Studies in Systems, Decision and Control 157

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Control of Discrete-Time Descriptor Systems

An Anisotropy-Based Approach

Springer

Studies in Systems, Decision and Control

Volume 157

Series editor

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ISSN 2198-4182 ISSN 2198-4190 (electronic)
Studies in Systems, Decision and Control
ISBN 978-3-319-78478-6 ISBN 978-3-319-78479-3 (eBook)
https://doi.org/10.1007/978-3-319-78479-3

Library of Congress Control Number: 2018935859

Mathematics Subject Classification (2010): 93E20, 93E25, 93C05, 93C55

Printed on acid-free paper

This Springer imprint is published by the registered company Springer International Publishing AG part of Springer Nature
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

Descriptor linear systems theory is an important part of control systems theory and has attracted much attention in the last decades. Many researchers pay a great attention to solving analysis and control design of descriptor systems. In the last 30 years, differential-algebraic equations have become a widely accepted tool for the modeling and simulation of constrained dynamical systems in numerous applications, such as mechanical multibody systems, electrical circuit simulation, chemical engineering, control theory, fluid dynamics, and many other areas.

Problems of sensitivity reduction or external disturbance attenuation are well-known in modern control theory. The mostly studied ones are LQG/$\mathscr{H}_2$ and $\mathscr{H}_\infty$ control problems. In LQG/$\mathscr{H}_2$ optimal theory the Gaussian white noise sequence is considered as the input disturbance. In discrete-time $\mathscr{H}_\infty$ control approach input disturbances are considered as sequences with limited power, i.e. the sequences are square summable. The discrete-time LQG/$\mathscr{H}_2$ and $\mathscr{H}_\infty$ control problems were successfully generalized on the class of descriptor systems. Anisotropy-based approach deals with the stationary random Gaussian signals with known mean anisotropy level $a \geq 0$, which has a sense of "spectral color" of the signal. Similar to $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms, anisotropic norm defines a performance index of the system from the input to output. The key feature of anisotropy-based approach is that anisotropic norm of the system lies between the scaled $\mathscr{H}_2$-norm and $\mathscr{H}_\infty$-norm. Anisotropy-based control theory allows to develop unified theoretical framework to performance analysis and control synthesis, which covers popular $\mathscr{H}_2$- and $\mathscr{H}_\infty$-approaches as limiting cases. From the practical point of view additional information about the input disturbance allows to expend less energy for control, and, at the same time, remove strong assumption that the input disturbance is white noise sequence.

This book addresses the original research on anisotropy-based analysis and control design theory for discrete-time descriptor systems. The book consists of seven chapters. The first chapter illustrates a variety of practical applications of descriptor systems. The aim of the second chapter is to provide a background material on linear discrete-time descriptor systems. The rest part of the book consists of authors' results on analysis and control of discrete-time descriptor systems in presence of colored noise.

First of all, we are grateful to a number of people who helped in reviewing and improving of the book, especially, Alexander B. Kurzhanski, Alexander S. Poznyak, and Boris T. Polyak for useful comments and remarks.

We also would like to acknowledge the financial support kindly provided by the Government of the Russian Federation through ITMO Postdoctoral Fellowship program (grant 074-U01) and the Russian Foundation for Basic Research (grant 16-38-00216, grant 17-08-00185).

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December 2017

Contents

Notations and Abbreviations

Symbols and Notations

$\mathbb{Z}$	Set of all integers
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}^n$	Set of all real vectors of dimension n
$\mathbb{R}^{m\times n}$	Set of all real matrices of dimension $m \times n$
$\mathbb{C}$	Set of all complex numbers
$\deg(f(x))$	Degree of the polynomial, $\deg(f(x)) = n$ if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, $a_n \neq 0$
$\det(A)$	Determinant of matrix A
$\mathrm{rank}(A)$	Rank of matrix A; if A is square, then $\mathrm{rank}(A)$ is equal to the number of nonzero eigenvalues of matrix A
$\mathrm{Tr}(A)$	Trace of matrix A: $\mathrm{Tr}(A) = \sum_j a_{jj} = \sum_j \lambda_j(A)$
A^{T}	Transpose of matrix A: $A^{\mathrm{T}} = (a_{ij})^{\mathrm{T}} = (a_{ji})$
I_n	Identity matrix of order $n \times n$
A^{-1}	Inverse matrix of matrix A: $AA^{-1} = A^{-1}A = I$
$\mathfrak{Ker}(W)$	Kernel of linear mapping W: $\mathfrak{Ker}(W) = \{x : x \in X, W(x) = 0\}$
$\mathfrak{Im}(W)$	Image of linear mapping W: $\Im\mathfrak{m}(W) = \{y : y = W(x), \forall x \in V\}$
$\mathfrak{span}(W)$	Linear span of linear mapping W
$\mathrm{diag}(A_i)$	Diagonal $(mn) \times (mn)$-matrix with elements A_i of dimension $m \times m$ on the main diagonal, $i = \overline{1, n}$
$\oplus$	Direct sum
$F^*(z)$	Transposed complex conjugate of $F(z)$: $F^*(z) = \overline{F}^{\mathrm{T}}(z)$
$\widehat{G}(\omega)$	Boundary value of transfer function $G(z)$: $\widehat{G}(\omega) = lim_{r\to 1-0}\, G(re^{i\omega})$
$\mathrm{Re}\lambda$	Real part of a complex number λ
$\\|G\\|_2$	$\mathscr{H}_2$ -norm of transfer function G
$\\|G\\|_\infty$	$\mathscr{H}_\infty$ -norm of transfer function G
$\\|Y\\|_{\mathscr{P}}$	Power norm of the signal Y
$\\|\\|G\\|\\|_a$	a-Anisotropic norm of transfer function G

$\Vert M \Vert$	Frobenius norm of matrix: $(\mathrm{Tr}\,(M^*M))^{1/2}$
$\mathbf{E}(W)$	Mathematical expectation of a random vector W
$\lambda(A)$	Range of matrix A: $\lambda(A) = \{\lambda_i : \det(\lambda_i I - A) = 0, i = \overline{1, n}\}$
$\rho(A)$	Spectral radius of matrix A: $\rho(A) = \max_i \vert\lambda(A)\vert$
A^+	Pseudo-inverse (Moore-Penrose inverse) matrix of matrix A, that satisfies the equations $AA^+A = A$, $A^+AA^+ = A^+$, $(AA^+)^T = AA^+$, $(A^+A)^T = A^+A$
$\{u(k)\}_{k\in\mathbb{Z}}$	Numeric sequence in the form $(\ldots, u(-k), \ldots, u(-1), u(0), u(1), \ldots, u(k), \ldots)$
$\mathbf{D}(\mathrm{P}\Vert M)$	Relative entropy (Kullback-Leibler information divergence) from probability measure P to probability measure M
$\mathbf{A}(w)$	Anisotropy of a random vector $w \in \mathbb{R}^m$
$\overline{\mathbf{A}}(W)$	Mean anisotropy of a random sequence $W = \{w(k)\}_{k\in\mathbb{Z}}$
$\overline{\mathbf{A}}(G)$	Mean anisotropy of a random sequence W, generated from the Gaussian white noise sequence V by the shaping filter G
Z^*	Hermitian conjugate of the matrix $Z = [z_{ij}] \in \mathbb{C}^{m\times n}$: $Z^* = [z_{ji}^*] \in \mathbb{C}^{n\times m}$
$\overline{\sigma}(A)$	Maximal singular value of the matrix A: $\overline{\sigma}(A) = \sqrt{\rho(A^*A)}$
$\mathrm{sym}(A)$	Symmetrization of matrix A: $\mathrm{sym}\,(A) = A + A^{\mathrm{T}}$
$(\mathbf{Z})[\mathbf{f}(\mathbf{k})]$	Laplace z-transform of a sequence $\{f(k)\}_{k\in\mathbb{Z}}$
∇	Laplace operator $\nabla = \frac{\partial}{\partial x}\overline{i} + \frac{\partial}{\partial y}\overline{j} + \frac{\partial}{\partial z}\overline{k}$

Abbreviations

BRL	Bounded real lemma
DAE	Differential-algebraic equations
FI	Full information
GDARE	Generalized discrete-time algebraic Riccati equation
GDARI	Generalized discrete-time algebraic Riccati inequality
LDTI	Linear discrete-time time-invariant (systems)
LFT	Linear fractional transformation
LMI	Linear matrix inequality
MOL	Method of lines
ODE	Ordinary differential equations
PDE	Partial differential equations
PDF	Probability density function
RMS	Root mean-square (gain)
SF	State feedback
SVD	Singular value decomposition

Introduction

A process has to have qualitative and quantitative description as prerequisite for solving control problems. The most common way of description is mathematical model of control object or the workflow.

Mathematical models of control systems are designed based on well-known laws of nature: physical, chemical, biological laws, etc.

These laws are usually described by differential and algebraic equations based on laws like Newton's second law or Kirchhoff's law. Plant models are normally done by differential or difference equations. However, in many cases, such description is not sufficient.

Attempts to represent systems only by differential or difference equations might result in a loss of relevant information and arrive to a description of the system in abstract variables, so called phase variables. It can create problems for practical realizations of controllers and diminish the quality of real object's control.

During design of mathematical model with real physical values, a designer has to account for the fact that system description could contain not only differential equations, but also algebraic constraints and relations.

As a rule, a system of algebraic-differential equations, describing control system, cannot be solved for the first derivative. This situation explains a new class of systems, called algebraic-differential systems. Algebraic-differential systems [1] are also called singular systems [2–4], generalized state-space systems (or generalized systems) [5], implicit systems [6, 7] or descriptor systems [8, 9]. The origin of descriptor systems theory dates back to the works of P. Dirac on generalized Hamiltonian systems [10–12]. In modern science the main idea discussed in these works, is called the differentiation index of semi-explicit descriptor systems.

A geometric method of studying the so-called constrained systems covered in the works of Dirac. This method has found its application in mechanics [13–21].

Mechanical systems, represented as descriptor systems, have become a subject of extensive research [22–24].

Further development of the theory on parametrized sets of bilinear form can be found in works of K. Weierstrass and L. Kronecker [25, 26]. F. R. Gantmacher used matrix pencils [27] to analyze matrices of linear normal systems with possible degeneracies of the main matrix coefficient.

A large number of works on descriptor systems also applies to the theory of electrical circuits. Presence of differential and algebraic equations in such systems involves a combination of differential equations describing the behavior of reactive elements, and algebraic relations, based on Kirchhoff's laws and characteristics of elements [28–31].

Mathematical theory of differential-algebraic systems began to develop in the 1970s independently in various fields of technology. We can mention the works of Gear [32], Takens [33], as well as monographs by Campbell [2, 3] and Petzold [34], released in the early 1980s. In these works, the main attention was focused on numerical aspects of descriptor systems modeling. Currently, much attention is paid to differential-algebraic systems in partial derivatives [35–37] and stochastic descriptor systems [38].

Descriptor systems have found their application in modeling the motion of aircrafts [39], chemical processes [40], circuit technique [8, 9], economic systems [41], description of interconnected systems of high order [42], technical systems [43], energy systems [44] and robotics [45].

Descriptor systems have some specific differences from systems described exclusively by differential or difference equations, which we will call normal systems.

Descriptor systems are characterized by the following properties [46, 47].

- The transfer function of a descriptor system may not necessarily be strictly proper.
- For arbitrary bounded initial conditions generalized functions:
  - can be in the solutions of algebraic-differential equations (impulsive behavior),
  - can depend on future for algebraic-difference equations (noncausal behavior).
- A solution of a linear algebraic-differential equation typically contains three components:
  - limited dynamic components, corresponding to the differential equations;
  - non-dynamic components, corresponding to the algebraic equations;
  - unlimited dynamic components from the set of generalized functions, the presence of which depends on the smoothness of the input signal and on initial conditions.

Study of descriptor systems is promising from the fundamental research point of view as well as for practical applications. Significant differences of descriptor systems from normal systems demanded development and generalization of mathematical tools.

A significant number of fundamental concepts and results for nominal systems have been successfully generalized for descriptor systems:

- solvability of algebraic-differential equations, study of controllability and observability [4];
- canonical, equivalent forms and representations of descriptor systems [48–50];
- minimal realizations [52–54];
- equivalence of systems [4, 47, 55, 56];
- regularity and regularization [57–62];
- stability and stabilization [63–66];
- modal control [4, 67–70];
- linear-quadratic optimal control [46, 71, 72];
- design of observers and filtering [73–77];
- Lyapunov's theorems and equations [78–81];
- model reduction [82, 83];
- $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control [66, 84–87].

Let us consider for a moment the tasks of LQG/$\mathscr{H}_2$ and $\mathscr{H}_\infty$ control. In this case, the control system is designed, assuming that some external disturbance is influencing the system.

Theory of design for linear-quadratic Gaussian controllers appeared at the end of 50s in 20th century and it is associated with the name of R. Kalman.

This theory provided a powerful tool for multidimensional control systems design with quadratic quality criterion [88].

The algorithm of control was designed with the assumption that systems are under disturbances in the form of Gaussian white noise.

This assumption reduced the design problem to the problem of minimizing the quality criterion, that is quadratic for control and state.

This problem can be reduced to the problem of $\mathscr{H}_2$-optimization, where $\mathscr{H}_2$-norm of the system's transfer function supports the quality criterion.

The most significant disadvantage of this approach is loss of stability of the system under a small perturbation in model description. This deficiency was investigated in [89].

For the closed-loop system, the design problem for stabilizing controllers that minimize $\mathscr{H}_\infty$-norm of the transfer function was stated and solved in [90], this problem got its further development in works [91–95].

Such a problem belongs to optimal control problems, where $\mathscr{H}_\infty$-norm of the transfer function is the quality criterion for the closed-loop system. Another important factor of the resulting control law in real life application is the degree of conservatism, it stands for the energy cost actuators of the control object, implementing the law. It is known that $\mathscr{H}_2$-controllers are not robust against the intensity of input disturbance [89], while $\mathscr{H}_\infty$-controllers are too conservative.

Among approaches allowing reduction of conservatism of controllers, is the approach in which the system is subjected to random disturbances with imprecisely known probability characteristics.

Additional information about the input disturbance allows to expend less energy for control, and, at the same time, remove the strong assumption that the input disturbance is white noise sequence.

This concept is related to an application of information-theoretical quality criteria, and it is called stochastic $\mathscr{H}_\infty$-optimization.

One of such information criteria is a stochastic norm of the closed-loop system. Stochastic norm is induced by power norm of random signals that belong to specific class of probability distributions. Anisotropic norm is a special case of a stochastic norm. This norm is used when input disturbance is a Gaussian random sequence with zero mean and bounded mean anisotropy [96, 97]. The latter is a measure of correlation for random vector components in a sequence or, in other words, a measure of random sequence deviation from Gaussian white noise sequence also known as “spectral color”.

Minimization of anisotropic norm of the transfer function for a closed-loop system in anisotropy-based controllers was first stated in [97] and solved in [98]. Paper [99] shows that a-anisotropic norm as a function of its parameter $a \geq 0$, has $\mathscr{H}_2$- and $\mathscr{H}_\infty$-norms as its limiting cases. This implies that design problem of anisotropy-based controllers includes classical problems of $\mathscr{H}_2$- and $\mathscr{H}_\infty$-optimization as limiting cases.

[100] shows that $\mathscr{H}_2$- and $\mathscr{H}_\infty$-controllers are limiting cases of anisotropy-based controller. This monograph introduces reader to the solutions of the anisotropy-based analysis and design for linear stationary descriptor systems.

The content of the book: Chapter 1 deals extensively with practical applications of descriptor systems. Chapter 2 is dedicated to theory basics of discrete-time descriptor systems. Chapter 3 is dedicated to anisotropy-based performance analysis using Riccati and convex optimization techniques. Chapter 4 deals with optimal anisotropy-based control for descriptor systems. Chapter 5 deals with suboptimal anisotropy-based control for descriptor systems. Chapter 6 develops anisotropy-based performance analysis with nonzero mean input sequences. Chapter 7 presents anisotropy-based analysis and robust control problems for uncertain descriptor systems.

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Chapter 1
Practical Application of Descriptor Systems

Mathematical Modeling Using Descriptor Systems

Systems whose variables describe some physical processes are called descriptor systems. Let's consider the following simple example of such systems.

The object of mass m shown in Fig. 1.1 has position $x(t)$ and speed $v(t)$. It is driven by force $F(t)$. Equations describing the system have the form:

$$\dot{x}(t) = v(t), \tag{1.1}$$

$$m\dot{v}(t) = F(t). \tag{1.2}$$

Introducing notations $x(t) = \xi_1(t)$ and $v(t) = \xi_2(t)$, these equations can be written in a state-space representation as

$$\begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F(t). \tag{1.3}$$

System (1.3) is descriptor because its variables describe physical processes. System (1.3) is nonsingular and can easily be rewritten in the form of the normal system.

However, there are situations when additional restrictions affect the physical system. Then a mathematical model of the plant or process includes not only differential but algebraic equations as well.

In general, a descriptor system is given by the relation:

$$\overline{F}(\dot{x}(t), x(t), t) = 0, \tag{1.4}$$

the output equation is given as

A. A. Belov et al., *Control of Discrete-Time Descriptor Systems*, Studies in Systems, Decision and Control 157, https://doi.org/10.1007/978-3-319-78479-3_1

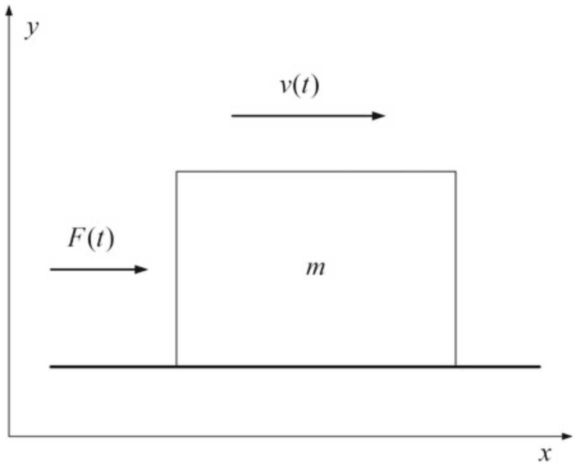


Fig. 1.1 Body moved by force $F(t)$.

$$h(x(t), t) = 0. \tag{1.5}$$

Before Gear's works appeared [1], descriptor systems in the form (1.4) had usually been rewritten by means of analytical transformations in ODE form:

$$\dot{y}(t) = g(y(t), t). \tag{1.6}$$

This made it possible to reduce the dimension of the original system; on the other hand, that representation required solving complex algebraic equations that in the general case had no analytical solutions for high-order systems.

Another possible way to get rid of algebraic equations was their differentiation in order to obtain an ordinary differential equation with the same number of variables as for the original system. The approach described above is time-consuming because of the need to use the implicit function theorem.

However, due to a possible change of basis, obtained state variables would have no physical meaning (i.e., they were abstract phase variables). Moreover, as a result of the numerical integration of the resulting system of ordinary differential equations, the solution could exceed the area limits defined by algebraic equations. Descriptor systems given in so-called special forms are the most studied at present. Take a closer look. Suppose that the relation in system (1.4) could be extracted with respect to the derivative $\dot{x}(t)$. Then descriptor system (1.4) and (1.5) is

$$\begin{aligned} E(x(t))\dot{x}(t) &= \widetilde{F}(x(t), u(t), t), \\ y(t) &= \widetilde{H}(x(t), u(t), t). \end{aligned} \tag{1.7}$$

Assuming that rank $\left(\frac{\partial E(x(t))}{\partial x}\right) = const$, we can rewrite (1.7) in the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)),$$
$$0 = f_2(x_1(t), x_2(t)).$$

This form is called *semi-explicit*.

A linear stationary system is also one of the special descriptions of descriptor systems and has the form:

$$E\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

As mentioned above, algebraic relations between variables do not allow us to solve the original equations with respect to the derivative. For example, in the case of linear stationary systems matrix E is singular; that is, rank $(E) < n$. This fact does not allow inverting matrix E and transforming the descriptor system to the system of ordinary differential equations.

Here we consider some practical applications related to the construction of mathematical models of processes and control plants in descriptor form.

1.1 Chemistry and Biology

Descriptor systems found their wide application in the fields of chemistry and biology. This is due to laws such as the transfer of mass and energy. We start a review of mathematical models with an isothermal reactor.

1.1.1 Isothermal Reaction in an Isothermal Batch Reactor System

A kinetic model describing the chemical reaction in an isothermal batch reactor system is obtained in [2]. The reaction occurs in an anhydrous, homogeneous, liquid phase catalyzed by a completely dissociated species.

The desired reaction is given by

$$HA + 2BM \rightarrow AB + MBMH$$

where AB is the desired product.

The reaction is initiated by adding the catalyst QM to the batch reactor containing two miscible reactants with reactant BM in excess.

The catalyst QM is initially assumed to be 100% dissociated to Q^+ and M^-ions. The following mechanism is proposed to describe the reaction.

Slow Kinetic Reactions

$$M^- + BM \Leftarrow_{k_{-1}} {}^{k_1} \Rightarrow MBM^-,$$
$$A^- + BM \ {}^{k_2} \Rightarrow ABM^-,$$
$$M^- + AM \Leftarrow_{k_{-3}} {}^{k_3} \Rightarrow ABM^-.$$

Rapid Acid-Base Reactions

$$MBMH \Leftarrow K_1 \Rightarrow MBM^- + H^+,$$
$$HA \Leftarrow K_2 \Rightarrow A^- + H^+,$$
$$HABM \Leftarrow K_3 \Rightarrow ABM^- + H^+.$$

In order to derive a model to account for these reactions, it is necessary to distinguish between the overall concentration of species and concentration of its neutral form. Overall concentrations are defined for three components based on neutral and ionic species.

$$[MBMH] = [(MBMH)_N] + [MBM^-],$$

$$[HA] = [(HA)_N] + [A^-],$$

$$[HABM] = [(HABM)_N] + [ABM^-],$$

where [ ] denotes concentration of the species in gmol/kg. By assuming the rapid acid-base reactions are at equilibrium, the equilibrium constants K_1, K_2, K_3 can be defined as follows.

$$K_1 = \frac{[MBM^-][H^+]}{[(MBMH)_N]},$$

$$K_2 = \frac{[A^-][H^+]}{[(HA)_N]},$$

$$K_3 = \frac{[ABM^-][H^+]}{[(HABM)_N]}.$$

Anionic species may then be represented by

$$[MBM^-] = \frac{K_1[MBMH]}{(K_1 + [H^+])},$$

$$[A^-] = \frac{K_2[HA]}{K_2 + [H^+]},$$

$$[ABM^-] = \frac{K_3[HABM]}{K_3 + [H^+]}.$$

Material balance equations for the three reactants in slow kinetic reactions yield

$$\frac{d[M^-]}{dt} = -k_1[M^-][BM] + k_{-1}[MBM^-] - k_3[M^-][AB] + k_{-3}[ABM^-],$$

$$\frac{d[BM]}{dt} = -k_1[M^-][BM] + k_{-1}[MBM^-] - k_2[A^-][BM],$$

$$\frac{d[AB]}{dt} = -k_3[M^-][AB] + k_{-3}[ABM^-].$$

As we can see from stoichiometry, rate expressions can also be written for the total species

$$\frac{d[MBMH]}{dt} = k_1[M^-][BM] + k_{-1}[MBM^-],$$

$$\frac{d[HA]}{dt} = k_2[A^-][BM],$$

$$\frac{d[HABM]}{dt} = k_2[A^-][BM] + k_3[M^-][AB] - k_{-3}[ABM^-].$$

An electroneutrality constraint gives the hydrogen ion concentration $[H^+]$ as

$$[H^+] + [Q^+] = [M^-] + [MBM^-] + [A^-] + [ABM^-].$$

We also assume

$$k_3 = k_1 \text{ and } k_{-3} = 0,5k_{-1}$$

in terms of similarities of the reacting species. Denoting the amount of the ith reactant by x_i, we get a mathematical model in the form:

$$\begin{aligned}
\dot{x}_1 &= -k_2x_2x_8,\\
\dot{x}_2 &= -k_1x_2x_6 + k_{-1}x_{10} - k_2x_2x_8,\\
\dot{x}_3 &= k_2x_2x_8 + k_1x_4x_6 - 0.5k_{-1}x_9,\\
\dot{x}_4 &= -k_1x_4x_6 + 0.5k_{-1}x_9,\\
\dot{x}_5 &= k_1x_2x_6 + k_{-1}x_{10},\\
\dot{x}_6 &= -k_1x_2x_6 - k_1x_4x_6 + k_{-1}x_{10} + 0.5k_{-1}x_9,\\
0 &= -x_7 + x_6 + x_8 + x_9 + x_{10} - Q^+,\\
0 &= -x_8(K_2 + x_7) + K_2x_1,\\
0 &= -x_9(K_3 + x_7) + K_3x_3,\\
0 &= -x_{10}(K_1 + x_7) + K_1x_5.
\end{aligned}$$

This mathematical model is a nonlinear descriptor system, and consists of six differential and four algebraic equations.

Parameter Q^+, that stands for the amount of catalyst in the reactor, is known, as paid before the reaction.

1.1.2 Chemical Reaction of Urethane

Consider a chemical process that takes place in the urethane reactor system. The reactions are given as follows.

$$A + B \rightarrow C,$$

$$A + C \rightleftharpoons D,$$

$$3A \rightarrow E,$$

where A is phenyl isocyanate, B is butanol, C is urethane, D is allophanate, and E is isocyanurate.

A mathematical model describing the reaction of isocyanate (n_1), butanol (n_2), urethane (n_3), allophanate (n_4), and isocyanurate (n_5) consists of three differential and three algebraic equations.

$$\begin{aligned}
\dot{n}_3 &= V(r_1 - r_2 + r_3),\\
\dot{n}_4 &= V(r_2 - r_3),\\
\dot{n}_5 &= Vr_4,\\
0 &= n_1 + n_3 + 2n_4 + 3n_5 - n_{a1} - n_{1ea}(t),\\
0 &= n_2 + n_3 + n_4 - n_{a2} - n_{2eb}(t),\\
0 &= n_6 - n_{a6} - n_{6ea}(t) - n_{6eb}(t),
\end{aligned}$$

where n_6 is the solvent of dimethylsulfoxide, $n_3(0) = 0$, $n_4(0) = 0$, $n_5(0) = 0$, and the following parameters are

$$V = \textstyle\sum_{i=1}^{6} \frac{M_i n_i}{\rho_i},\ k_1 = k_{ref1} \exp\left(-\frac{E_{a1}(1/T(t) - 1/T_{ref1}(t))}{R}\right),$$

$$r_1 = k_1 \frac{n_1 n_2}{V^2},\ k_2 = k_{ref2} \exp\left(-\frac{E_{a2}(1/T(t) - 1/T_{ref2}(t))}{R}\right),$$

$$r_2 = k_2 \frac{n_1 n_2}{V^2},\ k_3 = k_2/k_c,$$

$$r_3 = k_3 \frac{n_4}{V},\ k_4 = k_{ref4} \exp\left(-\frac{E_{a4}(1/T(t) - 1/T_{ref4}(t))}{R}\right),$$

$$r_4 = k_4 \frac{n_1^2}{V^2},\ k_c = k_{c2} \exp\left(-\frac{d_{h2}(1/T(t) - 1/T_{g2}(t))}{R}\right).$$

Two control feeds have the form of nonincreasing functions $\text{feed}_a(t)$ and $\text{feed}_b(t)$, and determine $n_{1ea} = n_{a1ea}\,\text{feed}_a(t)$, $n_{2eb} = n_{a2eb}\,\text{feed}_b(t)$, $n_{6ea} = n_{a6ea}\,\text{feed}_a(t)$, and $n_{6eb} = n_{a6eb}\,\text{feed}_b(t)$. Mole ratios of the active ingredients and the initial volume satisfy the constraints

$$0.1 \le MV_1 \le 10,$$

$$0 \le MV_2 \le 1000,$$

$$0 \le MV_3 \le 10,$$

$$0 \le g_a \le 0,8,$$

$$0 \le g_{aea} \le 0,9,$$

$$0 \le g_{aeb} \le 1,$$

$$0 \le V_a \le 0.00075,$$

and they are connected with the other parameters by the following algebraic relationships.

$$\begin{aligned}
MV_1(n_{a1} + n_{a1ea}) &= n_{a2} + n_{a2eb},\\
MV_2 n_{a1} &= n_{a1ea},\\
MV_3 n_{a1} &= n_{a2eb},\\
g_a(n_{a1}M_1 + n_{a2}M_2 + n_{a6}M_6) &= n_{a1}M_1 + n_{a2}M_2,\\
g_{aea}(n_{a1ea}M_1 + n_{a6ea}M_6) &= n_{a1ea}M_1,\\
g_{aeb}(n_{a2eb}M_2 + n_{a6eb}M_6) &= n_{a2eb}M_2,\\
V_a &= n_{a1}M_1/\rho_1 + n_{a2}M_2/\rho_2 + n_{a6}M_6/\rho_6,
\end{aligned}$$

that are nonlinear constraints. Numerical values of the parameters can be found in [3].

