mathcal{P}_e(\Omega)$ for each point $\omega \in \Omega$. Thus, $\Omega_{\mathcal{V}} = \Omega$, and the system (Ω, φ) is ergodically decomposable. \square

Theorem 4.5 (main theorem). *Tame \mathbb{N}_0 -systems (Ω, φ) are ergodically decomposable.*

Proof. An arbitrary projection $Q \in \text{ex Ker } \mathcal{K}_c$ is the W^*O -limit of some ergodic net $\{V_\alpha\} \subset \mathcal{L}(X^*)$. By Theorem 3.3 (i), there exists an ergodic operator sequence $V_{\alpha(n)} \xrightarrow{W^*O} Q$. The desired assertion with $\mathcal{V} = \{V_{\alpha(n)}\}$ follows from Proposition 4.4. \square

Now let us describe the structure of all possible decompositions of tame systems into ergodic components.

Lemma 4.6. *For a tame \mathbb{N}_0 -system (Ω, φ) , the operators $T \in \mathcal{K}_c(\Omega, \varphi)$ are determined by their values on Dirac measures.*

Proof. Under the assumptions of the lemma, the semigroup \mathcal{K}_c is a Fréchet–Urysohn compact set, and hence for each $T \in \mathcal{K}_c$ there exists a sequence $\{V_n\} \subseteq \text{co}\{V^n, n \geq 0\}$ converging to T in the W^*O -topology of the space $\mathcal{L}(X^*)$. If $U_n^* = V_n$, $x \in X$ and $\omega \in \Omega$, then $(x, V_n \delta_\omega) = (U_n x)(\omega)$ and $(U_n x)(\omega) \rightarrow \bar{x}(\omega)$. Here $\bar{x} \in X_1$, and $(x, V_n \mu) = (U_n x, \mu) \rightarrow (\bar{x}, \mu) = (x, T\mu)$ for each measure $\mu \in \mathcal{P}(\Omega)$ by the Lebesgue theorem. At the same time $\bar{x}(\omega) = (x, T\delta_\omega)$ for $\omega \in \Omega$, hence the operator T is completely defined by its values on $D(\Omega)$. \square

Lemma 4.6 is a refinement of a similar assertion [20, Theorem 3.5, (a1) \Rightarrow (a4)] where instead of the condition $(\Omega, \varphi) \in \mathcal{D}_{\text{tm}}$ the enveloping semigroup $E(\Omega, \varphi)$ is required to be metrizable.

Corollary 4.7. *If $(\Omega, \varphi) \in \mathcal{D}_{\text{tm}}$ and $Q_1|_{D(\Omega)} = Q_2|_{D(\Omega)}$ for $Q_1, Q_2 \in \text{Ker } \mathcal{K}_c$, then $Q_1 = Q_2$.*

Hence we readily find that in the tame case the condition $Q \in \text{ex Ker } \mathcal{K}_c$ is not only sufficient but also necessary for the relation $Q : D(\Omega) \rightarrow \mathcal{P}_e(\Omega)$, $Q \in \text{Ker } \mathcal{K}_c$ to hold. For $(\Omega, \varphi) \in \mathcal{D}_{\text{tm}}$, it is natural to define quasi-ergodic sets based on elements $Q \in \text{ex Ker } \mathcal{K}_c$ rather than convergent ergodic sequences \mathcal{V} . Namely, we set

$$\Omega_{\mu, Q} = \{\omega \in \Omega : Q\delta_\omega = \mu\}, \quad \mu \in \mathcal{P}_e(\Omega).$$

We see that Borel bi-invariant quasi-ergodic sets $\Omega_{\mu, Q}$ of full μ -measure form a partition Φ_Q of the phase space Ω . The set Λ of all ergodic sequences $\mathcal{V} \rightarrow Q$ splits into disjoint classes Λ_Q corresponding to distinct Q . The elements $\mathcal{V} \in \Lambda_Q$ define a relationship between the dynamics of the semicascade (Ω, φ) and the ergodic measures; namely, the asymptotic \mathcal{V} -distribution of each orbit $o(\omega)$ is determined by the measure $\mu = Q\delta_\omega$. There exists a one-to-one (by virtue of Corollary 4.7) correspondence between the projections $Q \in \text{ex Ker } \mathcal{K}_c$, the partitions Φ_Q of the phase space Ω into quasi-ergodic sets, and the partitions Λ_Q of Λ converging to the extreme points of the kernel of the semigroup $\mathcal{K}_c(\Omega, \varphi)$ of ergodic operator sequences.

5. SUPPLEMENT

Here we consider a number of typical examples of tame and untame \mathbb{N}_0 -dynamical systems. Let $I = [0, 1]$.

- (1) According to [7, Proposition 10.5] and [5, Sec. 9], every semicascade generated by a homeomorphism of I or \mathbb{S}^1 has a metrizable Ellis semigroup and hence is tame.
- (2) The left Bernoulli shift on the set $\Omega = \{0, 1\}^{\mathbb{N}_0}$ of sequences $\omega_0, \omega_1, \dots$ with the standard metric $\rho(\omega, \nu) = (1 + \min\{k : \omega_k \neq \nu_k\})^{-1}$ generates a untame \mathbb{N}_0 -system (Ω, φ) , which, however, admits tame subsystems (Θ, φ) . Here is an elegant description of these systems: every infinite set $L \subseteq \mathbb{N}_0$ contains an infinite subset $K \subseteq L$ such that the projection $\pi_K(\Theta)$ is a countable subset of $\{0, 1\}^K$ [8, Theorem 4.7].
- (3) The set of periodic points of the semicascade (I, φ) in the example in [1, pp. 147–149] is nonclosed, and every orbit $o(\omega), \omega \in I$, is either eventually periodic ($\varphi^k \omega = \varphi^{k+p} \omega$ for some $k \geq 0, p \geq 1$) or its limit points fill the classical Cantor set. This semicascade proves to be tame [14, Example 5.8 (c)].
- (4) On the other hand, every semicascade (I, φ) admitting periodic points with period that is not a power of 2 is not tame [14, Example 5.8 (e)].
- (5) A slight modification of the argument in [5, p. 2354] shows that the projective action of an arbitrary invertible operator $T \in GL(n, \mathbb{R}^n)$, $n \geq 2$, induces a tame semicascade on the sphere \mathbb{S}^{n-1} .