1.1.3 Evaporator

Consider a single-component system of phase equilibrium where there are gaseous and liquid phases. The plant is shown in Fig. 1.2. Vapor (denoted by subscript $_V$) and liquid (subscript $_L$) are in a heated container. There are two components with masses M_V, M_L, and temperatures T_V and T_L inside the tank. The system has a feed with a flow F. In this model, two volume balance equations are under consideration for the gaseous and liquid phases. Equations describing the dynamics of the process are discussed below.

Conservation Laws
Mass balance is described by

$$\dot{M}_V = E - V,$$
$$\dot{M}_L = F - E - L.$$

Energy balance is given in the form:

$$\dot{U}_V = Eh_{LV} - Vh_V + Q_E,$$
$$\dot{U}_L = Fh_F - Eh_{LV} - Lh_L + Q - Q_E.$$

Transfer equations of mass and energy are given below.

$$E = (k_{LV} + k_{VL})A(P^* - P), \tag{1.8}$$

$$Q_E = (u_{LV} + u_{VL})A(T_L - T_V), \tag{1.9}$$

where the subscript $_{LV}$ in transfer coefficients of mass and energy k_i and u_i means a transfer from liquid to vapor, and $_{VL}$ from vapor to liquid. The coefficients for $_{VL}$ and $_{LV}$ are usually different.

Balance of volumes equation is

$$V_V = V_T - V_L. \tag{1.10}$$

Control laws are

$$L = f_1(M_L, P) \text{ or } L = f_2(M_L).$$

The relation between the variables is expressed as

$$h_V = h_V(T_V, P),$$

$$h_L = h_L(T_L, P),$$

$$h_{LV} = h_{LV}(T_L, P),$$

$$h_F = h_F(T_F, P),$$

$$k_{LV} = k_{LV}(T_L, T_V, P),$$

$$k_{VL} = k_{VL}(T_L, T_V, P),$$

$$u_{LV} = u_{LV}(T_L, T_V, P),$$

$$u_{VL} = u_{VL}(T_L, T_V, P),$$

$$\rho_L = \rho_L(T_L, P).$$

Notations and variables shown in the example are the following.

M_V	Mass of vapor
U_V	Internal energy of vapor
F	Feed speed
L	Flow rate of liquid
T_V	Temperature of vapor
Q	Power of a heat source
P	System pressure
A	Surface area of the phase boundary
V_V	Vapor volume
h_V	Vapor enthalpy
h_F	Feed enthalpy
ρ_L	Liquid density
V_L	Liquid volume
M_L	Liquid mass
U_L	Internal energy of liquid
V	Vapor flow rate
E	Speed of phase separation
T_L	Liquid temperature
P^*	Vapor pressure
R	Universal gas constant
V_T	Tank volume
h_L	Liquid enthalpy
m_w	Molecular mass
h_{LV}	Vapor enthalpy at the interface
k_i	Exchange ratio weight
u_i	Heat transfer coefficient

1.1.4 Multispecies Food Chain

This problem concerns modeling of the multispecies ratio of predator-prey in a closed area [4]. We assume that the model consists of s individuals where the individuals with numbers $s/2+1, \ldots, s$ (predators) have an infinitely fast reaction. Then we have

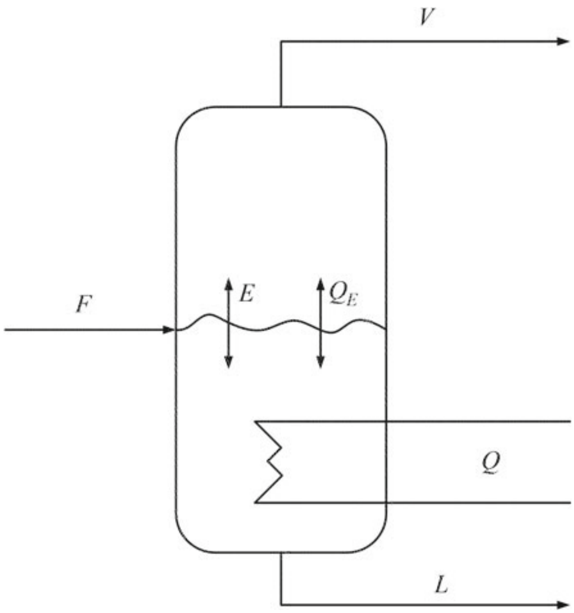
$$\frac{\partial c^i}{\partial t} = f_i(x, y, t, c) + d_i(c^i_{xx} + c^i_{yy}), \quad (i = 1, 2, \ldots, s/2), \tag{1.11}$$

$$0 = f_i(x, y, t, c) + d_i(c^i_{xx} + c^i_{yy}), \quad (i = s/2+1, \ldots, s), \tag{1.12}$$

where

$$f_i(x, y, t, c) = c^i(b_i + \sum_{j=1}^{s} a_{ij}c^j). \tag{1.13}$$

Fig. 1.2 Evaporator



c_x stands for the partial derivative for the corresponding variable.

In the general case, coefficients of mutual influence and interpenetration (a_{ij}, b_i, d_i) can depend on (x, y, t).

In the simplest case, if we consider a model consisting of p species of predators and p species of prey with the total number of individuals ($s = 2p$), located in vector c in series, the coefficients would be equal to

$$\begin{aligned} a_{ii} &= -1, \quad \forall i \\ a_{ij} &= -0.5 \cdot 10^{-6} \ \ (i \leq p, j > p), \\ a_{ij} &= 10^4 \ \ (i > p, j \leq p); \end{aligned}$$

the other coefficients are $a_{ij} = 0$,

$$\begin{aligned} b_i &= (1 + \alpha xy + \beta \sin(4\pi x)\sin(4\pi y)), \quad (i \leqslant p), \\ b_i &= -(1 + \alpha xy + \beta \sin(4\pi x)\sin(4\pi y)), \quad (i > p), \end{aligned}$$

$$d_i = 1, \quad (i < p), \tag{1.14}$$

$$d_i = 0.05, \quad (i > p). \tag{1.15}$$

All boundary conditions have zero derivatives on the normal. Parameters α and β are chosen positive in order to get stable solutions. Initial conditions are chosen in such a way that they satisfy boundary conditions and are close to the limits of the following type.

$$c^i = 10 + i(16x(1-x)(1-y))^2, \quad (i = \overline{1, s/2}),$$
$$c^i = -(b_i + \sum_{j=1}^{s/2} a_{ij}c^j)/a_{ii}, \quad (i = \overline{s/2+1, s}).$$

1.2 Economic Systems

A distinctive feature of theoretical interindustry dynamic models is the description of the relations “input – output” in the form of an interbranch balance matrix, where each product is represented by only one production method, and each method produces only one product. Benefits of dynamic interindustry among the economic dynamics models are determined by the following factors. First, they are detailed (disaggregated) analogues of reproduction of the social product and national income models. Second, they represent a generalization of the static (balance and optimization) interindustry models. Third, they serve as the theoretical and methodological basis for application of dynamic models with matrices of interindustry balance.

Example 1.1 The dynamic interindustry model, proposed by W. Leontief in the early 1950s, is a classic example of systems of difference and algebraic equations in the study of economic growth. This model is represented as disaggregation of simple reproduction elements of the social product dynamic model, where endogenous and exogenous macro variables are replaced by vectors, and technological macro parameters are given by matrices. The model has the form [5]

$$x(k) = Ax(k) + E(x(k+1) - x(k)) + d(k),$$

where $x(k)$ is the n-dimensional production vector of n sectors; $E(x(k+1) - x(k))$ is the overall amount for capacity expansion (see [6]), which often appears in the form of capital; $d(k)$ is the vector that includes demand or consumption; and $E \in \mathbb{R}^{n \times n}$ is the capital coefficient matrix.

$A \in \mathbb{R}^{n \times n}$ is an input-output (or production) matrix; $Ax(k)$ stands for the fraction of production required as input for the current production. The model's equation can be given as

$$Ex(k+1) = (I - A + E)x(k) - d(k).$$

In multisector economic systems, the increased production in one sector often needs investments in all other sectors of the economy. Moreover, in practical cases, only a few sectors can offer investment of capital to other sectors. Thus, most of the elements of matrix B are equal to zero, except for a few elements. E is often singular. In the general case, the system considered in this example is a typical discrete-time descriptor system.

1.3 Large-Scale Systems

In a computer simulation, representation of large-scale systems in descriptor form is quite attractive in the context of computation.

This is because transformation from the implicit descriptor form into a normal one requires matrix inversion. For large-scale systems it may take considerable time; in addition, the inversion of matrices can be ill-conditioned, which may lead to increased computational errors.

In some cases, such a transition to normal systems is not necessary. As an example consider the following system, given in descriptor form [7].

Example 1.2 [7] Consider a class of interconnected large-scale systems with subsystems of

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i a_i(t), \\ b_i(t) &= C_i x_i(t) + D_i a_i(t), \; i = 1, 2, \ldots, N, \end{aligned} \tag{1.16}$$

where $x_i(t)$, $a_i(t)$, $b_i(t)$ are substate, control input, and output of the ith subsystem, respectively. By denoting

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}, \; a(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_N(t) \end{bmatrix}, \; b(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_N(t) \end{bmatrix}.$$

$$\begin{aligned} A &= \mathrm{diag}(A_1, A_2, \ldots, A_N), \quad B = \mathrm{diag}(B_1, B_2, \ldots, B_N), \\ C &= \mathrm{diag}(C_1, C_2, \ldots, C_N), \; D = \mathrm{diag}(D_1, D_2, \ldots, D_N), \end{aligned}$$

the expression (1.16) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Ba(t), \\ b(t) &= Cx(t) + Da(t). \end{aligned} \tag{1.17}$$

Assume that the subsystems' interconnection is linear

$$\begin{aligned} a(t) &= L_{11} b(t) + L_{21} u(t) + R_{11} a(t) + R_{12} y(t), \\ y(t) &= L_{21} b(t) + L_{22} u(t) + R_{21} a(t) + R_{22} y(t) \end{aligned} \tag{1.18}$$

where $u(t)$ is the overall input of the large-scale system; $y(t)$ is its overall measurable output; L_{ij}, R_{ij}, $i, j = 1, 2$ are constant matrices of appropriate dimensions. Equations (1.17) and (1.18) form a large-scale system that cannot be given in the normal form. In fact, S.P. Singh and R.-W. Liu [8] and L.R. Petzold [9] have proved that the system composed by (1.17) and (1.18) could not be equivalent to a normal one. On the other hand, if we choose a state variable as

$$\begin{bmatrix} x^{\mathrm{T}} & a^{\mathrm{T}} & b^{\mathrm{T}} & y^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

we get a system

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{a}(t) \\ \dot{b}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} A & B & 0 & 0 \\ C & D & -I & 0 \\ 0 & R_{11}-I & L_{11} & R_{12} \\ 0 & R_{21} & L_{21} & R_{22}-I \end{bmatrix} \begin{bmatrix} x(t) \\ a(t) \\ b(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{12} \\ L_{22} \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x^{\mathrm{T}} & a^{\mathrm{T}} & b^{\mathrm{T}} & y^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

Consider interconnected subsystems via a simple example of parallel connection of two capacitors in more detail.

Example 1.3 Consider two capacitors with capacities C_1 and C_2, respectively. Voltage changes on both capacitors are described by an ordinary differential equation individually,

$$\dot{v}_1(t) = \frac{i_1(t)}{C_1}, \tag{1.19}$$

$$\dot{v}_2(t) = \frac{i_2(t)}{C_2}. \tag{1.20}$$

If these two subsystems are interconnected in parallel, we get the following equations:

$$\dot{v}_1(t) = \frac{i_1(t)}{C_1}, \tag{1.21}$$

$$\dot{v}_2(t) = \frac{i_2(t)}{C_2}, \tag{1.22}$$

$$0 = i_1(t) + i_2(t), \tag{1.23}$$

$$0 = v_1(t) - v_2(t). \tag{1.24}$$

This system has a differentiation index equal to 2, thus in order to get expressions for $\dot{i}_1(t)$ and $\dot{i}_2(t)$ it is necessary to differentiate the first three equations.

This leads to the appearance of the second derivatives $\ddot{v}_1(t)$ and $\ddot{v}_2(t)$. Differentiating (1.24) twice, we can remove variables $\ddot{v}_1(t)$ and $\ddot{v}_2(t)$. Thus, differential equations become solvable for $\dot{i}_1(t)$ and $\dot{i}_2(t)$. Combining differential equations for $\dot{i}_1(t)$ and $\dot{i}_2(t)$ with (1.21) and (1.22), one can rewrite the initial descriptor system in the standard state-space form. However, in modeling behavior of the system with a parallel connection of capacitors, voltages $v_1(t)$ and $v_2(t)$ cannot be selected independently of each other.

Even if we choose the initial values of voltages $v_1(0)$ and $v_2(0)$ equal and known, the unknown initial values $\dot{v}_1(0)$, $\dot{v}_2(0)$, $i_1(0)$, and $i_2(0)$ cannot be found, as the four equations with four variables obtained are degenerate with respect to these unknown variables.

Therefore the transition from a descriptor system to a normal one is not possible for this example.

1.4 Constrained Mechanical Systems

Constrained linear mechanical systems can be described as follows.

$$M\ddot{z}(t) + D\dot{z}(t) + Kz(t) = Lf(t) + J\lambda(t), \tag{1.25}$$

$$G\dot{z}(t) + Hz(t) = 0, \tag{1.26}$$

where $z(t) \in \mathbb{R}^n$ is the vector of coordinates, $f(t) \in \mathbb{R}^n$ is the input vector of forces, $\lambda(t) \in \mathbb{R}^q$ is the vector of Lagrangian multipliers, M is a matrix of inertial characteristics (usually, M is symmetric and positive definite), D is the damping and gyroscopic matrix, K is the stiffness and circulator matrix, L is the force distribution matrix, J is the Jacobian of the constraint, and G and H are the coefficient matrices of the constraint equation. All matrices in (1.25) and (1.26) are known and constant ones of appropriate dimensions. Relation (1.25) is a differential equation, whereas (1.26) is the constraint equation.

Assume that a linear combination of positions (C_p) and velocities (C_v) is measurable; then the output equation has the form

$$y(t) = C_p z(t) + C_v \dot{z}(t), \quad C_p, C_v \in \mathbb{R}^{m\times n}. \tag{1.27}$$

By choosing a state vector and an input vector as

$$x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \\ \lambda(t) \end{bmatrix} \text{ and } u(t) = f(t),$$

equations (1.25)–(1.27) can be given in the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1.28}$$

$$y(t) = Cx(t), \tag{1.29}$$

where

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ L \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I & 0 \\ -K & -D & J \\ H & G & 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_p & C_v & 0 \end{bmatrix}.$$

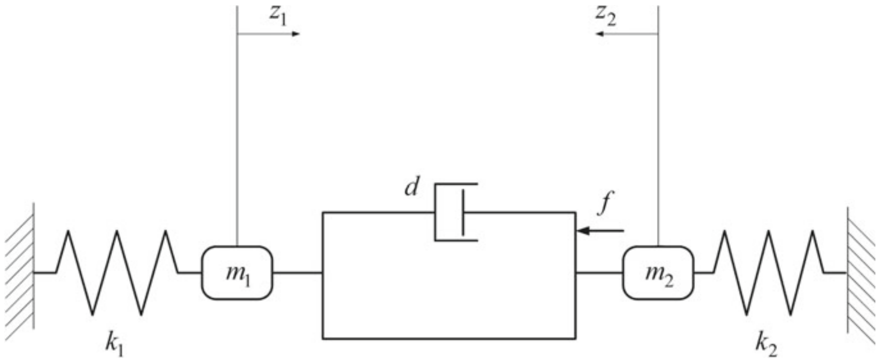


Fig. 1.3 Two connected one-mass oscillators

Consider some specific examples of constrained mechanical systems below.

1.4.1 Two Connected One-Mass Oscillators

Consider the mechanical system shown in Fig. 1.3 [10]. This system consists of two one-mass oscillators connected by a dashpot element. Let

$$m_1 = m_2 = 5\ \text{kg}\ ,\ d = 1\text{N} \cdot \text{s/m}\ ,\ k_1 = 2\ \text{N/m}\ ,\ k_2 = 1\ \text{N/m}.$$

Then equations (1.25)–(1.27) for this system can be obtained as follows.

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} f + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \lambda,$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Matrices are given as

$$M = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix},\ \ D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ \ K = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$L = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad J = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 \; 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \; 1 \end{bmatrix},$$

$$C_p = \begin{bmatrix} 1 \; 0 \end{bmatrix}, \quad C_v = \begin{bmatrix} 0 \; 0 \end{bmatrix}.$$

Based on these matrices, a linear descriptor system of the form (1.28) and (1.29) for the system can be readily written out.

1.4.2 Cart Pendulum

Now we consider a rigid pendulum of length l with point mass m_2 attached to a cart with mass m_1 that moves only in a horizontal direction. A cart is moved by external force u; g denotes gravitational acceleration. Denote by x_1, x_2, x_3 the cart position, horizontal position, and vertical position of the cart, and mass m_2, respectively. The motion of the system can be described by the Euler–Lagrange equations. The Lagrange function is given by

$$\mathscr{L}(x, \dot{x}, \lambda) = T(x, \dot{x}) - U(x) - \sum_{k=1}^{N} \lambda_k h_k(x),$$

where $T(x, \dot{x})$ denotes the kinetic energy, $U(x)$ denotes the potential energy, λ_k are elements of the vector of the Lagrange multipliers λ, and $h_k(x)$ denotes the constraints that restrict the motion of the system.

The Euler–Lagrange equation can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\widehat{x}}}\mathscr{L}(\widehat{x}, \dot{\widehat{x}})\right) - \frac{\partial}{\partial \widehat{x}}\mathscr{L}(\widehat{x}, \dot{\widehat{x}}) = F_{ex}, \tag{1.30}$$

where $\widehat{x} = \begin{bmatrix} x \; \lambda \end{bmatrix}^{\mathrm{T}}$, F_{ex} is some external force (Fig. 1.4).

The kinetic energy of the cart pendulum is defined as

$$T(x, \dot{x}) = \frac{1}{2}\left(m_1 \dot{x}_1^2 + m_2(\dot{x}_2^2 + \dot{x}_3^2)\right),$$

the potential energy is

$$U(x) = m_2 g x_3,$$

And the constraint is defined by

$$h(x) = (x_2 - x_1)^2 + x_3^2 - l^2 = 0.$$

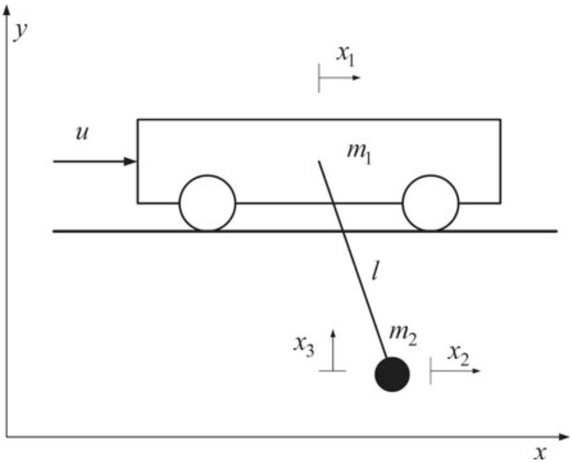


Fig. 1.4 Cart pendulum

Denoting $x_4 = \dot{x}_1$, $x_5 = \dot{x}_2$, and $x_6 = \dot{x}_3$, we have the following dynamical model of the cart pendulum.

$$\begin{aligned} \dot{x}_1 &= x_4, \\ \dot{x}_2 &= x_5, \\ \dot{x}_3 &= x_6, \\ m_1\dot{x}_4 &= 2\lambda(x_2 - x_1) + u, \\ m_2\dot{x}_5 &= -2\lambda(x_2 - x_1), \\ m_3\dot{x}_6 &= -2\lambda x_3 - m_2 g, \\ 0 &= (x_2 - x_1)^2 + x_3^2 - l^2. \end{aligned} \tag{1.31}$$

The measurable output is considered as a position of the pendulum. Hence, the output equation takes the form

$$y = Cx = \begin{bmatrix} 0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0 \end{bmatrix} x = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}.$$

Linearization of system (1.31) along the equilibrium point $x^0 = \begin{bmatrix} 0 & 0 & -l & 0 & 0 & 0 & \frac{m_2 g}{2l} \end{bmatrix}^{\mathrm{T}}$ gives us a system model of the form

$$\begin{aligned} E\delta\dot{x}(t) &= A\delta x(t) + Bu(t) + f(t), \\ \delta y(t) &= C\delta x(t), \end{aligned}$$

where matrices A, B, and C are constructed in the form (1.28) and (1.29) with

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \; K = \begin{bmatrix} \frac{m_2 g}{l} & -\frac{m_2 g}{l} & 0 \\ -\frac{m_2 g}{l} & \frac{m_2 g}{l} & 0 \\ 0 & 0 & \frac{m_2 g}{l} \end{bmatrix}, \; D = 0_{3\times 3},$$

$$J = 2l, \; H = \begin{bmatrix} 0 \; 0 \; -2l \end{bmatrix}, \; G = 0_{1\times 3}, \; L = \begin{bmatrix} 1 \; 0 \; 0 \end{bmatrix}^{\mathrm{T}}, \; f(t) = \begin{bmatrix} 0 \; 0 \; 0 \; 0 \; 0 \; -m_2 g \; 0 \end{bmatrix}^{\mathrm{T}};$$

$\delta x(t)$ and $\delta y(t)$ are deviations of $x(t)$ and $y(t)$, respectively.

1.4.3 Planar Crane Model

A crane is a mechanical system in plane Cartesian coordinates. It is represented in Fig. 1.5. The coordinates x and z represent a horizontal and vertical position of the load. The horizontal distance traveled by the cart of the crane from its initial position d, and the length of the rope lowered from the top of the crane r, are also state variables. State equations have the form:

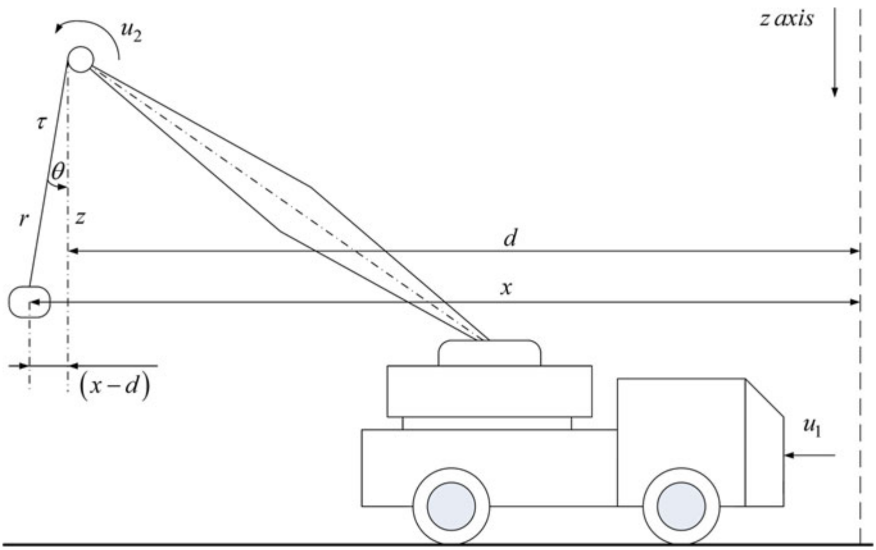


Fig. 1.5 Planar crane model

$$M_2\ddot{x} = -\tau \sin(\theta), \tag{1.32}$$
$$M_2\ddot{z} = -\tau \cos(\theta) + mg, \tag{1.33}$$
$$M_1\ddot{d} = -C_1\dot{d} + u_1 + \tau \sin(\theta), \tag{1.34}$$
$$J\ddot{r} = -C_2\dot{r} - C_3u_2 + C_3^2\tau, \tag{1.35}$$
$$0 = \theta - \operatorname{ctg}\left(\frac{x-d}{z}\right), \tag{1.36}$$
$$0 = r^2 - (x-d)^2 - z^2, \tag{1.37}$$
$$x = \varphi_1(t), \tag{1.38}$$
$$z = \varphi_2(t), \tag{1.39}$$

where M_1, M_2, m are the masses of the cart, the cable with the handling system, and the cargo, respectively. C_1, C_2, C_3 are known constants. J is the moment of inertia of the roller through which the cable is spanned. The tension of the cable τ and deflection angle of the rope from the vertical axis θ are algebraic variables. Control signals are the horizontal thrust of the cart u_1 and the torque on the shaft of the roller u_2.

1.5 Robotics

In this section, we consider the model of a three-link planar manipulator, which is a simplified model of a cleaning robot. Figure 1.6 shows a schematic illustration of a mobile manipulator cleaning the facade of a building [11]. The problem of development of this kind of manipulator is mentioned in [12, 13]. The mobile manipulator appertains to an important type of service robot widely utilized in disaster and emergency relief, construction, public services, and environmental protection [14].

A simplified scheme of a three-link planar manipulator is depicted in Fig. 1.7. This manipulator cleans the region between points A and B. The robot fulfills its task by repetitively moving the end-effector of the manipulator from point A to point B.

It is assumed that the cleaning flat surface is a rigid body and the end of the third arm is a smooth and rigid plate. Thus, there are two constraints on the motion of the robot:

- The restriction on the motion in the x direction, usually given as $x \leq 1$.
- Orthogonality of the third arm to the cleaning surface, which can be described by $\theta_1 + \theta_2 + \theta_3 = 0$.

Obviously, during the cleaning process these two constraints should be kept active.

Nonlinear Model

A motion of constrained robots can be easily modeled by the descriptor system framework. The problem of modeling free robot motion is studied in [15].

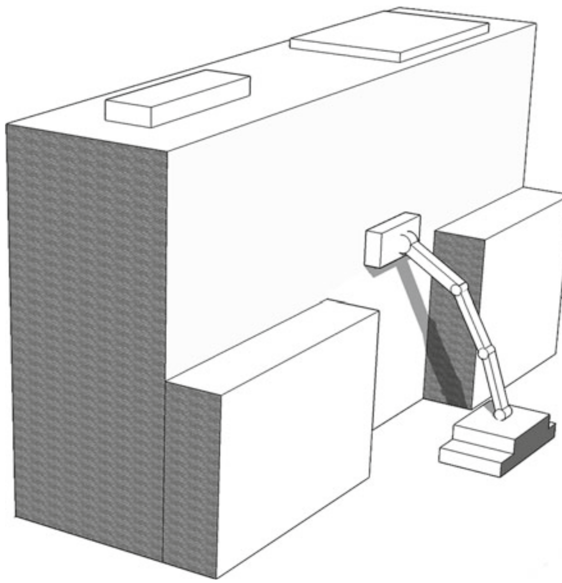


Fig. 1.6 Three-link planar manipulator

Dynamics of the three-link planar manipulator depicted in Fig. 1.7 is described by the following equations in joint coordinates.

$$M_\theta(\theta)\ddot{\theta} + C_\theta(\theta, \dot{\theta}) + G_\theta(\theta) = u_\theta + F_\theta^{\mathrm{T}}\lambda, \tag{1.40}$$

$$\psi_\theta(\theta) = 0, \tag{1.41}$$

where

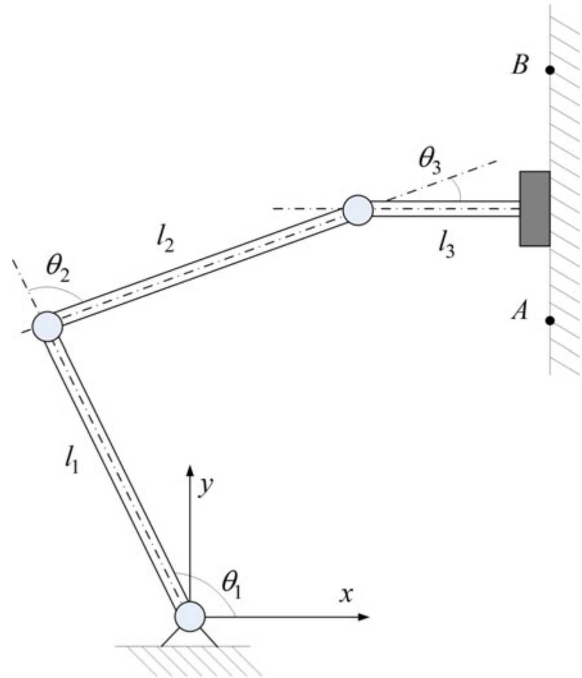
$$\theta = \begin{bmatrix} \theta_1 \ \theta_2 \ \theta_3 \end{bmatrix}^{\mathrm{T}}$$

is a vector of joint positions; $u_\theta \in \mathbb{R}^3$ is a vector of control torques applied at the joints. $F_\theta = \partial\phi_\theta/\partial\theta$, $\lambda \in \mathbb{R}^3$ is a vector of Lagrangian multipliers, and $F_\theta^{\mathrm{T}}\lambda$ is a generalized constraint force. The constraint function $\psi_\theta(\theta)$ is given by

$$\psi_\theta = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) - l \\ \theta_1 + \theta_2 + \theta_3 \end{bmatrix}, \tag{1.42}$$

where $M_\theta(\theta) \in \mathbb{R}^{3\times 3}$ is a mass matrix, given in the form $M_\theta(\theta) = [m_{ij}(\theta)]_{3\times 3}$,

Fig. 1.7 Simplified scheme of three-link manipulator



$$
\begin{aligned}
m_{11}(\theta) &= m_1 l_1^2 + m_2(l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta_1) + \\
&\quad + m_3(l_1^2 + l_2^2 + l_3^2 + 2l_1 l_2 \cos\theta_2) + \\
&\quad + m_3(2l_2 l_3 \cos\theta_3 + 2l_2 l_3 \cos(\theta_2 + \theta_3)), \\
m_{12}(\theta) &= m_2(l_2^2 + l_1 l_2 \cos\theta_2) + \\
&\quad + m_3(l_2^2 + l_3^2 + l_1 l_2 \cos\theta_2 + 2l_2 l_3 \cos\theta_3) + \\
&\quad + m_3 l_1 l_3 \cos(\theta_2 + \theta_3), \\
m_{22}(\theta) &= m_2 l_2^2 + m_3(l_2^2 + l_3^2 + 2l_2 l_3 \cos\theta_3), \\
m_{23}(\theta) &= m_3(l_3^2 + l_2 l_3 \cos\theta_3), \\
m_{33}(\theta) &= m_3 l_3^2.
\end{aligned}
$$

$C_\theta(\theta, \dot{\theta}) \in \mathbb{R}^3$ is the centrifugal and Coriolis vector, and is given by

$$C_\theta(\theta, \dot{\theta}) = C_I(\theta)\Theta_N + C_{II}(\theta)\Theta_S,$$

where

$$
\begin{aligned}
\Theta_N &= \left[\, \dot{\theta}_1\dot{\theta}_2 \;\; \dot{\theta}_1\dot{\theta}_3 \;\; \dot{\theta}_2\dot{\theta}_3 \,\right]^{\mathrm{T}}, \\
\Theta_S &= \left[\, \dot{\theta}_1^2 \;\; \dot{\theta}_2^2 \;\; \dot{\theta}_3^2 \,\right]^{\mathrm{T}},
\end{aligned}
$$

and

$$C_I(\theta) = [c_{I,ij}(\theta)]_{3\times 3},\ C_{II}(\theta) = [c_{II,ij}(\theta)]_{3\times 3}.$$

Coefficients $c_{I,ij}(\theta)$ and $c_{II,ij}(\theta)$ are equal to

$$\begin{aligned}
c_{I,11}(\theta) &= -2m_2 l_1 l_2 \sin\theta_2 - 2m_3 l_1 (l_2 \sin\theta_2 + l_3 \sin(\theta_2+\theta_3)),\\
c_{I,12}(\theta) &= -2m_3 l_3 (l_2 \sin\theta_3 + l_1 \sin(\theta_2+\theta_3)),\\
c_{I,13}(\theta) &= c_{I,12}(\theta),\\
c_{I,21}(\theta) &= c_{I,32}(\theta) = c_{I,33}(\theta) = 0,\\
c_{I,22}(\theta) &= c_{I,23}(\theta) = -c_{I,31}(\theta) = -2m_3 l_2 l_3 \sin\theta_3,\\
c_{II,11}(\theta) &= c_{II,22}(\theta) = c_{I,33}(\theta) = 0,\\
c_{II,21}(\theta) &= -c_{II,12}(\theta) = (m_2+m_3) l_1 l_2 \sin\theta_2 + m_3 l_1 l_3 \sin(\theta_2+\theta_3),\\
c_{II,31}(\theta) &= -c_{II,13}(\theta) = m_3 l_3 (l_2 \sin\theta_3 + l_1 \sin(\theta_2+\theta_3)),\\
c_{II,32}(\theta) &= -c_{II,23}(\theta) = m_3 l_2 l_3 \sin\theta_3.
\end{aligned}$$

$G_\theta(\theta) \in \mathbb{R}^3$ is a vector of gravity, which is given by

$$G_\theta(\theta) = \begin{bmatrix} g_1(\theta) & g_2(\theta) & g_3(\theta) \end{bmatrix}^{\mathrm{T}}$$

with

$$\begin{aligned}
g_1(\theta) &= g m_1 l_1 \cos\theta_1 + g m_2 (l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2)) +\\
&\quad + g m_3 (l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3)),\\
g_2(\theta) &= g m_2 l_2 \cos(\theta_1+\theta_2) +\\
&\quad + g m_3 (l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3)),\\
g_3(\theta) &= g m_3 l_3 \cos(\theta_1+\theta_2+\theta_3).
\end{aligned}$$

Linearized Model

It is useful to represent the dynamics of the manipulator in Cartesian coordinates. This is because the constraints on the environment as well as on the motion are often easily described in these coordinates. Let

$$z = \begin{bmatrix} x & y & \varphi \end{bmatrix}^{\mathrm{T}}$$

be a Cartesian vector representing position and orientation of the end effector. In Cartesian coordinates, the equations (1.40) and (1.41) are written as follows.

$$M_z(\theta)\ddot{z} + C_z(\theta,\dot{\theta}) + G_z(\theta) = u_z + F_z^{\mathrm{T}}\lambda, \tag{1.43}$$

$$\psi_z(\theta) = 0, \tag{1.44}$$

where

$$\begin{aligned} M_z(\theta) &= J^{-T}(\theta)M_\theta(\theta)J^{-1}(\theta), \\ G_z(\theta) &= J^{-T}(\theta)G_\theta(\theta), \\ C_z(\theta,\dot{\theta}) &= J^{-T}(\theta)\left[C_\theta(\theta,\dot{\theta}) - M_\theta(\theta)J^{-1}(\theta)\dot{J}(\theta)\dot{\theta}\right], \end{aligned}$$

and

$$u_z = J^{-T}(\theta)u_\theta,$$

where Jacobian $J(\theta)$ satisfies $\dot{z} = J(\theta)\dot{\theta}$ and is given by

$$J(\theta) = \begin{bmatrix} -l_1s_1 - l_2s_{12} - l_3s_{123} & -l_2s_{12} - l_3s_{123} & -l_3s_{123} \\ l_1c_1 + l_2c_{12} + l_3c_{123} & l_2c_{12} + l_3c_{123} & l_3c_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

with

$$\begin{aligned} s_1 &= \sin\theta_1,\ s_{12} = \sin(\theta_1+\theta_2),\ s_{123} = \sin(\theta_1+\theta_2+\theta_3), \\ c_1 &= \cos\theta_1,\ c_{12} = \cos(\theta_1+\theta_2),\ c_{123} = \cos(\theta_1+\theta_2+\theta_3). \end{aligned}$$

Due to relations

$$x = l_1c_1 + l_2c_{12} + l_3c_{123},\ \varphi = \theta_1+\theta_2+\theta_3,$$

we obtain

$$\psi_z(z) = F_0z - L_0,$$

where

$$F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ L_0 = \begin{bmatrix} l \\ 0 \end{bmatrix}.$$

Therefore,

$$F_z = \frac{\partial\psi_z(z)}{\partial z} = F_0.$$

This means that the considered robot model has linear constraints in Cartesian coordinates. This outcome is favorable in linearizing the model.

Choose the operating point of linearization as

$$z_\omega = \left[\,l\ \ l+\tfrac{\Delta l}{2}\ \ 0\,\right]^{\mathrm{T}},\ \dot{z}_\omega = \left[\,0\ \ \dot{y}_\omega\ \ 0\,\right]^{\mathrm{T}}.$$

Then the linearized model can be obtained as follows.

$$M_0\delta\ddot{z} + D_0\delta\dot{z} + K_0\delta z = S_0\delta u + F_0^{\mathrm{T}}\delta\lambda, \tag{1.45}$$

$$F_0\delta z = 0, \tag{1.46}$$

where

$$M_0 = M_z \mid_{z=z_\omega}, \tag{1.47}$$

$$D_0 = \frac{\partial C_z}{\partial z} \mid_{\substack{z = z_\omega \\ \dot{z} = \dot{z}_\omega}}, \tag{1.48}$$

$$K_0 = \frac{\partial C_z}{\partial z} \mid_{\substack{z = z_\omega \\ \dot{z} = \dot{z}_\omega}} + \frac{\partial G_z}{\partial z} \mid_{z=z_\omega}, \tag{1.49}$$

$$S_0 = J^{-T} \mid_{\theta=\theta_\omega}, \tag{1.50}$$

$$F_0 = F_z. \tag{1.51}$$

Here, $\delta u = u - u_\omega$, $\delta\lambda = \lambda - \lambda_\omega$, $\delta z = z - z_\omega$, angle θ_ω is determined by z_ω through inverse kinematics. Vector λ_ω is two-dimensional. The first element of λ_ω is chosen to be equal to the desired contact force in the x direction, and the second element is chosen to be zero. Then u_ω is suitably determined such that (1.45) remains balanced.

Define the state vector:

$$x^{\mathrm{T}} = \left[\delta z^{\mathrm{T}} \ \delta\dot{z}^{\mathrm{T}} \ \delta\lambda^{\mathrm{T}} \right]$$

where

$$\delta z = \left[\delta x \ \delta y \ \delta\varphi \right]^{\mathrm{T}} \text{ and } \delta\lambda = \left[\delta\lambda_1 \ \delta\lambda_2 \right]^{\mathrm{T}}.$$

Choosing δy, $\delta\lambda_1$, and $\delta\lambda_2$ as tracking outputs, system (1.45) and (1.46) can be written in the linear descriptor form

$$E\dot{x} = Ax + Bu, \tag{1.52}$$

$$y = Cx + Du, \tag{1.53}$$

where

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & M_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & I & 0 \\ -K_0 & -D_0 & F_0^{\mathrm{T}} \\ F_0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ S_0 \\ 0 \end{bmatrix}, \tag{1.54}$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = 0. \tag{1.55}$$

1.6 Electrical Networks

Following [16], consider an electrical network that consists of B branches connected to N nodes. Denote a current variable at each branch by $i_B(t)$ and a voltage variable at each node by $v_N(t)$. Equations of the circuit can be derived from the following Kirchhoff's laws.

1. The algebraic sum of currents in a node at any loop is zero.
2. The algebraic sum of voltage drops at any loop is zero.

Let the current be the flow of positive charge. Therefore we can define a current direction from a positive node to a negative one. The structure of the circuit can be described via a $B \times N$ network incidence matrix A. Element $A(i, j) = \pm 1$ if node j is the $\pm$ node for the ith branch.

Denote the vector of current variables by $i_B(t)$. Then, from Kirchhoff's current law, $A^{\mathrm{T}} i_B = 0$. By definition, the voltage drop across each branch is the difference between the voltage at the positive node and the voltage at the negative node. Branch and nodal voltages ($v_B(t)$ and $v_N(t)$) are connected by the expression $v_B(t) = A v_N(t)$.

Consider circuits with branches that include capacitors, inductors, and resistors. Energy can be stored as a charge or an electrical field in a capacitor, and as a magnetic field in an inductor. Resistors are used to reduce or increase power in a branch. Suppose that the relationship between current and voltage across branches is linear in such circuits. The voltage-current relation across a branch with a resistor satisfies Ohm's law $v_R(t) = R i_R(t)$ where positive definite matrix. For a linear capacitor and a linear inductor voltage-current relations are given as $i_C(t) = C\dot{v}_C(t)$ and $v_L(t) = L\dot{i}_L(t)$. Networks with linear resistors, capacitors, and inductors are commonly referred to as linear RLC-circuits.

There may also be voltage sources where $v_E(t) = e(t)$ for any current $i_E(t)$, and current sources where $i_S(t) = i(t)$ for any voltage $v_S(t)$.

Divide the incidence matrix, branch currents, and branch voltages as follows.

$$A = \begin{bmatrix} A_E \\ A_C \\ A_R \\ A_L \\ A_S \end{bmatrix}, \quad i_B(t) = \begin{bmatrix} i_E(t) \\ i_C(t) \\ i_R(t) \\ i_L(t) \\ i_S(t) \end{bmatrix}, v_B(t) = \begin{bmatrix} v_E(t) \\ v_C(t) \\ v_R(t) \\ v_L(t) \\ v_S(t) \end{bmatrix}.$$

The subscripts R, C, L, E, and S stand for a resistor, a capacitor, an inductor, a voltage source, and a current source, respectively. The circuit's equation can be written as

$$E\dot{x}(t) + Gx(t) = f(t), \tag{1.56}$$

where

$$E = \mathrm{diag}(0, C, 0, 0, 0, 0, 0, 0, L, 0, 0),$$

$$G = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_E \\ 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_C \\ 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_R \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & A_L \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & A_S \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & A_E^{\mathrm{T}} & A_C^{\mathrm{T}} & A_R^{\mathrm{T}} & A_L^{\mathrm{T}} & A_S^{\mathrm{T}} & 0 \end{bmatrix},$$

$$x(t) = \begin{bmatrix} v_B(t) \\ i_B(t) \end{bmatrix},$$

$$f(t) = \left[e(t)^{\mathrm{T}} \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; i(t)^{\mathrm{T}} \; 0 \right]^{\mathrm{T}}.$$

The inductance matrix L is diagonal if inductors are uncoupled, and $L \geqslant 0$ if they are mutually coupled. R and C are diagonal matrices with positive diagonal elements, and I is an identity matrix. Therefore system (1.56) is quite sparse and can be transformed to a semi-explicit form by changing variables. Equation (1.56) is still valid if E and G depend on time.

Example 1.4 Consider a simple circuit network [7, 17] as shown in Fig. 1.8, where voltage source $V_s(t)$ is a driver (control input), R, L, and C stand for the resistor, inductor, and capacity, respectively, as well as their quantities, and their voltages

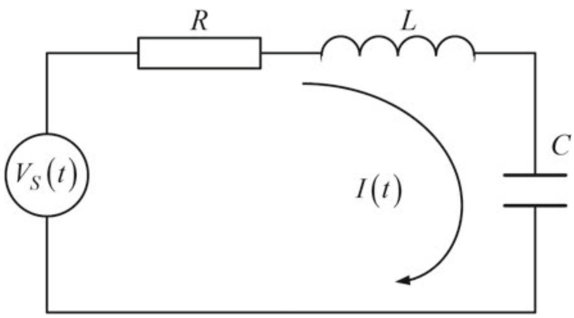


Fig. 1.8 RLC-circuit

are denoted by $V_R(t)$, $V_L(t)$, and $V_C(t)$, respectively. Then according to Kirchhoff's laws, we have the following circuit's equation (descriptor system).

$$\begin{bmatrix} L\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix} \begin{bmatrix} \dot{I}(t) \\ \dot{V}_L(t) \\ \dot{V}_C(t) \\ \dot{V}_R(t) \end{bmatrix} = \begin{bmatrix} 0\,1\,0\,0 \\ 1/C\,0\,0\,0 \\ -R\,0\,0\,1 \\ 0\,1\,1\,1 \end{bmatrix} \begin{bmatrix} I(t) \\ V_L(t) \\ V_C(t) \\ V_R(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} V_s(t). \tag{1.57}$$

Now we consider a linear circuit, composed only of voltage sources, resistors with constant resistance matrix R, and capacitors with constant capacitance matrix C. We suppose that the circuit is grounded; that is, there exists a path from each node along branches to the ground. This assumption leads to linear independence of rows of the incidence matrix A, so that rank $(A) = N$ and $N \geq B$. Note that each row of A contains only two nonzero entries, $+1$ and -1. It means that two rows of A are linearly dependent if and only if they are identical. In this case, the circuit contains a two-branch loop between these two nodes, and one branch could effectively be eliminated from the circuit. For a linear RC-circuit with voltage sources, the circuit must be connected, the voltage sources are independent, and there must be no loops containing only capacitors. Such kinds of systems can easily be reduced to the ODE system by straightforward algebraic computation. However, the sparsity of the original system is generally destroyed by this process.

Descriptor systems with impulsive modes arise in circuits containing differential amplifiers, which can be realized using operational amplifiers. An operational amplifier is a three-terminal device with two input terminals and one output terminal, as shown in Fig. 1.9. It is supposed that an ideal operational amplifier has no voltage drop or current across the input branch, and its gain is said to be infinite.

For example, the operational amplifier in Fig. 1.9 satisfies the relation $v_3 = K(v_1 - v_2)$ for $K \approx 10^5$. Consider a circuit with one differential amplifier as depicted in Fig. 1.10. If we assume that an operational amplifier is ideal, the circuit's equations lead to the relation $v_3 = -CR\dot{e}(t)$ where $e(t)$ is a voltage source. The solution to the circuit's equations involves at least one derivative of the input function. These equations have a differentiation index of DAE at least two. This index defines a

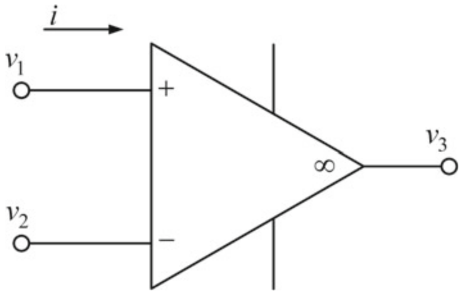


Fig. 1.9 Operational amplifier

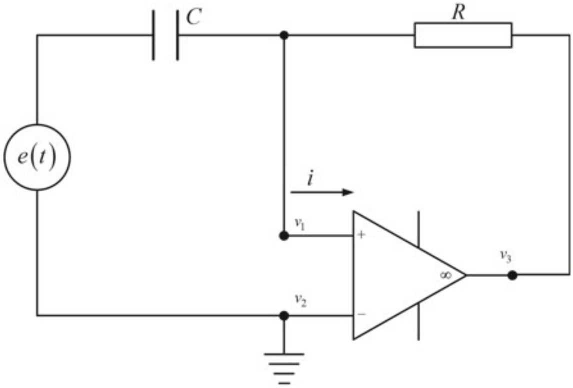


Fig. 1.10 Electrical circuit with one operational amplifier

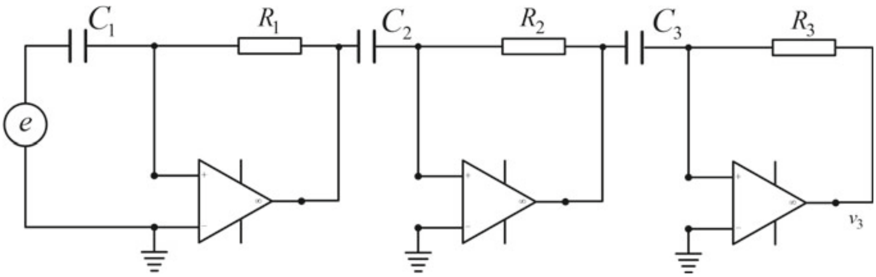


Fig. 1.11 Electrical circuit with cascade connection of operational amplifiers

minimum number of differentiations of the initial DAE that is required to obtain equivalent ordinary differential equations.

By cascading a series of differential amplifiers in a circuit as in Fig. 1.11, the index of the resulting system can be made arbitrarily high. The index of the system in Fig. 1.11 is at least four because $v_3 = -C_3R_3C_2R_2C_1R_1\dddot{e}(t)$.

1.7 Discretization of Partial Differential Equations

The method of solution of partial differential equations (PDEs) can lead to descriptor systems [4]. Such methods are the method of lines (MOL) and moving grids.

Numerical methods for solving PDEs usually involve substituting all derivatives by discrete difference approximations. MOL is also based on this discretization. However, MOL provides several advantages of the existing software. For parabolic PDEs, the typical MOL approach consists of discretization of the spatial derivatives. This can be made, for example, by finite differences, which convert the PDE system into an ODE initial conditions problem.

The MOL approach has two important advantages. The first is computational efficiency. Most ODE software is developed to be robust and computationally efficient. The second one is discretization only of spatial derivatives. This fact allows reducing the work required to develop a computer code.

As a rule, many MOL problems lead to an explicit ODE. However, many practical problems are more easily handled as algebraic-differential systems.

Example 1.5 As a first simple example consider the equation of MOL

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2},$$

defined in the region $t \geq 0$ and $0 \leq x \leq 1$ with boundary conditions given for $y(t, 0)$ and $y(t, 1)$, and initial conditions given for $y(0, x)$.

Taking a uniform spatial mesh of Δx, and mesh points $x_j = (j + 1)\Delta x$, $1 \leq j \leq (1/\Delta x) - 1 = N$, and using centered differences, we obtain the semi-explicit algebraic-difference equation for the variables $y_i(t) = y(t, x_i)$

$$\begin{aligned} \dot{y}_j - \frac{y_{j-1} - 2y_j + y_{j+1}}{(\Delta x)^2} &= 0, \; j = 2, \dots, N-1, \\ y_1 - y(t, 0) &= 0, \\ y_N - y(t, 1) &= 0. \end{aligned}$$

This particular problem is easily reformulated as an ODE, but this is not always the case.

Example 1.6 Consider the following equations for ignition of a single-component nonreacting gas in a closed cylindrical vessel in Lagrangian coordinates [16]

$$\frac{\partial T}{\partial t} - \frac{1}{\rho c_p}\frac{\partial p}{\partial t} = \frac{1}{c_p}\frac{\partial}{\partial \psi}\left(\rho r^2 \lambda \frac{\partial T}{\partial \psi}\right), \tag{1.58}$$

$$0 = \frac{\partial r}{\partial \psi} - \frac{1}{\rho r}, \tag{1.59}$$

$$0 = \frac{\partial p}{\partial \psi}, \tag{1.60}$$

$$0 = p - \rho\frac{RT}{W} \tag{1.61}$$

with conservation of mass

$$\int_0^R \rho r dr = \psi_R.$$

Because (1.61) can be used to evaluate ρ, we consider a descriptor system obtained by discretizing (1.58)–(1.60) with state variables r in the sense of spatial coordinates, T is temperature, and p stands for pressure. Boundary conditions are given at the center ($\psi = 0$)

$$r = 0,\ \frac{\partial T}{\partial \psi} = 0,$$

and at the vessel boundary ($\psi = \psi_R$)

$$r = R,\ T = T_w.$$

Notice that after discretization, time derivatives appear only in the equations derived from (1.58) and do not appear from (1.59) and (1.60). There are only two boundary conditions on r and T, but none on p. Boundary implicit conditions on p can also be given in algebraic form.

Now we show one more example of the system where the MOL and descriptor representation are appropriate.

Example 1.7 Consider the flow of an incompressible viscous fluid described by Navier–Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla\rho + \gamma\nabla^2 u, \tag{1.62}$$

$$\nabla \cdot u = 0, \tag{1.63}$$

where u is a velocity (u may have a dimension equal to two or three), p is a scalar pressure, and γ is a kinematic viscosity. Equation (1.62) is the momentum equation. Equation (1.63) is the incompressibility condition. After spatial discretization of the expressions with a finite difference or finite element method, the equations take the form

$$M\dot{U}(t) + (K + N(U(t)))U(t) + CP(t) = f(U(t), P(t)),$$
$$C^{\mathrm{T}}U(t) = 0,$$

where vectors $U(t)$ and $P(t)$ are approximations of $u(t, x)$ and $p(t, x)$, respectively, mass matrix M is either an identity matrix if the finite differences method is used or a symmetric positive definite matrix in the case of the finite elements method, discretization of the operator ∇ is C, and the forcing function f comes from the boundary conditions.

As can be seen from Example 1.7, PDEs can be easily approximated by virtue of algebraic-differential equations.

Conclusion

The variety of differential-algebraic systems is not limited only by the above-mentioned examples and applications. Different examples of technical systems can be found, for example, in [7, 9, 10, 16, 18].

The main purpose of this chapter is to show the features of descriptor systems and to give an idea of their use in different applications. Many of the considered examples can also be rewritten in the form of ordinary equations, but there can be difficulties in some transformations. In addition, the equations, written in descriptor variables, are more suitable from the modeling process visualization point of view, because in this case state variables represent physical processes in the system, and there is no need to use the inverse transform to obtain them.