Examples (3) and (5) show that tame systems can have a rather nontrivial phase dynamics.

Recently, using function–theoretic argument, Lebedev obtained a criteria that allows to distinguish tame and untame affine endomorphisms of the torus $\varphi : \omega \rightarrow A\omega + b$ ($\omega \in \mathbb{T}^d$, $d \geq 1$) with an integer matrix A and an arbitrary shift $b \in \mathbb{T}^d$. If $\det A = \pm 1$, then φ is an automorphism.

Theorem 5.1 (Lebedev [17]). *A semicascade (\mathbb{T}^d, φ) is tame if and only if $A^k = A^l$ for some $k, l \in \mathbb{N}_0$, $k \neq l$.*

If $\det A = \pm 1$, then the conclusion of the theorem is $A^k = \text{Id}$. In particular, the automorphism $\varphi : (\omega_1, \omega_2) \rightarrow (\omega_1 + \omega_2, \omega_2)$ of the torus \mathbb{T}^2 is not tame.

Acknowledgements. The author is thankful to V. Lebedev, H. Kreidler, and M. Megrelishvili for ideas, helpful suggestions, and inspiring discussions.

REFERENCES

- [1] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lect. Notes in Math., vol. 1513, Berlin: Springer-Verlag, 1992.
- [2] W. F. Eberlein, “Abstract ergodic theorems and weak almost periodic functions,” *Trans. Amer. Math. Soc.* **67**:1 (1949), 217–240.
- [3] T. Eisner, B. Farkas, M. Haase, and R. Nagel, *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics, vol. 272, Cham: Springer, 2015.
- [4] R. Ellis, *Lectures on Topological Dynamics*, New York: Benjamin, 1969.
- [5] E. Glasner, “Enveloping semigroups in topological dynamics,” *Topology Appl.* **154**:11 (2007), 2344–2363.
- [6] E. Glasner and M. Megrelishvili, “Representations of dynamical systems on Banach spaces not containing l_1 ,” *Trans. Amer. Math. Soc.*, **364**:12 (2012), 6395–6424.
- [7] E. Glasner and M. Megrelishvili, “Hereditarily non-sensitive dynamical systems and linear representations,” *Colloq. Math.* **104**:2 (2006), 223–283.

- [8] E. Glasner and M. Megrelishvili, “More on tame dynamical systems,” in *Ergodic Theory and Dynamical Systems in their Interactions with Arithmetics and Combinatorics*, Ed. by S. Ferenczi, J. Kulaga-Przymus, M. Lemanczyk, Lect. Notes in Math., vol. 2213, Cham: Springer, 2018, 351–392.
- [9] W. Huang, “Tame systems and scrambled pairs under an abelian group action,” *Ergod. Theory Dynam. Systems* **26**:5 (2006), 1549–1567.
- [10] A. Iwanik, “On pointwise convergence of Cesàro means and separation properties for Markov operators on $C(X)$,” *Bull. Acad. Polon. Sci., Ser. Sci. Math.* **29**:9–10 (1981), 515–520.
- [11] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia Math. Appl., vol. 54, Cambridge: Cambridge Univ. Press, 1995.
- [12] Y. Katznelson and B. Weiss, “When all points are recurrent/generic,” in *Ergodic Theory and Dynamical Systems I*, Ed. by A. Katok, Boston: Birkhäuser, 1981, 195–210.
- [13] D. Kerr and H. Li, “Independence in topological and C^* -dynamics,” *Math. Ann.* **338**:4 (2007), 869–926.
- [14] A. Köhler, “Enveloping semigroups for flows,” *Proc. Roy. Irish. Acad.* **95A**:2 (1995), 179–191.
- [15] H. Kreidler, “Compact operator semigroups applied to dynamical systems,” *Semigroup Forum*, Published online 18 July 2018, 1–25; arXiv:1703.05014v3.
- [16] N. Kryloff and N. Bogoliouboff, “La theorie generale de la mesure dans son application l’etude des systemes dynamiques de la mecanique non lineaire,” *Ann. of Math.* **38**:1 (1937), 65–113.
- [17] Vladimir Lebedev, “Tame semicascades and cascades generated by affine self-mappings of the d -torus”, arXiv:1806.06386v2.
- [18] J. C. Oxtoby, “Ergodic sets”, *Bull. Amer. Math. Soc.* **58**:2 (1952), 116–136.
- [19] A. V. Romanov, “Weak* convergence of operator means,” *Izv. Math.* **75**:6 (2011), 1165–1183.
- [20] A. V. Romanov, “Ergodic properties of discrete dynamical systems and enveloping semigroups,” *Ergod. Theory Dynam. Systems* **36**:1 (2016), 198–214.
- [21] H. P. Rosenthal, “A characterization of Banach spaces containing l^1 ,” *Proc. Natl. Acad. Sci. USA* **71**:6 (1974), 2411–2413.

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, MOSCOW, RUSSIA
E-mail address: av.romanov@hse.ru