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Chapter 2
Basics of Discrete-Time Descriptor Systems Theory

2.1 Equivalent Forms of Descriptor Systems

The state-space representation of a LDTI descriptor system P is

$$Ex(k+1) = Ax(k) + Bf(k), \tag{2.1}$$
$$y(k) = Cx(k) + Df(k) \tag{2.2}$$

where $x(k) \in \mathbb{R}^n$ is the state, $f(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are the input and output signals, respectively, $k \in \mathbb{Z}$, $k \geqslant 0$. E, A, B, C, D are constant real matrices of appropriate dimensions, and rank $E = r < n$.

We also use the following denotation for system (2.1) and (2.2).

$$P = \left[E, \, \begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{2.3}$$

Mathematical models of control systems depend on the choice of state variables. Obviously, this choice is not unique. This leads to nonuniqueness of the system's model. In this section, the relationship between state variables and models is examined.

Definition 2.1 Two realizations $\left[E, \, \begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. and $\left[\overline{E}, \, \begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array}\right]$ are called *restricted system equivalent* if there exist such nonsingular matrices $\overline{W}$ and $\overline{V}$, that

$$E = \overline{W}\,\overline{E}\,\overline{V}, \; A = \overline{W}\,\overline{A}\,\overline{V}, \; B = \overline{W}\,\overline{B}, \; C = \overline{C}\,\overline{V}$$

with coordinate transformation $x(k) = \overline{V}\overline{x}(k)$. Matrix pair $(\overline{W}, \;\; \overline{V})$ is called *system equivalence transformation*.

Definition 2.1 is also true for continuous-time descriptor systems.

A. A. Belov et al., *Control of Discrete-Time Descriptor Systems*, Studies in Systems, Decision and Control 157, https://doi.org/10.1007/978-3-319-78479-3_2

Example 2.1 ([1]) Consider a simple RLC-circuit as shown in Fig. 1.8. Recall that its mathematical model can be written in the form:

$$\begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{I}(t) \\ \dot{V}_L(t) \\ \dot{V}_C(t) \\ \dot{V}_R(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/C & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I(t) \\ V_L(t) \\ V_C(t) \\ V_R(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} V_S(t). \qquad (2.4)$$

On the other hand, if we choose the following state variables:

$$\widetilde{x} = \begin{bmatrix} I(t) \\ V_R(t) + V_L(t) \\ V_R(t) + V_L(t) + V_C(t) \\ V_R(t) \end{bmatrix}, \qquad (2.5)$$

the system will take the form

$$\begin{bmatrix} L & 0 & 0 & 0 \\ L & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ L & 0 & 0 & 0 \end{bmatrix} \dot{\widetilde{x}}(t) = \begin{bmatrix} 0 & 1 & 1 & -1 \\ -1/C & 1 & 1 & -1 \\ 1/C - 2R & 0 & 0 & 2 \\ -R & 1 & 0 & 0 \end{bmatrix} \widetilde{x}(t) + \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} V_S(t). \qquad (2.6)$$

Both equations (2.4) and (2.6) are mathematical models that describe the behavior of the circuit shown in Fig. 1.8. Despite the fact that Eqs. (2.4) and (2.6) are different, it is easy to check that the systems are equivalent; transformation matrices are the following.

$$\overline{W} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \ \overline{V} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Remark 2.1 A continuous-time descriptor system of the form

$$E_c\dot{x}(t) = A_c x(t) + B_c u(t)$$

can be easily transformed to a discrete-time system, driven by

$$E_d\widetilde{x}(k+1) = A_d\widetilde{x}(k) + B_d\widetilde{u}(k)$$

with the matrices:

$$E_d = E_c, \ A_d = hA_c + E_c, \ B_d = hB_c.$$

Here h stands for the step of discretization.

Now we give some definitions from the theory of matrices necessary for further presentation.

Definition 2.2 Square matrix N is called the *nilpotent of index* h, if $N^h = 0$, and $N^i \neq 0$, $i = \overline{1, (h-1)}$.

Definition 2.3 For any two given matrices $E, A \in \mathbb{R}^{n \times n}$ a matrix pair (E, A) is called *regular* if there exists a constant scalar $\lambda \in \mathbb{C}$, for which $\det(\lambda E + A) \neq 0$.

As shown in [1], a matrix pair (E, A) is regular if and only if there exist two nonsingular matrices $\overline{W}$ and $\overline{V}$ such that

$$\overline{W} E \overline{V} = \mathrm{diag}(I_r, N), \ \overline{W} A \overline{V} = \mathrm{diag}(A_1, I_{n-r}) \tag{2.7}$$

where rank $(E) = r$, $A_1 \in \mathbb{R}^{r \times r}$, $N \in \mathbb{R}^{(n-r)\times(n-r)}$ is a nilpotent matrix.

Unfortunately, existence of equivalent form (2.7) is connected with nilpotent transformations and difficult to check. Verification of the existence and uniqueness of solutions to descriptor systems can be carried out on the basis of the Definition 2.3. Thus, to guarantee the existence and uniqueness of solutions for system (2.1), we assume that E and A are square matrices, and the pair (E, A) is regular.

The most used equivalent forms of descriptor systems are discussed below.

2.1.1 Weierstrass Canonical Form

For any regular descriptor system (2.1) and (2.2) there are two nonsingular matrices $\overline{W}$ and $\overline{V}$ such that equations (2.1) and (2.2) are equivalently described by

$$\begin{aligned} x_1(k+1) &= A_1 x_1(k) + B_1 f(k), \\ y_1(k) &= C_1 x_1(k), \end{aligned} \tag{2.8}$$

$$\begin{aligned} N x_2(k+1) &= x_2(k) + B_2 f(k), \\ y_2(k) &= C_2 x_2(k), \end{aligned} \tag{2.9}$$

$$y(k) = C_1 x_1(k) + C_2 x_2(k) + Df(k) = y_1(k) + y_2(k) + Df(k) \tag{2.10}$$

with the change of variables

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \overline{V}^{-1} x(k), \ x_1(k) \in \mathbb{R}^r, \ x_2(k) \in \mathbb{R}^{n-r} \tag{2.11}$$

and

$$\begin{aligned} \overline{W} E \overline{V} = \operatorname{diag}(I_r, N),\ \overline{W} A \overline{V} = \operatorname{diag}(A_1, I_{n-r}), \\ \overline{W} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ C \overline{V} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \end{aligned} \tag{2.12}$$

where $N \in \mathbb{R}^{(n-r)\times(n-r)}$ is a nilpotent matrix.

Equations (2.8)–(2.10) determine canonical form, commonly called standard decomposition. For this form, subsystems (2.8) and (2.9) are called slow and fast subsystems, respectively, and $x_1(t)$ and $x_2(t)$ are slow and fast states, respectively.

Definition 2.4 The nilpotency index of N is called the *index of matrix pencil* $(\lambda E - A)$.

Typically, matrices $\overline{W}$ and $\overline{V}$ that put the system into canonical form are not unique, and thus matrices A_1, B_1, B_2, C_1, C_2, and N are not unique. Suppose that $\widehat{W}$ and $\widehat{V}$ are nonsingular, and system (2.1) and (2.2) is transformed into canonical form. In other words, (2.1) and (2.2) is a system of restricted equivalence for the system

$$\begin{aligned} \overline{x}_1(k+1) &= \overline{A}_1 \overline{x}_1(k) + \overline{B}_1 \overline{f}(k), \\ \overline{y}_1(k) &= \overline{C}_1 \overline{x}_1(k), \end{aligned} \tag{2.13}$$

$$\begin{aligned} \overline{N}\overline{x}_2(k+1) &= \overline{x}_2(k) + \overline{B}_2 \overline{f}(kt), \\ \overline{y}_2(k) &= \overline{C}_2 \overline{x}_2(k), \end{aligned} \tag{2.14}$$

$$y(k) = C_1 x_1(k) + C_2 x_2(k) + D\overline{f}(k) = y_1(k) + y_2(k) + D\overline{f}(k) \tag{2.15}$$

with change of variables

$$\widehat{V}^{-1} x(k) = \begin{bmatrix} \overline{x}_1(k) \\ \overline{x}_2(k) \end{bmatrix}.$$

It is shown in [1], that systems (2.8)–(2.10) and (2.13)–(2.15) are standard decompositions of the system (2.1) and (2.2) if and only if

$$\dim x_1(k) = \dim \overline{x}_1(k)$$

and there exist two nonsingular matrices $T_1 \in \mathbb{R}^{r\times r}$, $T_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ such that

$$\overline{W} = \operatorname{diag}(T_1, T_2)\widehat{W}, \quad \overline{V} = \widehat{V}\operatorname{diag}(T_1, T_2).$$

This implies that this form is unique up to the similarity transformations.

If A_1 is given in Jordan canonical form, then system (2.8)–(2.10) is called *Weierstrass canonical form*. Weierstrass canonical form plays an important role in time domain analysis of descriptor systems. The regularity property (i.e., existence and uniqueness of solution) of a descriptor system is connected with the existence of the Weierstrass canonical form.

Example 2.2 Consider a RLC-circuit from Example 1.4. Suppose $L = 1$, $R = 100$, $C = 0.1$, and an output equation is

$$y(t) = V_c(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x(t),$$

$$x(t) = \begin{bmatrix} I(t) & V_L(t) & V_C(t) & V_R(t) \end{bmatrix}^{\mathrm{T}},$$

thus the descriptor system can be written in the form

$$\begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0&1&0&0 \\ 10&0&0&0 \\ -100&0&0&1 \\ 0&1&1&1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} V_s(t), \tag{2.16}$$
$$y(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x(t).$$

Choose the following transformation of coordinates.

$$\overline{W}^{-1} x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \; x_1(t) \in \mathbb{R}^2, \; x_2(t) \in \mathbb{R}^2,$$

$$\overline{W} = \begin{bmatrix} 1&0&1&-1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}, \overline{V} = \begin{bmatrix} 1&0&0&0 \\ -100&-1&1&1 \\ 0&1&0&0 \\ 100&0&1&0 \end{bmatrix}.$$

Then the initial system takes the form

$$\dot{x}_1(t) = \begin{bmatrix} -100 & -1 \\ 10 & 0 \end{bmatrix} x_1(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_s(t),$$
$$0 = x_2(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} V_s(t),$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 & 0 \end{bmatrix} x_2(t).$$

System (2.16) can be easily transformed into a discrete-time system, using discretization from Remark 2.1.

2.1.2 SVD Equivalent Form

Let $r = \operatorname{rank}(E)$. From matrix theory it is known that there are two nonsingular matrices $\widetilde{W}$ and $\widetilde{V}$ that $\widetilde{W}E\widetilde{V} = \operatorname{diag}(I_r, 0)$. Applying the coordinate transformation $\widetilde{V}^{-1}x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$, $x_1(k) \in \mathbb{R}^r$, $x_2(k) \in \mathbb{R}^{n-r}$, system (2.1) and (2.2) is equivalent to the system:

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1 f(k), \tag{2.17}$$

$$0 = A_{21}x_1(k) + A_{22}x_2(k) + B_2 f(k), \tag{2.18}$$

$$y(k) = C_1x_1(k) + C_2x_2(k) + Df(k) \tag{2.19}$$

where

$$\widetilde{W}A\widetilde{V} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \widetilde{W}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C\widetilde{V} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{2.20}$$

System (2.17)–(2.19) is the second equivalent form (singular value decomposition, SVD, equivalent form) of system (2.1) and (2.2). Transformation matrices $\widetilde{W}$ and $\widetilde{V}$ are not the only thing that leads to nonuniqueness of SVD equivalent form. Two SVD forms can have a quite complex relation.

Matrices $\widetilde{W}$ and $\widetilde{V}$ can be found, for example, from the SVD

$$E = U\operatorname{diag}(S, 0)H^{\mathrm{T}}. \tag{2.21}$$

Here U and H are real orthogonal matrices and S is a diagonal $r \times r$-matrix, that is formed by nonzero singular values of the matrix E

$$\widetilde{W} = \operatorname{diag}(S^{-1}, I_{n-r})U^{\mathrm{T}}, \qquad \widetilde{V} = H. \tag{2.22}$$

The SVD equivalent form accurately reflects the physical meaning of descriptor systems. Equation (2.17) is the difference and forms a dynamic subsystem, and equation (2.18) is algebraic and reflects the relation between subsystems. Thus a descriptor system can be considered as a complex system formed by several interconnected subsystems. Furthermore, states $x_1(k)$ and $x_2(k)$ are in two different subspaces: one subspace has dynamic properties (described by differential equations), and another subspace contains restrictions, relation and sequencing (described by algebraic equations).

Example 2.3 Consider the system from Example 2.2. Applying the coordinate transformation

$$\widetilde{V}^{-1}x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \; x_1(t) \in \mathbb{R}^2, \; x_2(t) \in \mathbb{R}^2,$$

$$\widetilde{W} = I_4,\ \widetilde{V} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

system (2.16) is equivalent to the system:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 \\ 10 & 0 \end{bmatrix} x_1(t) + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} x_2(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} V_s(t), \\ 0 &= \begin{bmatrix} -100 & 0 \\ & 0 & 1 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x_2(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} V_s(t), \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 & 0 \end{bmatrix} x_2(t). \end{aligned} \tag{2.23}$$

2.1.3 Generalized Upper Triangular Equivalent Form

Under the regularity assumption on system (2.1), there always exist such matrices [2] V and U, that

$$E = V \begin{bmatrix} E_f & E_u \\ 0 & E_\infty \end{bmatrix},\ A = V \begin{bmatrix} A_f & A_u \\ 0 & A_\infty \end{bmatrix}. \tag{2.24}$$

Matrix pencil $(\lambda E_f - A_f)$ is quasi-triangular and contains only finite eigenvalues (eigenvalues attributable to a differential subsystem), and matrix pencil $(\lambda E_\infty - A_\infty)$ is triangular and contains only infinite eigenvalues. It is obvious that matrices E_f and A_∞ are nonsingular, and matrix E_∞ is a nilpotent. The generalized upper triangular form is a special case of the generalized Schur form for regular matrix pencils. This form is convenient for solving generalized Lyapunov equations arising in stability analysis of descriptor systems, and also in computation of controllability and observability Gramians, which are mentioned below.

2.2 Discrete-Time Descriptor Systems on a Finite Horizon

Consider a LDTI descriptor system described by

$$Ex(k+1) = Ax(k) + Bu(k), \tag{2.25}$$

$$y(k) = Cx(k) \tag{2.26}$$

where $x(k) \in \mathbb{R}^n$ is a state vector, $u(k) \in \mathbb{R}^m$ is a control signal, $y(k) \in \mathbb{R}^p$ an output; $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ are constant matrices. Matrix E is singular ($\operatorname{rank}(E) < n$). Consider system (2.25) and (2.26) on a finite horizon $[0, L]$; $L \geq n$ is a fixed finite number.

For input values $u(0), u(1), \ldots, u(L)$ states $x(0), x(1), \ldots, x(L)$ of system (2.25) can be found from the relation

$$\begin{bmatrix} -A & E & & & \\ & -A & E & & \\ & & \ddots & & \\ & & & E & \\ & & & -A & E \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(L-1) \\ x(L) \end{bmatrix} = \begin{bmatrix} Bu(0) \\ Bu(1) \\ \vdots \\ Bu(L-1) \\ Bu(L) \end{bmatrix}. \tag{2.27}$$

The left multiplier in (2.27) is an $nL \times n(L+1)$-matrix. Thus, equation (2.27) has n independent solutions if they exist. If the initial and final conditions are different in at least one point and give rise to different solutions of the given system, then these conditions are called full.

Luenberger [3] noticed that if the pair (E, A) of system (2.25) is regular (i.e., $\det(zE - A) \neq 0$), there exists a full state such that a vector $x(k)$ is uniquely determined by initial and terminal states $x(0)$ and $x(L)$, and an input signal $u(k)$. Hereinafter, $k = \overline{0, L}$.

For normal systems, full state is determined by initial conditions only.

In accordance with regularity of the matrix pair (E, A), there are two nonsingular matrices $\overline{W}$ and $\overline{V}$ such that

$$x_1(k+1) = A_1 x_1(k) + B_1 u(k), \tag{2.28}$$

$$N x_2(k+1) = x_2(k) + B_2 u(k), \tag{2.29}$$

$$y(k) = C_1 x_1(k) + C_2 x_2(k). \tag{2.30}$$

In this system, (2.28) is a direct recurrent equation, where the state at each step is unique, determined only by the initial conditions $x_1(0)$ and the input signal $u(\cdot)$, and has the form

$$x_1(k) = A_1^k x_1(0) + \sum_{i=0}^{k-1} A_1^{k-i-1} B_1 u(i). \tag{2.31}$$

Equation (2.29) determines backward recursion, where the state uniquely defines the terminal state $x_2(L)$ and $u(\cdot)$ in accordance with the expression

$$x_2(k) = N^{L-k} x_2(L) - \sum_{i=0}^{L-k-1} N^i B_2 u(k+i). \tag{2.32}$$

Equations (2.31) and (2.32) show how the initial state $x_1(0)$ and the final state $x_2(L)$ form the full state in accordance with which the solution is defined as

$$x(k) = \overline{V} \begin{bmatrix} I \\ 0 \end{bmatrix} \left(A_1^k x_1(0) + \sum_{i=0}^{k-1} A_1^{k-i-1} B_1 u(i) \right) +$$

$$+\overline{V} \begin{bmatrix} 0 \\ I \end{bmatrix} \left(N^{L-k} x_2(L) - \sum_{I=0}^{L-k-1} N^i B_2 u(k+i) \right),$$

$$y(k) = Cx(k).$$

Example 2.4 Consider the system on a finite horizon, given as [1]

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(k+1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0.3 & 0.9 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 2 \\ -0.5 \end{bmatrix} u(k), \tag{2.33}$$

$$y(k) = \begin{bmatrix} 0 & 2 & 1 & 0 \end{bmatrix} x(k), \tag{2.34}$$

Denote $x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$, where $x_1(k), x_2(k) \in \mathbb{R}^2$. Then system (2.33) and (2.34) can be given in the form

$$x_1(k+1) = \begin{bmatrix} 1 & 1 \\ 0.3 & 0.9 \end{bmatrix} x_1(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \tag{2.35}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_2(k+1) = x_2(k) + \begin{bmatrix} 2 \\ -0.5 \end{bmatrix} u(k), \tag{2.36}$$

$$y(k) = \begin{bmatrix} 0 & 2 \end{bmatrix} x_1(k) + \begin{bmatrix} 1 & 0 \end{bmatrix} x_2(k), \tag{2.37}$$

Then the solution of this system is defined by expressions

$$x_1(k) = A_1^k x_1(0) + \sum_{i=0}^{k-1} A_1^{k-i-1} B_1 u(i),\ 0 \le k \le L,$$

$$x_2(k) = N^{L-k} x_2(L) + \sum_{i=0}^{L-k-1} N^i B_2 u(k+i) =$$

$$= \begin{cases} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_2(L) - \begin{bmatrix} 2 \\ -0.5 \end{bmatrix} u(k),\ k = L-1; \\ \begin{bmatrix} -0.5 \\ 0 \end{bmatrix} u(k+1) - \begin{bmatrix} 2 \\ -0.5 \end{bmatrix} u(k),\ 0 \le k \le L-2. \end{cases}$$

This shows that $x_2(k)$ does not depend on terminal state when $k \le (L-2)$.

The state of the system at each moment is determined not only by initial conditions and current input, as in case of normal systems of differential equations, but also by terminal conditions and future values of the input signal until the moment L.

For simplicity, we assume that system (2.25) and (2.26) is given in Weierstrass canonical form.

The following definitions are given for system (2.25).

Definition 2.5 System (2.25) on a finite horizon is called *causal*, if its state $x(k)$, $(0 \le k \le L)$ at each moment k is defined only by initial conditions $x(0)$ and formed by input signal $u(0), u(1), \dots, u(k)$; otherwise the system is called *noncausal*.

Causality is a very important property. Thus causal systems, defined on a finite horizon, constitute an important class of systems, the advantage of which is a simple physical implementation.

If follows from (2.32) that system (2.25) is causal if and only if $\deg(\det(zE - A)) = \text{rank}\,(E)$, or $N = 0$; that is, the system has no poles at infinity.

In the next section we discuss controllability and observability of system (2.25) and (2.26).

2.3 Controllability of Discrete-Time Descriptor Systems

Previously it was shown that in contrast to normal systems descriptor systems have several types of controllability and observability [1, 2, 4]. We consider these concepts in more detail.

2.3.1 C-Controllability (Complete Controllability)

As mentioned earlier, descriptor systems are not always causal, thus the state of the system at the current time k depends not only on initial conditions (for a direct subsystem) but also on terminal conditions (for a backward subsystem).

Definition 2.6 Known initial conditions for the direct subsystem and terminal conditions for the backward subsystem are called *boundary conditions*. They are denoted as $\begin{bmatrix} x_1(0) \\ x_2(L) \end{bmatrix}$.

Definition 2.7 System (2.25) is called *complete controllable (C-controllable)* if for any boundary conditions $\begin{bmatrix} x_1(0) \\ x_2(L) \end{bmatrix}$ and $w \in \mathbb{R}^n$ there exists such a moment k_1, $0 \le k_1 \le L$, and control signal $u(0), u(1), \dots, u(L)$, that $x(k_1) = w$.

Thus, C-controllability is determined at each point [1]. According to the assumption on C-controllability, state $x(k)$ can fill the entire state vector. Therefore system (2.25) is C-controllable if and only if its forward and backward recursions are controllable. As the state $x_1(k)$ is controlled by input sequences $u(k)$, $u(k+1), \ldots, u(L-1)$, it is possible to select the appropriate control input for certain components $x_1(k)$ and $x_2(k)$. If the system is written in Weierstrass canonical form, then C-controllability property is equivalent to the following rank conditions [1]

$$\operatorname{rank}[B_1, A_1B_1, \ldots, A_1^{r-1}B_1] = r,$$
$$\operatorname{rank}[B_2, NB_2, \ldots, N^{n-r-1}B_2] = n - r.$$

2.3.2 R-Controllability (Controllability in the Initial Reachable Set)

For any fixed terminal conditions $x_2(L) \in \mathbb{R}^{n-r}$ introduce a notation $R(x_2(L))$ for description of reachable set for (2.25) with an arbitrary initial state, which we simply call the initial reachable set

$$R(x_2(L)) = \{w | w \in \mathbb{R}^n, \exists x_1(0), 0 \le k_1 \le L,$$
$$u(0), u(1), \ldots, u(L) : x(k_1) = w\}.$$

Evidently, the initial reachable set $R(x_2(L))$ depends on $x_2(L)$. For different $x_2(L)$ reachable sets $R(x_2(L))$ can be different.

Definition 2.8 System (2.25) is called *controllable in the initial reachable set (R-controllable)* if for any fixed terminal condition $x_2(L)$ the system's state from any initial condition can be transferred to any point of $R(x_2(L))$ at a finite time by a control input.

R-controllability guarantees controllability for any condition from the initial reachable set. System (2.25) is R-controllable if and only if subsystem (2.28) is controllable [1]; that is,

$$\operatorname{rank}[B_1, A_1B_1, \ldots, A_1^{r-1}B_1] = r.$$

2.3.3 Y-Controllability (Causal Controllability)

While selecting a control in the form of feedback as

$$u(k) = Kx(k) + v(k), \tag{2.38}$$

where $K \in \mathbb{R}^{m\times n}$ is a constant matrix, $v(k)$ is a new control signal, the closed-loop system (2.25) is given as

$$Ex(k+1) = (A + BK)x(k) + Bv(k). \tag{2.39}$$

Here $k = \overline{0, L}$.

Definition 2.9 System (2.25) is called *causally controllable (Y-controllable)* if there exists a state feedback (2.38) such that the closed-loop system (2.39) is causal.

In real systems, violation of causality principle can yield a lot of troubles when solving problems of control, identification, and state estimation. Y-controllability provides the ability to control causality with feedback condition (2.38). In [1] it is shown that Y-controllability condition can easily be checked by the following rank equality

$$\operatorname{rank} \begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = \operatorname{rank}(E) + n.$$

In [2], the following equivalent definitions of all types of controllability are given.

Definition 2.10 1. System (2.25) or the triple (E, A, B) is called *R-controllable* if

$$\operatorname{rank}[\lambda E - A,\ B] = n \quad \text{for any finite} \quad \lambda \in \mathbb{C}. \tag{2.40}$$

2. System (2.25) or the triple (E, A, B) is called *Y-controllable* if

$$\operatorname{rank}[E,\ AK_E,\ B] = n, \quad \text{where} \quad K_E = \mathfrak{span}(\mathfrak{Ker} E). \tag{2.41}$$

3. System (2.25) or the triple (E, A, B) is called *C-controllable* if (2.40) and $\operatorname{rank}[E,\ B] = n$ are satisfied.

C-controllability implies that for any initial condition $x(0) \in \mathbb{R}^n$ and terminal condition $x(L) \in \mathbb{R}^n$ there exists a control input $u(k)$ which transfers the system from the state $x(0)$ into $x(L)$ for a finite time. This concept is given in [5], and it agrees with the definition of controllability from [1], introduced above.

R-controllability ensures that for any given finite values of $x(0), x(L) \in \mathscr{X}$ where

$$\mathscr{X} = \left\{ x \in \mathbb{R}^n : (I - P_r)x = \sum_{j=0}^{h-1} F_{-j-1} B w(j),\ w(j) \in \mathbb{R}^m \right\}, \tag{2.42}$$

$$P_r = \overline{V}^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} \overline{V}, \quad F_k = \overline{V}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{bmatrix} \overline{W}^{-1}, \tag{2.43}$$

matrices $\overline{W}$ and $\overline{V}$ transform the initial system to (2.28) and (2.29), and h is a nilpotency index of N, there is a control sequence $u(k)$ that transfers the system from the state $x(0)$ into $x(L)$ for a finite time. If $E = I$, R-controllability coincides with C-controllability and stands for classical controllability of normal systems [6].

Y-controllability means that for any initial condition $x(0) \in \mathbb{R}^n$ there exists a feedback control of the form $u(k) = Fx(k) + v(k)$, where $F \in \mathbb{R}^{m \times n}$ is a state-feedback gain, and $v(k) \in \mathbb{R}^m$ is a new control input, such that the closed-loop system is causal. Note that descriptor system (2.25) with the matrix pencil $(\lambda E - A)$ with nilpotency index at most one is Y-controllable.

2.4 Observability of Discrete-Time Descriptor Systems

Controllability, R-controllability, and Y-controllability define the ability to change the state of the system via control input signals. Observability is dual to controllability, which describes the possibility of restoring the system state vector using measurements $y(k)$ where $k = \overline{0, L}$.

As shown earlier, state $x(k)$, is fully defined by terminal condition $\begin{bmatrix} x_1(0) \\ x_2(L) \end{bmatrix} \in \mathbb{R}^n$ and the control signal $u(k)$. As control input $u(k)$ is a known vector, observability of (2.25) and (2.26) is, in fact, a possibility to recover the complete condition $\begin{bmatrix} x_1(0) \\ x_2(L) \end{bmatrix} \in \mathbb{R}^n$ via measurements of $y(k)$.

Definition 2.11 1. System (2.25) and (2.26) is called *observable* if its state $x(k)$ is uniquely defined by $\{u(i), y(i), i = 0, 1, \ldots, L\}$ at any time k.
2. System (2.25) and (2.26) is called *R-observable*, if it is observable in the initial reachable set $R(x_2(L))$ for any terminal condition $x_2(L) \in \mathbb{R}^{n-r}$.
3. System (2.25) and (2.26) is called *causally observable (Y-observable)*, if its state vector $x(k)$ is determined uniquely by initial condition $x_1(0)$, control input $u(i)$, $i = \overline{1, k}$, and the measurable output $y(i)$, $i = 0, 1, \ldots, L$ at any time k.

Dual to controllability, the rank criteria of observability, R-observability, and Y-observability for descriptor systems can be formulated as follows [1].

Let the system be given in Weierstrass canonical form, then

1. System (2.25) and (2.26) is observable if and only if

$$\operatorname{rank}\begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{r-1} \end{bmatrix} = r \quad \text{and} \quad \operatorname{rank}\begin{bmatrix} C_2 \\ C_2 N \\ \vdots \\ C_2 A^{n-r-1} \end{bmatrix} = n - r.$$

2. System (2.25) and (2.26) is R-observable if and only if

$$\operatorname{rank}\begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{r-1} \end{bmatrix} = r.$$

System (2.25) and (2.26) is causally observable (Y-observable) if and only if

$$\operatorname{rank}\begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = n + \operatorname{rank}(E).$$

2.5 Discrete-Time Descriptor Systems on an Infinite Horizon

In this section, we discuss the main concepts and definitions of discrete-time descriptor systems, defined on an infinite horizon. Such concepts are characteristics of discrete-time descriptor systems in frequency and time domains, controllability, and observability properties. It is shown, that the solution of a system on a finite horizon differs from the solution on an infinite horizon.

Consider a LDTI descriptor system in the form

$$Ex(k+1) = Ax(k) + Bu(k), \tag{2.44}$$
$$y(k) = Cx(k) + Du(k), \tag{2.45}$$

Here $k = 0, 1, 2, \dots$.

In (2.44) and (2.45) $x(k) \in \mathbb{R}^n$ is a state vector, $u(k) \in \mathbb{R}^m$ is a control signal, $y(k) \in \mathbb{R}^p$ is a measurable output; and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ are constant matrices. Matrix E is singular; that is, ($\operatorname{rank}(E) = r < n$). The matrix pair (E, A) is supposed to be regular. Without loss of generality we assume that $D = 0$. Indeed, if $D \neq 0$, then we can consider an extended descriptor system

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \xi(k+1) = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \xi(k) + \begin{bmatrix} B \\ D_2 \end{bmatrix} u(k), \tag{2.46}$$
$$y(k) = \begin{bmatrix} C & -D_1 \end{bmatrix} \xi(k), \tag{2.47}$$

where $D = D_1 D_2$ is a factorization of D; for example, $D_1 = I$ and $D_2 = D$. System (2.44) and (2.45) is equivalent to (2.46) and (2.47) if $\xi(k)$ with control input $u(k)$ is the solution of the initial system. This condition holds if and only if $\xi(k) = \begin{bmatrix} x(k) \\ -D_2 u(k) \end{bmatrix}$.

The difference between a descriptor system on an infinite horizon (2.44) and (2.45) and a descriptor system on a finite horizon (2.25) and (2.26) is that system (2.25) and (2.26) has both initial and finite conditions, whereas system (2.44) and (2.45) has only initial conditions.

2.5.1 Time Domain Analysis

Because system (2.44) is regular, there are two nonsingular matrices $\overline{W}$ and $\overline{V}$ such that system (2.44) and (2.45) is restricted system equivalence in Weierstrass canonical form, where

$$x(k) = \overline{V}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},\ x_1(k) \in \mathbb{R}^r,\ x_2(k) \in \mathbb{R}^{n-r},$$

and it takes the form

$$\begin{aligned} x_1(k+1) &= A_1 x_1(k) + B_1 u(k), \\ y_1(k) &= C_1 x_1(k), \end{aligned} \tag{2.48}$$

$$\begin{aligned} N x_2(k+1) &= x_2(k) + B_2 u(k), \\ y_2(k) &= C_2 x_2(k), \end{aligned} \tag{2.49}$$

$$y_1(k) = y_1(k) + y_2(k),\ k = 0, 1, 2, \ldots, \tag{2.50}$$

and

$$\begin{aligned} &\overline{W} E \overline{V} = \operatorname{diag}(I_r, N),\ \overline{W} A \overline{V} = \operatorname{diag}(A_1, I_{n-r}), \\ &\overline{W} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C\overline{V} = \begin{bmatrix} C_1, C_2 \end{bmatrix}, \end{aligned} \tag{2.51}$$

where $A_1 \in \mathbb{R}^{r\times r}$ and $N \in \mathbb{R}^{(n-r)\times(n-r)}$ is a nilpotent of index h. In (2.48) and (2.49) measurements of y_1 and y_2 are inaccessible.

Direct subsystem (2.48) is a LDTI normal system, which state is defined as

$$x_1(k) = A_1^k x_1(0) + \sum_{i=0}^{k-1} A_1^{k-i-1} B_1 u(i). \tag{2.52}$$

In this subsystem, there is a causal relation between state and control.

The backward subsystem (2.49) is the inverse recursion of state. By left multiplying it on $N^0 = I$, N^1, N^2, …, N^{h-1}, we have

$$x_2(k) = N x_2(k+1) - B_2 u(k),$$

$$N x_2(k+1) = N^2 x_2(k+2) - N B_2 u(k+1),$$

$$N^2 x_2(k+2) = N^3 x_2(k+3) - N^2 B_2 u(k+2),$$

$$\cdots\cdots\cdots\cdots$$

$$N^{h-1} x_2(k+h-1) = N^h x_2(k+h) - N^{h-1} B_2 u(k+h-1).$$

Taking into account $N^h = 0$, the state of the backward subsystem can be defined by

$$x_2(k) = -\sum_{i=0}^{h-1} N^i B_2 u(k+i), \tag{2.53}$$

which shows that in order to determine the state $x_2(k)$ it is necessary to know future values of the control signal $u(i)$, $k \le i \le k+h$.

Comparison of (2.32) with (2.53), allows us to conclude that they are identical for $k \le L-h$. Differences begin to appear when $L-h < k \le L$. Thus, if we consider (2.44) as a limiting case of (2.25), equation (2.53) is a limiting case of (2.32).

If we put together (2.51)–(2.53), we get a full state of system (2.44)

$$\begin{aligned} x(k) = \overline{V}\begin{bmatrix} I \\ 0 \end{bmatrix} x_1(k) + \overline{V}\begin{bmatrix} 0 \\ I \end{bmatrix} x_2(k) = \\ = \overline{V}\begin{bmatrix} I \\ 0 \end{bmatrix}\left(A_1^k x_1(0) + \textstyle\sum_{i=0}^{k-1} A_1^{k-i-1} B_1 u(i)\right) - \overline{V}\begin{bmatrix} 0 \\ I \end{bmatrix}\textstyle\sum_{i=0}^{h-1} N^i B_2 u(k+i) = \\ = \overline{V}\begin{bmatrix} I \\ 0 \end{bmatrix}\left(A_1^k \begin{bmatrix} I \\ 0 \end{bmatrix} \overline{V}^{-1} x(0) + \textstyle\sum_{i=0}^{k-1} A_1^{k-i-1} B_1 u(i)\right) - \\ -\overline{V}\begin{bmatrix} 0 \\ I \end{bmatrix}\textstyle\sum_{I=0}^{h-1} N^i B_2 u(k+i). \end{aligned} \tag{2.54}$$

It is well known that causality is performed for LDTI normal systems, although descriptor systems are not always causal. In order to determine the current state of a descriptor system, the future values of the input signal are often required. In fact, causality may not appear in real physical systems. For example, the fundamental dynamic Leontief model of economic systems is described by the equation

$$x(k) = Ax(k) + E(x(k+1) - x(k)) + d(k),$$

where $d(k)$ is an input signal, which may include various elements such as demand. The purpose of production is the sale of goods. But there is a temporary delay between these two moments. Thus the possible future consumption of goods (input) is often used to estimate production at the current time (state). The absence of the causality principle is true for systems where state variables are deployed in space, not in time. Noncausality is one of the important characteristics of discrete-time descriptor systems.

In discrete-time descriptor systems there are several types of causality.

- *Causality between state and control input*

  Without loss of generality we suppose that system (2.44) and (2.45) is given in the form (2.48)–(2.50). It is obvious that causality is present in system (2.48)–(2.50), if its backward subsystem (2.49) is causal. It follows from (2.53) that causality between state and control exists if and only if

$$NB_2 = 0. \tag{2.55}$$

- *Causality between measurable output and control input*

  Because the input–output relation is defined only for controllable and observable systems, it is no loss of generality to assume that the triple (N, B_2, C_2) is controllable and observable. Thus causality between $y(k)$ and $u(k)$ exists only when such a ratio exists between $y_2(k)$ and $u(k)$, and

$$y_2(k) = C_2x_2(k) = -\sum_{i=0}^{h-1} C_2N^iB_2u(k+i), \tag{2.56}$$

  therefore causality between $y_2(k)$ and $u(k)$ exists if and only if

$$C_2N^iB_2 = 0,\ \ i = 1, 2, \dots, h-1,$$

  that is,

$$\begin{bmatrix} C_2 & C_2N & \cdots & C_2N^{h-1} \end{bmatrix}^{\mathrm{T}} N \begin{bmatrix} B_2 & NB_2 & \cdots & N^{h-1}B_2 \end{bmatrix} = 0. \tag{2.57}$$

  Controllability and observability of the triple (N, B_2, C_2) gives $N = 0$.

Another difference between normal and descriptor systems in discrete time is that descriptor systems do not always have solutions for any initial conditions. This fact is easy to show using (2.54) when $k = 0$. We have

$$x(0) = \overline{V}\begin{bmatrix} I \\ 0 \end{bmatrix}\begin{bmatrix} I & 0 \end{bmatrix}\overline{V}^{-1}x(0) - \overline{V}\begin{bmatrix} 0 \\ I \end{bmatrix}\sum_{i=0}^{h-1} N^iB_2u(i)$$

or, in another form,

$$\begin{bmatrix} 0 & I \end{bmatrix}\overline{V}^{-1}x(0) = \sum_{i=0}^{h-1} N^iB_2u(i), \tag{2.58}$$

that are called the *consistent initial* conditions which state $x(0)$ satisfies.

If we pay attention to (2.53), we can see that consistency of the initial conditions is achieved only by a backward subsystem. As descriptor systems can describe a class of systems of high order, consisting of several interconnected subsystems, their state $x_2(k)$ is more like a pseudo-state in the sense that it only reflects connections between subsystems. As expected, the initial state loses its physical meaning as the normal initial state of the system, because it determines coupling between the subsystems at zero time.

For example, considering $N = 0$, expression (2.53) can be written as

$$x_2(k) = -B_2u(k),$$

that is just a linear combination of control signal $u(k)$ at the current time.

Let I_o denote a set of consistent initial conditions in the form:

$$I_o = \left\{ x(0) \in \mathbb{R}^n | \begin{bmatrix} 0 & I \end{bmatrix} \overline{V}^{-1} x(0) = \sum_{i=0}^{h-1} N^i B_2 u(i) \right\}. \tag{2.59}$$

When $u(k) = 0$, free motion of system (2.44) is defined by the expression

$$\begin{aligned} x_1(k) &= A_1^k x_1(0), \\ x_2(k) &= 0. \end{aligned} \tag{2.60}$$

The state of the backward subsystem $x_2(k)$ is equal to zero.

Equations (2.54) and (2.58) do not only show the differences between normal and descriptor systems in discrete time but also demonstrate the differences between continuous-time and discrete-time descriptor systems [1].

A continuous-time descriptor system is given as

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{2.61}$$

It has a solution

$$\begin{aligned} x(t) = \overline{V} \begin{bmatrix} I \\ 0 \end{bmatrix} \mathrm{e}^{A_1 t} x_1(0) + \textstyle\int_0^t \mathrm{e}^{A_1(t-\tau)} B_1 u(\tau) d\tau - \\ - \overline{V} \begin{bmatrix} 0 \\ I \end{bmatrix} \textstyle\sum_{i=0}^{h-1} \delta^{(i)}(0) N^{i+1} x_2(0) - \overline{V} \begin{bmatrix} 0 \\ I \end{bmatrix} \textstyle\sum_{i=0}^{h-1} N^i B_2 u^{(i)}(t) \end{aligned} \tag{2.62}$$

where $\delta^{(i)}(t)$ is the ith derivative of the Dirac function $\delta(t)$.

Comparison (2.54) with (2.62) shows that the presence of impulse components in the state equation of a continuous-time system entails a lack of causality in the state equation of a discrete-time system. Impulse components and derivatives of the input signal, which are contained in the state equation of a continuous system, indicate noncausal behavior (it is necessary to know the derivative of the input signal). Moreover, the expression $u(k+i)$ in the state equation of a discrete-time system simply matches the expression $\frac{\mathrm{d}^i u}{\mathrm{d}t^i}(t)$ in the continuous case.

Definition 2.12 Consider system (2.44) and (2.45)

1. The system is called *controllable (R-controllable and Y-controllable)* if for any sufficiently large $L > n$ the finite time series of (2.27) are controllable (R-controllable and Y-controllable).
2. The system is called *observable (R-observable and Y-observable)* if the finite time series of (2.27) are observable (R-observable and Y-observable) for any sufficiently large $L > n$.

Obviously, the two definitions are generalizations of corresponding notions for normal systems. They are introduced to system (2.44) and do not depend on L.

2.5.2 Frequency Domain Analysis

Thus, consider a discrete-time descriptor system

$$Ex(k+1) = Ax(k) + Bu(k), \tag{2.63}$$

$$y(k) = Cx(k). \tag{2.64}$$

As the matrix pair (E, A) is regular, system (2.63) and (2.64) can be transformed into Weierstrass canonical form [7]. There exist nonsingular matrices $\overline{W}$ and $\overline{V}$ that

$$E = \overline{W} \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} \overline{V} \text{ and } A = \overline{W} \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} \overline{V}, \tag{2.65}$$

where J and N are matrices in the Jordan canonical form; besides in addition N is a nilpotent of index h. The values n_f and n_∞ are dimensions of subspaces $(\lambda E - A)$, corresponding to finite and infinite eigenvalues. Also, $n_f = r = \text{rank}\,(E)$. Matrices

$$P_r = \overline{V}^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} \overline{V},\ P_l = \overline{W} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} \overline{W}^{-1} \tag{2.66}$$

are spectral projections of the matrix pencil $(\lambda E - A)$ onto the left and right root subspaces, that are responsible for finite eigenvalues.

Using Weierstrass canonical transformation (2.65), we get the following decomposition of the generalized resolvent into the Laurent series

$$(\lambda E - A)^{-1} = \sum_{k=-\infty}^{\infty} F_k \lambda^{-k-1}, \tag{2.67}$$

where coefficients F_k are

$$F_k = \begin{cases} \overline{V}^{-1} \begin{bmatrix} J^k & 0 \\ 0 & 0 \end{bmatrix} \overline{W}^{-1},\ k = 0, 1, 2, \ldots \\ \overline{V}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{bmatrix} \overline{W}^{-1},\ k = -1, -2, \ldots \end{cases} \tag{2.68}$$

Note that $F_k = 0$ for $k < -h$, where h is the nilpotency index of N. Matrices F_k are called *fundamental matrices*. They are important for discrete-time descriptor systems (2.63).

It is well known [1, 4] that if the matrix pair (E, A) is regular, and initial conditions $x(0)$ are consistent, we have

$$(I - P_r)x(0) = \sum_{j=0}^{h-1} F_{-j-1}Bu(j).$$

It means that discrete-time descriptor system (2.63) has a unique solution $x(k)$ for all $k \geq 0$. Using fundamental matrices F_k, this solution can be represented by the expression:

$$x(k) = F_k Ex(0) + \sum_{j=0}^{k+h-1} F_{k-j-1}Bu(j), k \geq 0. \tag{2.69}$$

It is easy to show that the solution belongs to the region (2.42), called the solution space for descriptor systems (2.63) and (2.64). Moreover, equation (2.69) shows that to determine $x(k)$ it is necessary to know not only the previous control values $u(j)$, $j \leq k$, but also future values of the control $u(j), k < j \leq k+h-1$. Such a property is called noncausality of discrete-time descriptor systems. For a causal system (2.63) the solution $x(k)$ is completely determined by the initial conditions $x(0)$ and control $u(0), u(1), \ldots, u(k)$. It is obvious that system (2.63) is causal if it has the nilpotency index equal to one.

2.5.3 Transfer Functions and Realizations

Consider a two-way **Z**-transform [8], that transfers a sequence $\{f(k)\}_{k\in\mathbb{Z}}$, where $f(k) \in \mathbb{R}^n$, into the function $F(z)$ of a complex variable z defined by the expression

$$F(z) = \mathbf{Z}[f(k)] = \sum_{k=-\infty}^{\infty} f(k)z^{-k}.$$

Complex variable z is called the frequency in the discrete case. Applying the **Z**-transform to descriptor system (2.63), we get $Y(z) = C(zE - A)^{-1}BU(z)$, where $U(z)$ and $Y(z)$ are **Z**-transform of sequences $\{u(k)\}_{k\in\mathbb{Z}}$ and $\{y(k)\}_{k\in\mathbb{Z}}$, respectively.

Definition 2.13 The rational matrix function $P(z) = C(zE - A)^{-1}B$ is called a *transfer function* of a discrete-time descriptor system (2.63) and (2.64).

The transfer function is a ratio between **Z**-transforms of input $u(k)$ and output $y(k)$. In other words, the transfer function $P(z)$ describes the input–output behavior of system and (2.64) in the frequency domain.

For any rational matrix transfer function $P(z)$ there exist such matrices E, A, B, and C, that $P(z) = C(zE - A)^{-1}B$; see [1]. Descriptor system (2.63) and (2.64) with these matrices is called the realization of $P(z)$. Note that the realization of $P(z)$ is not unique in the general case [1].

A characteristic of system (2.63) and (2.64) is said to be an input–output invariant if it does not depend on the transformation of equivalent systems. The transfer function is an input–output invariant, because

$$P(z) = C(zE - A)^{-1}B = \widehat{C}\widehat{V}\widehat{V}^{-1}(z\widehat{E} - \widehat{A})^{-1}\widehat{W}^{-1}\widehat{W}\widehat{B} = \widehat{C}(z\widehat{E} - \widehat{A})^{-1}\widehat{B}.$$

2.5.4 Impulse and Frequency Characteristics

Using (2.67) the transfer function can be expanded by a Laurent series [8] at the point $z = \infty$ in the following way.

$$P(z) = \sum_{k=-\infty}^{\infty} P_k z^{-k}, \tag{2.70}$$

where $P_k = CF_{k-1}B$ and F_k are defined by the formula (2.68). The sequence $\{P_k\}_{k\in\mathbb{Z}}$ determines an impulse transfer function of the discrete-time descriptor system (2.63) and (2.64). It is easy to see that the transfer function $G(z)$ represents **Z**-transformation of the impulse response function. Note that $P_k = 0$ for all $k \leq -h$, where h is the nilpotency index of the matrix N. The physical impulse transfer function of system (2.63) and (2.64) can be interpreted in the following way.

Consider the system of difference equations of the form

$$EX_{k+1} = AX_k + BU_k,\ Y_k = CX_k, \tag{2.71}$$

where $X_k \in \mathbb{R}^{n\times m}$, $U_k \in \mathbb{R}^{m\times m}$, and $Y_k \in \mathbb{R}^{p\times m}$. For the impulse pulse input $U_k = \delta_{0,k}I$, where $\delta_{j,k}$ means the Kronecker symbol, system (2.71) takes the form $Y_k = CF_{k-1}B = G_k$. Thus, elements P_k of the impulse transfer function of the system (2.63) and (2.64) coincide with the output matrices of the difference equation (2.71) caused by the impulse input.

Definition 2.14 The transfer function $P(z)$ is called *proper* if $\lim_{z\to\infty} P(z) < \infty$, and *improper* otherwise. If $\lim_{z\to\infty} P(z) = 0$, then $P(z)$ is called *strictly proper*.

Taking into account (2.70), the transfer function $G(z)$ can be decomposed into components in the form $P(z) = P_{sp}(z) + P_{ip}(z)$, where $P_{sp}(z) = \sum_{k=1}^{\infty} P_k z^{-k}$ and $P_{ip}(z) = \sum_{k=0}^{h-1} P_{-k} z^k$ are strictly proper and polynomial parts of $P(z)$, respectively. The transfer function $P(z)$ is strictly proper if and only if $P_k = 0$ for all $k \leq 0$. Moreover, $P(z)$ is proper if and only if $P_k = 0$ for $k \leq -1$. Obviously, if the matrix pencil $(\lambda E - A)$ is of index at most one, $P(z)$ is proper.

Note that the causal descriptor system (2.63) and (2.64) has a proper transfer function $P(z)$. However, system (2.63) and (2.64) with the proper transfer function $P(z)$ is not necessarily causal.

Example 2.5 Descriptor system (2.63) and (2.64) with matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = C^{\mathrm{T}}$$

is not causal, despite the fact that its transfer function

$$P(z) = \frac{2 - z}{z - 1}$$

is proper.

As in the standard state-space case [9], frequency characteristics of the discrete-time descriptor system (2.63) and (2.64) are determined by the values of its transfer function on the unit circle $P(\mathrm{e}^{i\omega})$. It follows from (2.70) that

$$P(\mathrm{e}^{i\omega}) = \sum_{k=-\infty}^{\infty} P_k \mathrm{e}^{-i\omega k}; \tag{2.72}$$

that is, the impulse transfer function P_k, $k \in \mathbb{Z}$, is the sequence of Fourier coefficients [8, 10] of the frequency characteristics $P(\mathrm{e}^{i\omega})$. System (2.63) and (2.64) with an input sequence $\mathrm{e}^{i\omega k}u(0),\; k \in \mathbb{Z}$ where $\omega \in \mathbb{R}$ and $u(0) \in \mathbb{R}^m$ has the output

$$\begin{aligned} y(k) &= \sum_{j=-\infty}^{\infty} C F_{k-j-1} B \mathrm{e}^{i\omega j} u(0) = \\ &= \left(\sum_{j=-\infty}^{\infty} P_j \mathrm{e}^{-i\omega j} \right) \left(\mathrm{e}^{i\omega k} u(0) \right) = P(\mathrm{e}^{i\omega}) \left(\mathrm{e}^{i\omega k} u(0) \right). \end{aligned}$$

Thus the frequency characteristic $P(\mathrm{e}^{i\omega})$ establishes a relationship between the input sequence and the output sequence of system (2.63) and (2.64).

2.5.5 *Controllability and Observability Gramians*

Consider matrices of causal controllability and observability, given by expressions

$$C_+ = [F_0 B, \ldots, F_k B \ldots] \tag{2.73}$$

$$O_+ = \left[F_0^{\mathrm{T}} C^{\mathrm{T}}, \ldots, F_k^{\mathrm{T}} C^{\mathrm{T}} \ldots \right]^{\mathrm{T}} \tag{2.74}$$

where matrices F_k are defined by (2.68). Suppose that a matrix pair (E, A) is stable.

Definition 2.15 1. *Causal controllability Gramian* of system (2.63) and (2.64) is given in the form

$$G_{dcc} = C_+ C_+^{\mathrm{T}} = \sum_{k=0}^{\infty} F_k B B^{\mathrm{T}} F_k^{\mathrm{T}}, \tag{2.75}$$

2. *Causal observability Gramian* has the form

$$G_{dco} = O_+ O_+^{\mathrm{T}} = \sum_{k=0}^{\infty} F_k^{\mathrm{T}} C^{\mathrm{T}} C F_k, \tag{2.76}$$

In turn, noncausal controllability and observability matrices are defined as

$$C_- = \left[F_{-h} B, \ldots, F_{-1} B\right] \tag{2.77}$$

$$O_- = \left[F_{-h}^{\mathrm{T}} C^{\mathrm{T}}, \ldots, F_{-1}^{\mathrm{T}} C^{\mathrm{T}}\right]^{\mathrm{T}}, \tag{2.78}$$

respectively.

Definition 2.16 1. The matrix:

$$G_{dnc} = C_- C_-^{\mathrm{T}} = \sum_{k=-h}^{-1} F_k B B^{\mathrm{T}} F_k^{\mathrm{T}} \tag{2.79}$$

is called the *noncausal controllability Gramian* of system (2.63) and (2.64)

2. The matrix

$$G_{dno} = O_- O_-^{\mathrm{T}} = \sum_{k=-h}^{-1} F_k^{\mathrm{T}} C^{\mathrm{T}} C F_k \tag{2.80}$$

is called the *noncausal observability Gramian*.

Definition 2.17 The *controllability Gramian* of a discrete-time descriptor system (2.63) and (2.64) is determined by the formula $G_{dc} = G_{dcc} + G_{dnc}$, and the *observability Gramian* is $G_{do} = G_{dco} + G_{dno}$.

If $E = I$, then $G_{dcc} = G_{dc}$ and $G_{dco} = G_{do}$ are controllability and observability Gramians of discrete-time normal systems [2].

The following theorem holds true.

Theorem 2.1 [2] *Consider a discrete-time descriptor system given in the form* (2.63) *and* (2.64). *Let the matrix pair* (E, A) *be stable. Then*

1. *Causal controllability and observability Gramians* G_{dcc} *and* G_{dco} *are the unique symmetric positive definite solutions of the projected generalized discrete-time Lyapunov equations*

$$\begin{gathered} A G_{dcc} A^{\mathrm{T}} - E G_{dcc} E^{\mathrm{T}} = -P_l B B^{\mathrm{T}} P_l^{\mathrm{T}}, \\ G_{dcc} = P_r G_{dcc} P_r^{\mathrm{T}} \end{gathered} \tag{2.81}$$

and

$$\begin{aligned} A^{\mathrm{T}} G_{dco} A - E^{\mathrm{T}} G_{dco} E &= -P_r^{\mathrm{T}} C^{\mathrm{T}} C P_r, \\ G_{dco} &= P_l^{\mathrm{T}} G_{dco} P_l, \end{aligned} \tag{2.82}$$

respectively.

2. *Noncausal controllability and observability Gramians G_{dnc} and G_{dno} are the only symmetric positive definite solutions of the projected generalized discrete-time Lyapunov equations*

$$\begin{aligned} A G_{dnc} A^{\mathrm{T}} - E G_{dnc} E^{\mathrm{T}} &= (I - P_l) B B^{\mathrm{T}} (I - P_l)^{\mathrm{T}}, \\ P_r G_{dnc} P_r^{\mathrm{T}} &= 0 \end{aligned} \tag{2.83}$$

and

$$\begin{aligned} A^{\mathrm{T}} G_{dno} A - E^{\mathrm{T}} G_{dno} E &= (I - P_r)^{\mathrm{T}} C^{\mathrm{T}} C (I - P_r), \\ P_l^{\mathrm{T}} G_{dno} P_l &= 0, \end{aligned} \tag{2.84}$$

respectively.

3. *Controllability and observability Gramians G_{dc} and G_{do} are the unique symmetric positive definite solutions of the projected generalized discrete-time Lyapunov equations*

$$\begin{aligned} A G_{dc} A^{\mathrm{T}} - E G_{dc} E^{\mathrm{T}} &= -P_l B B^{\mathrm{T}} P_l^{\mathrm{T}} + (I - P_l) B B^{\mathrm{T}} (I - P_l)^{\mathrm{T}}, \\ G_{dc} &= (I - P_r) G_{dc} (I - P_r)^{\mathrm{T}} \end{aligned} \tag{2.85}$$

and

$$\begin{aligned} A^{\mathrm{T}} G_{do} A - E^{\mathrm{T}} G_{do} E &= -P_r^{\mathrm{T}} C^{\mathrm{T}} C P_r + (I - P_r)^{\mathrm{T}} C^{\mathrm{T}} C (I - P_r), \\ G_{do} &= (I - P_l)^{\mathrm{T}} G_{do} (I - P_l), \end{aligned} \tag{2.86}$$

respectively.

The proof of this theorem can be found in [2, 11].

2.6 Stability of Discrete-Time Descriptor Systems

Consider a LDTI descriptor system given by the equations:

$$E x(k+1) = A x(k) + B u(k), \tag{2.87}$$

$$y(k) = C x(k) \tag{2.88}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control signal, $y(k) \in \mathbb{R}^p$ is the measurable output. The matrix $E \in \mathbb{R}^{n \times n}$ can be singular. We assume that $\operatorname{rank}(E) = r \le n$. A, B, C are known real matrices of appropriate dimensions.

Definition 2.18 The discrete-time system (2.87) is called *autonomically globally stable* or, simply, *stable* if for $u(k) = 0$ the following inequality is satisfied.

$$\|x(k)\| \leq \alpha\beta^k \|x(0)\|, \; k \in \mathbb{Z}, \; k \geqslant 0 \; \alpha > 0, \; 0 < \beta < 1$$

for any consistent initial condition $x(0)$.

Definition 2.19 System (2.87) is called *admissible* if it is regular, causal, and stable.

Definition 2.20 The discrete-time system (2.87) is called *asymptotically stable* if $\lim_{k\to\infty} x(k) = 0$ for any solutions of the system $Ex(k+1) = Ax(k)$.

For descriptor systems, written in various equivalent forms, it is easy to obtain the following relations.

Let system (2.87) and (2.88) be given in Weierstrass canonical form. Then

- The pair (E, A) is causal if and only if $N = 0$.
- The pair (E, A) is stable if and only if $\rho(A_1) < 1$.
- The pair (E, A) is admissible if and only if $N = 0$ and $\rho(A_1) < 1$.

Let system (2.87) and (2.88) be given in SVD equivalent form. Then

- The pair (E, A) is causal if and only if A_{22} is nonsingular.
- The pair (E, A) is admissible if and only if A_{22} is nonsingular and $\rho(A_{11} - A_{12}A_{22}^{-1}A_{21}) < 1$.

Example 2.6 Let the system have the state-space representation:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} f(k), \tag{2.89}$$

The solution is defined as

$$\begin{aligned} x_2(k) &= -f(k), \\ x_1(k) &= -f(k) - f(k+1). \end{aligned}$$

System (2.89) is noncausal. The transformation matrices $\overline{W}$ and $\overline{V}$ for system (2.89) can be selected as $\overline{W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\overline{V} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then $\overline{W}E\overline{V} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\overline{W}A\overline{V} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Here $A_{22} = 0$.

It is obvious that asymptotic stability of a descriptor system (2.87) can be described in terms of the generalized spectrum of the matrix pencil $(\lambda E - A)$. Note also that infinite eigenvalues do not affect asymptotic stability of the system (2.87). The conditions introduced above are based on the decomposition of the system's matrices, which can lead to computational difficulties. Given this, we suggest another form of admissibility conditions expressed by the following theorems. These conditions involve no equivalent transformations.

The following theorem gives strict admissibility conditions for a discrete-time descriptor system in terms of LMI.

Theorem 2.2 ([12] *The system* (2.87) *is admissible if and only if there exist such matrices* $R > 0$ *and* Q *that the following inequality holds.*

$$A^{\mathrm{T}}RA - E^{\mathrm{T}}RE + QS^{\mathrm{T}}A + A^{\mathrm{T}}SQ^{\mathrm{T}} < 0, \tag{2.90}$$

where $S \in \mathbb{R}^{n\times(n-r)}$ *is a full column rank matrix, satisfying the condition* $E^{\mathrm{T}}S = 0$.

Note that in the case of $E = I$ system (2.87) becomes nonsingular; it is easy to show that $S = 0$. Theorem 2.2 coincides with Lyapunov's theorem in [13].

The above theorem demonstrates the LMI-based approach to check admissibility of descriptor system (2.87). Another approach is based on the solution of the generalized discrete-time Lyapunov equation.

Theorem 2.3 ([2]) *A regular discrete-time descriptor system* (2.87) *is admissible if there exists a solution* $X = X^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ *of the generalized Lyapunov equation*

$$A^{\mathrm{T}}XA - E^{\mathrm{T}}XE + Q = 0, \tag{2.91}$$

satisfying the condition $E^{\mathrm{T}}XE \geqslant 0$ *for some matrix* $Q = Q^{\mathrm{T}} > 0$.

Note that the discretized model of the continuous descriptor system, obtained by relation

$$E\frac{x(k+1) - x(k)}{h} = Ax(k) \tag{2.92}$$

where h is a discretization step of the continuous-time system $E\dot{x}(t) = Ax(t)$, is admissible if matrix pair $(E, E + Ah)$ is admissible.

Conclusion

In this chapter, basic concepts of linear discrete-time descriptor systems theory are discussed. Unlike continuous-time systems, discrete-time ones should be considered not only on an infinite, but also on a finite, horizon. It is shown that the solution of systems defined on finite and infinite horizons are substantially different. The solution of a discrete-time system on a finite horizon is completely determined by its terminal conditions, some of which are initial and determine the position of the system at the initial moment, and the other part stands for terminal conditions and determines the position of another part of the system for a finite time.

In the case of an infinite horizon, another feature of discrete-time descriptor systems appears. The solution of the system can depend on future values of the input signal. Thus, noncausal behavior of the system (dependence on future values of the input signal) is analogous to the impulsive behavior of a continuous-time system.

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Chapter 3
Anisotropy-Based Analysis of LDTI Descriptor Systems

3.1 Preliminaries of Anisotropy-Based Control Theory

3.1.1 Anisotropy of the Random Vector and Mean Anisotropy of the Signal

Let $W = \{w(k)\}_{k\in\mathbb{Z}}$ be a stationary sequence of random m-dimensional vectors. Assembling the elements of W, associated with the interval $[0, N-1]$, into a random vector $W_{0:N-1} = \begin{bmatrix} w_0 \\ \vdots \\ w_{N-1} \end{bmatrix}$, we assume that $W_{0:N-1}$ is absolutely continuously distributed for every $N > 0$.

Definition 3.1 *Anisotropy* $\mathbf{A}(W_{0:N-1})$ is defined as the minimal value of relative entropy [1] with respect to the Gaussian distributions in $\mathbb{R}^m$ with zero mean and scalar covariance matrix described by

$$\mathbf{A}(W_{0:N-1}) = \frac{m}{2}\ln\left(\frac{2\pi\mathrm{e}}{m}\mathbf{E}(|W_{0:N-1}|^2)\right) - h(W_{0:N-1})$$

where $h(W_{0:N-1}) = \mathbf{E}\ln f(W_{0:N-1}) = -\int\limits_{\mathbb{R}^m} f(w)\ln f(w)dw$ is a differential entropy, $f : \mathbb{R}^{mN} \longrightarrow \mathbb{R}_+$ is the probability density function (PDF) of the vector $W_{0:N-1}$.

Definition 3.2 *Mean anisotropy of the sequence* W is defined by the expression

$$\overline{\mathbf{A}}(W) = \lim_{N\to+\infty}\frac{\mathbf{A}(W_{0:N-1})}{N}. \tag{3.1}$$

A. A. Belov et al., *Control of Discrete-Time Descriptor Systems*, Studies in Systems, Decision and Control 157, https://doi.org/10.1007/978-3-319-78479-3_3

It is shown in [2] that

$$\overline{\mathbf{A}}(W) = \mathbf{A}(w_0) + \mathbf{I}(w_0;\,(w_k)_{k<0}) \tag{3.2}$$

where $\mathbf{I}(w_0;\,(w_k)_{k<0}) = \lim_{s\to-\infty}\mathbf{I}(w_0;\,W_{s:-1})$ is the Shannon mutual information [3] between w_0 and the past history $(w_k)_{k<0}$ of the sequence W.

Now suppose that W is generated from the Gaussian white noise sequence $V = \{v(k)\}_{k\in\mathbb{Z}}$ with zero mean and identity covariance matrix by an admissible shaping filter $G(z) = C_G(zE_G - A_G)^{-1}B_G + D_G$; that is, the pair (E_G, A_G) is admissible. Then,

$$\mathbf{I}(w_0;\,(w_k)_{k<0}) = \frac{1}{2}\ln\det(\mathbf{cov}(w_0)\mathbf{cov}^{-1}(\tilde{w}_0)) \tag{3.3}$$

where

$$\tilde{w}_0 = w_0 - \mathbf{E}(w_0|(w_k)_{k<0}) \tag{3.4}$$

is the error of the mean-square optimal prediction of w_0 by the past history $(w_k)_{k<0}$, provided by the conditional expectation.

Suppose $w(j) = \sum_{k=0}^{+\infty} g(k)v(j-k),\ j\in\mathbb{Z}$. The impulse response of the filter $g(k)\in\mathbb{R}^{m\times m}$ is assumed to be square summable over $k \geqslant 0$, ensuring mean square convergence of the series.

Transfer function of the filter $G(z) = \sum_{k=0}^{+\infty} g(k)z^k$ is supposed to belong to Hardy space $\mathscr{H}_2{}^{m\times m}$, that is, the space of matrix-valued functions, analytic in the disc $|z| < 1$ on the complex plane. Space is equipped with the $\mathscr{H}_2$ -norm, defined by

$$\|G\|_2 = \left(\sum_{k=0}^{+\infty}\mathrm{Tr}\,(g(k)g^{\mathrm{T}}(k))\right)^{1/2} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Tr}\,S(\omega)d\omega\right)^{1/2} \tag{3.5}$$

where $S(\omega) = \widehat{G}^*(\omega)\widehat{G}(\omega)$, $(-\pi \leqslant \omega \leqslant \pi)$ is a spectral density of W, and $\widehat{G}(\omega) = G(\mathrm{e}^{i\omega})$ is the boundary value of the transfer function $G(z)$.

The covariance matrix of the prediction error (3.4) and the spectral density $S(\omega)$ are related by the Kolmogorov-Szegö formula as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\det S(\omega)d\omega = \ln\det\mathbf{cov}(\tilde{w}_0). \tag{3.6}$$

Using the (3.2)–(3.4), Szegö limit theorem [4], and (3.6), the mean anisotropy (3.1) of the stationary Gaussian random sequence $W = GV$ can be computed in terms of spectral density $S(\omega)$ and the $\mathscr{H}_2$ -norm of the shaping filter G .

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det\frac{mS(\omega)}{\|G\|_2^2}d\omega = -\frac{1}{4\pi}\ln\det\frac{m\mathbf{cov}(\tilde{w}_0)}{\|G\|_2^2}. \tag{3.7}$$

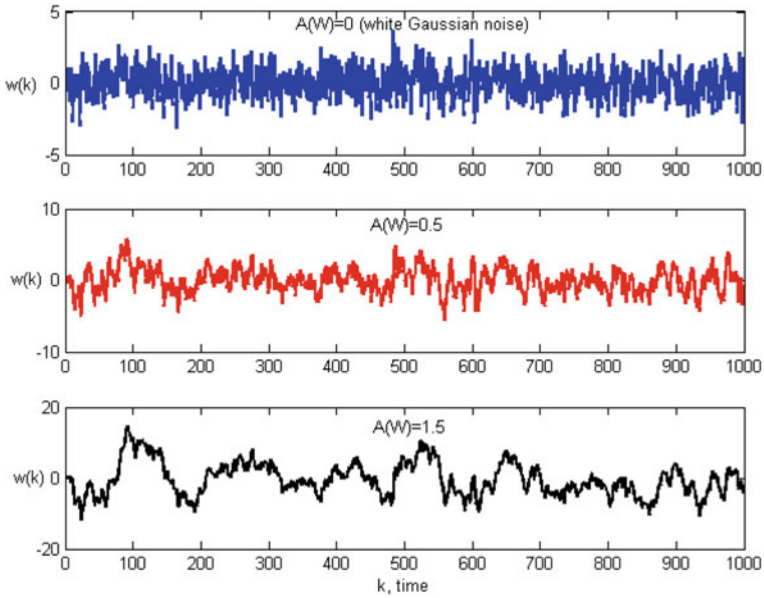

Fig. 3.1 Random signals with different mean anisotropy level $\overline{\mathbf{A}}(W)$

The mean anisotropy of the signal is the Kullback-Leibler information divergence from the PDF of the signal to the PDF of the white Gaussian noise sequence. It characterizes divergence between the signal and white Gaussian noise sequence. For more information see [2, 5] (Fig. 3.1).

Remark 3.1 The mean anisotropy of random sequence W, generated by shaping filter $G(z)$, is fully defined by its parameters; thus the notations $\overline{\mathbf{A}}(G)$ and $\overline{\mathbf{A}}(W)$ are equivalent.

For more details, see [6].

3.2 System Norms

Consider a descriptor system of the form

$$Ex(k+1) = Ax(k) + Bf(k), \tag{3.8}$$

$$y(k) = Cx(k) + Df(k). \tag{3.9}$$

System (3.8)–(3.9) is equal to its transfer function

$$P(z) = C(zE - A)^{-1}B + D.$$

Now we describe norms of the transfer function $P(z)$, widely used in control theory [7].

3.2.1 $L_2^{p\times m}$- and $\mathscr{H}_2$-Norms

Let $\mathbb{L}_2^{p\times m}(\Gamma)$ be a space of matrix-valued functions $P : \Gamma \to \mathbb{C}^{p\times m}$ that have finite $\mathbb{L}_2^{p\times m}(\Gamma)$-norm

$$\|P\|_{\mathbb{L}_2^{p\times m}(\Gamma)} = \left(\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Tr}\left(P^*(\mathrm{e}^{i\omega})P(\mathrm{e}^{i\omega})\right) d\omega\right)^{\frac{1}{2}}. \tag{3.10}$$

The subspace of $\mathbb{L}_2^{p\times m}(\Gamma)$, denoted by $\mathscr{H}_2$, consists of all rational transfer functions that are analytic in the exterior of the closed unit disk. The $\mathscr{H}_2$-norm of the transfer function $P(z) \in \mathscr{H}_2$ is defined by

$$\|P\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Tr}\left(P^*(\mathrm{e}^{i\omega})P(\mathrm{e}^{i\omega})\right) d\omega\right)^{\frac{1}{2}} = \left(\frac{1}{2\pi}\int_0^{2\pi} \|P(\mathrm{e}^{i\omega})\| d\omega\right)^{\frac{1}{2}}.$$

If $P(z)$ is strictly proper, and the matrix pencil $(\lambda E - A)$ is d-stable (i.e., $\rho(E, A) < 1$), then $P(z) \in \mathscr{H}_2$. On the other hand, if $P(z) \in \mathscr{H}_2$, then $P(z)$ is strictly proper, but $(\lambda E - A)$ is not necessarily d-stable.

Recall that, in control theory, a *proper transfer function* is a transfer function in which the numerator degree does not exceed the degree of the denominator. A *strictly proper transfer function* is a transfer function where the degree of the numerator is less than the degree of the denominator.

Example 3.1 Let the system be given as

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = C^{\mathrm{T}}.$$

Transfer function $P(z) = \frac{2}{2z-1} \in \mathscr{H}_2$ is proper, but the matrix pencil $(\lambda E - A)$ is not d-stable.

The transfer function $P(z)$ of system (3.8) and (3.9) may not be proper. In this case, if the matrix pencil $(\lambda E - A)$ has no eigenvalues on the unit circle, then $P(z) \in \mathbb{L}_2^{p\times m}(\Gamma)$.

$\mathscr{H}_2$-*Norm Computation*

Now consider the Hilbert space $l_2^{p\times m}(\mathbb{Z})$ that contains matrix-valued sequences $S = \{S_k\}_{k\in\mathbb{Z}}$, $S_k \in \mathbb{R}^{p\times m}$ which have a bounded $l_2^{p\times m}(\mathbb{Z})$-norm

$$\|S\|_{l_2^{p\times m}(\mathbb{Z})} = \left(\sum_{k=-\infty}^{\infty} \mathrm{Tr}\,(S_k^{\mathrm{T}} S_k)\right)^{\frac{1}{2}} = \left(\sum_{k=-\infty}^{\infty} \|S_k\|_F^2\right)^{\frac{1}{2}}.$$

Using the Parseval identity, from (2.67) we get

$$\|P\|_{\mathbb{L}_2^{p\times m}(\Gamma)} = \|P\|_{l_2^{p\times m}(\mathbb{Z})} = \left(\sum_{k=-\infty}^{\infty} \|P_k\|_F^2\right)^{\frac{1}{2}} \tag{3.11}$$

where $P = \{P_k\}_{k\in\mathbb{Z}}$ is an impulse transfer function of the system (3.8) and (3.9). Moreover, if the matrix pencil $(\lambda E - A)$ is d-stable, then substituting $P_k = C\mathcal{P}_{k-1}B$ into (3.11), we have

$$\begin{gathered}\|P\|^2_{\mathbb{L}_2^{p\times m}(\Gamma)} = \textstyle\sum_{k=-\infty}^{\infty} \mathrm{Tr}\,(B^{\mathrm{T}}\mathcal{P}_{k-1}^{\mathrm{T}}C^{\mathrm{T}}C\mathcal{P}_{k-1}B) = \\ = \textstyle\sum_{k=-\infty}^{\infty} \mathrm{Tr}\,(C\mathcal{P}_{k-1}BB^{\mathrm{T}}\mathcal{P}_{k-1}^{\mathrm{T}}C^{\mathrm{T}}) = \mathrm{Tr}\,(B^{\mathrm{T}}\mathcal{P}_{do}B) = \mathrm{Tr}\,(C\mathcal{P}_{dc}C^{\mathrm{T}}).\end{gathered}$$

This relation leads to the algorithm of $\mathbb{L}_2^{p\times m}(\Gamma)$-norm computation of the transfer function $P(z)$ with a d-stable matrix pencil $(\lambda E - A)$ [7].

Consider the following projected generalized discrete-time algebraic Lyapunov equation.

$$A^{\mathrm{T}}XA - E^{\mathrm{T}}XE = -P_r^{\mathrm{T}}C^{\mathrm{T}}CP_r + (I - P_r)^{\mathrm{T}}C^{\mathrm{T}}C(I - P_r), \tag{3.12}$$

$$P_l^{\mathrm{T}}X = XP_l. \tag{3.13}$$

Matrices P_r and P_l are defined in (2.66). We have to find the solution of (3.12)–(3.13) $X = L^{\mathrm{T}}L$.

- Transform the matrices E and A to the upper-triangular form

$$E = V\begin{bmatrix} E_f & E_u \\ 0 & E_\infty \end{bmatrix}U^{\mathrm{T}},\ A = V\begin{bmatrix} A_f & A_u \\ 0 & A_\infty \end{bmatrix}U^{\mathrm{T}} \tag{3.14}$$

  where the matrix pencil $(\lambda E_f - A_f)$ is quasi-triangular and has only finite eigenvalues. The pencil $(\lambda E_\infty - A_\infty)$ is triangular and has infinite generalized eigenvalues.
- Solve the generalized Sylvester equation

$$E_fY - ZE_\infty = -E_u, \tag{3.15}$$

$$A_fY - ZA_\infty = -A_u. \tag{3.16}$$

- Compute the matrix $CU = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$.
- Using the generalized Hammarling algorithm, find the Cholesky decomposition $U_{X_{11}}$ for the solution $X_{11} = U_{X_{11}}^{\mathrm{T}}U_{X_{11}}$ of the regular generalized discrete-time Lyapunov equation

$$A_f^{\mathrm{T}} X_{11} A_f - E_f^{\mathrm{T}} X_{11} E_f = -C_1^{\mathrm{T}} C_1. \tag{3.17}$$

- Using the generalized Hammarling algorithm, find the Cholesky decomposition U_{X_∞} for the solution $X_\infty = U_{X_\infty}^{\mathrm{T}} U_{X_\infty}$ of the regular generalized discrete-time Lyapunov equation

$$A_\infty^{\mathrm{T}} X_\infty A_\infty - E_\infty^{\mathrm{T}} X_\infty E_\infty = (C_1 Y + C_2)^{\mathrm{T}} (C_1 Y + C_2). \tag{3.18}$$

- The full-rank matrix L can be found from QR-decomposition, applying Householder and Givens transformations

$$\begin{bmatrix} U_{X_{11}} & -U_{X_{11}} Z \\ 0 & U_{X_\infty} \end{bmatrix} V^{\mathrm{T}} = Q \begin{bmatrix} L \\ 0 \end{bmatrix}. \tag{3.19}$$

Remark 3.2 When system (3.8) and (3.9) is admissible, the simplified method of $\mathscr{H}_2$ -norm computation can be applied. The procedure consists of the following steps.

1. Calculate matrices W_1 and V_1 and transform the initial system into the singular value decomposition (SVD) equivalent form with

$$W_1 A V_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

2. Construct matrices W_2 and V_2 as follows.

$$W_2 = \begin{bmatrix} I_r & -A_{12} A_{22}^{-1} \\ 0 & I_{n-r} \end{bmatrix}, \quad V_2 = \begin{bmatrix} I_r & 0 \\ -A_{22}^{-1} A_{21} & A_{22}^{-1} \end{bmatrix}.$$

3. Define $\overline{W} = W_1 W_2$ and $\overline{V} = V_1 V_2$. Matrices $\overline{W}$ and $\overline{V}$ transform the initial system to Weierstrass canonical form with

$$\overline{W} E \overline{V} = \mathrm{diag}(I_r, 0), \ \overline{W} A \overline{V} = \mathrm{diag}(A_1, I_{n-r}),$$
$$\overline{W} B = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \ C \overline{V} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix},$$

   After transformations, the generalized projected Lyapunov equation (3.12) can be rewritten as

$$A_1^{\mathrm{T}} G_{co} A_1 - G_{co} = -\bar{C}_1^{\mathrm{T}} \bar{C}_1. \tag{3.20}$$

   The $\mathscr{H}_2$ -norm of the system can be computed as

$$\|P\|_2 = \sqrt{\mathrm{Tr}\left(\bar{B}_1^{\mathrm{T}} G_{co} \bar{B}_1 + (D - \bar{C}_2 \bar{B}_2)^{\mathrm{T}} (D - \bar{C}_2 \bar{B}_2)\right)}.$$

Example 3.2 Let the system be given by the following matrices.

$$E = \begin{bmatrix} 3 & 0 & 2 & -5 \\ 0 & 3 & -2 & 2 \\ 2 & 2 & 0 & -2 \\ 2 & -4 & 4 & -6 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.7 & -3.25 & -0.7 & 0 \\ 1.8 & 0.4 & -6.4 & 2.6 \\ 1 & -1.9 & -5.4 & 2.4 \\ -0.6 & -2.7 & 5.4 & -2.8 \end{bmatrix},$$

$$B = \begin{bmatrix} 2.1 & -0.8 \\ 0.2 & 1 \\ 1.5 & 1.3 \\ 0 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1.5 & 0 & -2 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Matrices $\overline{W}$ and $\overline{V}$ have the form:

$$\overline{W} = \begin{bmatrix} 1 & 0 & 2.2847 & -2.1209 \\ 0 & 1 & 1.4593 & 5.2108 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{W} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.1713 & -0.2585 & -3.2481 & 0.6194 \\ -0.1705 & -0.0758 & -1.7412 & 1.1245 \end{bmatrix},$$

The system is transformed into Weierstrass canonical form with

$$A_1 = \begin{bmatrix} 0.2371 & -0.8616 \\ -0.0031 & -0.5371 \end{bmatrix}, \quad \overline{W}B = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} = \left[\begin{array}{cc} -0.4428 & -6.0760 \\ -5.6722 & 12.7570 \\ \hline -0.2187 & 0.0877 \\ -0.6087 & 2.4927 \end{array}\right],$$

$$C\overline{V} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 0.1568 & 0.1570 & 3.0786 & 0.5697 \end{array}\right].$$

Solution G_{co} of equation (3.20) takes the form

$$G_{co} = \begin{bmatrix} 0.0260 & 0.0173 \\ 0.0173 & 0.0843 \end{bmatrix}.$$

Finally, $\|P\|_2 = 4.4$.

3.2.2 $L_\infty^{p\times m}$- and $\mathscr{H}_\infty$-Norms

Let $\mathbb{L}_\infty^{p\times m}(\Gamma)$ (where Γ is a unit circle on the complex plane) be a space of matrix-valued functions $P : \Gamma \to \mathbb{C}^{p\times m}$ that are essentially bounded on Γ. The subspace of $\mathbb{L}_\infty^{p\times m}(\Gamma)$, denoted by $\mathscr{H}_\infty$, consists of all rational transfer functions that are analytic in the exterior of the closed unit disk. Therefore the $\mathscr{H}_\infty$-norm of the transfer function $P(z) \in \mathscr{H}_\infty$ is defined by

$$\|P\|_\infty = \sup_{\omega\in[0,2\pi]} \overline{\sigma}\left(P(\mathrm{e}^{i\omega})\right) = \sup_{\omega\in[0,2\pi]} \|(P(\mathrm{e}^{i\omega})\|_2.$$

Evidently, the $\mathscr{H}_\infty$-norm of the transfer function $P(z)$ is finite if and only if $P(z) \in \mathbb{L}_\infty^{p\times m}(\Gamma)$ is proper.

For LDTI descriptor system given in the form (3.8) and (3.9) the following bounded real lemma can be formulated. This lemma allows computing the $\mathscr{H}_\infty$-norm of the system. Its formulation can be found in [8].

Lemma 3.1 *For a given real value* $\gamma > 0$ *system* (3.8)–(3.9) *is admissible, and its transfer function*

$$P(z) = C(zE - A)^{-1}B + D \tag{3.21}$$

satisfies the condition

$$\|P\|_\infty < \gamma \tag{3.22}$$

if and only if there exists a matrix $\widetilde{R} = \widetilde{R}^{\mathrm{T}}$, *such that the following LMIs hold true.*

$$E^{\mathrm{T}}\widetilde{R}E \geq 0, \tag{3.23}$$

$$\begin{bmatrix} A^{\mathrm{T}}\widetilde{R}A - E^{\mathrm{T}}\widetilde{R}E + C^{\mathrm{T}}C & A^{\mathrm{T}}\widetilde{R}B + C^{\mathrm{T}}D \\ B^{\mathrm{T}}\widetilde{R}A + D^{\mathrm{T}}C & B^{\mathrm{T}}\widetilde{R}B + D^{\mathrm{T}}D - \gamma^2 I \end{bmatrix} < 0. \tag{3.24}$$

$\mathscr{H}_\infty$-Norm Computation

Conditions of Lemma 3.1 can be used in $\mathscr{H}_\infty$-norm calculation. Denoting $\xi = \gamma^2$ the problem of $\mathscr{H}_\infty$-norm calculation is to find

$$\xi_* = \inf \xi$$

on the set

$$\{\xi,\ \widetilde{R}\}$$

that satisfies (3.23)–(3.24). If the minimum value ξ_* is found, then the $\mathscr{H}_\infty$-norm of system (3.21) can be approximately calculated as

$$\|P\|_\infty \approx \sqrt{\xi_*}. \tag{3.25}$$

Example 3.3 The system parameters are taken from Example 3.2.

Compute the $\mathscr{H}_\infty$-norm of the transfer function, using a bounded real lemma for descriptor systems. Matrix R, which corresponds to the minimal value of the parameter γ, is equal to

$$R = \begin{bmatrix} 0.1120 & -9.0655 & 5.1750 & -5.6424 \\ -9.0655 & -12.2041 & 13.4356 & -1.5472 \\ 5.1750 & 13.4356 & 11.4013 & 5.1043 \\ -5.6424 & -1.5472 & 5.1043 & 2.8984 \end{bmatrix}.$$

The system's norm is $\gamma = 9.0478$. Note that the exact value of the $\mathscr{H}_\infty$-norm for the equivalent normal system is $\gamma_{eq} = 9.0386$.

3.2.3 a-Anisotropic Norm

Consider an admissible linear discrete-time descriptor system P written in a state-space representation

$$Ex(k+1) = Ax(k) + Bw(k), \tag{3.26}$$
$$y(k) = Cx(k) + Dw(k). \tag{3.27}$$

$W = \{w(k)\}_{k\in\mathbb{Z}}$ is a stationary Gaussian sequence of m-dimensional random vectors with a bounded mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$ ($a \geqslant 0$) and zero mean, $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^p$. Recall that $\operatorname{rank}(E) = r < n$.

For a given system P with the input signal $W = \{w(k)\}_{k\in\mathbb{Z}}$ the root mean-square (RMS) gain is defined as [6, 9]

$$Q(P, W) = \frac{\|Y\|_{\mathscr{P}}}{\|W\|_{\mathscr{P}}} \tag{3.28}$$

where

$$\|Y\|_{\mathscr{P}} = \sqrt{\lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \mathbf{E}|y(k)|^2}$$

is power norm of the square summable stationary sequence $Y = \{y(k)\}_{k\in\mathbb{Z}}$.

Definition 3.3 For a given parameter $a \geqslant 0$ *a-anisotropic norm of the system* P (3.26) and (3.27) is defined by

$$|\!|\!| P |\!|\!|_a = \sup_{\overline{\mathbf{A}}(W)\leqslant a} Q(P, W). \tag{3.29}$$

Thus the a-anisotropic norm $|\!|\!| P |\!|\!|_a$ describes the "stochastic gain" of the system P with respect to W.

Sequence $W = \{w(k)\}_{k\in\mathbb{Z}}$ can be generated as [9]

$$w(k) = C_g x(k) + D_g v(k) \tag{3.30}$$

where $x(k)$ is the state of the system (3.26).

Using (3.30), we can choose C_g and D_g such that filter G with a state-space representation

$$Ex(k+1) = (A + BC_g)x(k) + BD_g v(k), \tag{3.31}$$
$$w(k) = C_g x(k) + D_g v(k) \tag{3.32}$$

is admissible.

The ratio of power norms of outputs of systems (3.26) and (3.27), and (3.31) and (3.32) may be written as

$$\frac{\|Y\|_{\mathscr{P}}}{\|W\|_{\mathscr{P}}} = \frac{\|PG\|_2}{\|G\|_2}.$$

Therefore the anisotropic norm can be defined as follows.

Definition 3.4 For a given scalar value $a \geqslant 0$ *a-anisotropic norm of system* P can be defined as

$$|\!|\!| P |\!|\!|_a = \sup_{G \in \mathbf{G}_a} \frac{\|PG\|_2}{\|G\|_2}. \tag{3.33}$$

The right-hand side of the expression (3.33) stands for the maximal value of the system's gain (the ratio of power norms of output Y and input W) against the class of shaping filters

$$\mathbf{G}_a = \left\{ G \in \mathscr{H}_2^{m \times m} : \overline{\mathbf{A}}(G) \leqslant a \right\}.$$

Definition 3.5 The set $\overline{\mathbf{G}}_a$, for which the supremum from (3.33) is reached, is called the set of worst-case shaping filters, and filter $G \in \overline{\mathbf{G}}_a$ is called the worst-case shaping filter.

a-Anisotropic Norm Computation

As system P, given in the state space by (3.26) and (3.27), is admissible, there exist such nonsingular matrices $\widetilde{W}$ and $\widetilde{V}$ that

$$\widetilde{W} E \widetilde{V} = \mathrm{diag}(I_r, 0).$$

Thus system (3.26) and (3.27) can be transformed to SVD equivalent form (see Chap. 2)

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1 w(k), \tag{3.34}$$
$$0 = A_{21}x_1(k) + A_{22}x_2(k) + B_2 w(k), \tag{3.35}$$
$$y(k) = C_1 x_1(k) + C_2 x_2(k) + Dw(k), \tag{3.36}$$

matrix A_{22} in (3.35) is nonsingular. Matrices $\widetilde{W}$ and $\widetilde{V}$ can be found from SVD decomposition (2.21) and (2.22). The nonsingularity of matrix A_{22} allows the expression of $x_2(k)$ from equation (3.35)

$$x_2(k) = -A_{22}^{-1}(A_{21}x_1(k) + B_2 w(k)). \tag{3.37}$$

Substitution of (3.37) into (3.34) leads to

$$x_1(k+1) = \widetilde{A}x_1(k) + \widetilde{B}w(k), \tag{3.38}$$

$$y(k) = \widetilde{C}x_1(k) + \widetilde{D}w(k) \tag{3.39}$$

where

$$\begin{aligned} \widetilde{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21}, & \widetilde{B} &= B_1 - A_{12}A_{22}^{-1}B_2, \\ \widetilde{C} &= C_1 - C_2A_{22}^{-1}A_{21}, & \widetilde{D} &= D - C_2A_{22}^{-1}B_2. \end{aligned} \tag{3.40}$$

The obtained system (3.38) and (3.39) is equal to the initial descriptor system (3.26) and (3.27) in the sense of an input-output operator. But its state vector $x_1(k)$ of lower dimension is driven by the normal difference equation.

Now we formulate the following theorem for anisotropic norm computation [10].

Theorem 3.1 *Let system P in the form* (3.26) *and* (3.27) *be admissible. Then its a-anisotropic norm can be computed as the anisotropic norm of the equivalent normal system* (3.38) *and* (3.39)*, using the expression*

$$|\!|\!| P |\!|\!|_a = \left(\frac{1}{q} \left(1 - \frac{m}{\mathrm{Tr}\,(L\Pi L^{\mathrm{T}} + \Sigma)} \right) \right)^{1/2}$$

where $q \in [0, \|P\|_\infty^{-2})$. The matrices Π, L, and Σ are found from the solution of the following three expressions.

Algebraic Riccati equation

$$\begin{aligned} R &= \widetilde{A}^{\mathrm{T}} R \widetilde{A} + q\widetilde{C}^{\mathrm{T}}\widetilde{C} + L^{\mathrm{T}}\Sigma^{-1}L, \\ L &= \Sigma(\widetilde{B}^{\mathrm{T}} R \widetilde{A} + q\widetilde{D}^{\mathrm{T}}\widetilde{C}), \\ \Sigma &= (I_m - \widetilde{B}^{\mathrm{T}} R \widetilde{B} - q\widetilde{D}^{\mathrm{T}}\widetilde{D})^{-1} \end{aligned}$$

(with the additional condition $\rho(\widetilde{A} + \widetilde{B}L) < 1$)

Lyapunov equation

$$\Pi = (\widetilde{A} + \widetilde{B}L)\Pi(\widetilde{A} + \widetilde{B}L)^{\mathrm{T}} + \widetilde{B}\Sigma\widetilde{B}^{\mathrm{T}},$$

and the special type equation

$$a = -\frac{1}{2} \ln\det\left(\frac{m\,\Sigma}{\mathrm{Tr}\,(L\Pi L^{\mathrm{T}} + \Sigma)} \right).$$

The matrices $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, and $\widetilde{D}$ are defined by (3.40).

Note that we use parameters of the worst-case shaping filter while computing the anisotropic norm of descriptor systems according to Theorem 3.1, that is,

$$G = \left[\begin{array}{c|c} \widetilde{A} + \widetilde{B}L & \widetilde{B}\Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array}\right].$$

This filter is designed for the equivalent normal system; its state vector $x_1(k)$ has lower dimension than the initial descriptor system. This feature has to be taken into account when constructing anisotropic control laws for descriptor systems.

Example 3.4 Compute the anisotropic norm for the system, given in Example 3.2.

Here $\operatorname{rank}(E) = 2$. To transform the system into the SVD equivalent form we use the following matrices.

$$\widetilde{W} = \begin{bmatrix} -0.5228 & 0.3220 & -0.1338 & -0.7779 \\ -0.5233 & -0.4659 & -0.6595 & 0.2723 \\ 0.1736 & 0.7599 & -0.5536 & 0.2932 \\ -0.6501 & 0.3190 & 0.4906 & 0.4846 \end{bmatrix},$$

$$\widetilde{V} = \begin{bmatrix} -0.0292 & -0.1028 & -0.7935 & 0.1742 \\ 0.0328 & -0.1669 & 0.4113 & 0.2662 \\ -0.0413 & 0.0427 & 0.2347 & 0.8396 \\ 0.0705 & 0.0601 & -0.3822 & 0.4403 \end{bmatrix},$$

which transform matrix E to the form

$$\widetilde{W} E \widetilde{V} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Matrix A is converted to

$$\widetilde{W} A \widetilde{V} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 0.5249 & -0.8132 & -0.4367 & 2.1266 \\ -0.4712 & 0.0136 & 4.1626 & -6.9268 \\ \hline -0.0338 & -0.0947 & -0.4369 & 0.2406 \\ 0.0993 & -0.0792 & -0.6765 & 1.2619 \end{array}\right],$$

$$\widetilde{W} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{cc} -1.2341 & -0.9895 \\ -2.1814 & -0.3600 \\ \hline -0.2187 & 0.0877 \\ -0.6087 & 2.4927 \end{array}\right],$$

$$C\widetilde{V} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 0.1094 & -0.1795 & -1.7304 & 1.4597 \end{array}\right].$$

Matrix A_{22} is not singular; therefore the system is causal. Moreover, matrix $\widetilde{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ has the following spectral radius $\rho(\widetilde{A}) = 0.5405 < 1$; it means

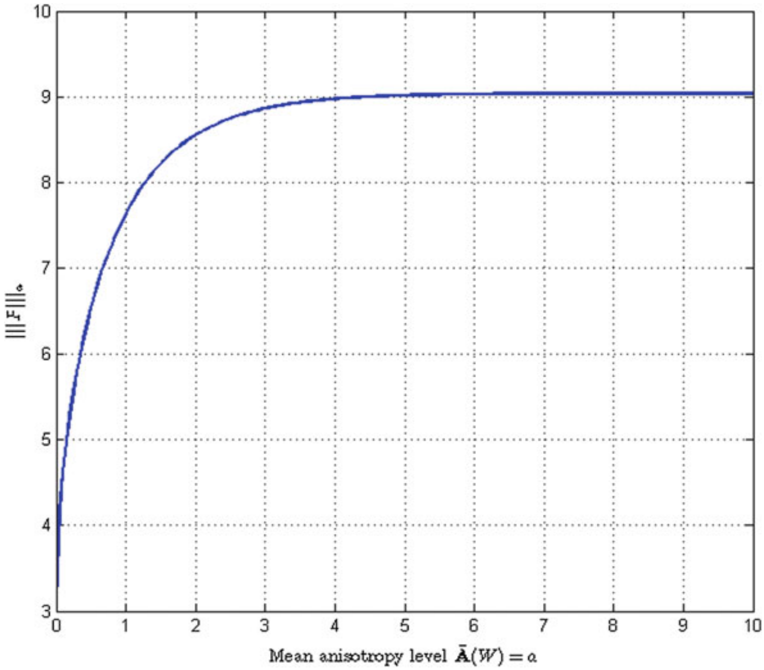

Fig. 3.2 Relation between the a-anisotropic norm of the system P and mean anisotropy level of the input disturbance a

that the system is stable. Consequently, the SVD equivalent form is represented by an asymptotically stable normal system (3.38) and (3.39), which matrices are computed according to the expressions (3.40):

$$\widetilde{A} = \begin{bmatrix} 0.2371 & -0.8616 \\ -0.0031 & -0.5371 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} -0.4428 & -6.0760 \\ -5.6722 & 12.7570 \end{bmatrix},$$
$$\widetilde{C} = \begin{bmatrix} 0.1568 & 0.1570 \end{bmatrix}, \quad \widetilde{D} = \begin{bmatrix} 2.0200 & -0.6900 \end{bmatrix}.$$

The results of a-anisotropic norm computation $|||P|||_a$ according to Theorem 3.1 for different values a are depicted in Fig. 3.2. When $a = 0$ the value of the anisotropic norm is equal to

$$|||P|||_a = 3.1112 = \frac{4.4}{\sqrt{2}} = \frac{\|P\|_2}{\sqrt{2}}.$$

3.3 Anisotropy-Based Performance Analysis

The method described above for anisotropic norm computation is useful only for admissible systems. However, it is preferable to check admissibility of a system simultaneously with anisotropic norm computation. Such methods are more useful for solving not only the analysis problem but also that of control design. The following section describes an approach to anisotropy-based performance analysis, which is used for solving control design problems.

Problem Statement

A linear discrete-time stationary descriptor system in a state-space representation is given by

$$Ex(k+1) = Ax(k) + Bw(k), \tag{3.41}$$

$$y(k) = Cx(k) + Dw(k) \tag{3.42}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^m$ is a random stationary sequence with bounded mean anisotropy $\overline{\mathbf{A}}(W) \leqslant a$ $(a \geqslant 0)$, $y(k) \in \mathbb{R}^p$ is a measurable output, E, A, B, C, and D are known matrices of appropriate dimensions. Matrix E is singular; that is, $\operatorname{rank}(E) = r < n$. Hereinafter, we suppose that the following rank condition is true for the system (3.41).

$$\operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} = \operatorname{rank}(E). \tag{3.43}$$

System (3.41) and (3.42) is equal to $P \in \mathscr{H}_\infty{}^{p\times m}$, given by its transfer function

$$P(z) = C(zE - A)^{-1}B + D.$$

We also use the denotation [11]

$$P = \left[E, \begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Let system P be admissible. Suppose that the input sequence W is a stationary sequence of random Gaussian vectors with bounded mean anisotropy $\overline{\mathbf{A}}(W) \leqslant a$; it means that W is generated from the m-dimensional Gaussian white noise sequence V with zero mean and identity covariance matrix by an unknown stable shaping filter G from the set $\mathbf{G}_a$. For system P given by (3.41) and (3.42), for the known mean anisotropy boundary value $a \geqslant 0$, and a value $\gamma > 0$ we have to obtain the conditions of anisotropic norm $|\!|\!| P |\!|\!|_a$ boundedness by γ.

3.3.1 Riccati Equations Technique

In order to prove an anisotropy-based bounded real lemma we have to describe some results from the theory of all-pass systems.

Definition 3.6 A system with a transfer function $P(z)$ such that $P^*P = I_m$ is called the *all-pass system.*

Lemma 3.2 [11] *System* (3.41) *and* (3.42) *is the all-pass system if there exists* $\widehat{R} = \widehat{R}^{\mathrm{T}}$ *that satisfies the conditions* $E^{\mathrm{T}}\widehat{R}E \geqslant 0$ *and*

$$\begin{aligned} B^{\mathrm{T}}\widehat{R}B + D^{\mathrm{T}}D &= I_m, \\ B^{\mathrm{T}}\widehat{R}A + D^{\mathrm{T}}C &= 0, \\ A^{\mathrm{T}}\widehat{R}A + C^{\mathrm{T}}C - E^{\mathrm{T}}\widehat{R}E &= 0. \end{aligned}$$

The conditions of anisotropic norm boundedness for descriptor systems can be formulated as follows.

Theorem 3.2 *Let* $P \in \mathscr{H}_\infty{}^{p\times m}$ *be an admissible system with a state-space representation* (3.41) *and* (3.42) *where* $\rho(E, A) < 1$. *For given scalar quantities* $a \geqslant 0$ *and* $\gamma > 0$ *the a-anisotropic norm is bounded by* γ, *that is,* $|||P|||_a \leqslant \gamma$ *if and only if there exists* $q \in [0, \min(\gamma^{-2}, \|P\|_\infty^{-2}))$ *such that the inequality*

$$-\frac{1}{2}\ln\det((1 - q\gamma^2)\Sigma) \geqslant a \tag{3.44}$$

is satisfied for matrix Σ *associated with the stabilizing solution*[1] $\widehat{R} = \widehat{R}^{\mathrm{T}}$ *of the algebraic Riccati equation*

$$E^{\mathrm{T}}\widehat{R}E = A^{\mathrm{T}}\widehat{R}A + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L, \tag{3.45}$$

$$L = \Sigma(B^{\mathrm{T}}\widehat{R}A + qD^{\mathrm{T}}C), \tag{3.46}$$

$$\Sigma = (I_m - B^{\mathrm{T}}\widehat{R}B - qD^{\mathrm{T}}D)^{-1}, \tag{3.47}$$

in addition, $E^{\mathrm{T}}\widehat{R}E \geqslant 0$.

Proof The power norm ratio $\|PG\|_2/\|G\|_2$ on the right-hand side of (3.33) and mean anisotropy $\overline{\mathbf{A}}(G)$ in (3.7) are both invariant under the scaling of the shaping filter G. Assuming system P to be fixed, they are completely specified by the normalized spectral density [12]:

$$\Pi(\omega) = \frac{mS(\omega)}{\|G\|_2^2} = \frac{2\pi mS(\omega)}{\int\limits_{-\pi}^{\pi} \operatorname{Tr} S(v)dv}, \tag{3.48}$$

[1]The stabilizing solution of the Riccati equation (3.45) stands for solution $\widehat{R}$, that the pair $(E, A + BL)$ is admissible.

then

$$\overline{\mathbf{A}}(G) = \alpha(\Pi) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det \Pi(\omega)d\omega, \tag{3.49}$$

$$\frac{\|PG\|_2}{\|G\|_2} = \nu(\Pi) = \left(\frac{1}{2\pi m}\int_{-\pi}^{\pi} \mathrm{Tr}\,(\Lambda(\omega)\Pi(\omega))d\omega\right)^{1/2} \tag{3.50}$$

where function $\Pi(\omega)$, defined on the interval $[-\pi, \pi)$ by (3.48), takes values from the set of positive definite Hermitian m-dimensional matrices and satisfies the condition

$$\int_{-\pi}^{\pi} \mathrm{Tr}\,\Pi(\omega)d\omega = 2\pi m.$$

Let the function $\Lambda(\omega)$ be given by

$$\Lambda(\omega) = \widehat{P}^*(\omega)\widehat{P}(\omega). \tag{3.51}$$

Note that the squared functional $\nu^2(\Pi)$ is linear on the variable $\Pi(\omega)$, and $\alpha(\Pi)$ is strictly convex with respect to $\Pi(\omega)$. Strict convexity of α follows from strict concavity of the function $\ln\det(\cdot)$ considered on a convex cone of positive definite matrices [13]. Strict convexity of $\alpha(\Pi)$ can also be obtained directly from the positive definiteness of its second variation

$$\delta^2\alpha(\Pi) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{Tr}\,(\Pi^{-1}(\omega)\delta\Pi(\omega)\Pi^{-1}(\omega)\delta\Pi(\omega))d\omega =$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} \|\Pi^{-1/2}(\omega)\delta\Pi(\omega)\Pi^{-1/2}(\omega)\|^2 d\omega \tag{3.52}$$

where $\delta\Pi(\omega)$ is a variation of Π, and $\|M\| = (\mathrm{Tr}\,(M^*M))^{1/2}$ denotes the Frobenius norm of the matrix. In the equation (3.52) we used the property of matrix trace

$$\ln\det\Xi = \mathrm{Tr}\ln\Xi$$

and the first variation of the inverse nonsingular matrix

$$\delta(\Xi^{-1}) = -\Xi^{-1}(\delta\Xi)\Xi^{-1}.$$

Thus the minimum value of the mean anisotropy of disturbance W, necessary to achieve the given level $\gamma > 0$ for the power norm ratio of the system, is

$$\min_{\nu(\Pi)\geqslant\gamma} \alpha(\Pi) = -\frac{1}{4\pi} \max_{\nu^2(\Pi)\geqslant\gamma^2} \int_{-\pi}^{\pi} \ln\det \Pi(\omega)d\omega. \tag{3.53}$$

By using the Lagrange multipliers method, the first minimum in (3.53) is shown to be achieved at a spectral density proportional to

$$S_q(\omega) = (I_m - q\Lambda(\omega))^{-1} \tag{3.54}$$

where q is a subsidiary variable satisfying $0 \leqslant q < \|P\|_\infty^{-2}$.

Accordingly, functions

$$\mathscr{A}(q) = \alpha(\Pi_q),\ \mathscr{N}(q) = \nu(\Pi_q) \tag{3.55}$$

are defined by evaluating the functionals $\alpha(\Pi)$ and $\nu(\Pi)$ from (3.49) and (3.50) at the normalized spectral density

$$\Pi_q(\omega) = \frac{2\pi m S_q(\omega)}{\int_{-\pi}^{\pi} \operatorname{Tr} S_q(v)dv}, \tag{3.56}$$

obtained by substituting (3.54) into (3.48). Excluding from consideration the trivial case when the function Λ in (3.51) is a constant matrix, $\mathscr{A}(q)$ and $\mathscr{N}(q)$ are both strictly increasing over q (see [6, 9]). This allows the minimum value of the mean anisotropy in (3.53) to be computed as $\mathscr{A}(\mathscr{N}^{-1}(\gamma))$ where $\mathscr{N}^{-1}(\gamma)$ denotes the functional inverse of $\mathscr{N}(q)$. Therefore, the inequality $|||P|||_a \leqslant \gamma$ is equivalent to $\mathscr{A}(\mathscr{N}^{-1}(\gamma)) \geqslant a$. Now (3.54) implies that $\Lambda(\omega) = (I_m - S_q(\omega)^{-1})/q$ and, hence

$$\frac{1}{2\pi m}\int_{-\pi}^{\pi} \operatorname{Tr}\left(\Lambda(\omega)S_q(\omega)\right)d\omega = \frac{1}{q}\left(\frac{1}{2\pi m}\int_{-\pi}^{\pi}\operatorname{Tr} S_q(\omega)d\omega - 1\right),$$

which, in combination with the definition of the function $\mathscr{N}(q)$ via (3.50), (3.55), and (3.56) yields

$$\frac{1}{2\pi m}\int_{-\pi}^{\pi}\operatorname{Tr} S_q(\omega)d\omega = \frac{1}{1 - q\mathscr{N}^2(q)}. \tag{3.57}$$

Substituting (3.49), (3.56), and (3.57) into (3.55), we get function $\mathscr{A}(q)$ in the form

$$\mathscr{A}(q) = \mathfrak{A}(q, \mathscr{N}(q)) \tag{3.58}$$

where

$$\mathfrak{A}(q,\gamma) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det S_q(\omega)d\omega - \frac{m}{2}\ln(1-q\gamma^2). \tag{3.59}$$

Because $-\ln(1-q\gamma^2)$ is monotonically increasing for $\gamma \in [0, 1/\sqrt{q})$, then so is $\mathfrak{A}(q,\gamma)$. A remarkable property of the function $\mathfrak{A}(q,\gamma)$ is that it achieves its maximum with respect to q at the point $q = \mathscr{N}^{-1}(\gamma)$, where, in view of (3.58), it coincides with function $\mathscr{A}(q)$:

$$\max_{0\leqslant q<\|P\|_\infty^{-2}} \mathfrak{A}(q,\gamma) = \mathfrak{A}(\mathscr{N}^{-1}(\gamma),\gamma) = \mathscr{A}(\mathscr{N}^{-1}(\gamma)). \tag{3.60}$$

The significance of this property for formulating the criterion of boundedness $|\!|\!| P |\!|\!|_a \leqslant \gamma$ is explained by expression (3.60) implying the equivalence between realization of the inequality $\mathscr{A}(\mathscr{N}^{-1}(\gamma)) \geqslant a$ and the existence of the parameter $q \in [0, \|P\|_\infty^{-2})$, satisfying $\mathfrak{A}(q,\gamma) \geqslant a$. Therefore, $|\!|\!| P |\!|\!|_a \leqslant \gamma$ if $\mathfrak{A}(q,\gamma) \geqslant a$ for some $q \in [0, \|P\|_\infty^{-2})$.

Property (3.60) is verified by differentiating the function $\mathfrak{A}(q,\gamma)$ from (3.59) with respect to its first argument:

$$\begin{aligned}
&\frac{\partial\mathfrak{A}(q,\gamma)}{\partial q} = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\partial \ln\det(I_m - q\Lambda(\omega))}{\partial q} d\omega + \frac{m\gamma^2}{2(1-q\gamma^2)} = \\
&-\frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{Tr}\left(\Lambda(\omega)S_q(\omega)\right)d\omega + \frac{m\gamma^2}{2(1-q\gamma^2)} = -\frac{m\mathscr{N}^2(q)}{2(1-q\mathscr{N}^2(q))} + \frac{m\gamma^2}{2(1-q\gamma^2)} = \\
&\qquad = \frac{m(\gamma^2 - \mathscr{N}^2(q))}{2(1-q\gamma^2)(1-q\mathscr{N}^2(q))}.
\end{aligned} \tag{3.61}$$

Function $\mathscr{N}(q)$ is strictly monotonic; the representation (3.61) implies that $\frac{\partial\mathfrak{A}(q,\gamma)}{\partial q}$ is positive for $q < \mathscr{N}^{-1}(\gamma)$ and negative for $q > \mathscr{N}^{-1}(\gamma)$. Now we have to represent the inequality $\mathfrak{A}(q,\gamma) \geqslant a$ for the function (3.59) in terms of the state-space dynamics of the system P. Note that (3.54) describes the parametric set of the worst-case spectral densities of the input disturbance W for the admissible values of q. Because the subsidiary variable q is fixed for the rest of the proof, we use the notation

$$S_\star(\omega) = (I_m - q_\star\Lambda(\omega))^{-1} \tag{3.62}$$

where $q_\star = \mathscr{A}^{-1}(a)$.

Now we get a state-space representation of the worst-case input disturbance $W_\star$ with a spectral density $S_\star$. In view of (3.51), the relation (3.62) is equivalent to

$$\widehat{\Theta}^*(\omega)\widehat{\Theta}(\omega) = I_m,\ -\pi \leqslant \omega < \pi \tag{3.63}$$

where $\widehat{\Theta} = \begin{bmatrix} \sqrt{q_\star}\widehat{P}(\omega) \\ \widehat{G}_\star^{-1}(\omega) \end{bmatrix}$.

Here, $\widehat{G}_\star$ is a shaping filter which, in accordance with [9], factorizes the worst-case spectral density (3.62) as $S_\star = \widehat{G}_\star^* \widehat{G}_\star$. The condition (3.63) means that the system $\widehat{\Theta}$ is the all-pass system.

Let $L \in \mathbb{R}^{m\times n}$ be a matrix such that the pair $(E, A+BL)$ is admissible; $\Sigma \in \mathbb{R}^{m\times m}$ is a positive definite symmetric matrix. We consider the worst-case input disturbance $W_\star = G_\star V$, which can be generated as

$$w_\star(k) = Lx(k) + \Sigma^{1/2}v(k). \tag{3.64}$$

Find such matrices L and Σ, that the input disturbance $W_\star$ is the worst case. A state-space representation of the shaping filter $G_\star$ is

$$G_\star = \left[E, \frac{A+BL}{L} \middle| \frac{B\Sigma^{1/2}}{\Sigma^{1/2}} \right].$$

Because $G_\star$ is invertible, its inverse is described by

$$G_\star^{-1} = \left[E, \frac{A}{-\Sigma^{-1/2}L} \middle| \frac{B}{\Sigma^{-1/2}} \right].$$

A state-space representation of the closed-loop system Θ is

$$\Theta = \left[E, \begin{array}{c|c} A & B \\ \hline q^{1/2}C & q^{1/2}D \\ -\Sigma^{-1/2}L & \Sigma^{-1/2} \end{array} \right].$$

According to Lemma 3.2, there exists a matrix $\widehat{R} = \widehat{R}^{\mathrm{T}}$, satisfying the condition $E^{\mathrm{T}}\widehat{R}E \geqslant 0$ such that

$$B^{\mathrm{T}}\widehat{R}B + \left[q^{1/2}D^{\mathrm{T}} \ (\Sigma^{-1/2})^{\mathrm{T}} \right] \begin{bmatrix} q^{1/2}D \\ \Sigma^{-1/2} \end{bmatrix} = I_m, \tag{3.65}$$

$$B^{\mathrm{T}}\widehat{R}A + \left[q^{1/2}D^{\mathrm{T}} \ (\Sigma^{-1/2})^{\mathrm{T}} \right] \begin{bmatrix} q^{1/2}C \\ -\Sigma^{-1/2}L \end{bmatrix} = 0, \tag{3.66}$$

$$A^{\mathrm{T}}\widehat{R}A + \left[q^{1/2}C^{\mathrm{T}} \ -L^{\mathrm{T}}(\Sigma^{-1/2})^{\mathrm{T}} \right] \begin{bmatrix} q^{1/2}C \\ -\Sigma^{-1/2}L \end{bmatrix} - E^{\mathrm{T}}\widehat{R}E = 0. \tag{3.67}$$

Inasmuch as Σ is a positive definite symmetric matrix, from equations (3.65) and (3.66) we get

$$\Sigma = (I_m - B^{\mathrm{T}}\widehat{R}B - qD^{\mathrm{T}}D)^{-1}, \tag{3.68}$$

$$L = \Sigma(B^{\mathrm{T}}\widehat{R}A + qD^{\mathrm{T}}C). \tag{3.69}$$

These equations coincide with the equalities (3.46) and (3.47). The expression (3.67) can be rewritten as

$$E^{\mathrm{T}}\widehat{R}E = A^{\mathrm{T}}\widehat{R}A + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L. \tag{3.70}$$

The worst-case input disturbance is described by (3.64), where $\{v_k\}_{k\in\mathbb{Z}}$ is a white noise sequence with an identity covariance matrix; thus the prediction error (3.4) takes the form $\tilde{w}(0) = \Sigma^{1/2}v(0)$ and, hence, $\mathbf{cov}(\tilde{w}(0)) = \Sigma$. Therefore, in combination with Kolmogorov-Szegö formula (3.6), we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det S_\star(\omega)d\omega = \ln\det \Sigma.$$

Substituting this equation into (3.59), we obtain

$$\mathfrak{A}(q,\gamma) = -\frac{1}{2}\ln\det\left((1-q\gamma^2)\Sigma\right).$$

Hence, the condition $\mathfrak{A}(q,\gamma) \geqslant a$ is equivalent to the inequality (3.44) for the matrix Σ, associated with the generalized Riccati equation (3.45)–(3.47).

The theorem is proved. ■

Example 3.5 Consider the system (3.41) and (3.42) with the following parameters:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.3500 & 1.0000 & -0.7800 \\ 0.6300 & -0.1100 & 0.9700 \\ 0.6177 & 0.8038 & 0.7851 \end{bmatrix}, \quad B = \begin{bmatrix} -0.3 \\ 0.1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.70 & 2.00 & -1.56 \end{bmatrix}, \quad D = [0.6].$$

The system is admissible; that is, $\rho(E, A) = 0.9799$, and the rank condition (3.43) holds true. The $\mathscr{H}_\infty$-norm of the transfer function $\|P\|_\infty$ is equal to 6.8364.

To satisfy the conditions of Theorem 3.2 for given a and γ, parameter q should satisfy the condition $q \in [0, \min(\gamma^{-2}, \|P\|_\infty^{-2}))$ where $\|P\|_\infty^{-2} = 0.0214$, and the inequality $E^{\mathrm{T}}\widehat{R}E \geqslant 0$ should be true for the matrix $\widehat{R}$.

Consider the results for different values of γ. For $a = 0.1$ the results are given in Table 3.1. The anisotropic norm is equal to $|||P|||_a = 3.1537$.

As we can see, the conditions of Theorem 3.2 are satisfied for $|||P|||_a < \gamma$; for $|||P|||_a > \gamma$ the conditions not only on q, but also on $\widehat{R}$ get broken.

Therefore the conditions of Theorem 3.2 can be used for anisotropic norm computation with a given accuracy.

Table 3.1 Checking the conditions of Theorem 3.2 for $a = 0.1$

γ	3.170	3.160	3.150
$[0, \min(\gamma^{-2}, \Vert P\Vert_\infty^{-2}))$	$[0, 0.0214)$		
q	0.0209	0.0213	-0.0001ı+0.0214
$E^{\mathrm{T}}\widehat{R}E$	$\begin{bmatrix} 0.2508 & 0.4703 & 0 \\ 0.4703 & 3.3776 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.2538 & 0.4818 & 0 \\ 0.4818 & 3.5087 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.0018\imath+0.2577 & -0.0123\imath+0.4993 & 0 \\ -0.0123\imath+0.4993 & -0.1636\imath+3.7194 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

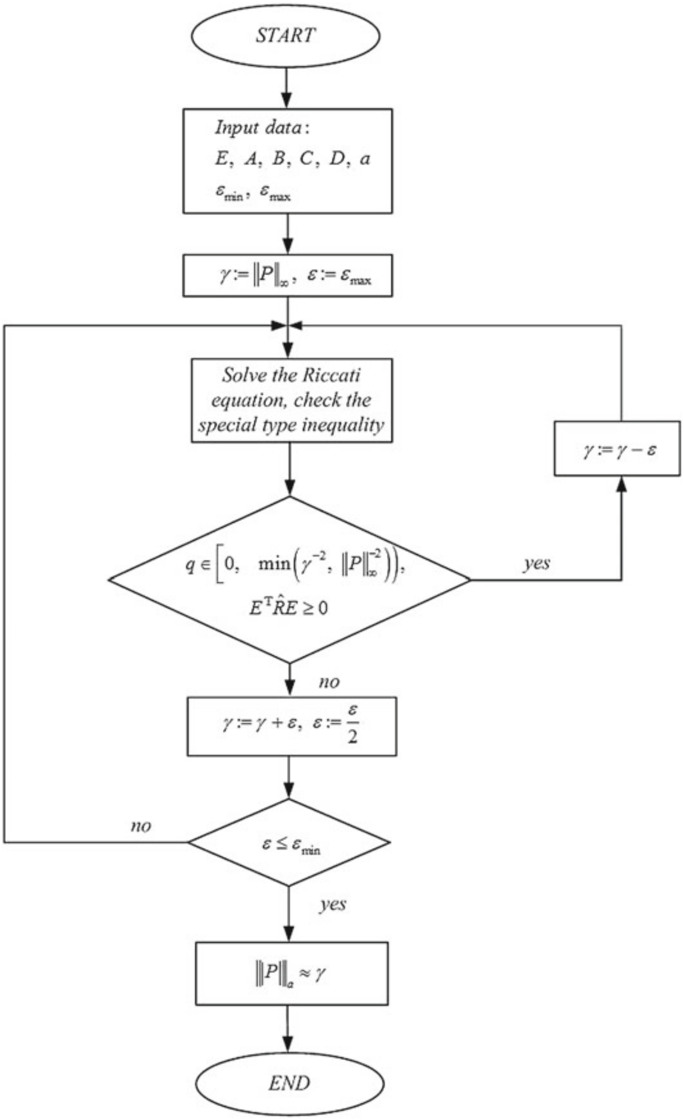

Fig. 3.3 Block diagram of anisotopic norm computation algorithm

Anisotropic Norm Computation on Basis of an Anisotropy-Based Bounded Real Lemma

Using the conditions of an anisotropy-based bounded real lemma (Theorem 3.2), one can compute the anisotropic norm with prescribed accuracy $\varepsilon_{\min}$. A block diagram of the algorithm is shown in Fig. 3.3 where $\varepsilon_{\max}$ is the accuracy, taken from the previous step.

3.3.2 Convex Optimization Approach

In this section, conditions of an anisotropy-based bounded real lemma are given in terms of LMIs, which form convex constraints. To solve the analysis problem we formulate sufficient conditions of anisotropic norm boundedness for a descriptor system (3.41) and (3.42).

Theorem 3.3 *Let $W = \{w(k)\}_{k\in\mathbb{Z}}$ be a stationary Gaussian random sequence whose mean anisotropy does not exceed the given value $a \geqslant 0$. System $P \in \mathscr{H}_\infty{}^{p\times m}$ with a state-space representation (3.41) and (3.42) is admissible and its a-anisotropic norm is bounded by a positive scalar $\gamma > 0$; that is,*

$$|\!|\!| P |\!|\!|_a < \gamma$$

if there exist a scalar $q \in (0, \min(\gamma^{-2}, \|P\|_\infty))$, and a symmetric matrix R satisfying

$$ERE^{\mathrm{T}} \geqslant 0,$$

$$-(\det(I_m - B^T RB - qD^T D))^{1/m} < -(1 - q\gamma^2)\mathrm{e}^{2a/m}, \tag{3.71}$$

$$\begin{bmatrix} A^{\mathrm{T}}RA - E^{\mathrm{T}}RE & A^{\mathrm{T}}RB \\ B^{\mathrm{T}}RA & B^{\mathrm{T}}RB - I_m \end{bmatrix} + q \begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0. \tag{3.72}$$

Proof Use the following denotations.

$$\Sigma = (I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D)^{-1} \text{ and } L = \Sigma(B^{\mathrm{T}}RA + qD^{\mathrm{T}}C).$$

Proof of this theorem includes the following steps.

1. First show that inequality (3.44) can be rewritten in the form (3.71).
2. Prove that (3.72) holds true for an admissible system P.
3. Show that the pair $(E, A + BL)$ is admissible. Note that the pair $(E, A + BL)$ is connected with the worst-case shaping filter [9].

Using logarithm properties, we transform inequality (3.44) and get

$$-\ln\det((1 - q\gamma^2)\Sigma) > 2a,$$

$$-\ln\det\Sigma - \ln\det(I_m(1 - q\gamma^2)) > 2a,$$

then

$$-\ln\det\Sigma > m\ln(1 - q\gamma^2) + 2a,$$

$$-1/m\ln\det\Sigma > \ln\left(\mathrm{e}^{2a/m}(1 - q\gamma^2)\right),$$

$$(\det \Sigma)^{-1/m} > (1 - q\gamma^2)\mathrm{e}^{2a/m}.$$

By multiplication on (-1) we get

$$-(\det \Sigma)^{-1/m} < -(1 - q\gamma^2)\mathrm{e}^{2a/m}.$$

Remark 3.3 It is known that [14]

1. Function $(\det \Psi)^p$ of $(m \times m)$-matrix $\Psi = \Psi^{\mathrm{T}} \geqslant 0$ is concave over its argument for any $0 \leqslant p \leqslant \frac{1}{m}$.
2. Function $(\det \Psi)^{\frac{1}{m}}$ of the $(m \times m)$-matrix $\Psi = \Psi^{\mathrm{T}} \geqslant 0$ is the geometric mean of its eigenvalues $\sqrt[m]{\lambda_1(\Psi)\ldots\lambda_m(\Psi)}$.
3. Set
$$\{(\lambda_1,\ \lambda_2,\ t) \in \mathbb{R}^3 | \lambda_1,\ \lambda_2 \geqslant 0,\ t \leqslant \sqrt{\lambda_1\lambda_2}\}$$
can be represented as a second-order cone
$$\left\{(\lambda_1,\ \lambda_2,\ t) | \exists\tau : t \leqslant \tau; \tau \geqslant 0,\ \left\| \begin{bmatrix} \tau \\ \frac{\lambda_1 - \lambda_2}{2} \end{bmatrix} \right\|_2 \leqslant \frac{\lambda_1 + \lambda_2}{2} \right\},$$
and the set $\{(\lambda_1, \ldots, \lambda_{2^l}, t) \in \mathbb{R}^{2^l+1} | \lambda_i \geqslant 0, I = 1, \ldots, 2^l, t \leqslant (\lambda_1\lambda_2\ldots\lambda_{2^l})^{1/2^l}\}$, $l \in \mathbb{N}$ is an intersection of a finite number of second-order cones.
4. If $p \in \mathbb{R}$ is a rational number $0 \leq p \leq \frac{1}{m}$, then a convex function $(\det \Psi)^p$ of the $(m \times m)$-matrix $\Psi = \Psi^{\mathrm{T}} \geq 0$ can be represented in the LMI form. Namely, the set
$$\{(\Psi, t) | \Psi = \Psi^{\mathrm{T}} \geq 0, t \leq (\det \Psi)^p\}$$
can be represented as
$$\left\{(\Psi, t) | \Psi = \Psi^{\mathrm{T}} \geq 0,\ \begin{bmatrix} \Psi & \Delta \\ \Delta^{\mathrm{T}} & \operatorname{diag}\Delta \end{bmatrix} \geq 0,\ t \leq (\delta_1 \ldots \delta_m)^p \right\}$$
where Δ is the lower triangular $(m \times m)$-matrix, constructed of auxiliary variables with elements δ_i, $i = 1 \ldots m$ on the main diagonal. The subgraph of the concave term $t \leq (\delta_1 \ldots \delta_m)^p$ can be expressed via a second-order cone, and, hence, as LMI. See [14] for more details.

For $(I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D) > 0$, using the Schur complement [15], rewrite the inequality (3.72) in the form

$$\begin{aligned} &A^{\mathrm{T}}RA - E^{\mathrm{T}}RE + qC^{\mathrm{T}}C + \\ &\quad + (ARB^{\mathrm{T}} + qC^{\mathrm{T}}D)(I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D)^{-1}(BRA^{\mathrm{T}} + qD^{\mathrm{T}}C) < 0. \end{aligned} \tag{3.73}$$

System P is admissible; thus there exists such R that (3.73) (or equivalently (3.72)) holds true. Now if (3.73) is satisfied then there exists a nonnegative matrix Ξ such that

$$A^{\mathrm{T}}RA - E^{\mathrm{T}}RE + qC^{\mathrm{T}}C + (ARB^{\mathrm{T}} + qC^{\mathrm{T}}D)(I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D)^{-1}\times \\ \times (BRA^{\mathrm{T}} + qD^{\mathrm{T}}C) + \Xi = 0. \tag{3.74}$$

Stationary sequence W is generated from the Gaussian white noise sequence by the shaping filter G with a state-space representation

$$G = \left[E, \begin{array}{c|c} A+BL & B\Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array}\right].$$

Prove that the pair $(E, A+BL)$ is admissible.

$$(A+BL)^{\mathrm{T}}\tilde{R}(A+BL) - E^{\mathrm{T}}\tilde{R}E = A^{\mathrm{T}}\tilde{R}A - E^{\mathrm{T}}\tilde{R}E + \\ + L^{\mathrm{T}}B^{\mathrm{T}}\tilde{R}A + A^{\mathrm{T}}\tilde{R}BL + L^{\mathrm{T}}B^{\mathrm{T}}\tilde{R}BL.$$

System (3.41) is admissible. According to the conditions of Theorem 2.2 there exists a matrix $\widehat{R}$ such that

$$A^{\mathrm{T}}\widehat{R}A - E^{\mathrm{T}}\widehat{R}E + L^{\mathrm{T}}B^{\mathrm{T}}\widehat{R}A + A^{\mathrm{T}}\widehat{R}BL < 0.$$

It means that there exists a matrix $\tilde{R}$ such that

$$A^{\mathrm{T}}\tilde{R}A - E^{\mathrm{T}}\tilde{R}E + L^{\mathrm{T}}B^{\mathrm{T}}\tilde{R}A + A^{\mathrm{T}}\tilde{R}BL + L^{\mathrm{T}}B^{\mathrm{T}}\tilde{R}BL < 0$$

holds true.

This completes the proof. ■

Remark 3.4 Convex conditions (3.71) and (3.72) cannot be directly applied for minimal value of γ computation because of the product of q and γ^2 on the right-hand side of (3.71).

Transform the inequalities (3.71) and (3.72). Define $\eta = q^{-1}$ and $\xi = \gamma^2$. Multiplying both inequalities on η and substituting ξ conditions of Theorem 3.3 can be rewritten as

$$\eta - (\mathrm{e}^{-2a}\det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} < \xi, \tag{3.75}$$

$$\begin{bmatrix} A^{\mathrm{T}}\Phi A - E^{\mathrm{T}}\Phi E + C^{\mathrm{T}}C & A^{\mathrm{T}}\Phi B + C^{\mathrm{T}}D \\ B^{\mathrm{T}}\Phi A + D^{\mathrm{T}}C & B^{\mathrm{T}}\Phi B + D^{\mathrm{T}}D - \eta I_m \end{bmatrix} < 0, \tag{3.76}$$

$$E^{\mathrm{T}}\Phi E \geqslant 0 \tag{3.77}$$

where $\Phi = \eta R$. Conditions (3.75) and (3.76) are linear on ξ.

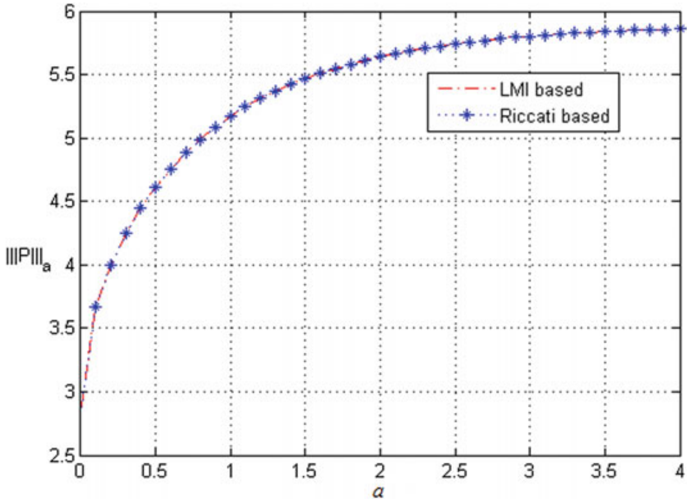

Fig. 3.4 Anisotropic norm of the system computed using Riccati- and LMI-based algorithms

They allow us to calculate the minimum value of γ by solving the following convex optimization problem: to find $\xi_* = \inf \xi$ on the set $\{\Phi, \eta, \xi\}$ that satisfies (3.75)–(3.77). If the minimum value ξ_* is found, then the a-anisotropic norm of the system P can be approximately calculated as

$$|||P|||_a \approx \sqrt{\xi_*}.$$

Example 3.6 The system is described by equations

$$E = \begin{bmatrix} 3 & 0 & 2 & -5 \\ 0 & 3 & -2 & 2 \\ 2 & 2 & 0 & -2 \\ 2 & -4 & 4 & -6 \end{bmatrix}, \quad A = \begin{bmatrix} 0.7 & -3.25 & -0.7 & 0 \\ 1.8 & 0.4 & -6.4 & 2.6 \\ 1 & -1.9 & -5.4 & 2.4 \\ -0.6 & -2.7 & 5.4 & -2.8 \end{bmatrix}, \quad B = \begin{bmatrix} 3.2 & -3.5 \\ 2.5 & -7.9 \\ 3.8 & -7.6 \\ -1.2 & 8.2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.2 & 0.4 & 0.45 & 0.6 \end{bmatrix}, \quad D = \begin{bmatrix} 0.2 & 1 \end{bmatrix}.$$

Rank condition (3.43) holds true. The results of anisotropic norm computation using an LMI-based algorithm and Riccati-based algorithms are presented in Fig. 3.4. Figure 3.5 shows an absolute error between the obtained values. Simulation results show that the suggested algorithm allows computing the anisotropic norm of descriptor systems with high accuracy. This method can be applied without using algebraic transformation for the given system in contrast to the algorithm in Sect. 3.2.3. For more information, see [10].

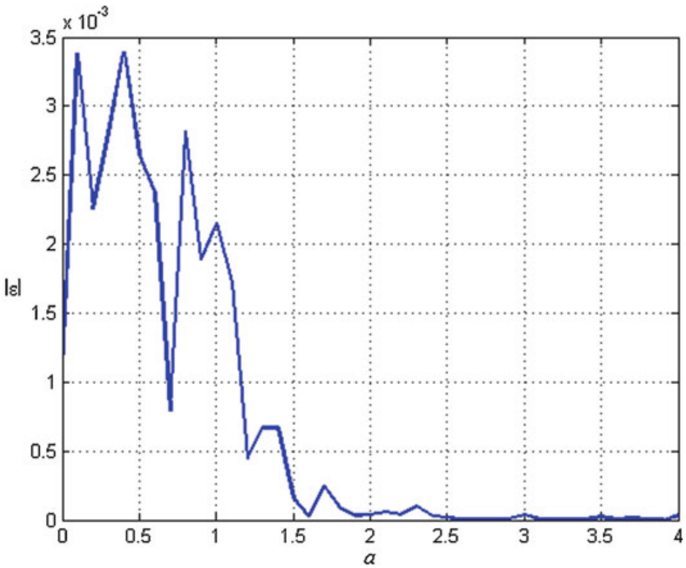

Fig. 3.5 Absolute error of anisotropic norm computation between LMI- and Riccati-based algorithms

Limiting Cases

Consider the conditions of Theorem 3.3 for two limiting cases when mean anisotropy level a of the input disturbance goes to zero and to infinity. As $\mathscr{H}_2$ - and $\mathscr{H}_\infty$ -norms are limiting cases of the anisotropic norm for $a = 0$ and $a \to \infty$, one can expect that the inequalities (3.71) and (3.72) transform to the criteria of boundedness of the scaled $\mathscr{H}_2$ -norm and $\mathscr{H}_\infty$ -norm of system P.

Prove the following lemma, which allows us to compute the $\mathscr{H}_2$ -norm of descriptor systems.

Lemma 3.3 *Let T be a solution of the following generalized Lyapunov equation*

$$A^{\mathrm{T}} T A - E^{\mathrm{T}} T E + C^{\mathrm{T}} C = 0 \tag{3.78}$$

for system (3.41) *and* (3.42)*. Then the $\mathscr{H}_2$ -norm of system* (3.41) *and* (3.42) *can be computed as*

$$\|P\|_2 = \sqrt{\operatorname{Tr}\,(B^{\mathrm{T}} T B + D^{\mathrm{T}} D)}. \tag{3.79}$$

Proof Condition (3.43) means that there exists a matrix $\overline{W}$ such that

$$\overline{W} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

and, therefore, the condition $B_1 B_2^{\mathrm{T}} = 0$ holds. Hence, the solution of the generalized Lyapunov equation (3.78) defines a observability Gramian of system (3.41) and (3.42). Using the definition of an $\mathscr{H}_2$-norm we get (3.79) that completes the proof. ■

Now we give the conditions of $\mathscr{H}_\infty$-norm boundedness for descriptor systems [16].

Lemma 3.4 *The following statements are equivalent.*

1. *A is a stable matrix,* $\|C(zE - A)^{-1}B + D\|_\infty < \gamma$.
2. *There exists a matrix* $R = R^{\mathrm{T}}$ *such that* $E^{\mathrm{T}}RE > 0$ *and*

$$\begin{aligned} A^{\mathrm{T}}RA - E^{\mathrm{T}}RE + C^{\mathrm{T}}C + (A^{\mathrm{T}}RB + C^{\mathrm{T}}D)\times \\ \times (\gamma^2 I_m - B^{\mathrm{T}}RB - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}RA + D^{\mathrm{T}}C) < 0 \end{aligned} \tag{3.80}$$

where $\gamma^2 I_m - B^{\mathrm{T}}RB - D^{\mathrm{T}}D > 0$.

Consider now limiting cases of anisotropic norm boundedness conditions for $a = 0$ and $a \to \infty$.

1. If $a = 0$, then the inequality (3.75) is equal to

$$\eta - (\det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} < \gamma^2. \tag{3.81}$$

   Taking into account the relation between arithmetic and geometric means [17], we get

$$(\det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} \leqslant \frac{1}{m}\mathrm{Tr}\,(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D)).$$

   The inequality (3.81) leads to

$$\mathrm{Tr}\,(B^{\mathrm{T}}\Phi B + D^{\mathrm{T}}D) < m\gamma^2. \tag{3.82}$$

   It is easy to verify that the inequality (3.76) is satisfied if

$$A^{\mathrm{T}}\Phi A - E^{\mathrm{T}}\Phi E + C^{\mathrm{T}}C < 0. \tag{3.83}$$

   The conditions (3.82) and (3.83) are equivalent to the inequality

$$\frac{1}{\sqrt{m}}\|P\|_2 < \gamma.$$

2. If $a \to \infty$, then $\eta \to \gamma^2$, and condition (3.75) is violated. For $\bar{\Phi} = \gamma\Phi$ it is easy to notice that the inequality (3.76) is equivalent to (3.80). Thus for $a \to \infty$ the condition $|||P|||_a < \gamma$ coincides with the condition $\|P\|_\infty < \gamma$.

3.3.3 Novel Anisotropy-Based Bounded Real Lemma: Strict Conditions

The result obtained in the previous section deals with nonstrict LMIs. This may lead to some inaccuracies while computing the a-anisotropic norm of a descriptor system. Moreover, inequality (3.80) leads to nonlinearities in control design problems. The following results represent strict inequalities that can be used for solving the control design problem.

Novel Bounded Real Lemma for Normal Systems

Before we formulate a novel anisotropy-based bounded real lemma for descriptor systems, we consider the standard state-space case:

$$x(k+1) = Ax(k) + Bw(k), \tag{3.84}$$

$$y(k) = Cx(k) + Dw(k) \tag{3.85}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ is a random stationary sequence with known mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$, $y(k) \in \mathbb{R}^q$ is an observable output, and $A,\ B,\ C$, and D are constant real matrices of appropriate dimensions. The transfer function of system (3.84) and (3.85) is defined by

$$T(z) = C(zI_n - A)^{-1}B + D.$$

We suppose that system (3.84) and (3.85) is stable, and constants $a \geqslant 0$ and $\gamma > 0$ are known. The problem is to satisfy the inequality

$$|\!|\!| T |\!|\!|_a < \gamma.$$

The following lemma gives the answer to this problem [18].

Lemma 3.5 *Let system* (3.84) *and* (3.85) *with a transfer function* $T(z) \in \mathscr{H}_\infty{}^{q\times m}$ *be stable. For the given scalar values* $a \geqslant 0$ *and* $\gamma > 0$ *the* a*-anisotropic norm is bounded by a given scalar value* γ*; that is,*

$$|\!|\!| T |\!|\!|_a < \gamma$$

if there exist such scalar value $\eta > \gamma^2$ *and* $n \times n$*-matrix* $\Phi = \Phi^{\mathrm{T}} > 0$ *that the following inequalities hold true.*

$$\eta - (\mathrm{e}^{-2a}\det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} < \gamma^2, \tag{3.86}$$

$$\begin{bmatrix} A^{\mathrm{T}}\Phi A - \Phi + C^{\mathrm{T}}C & A^{\mathrm{T}}\Phi B + C^{\mathrm{T}}D \\ B^{\mathrm{T}}\Phi A + D^{\mathrm{T}}C & B^{\mathrm{T}}\Phi B + D^{\mathrm{T}}D - \eta I_m \end{bmatrix} < 0. \tag{3.87}$$

Based on Lemma 3.5, we introduce the following theorem.

Theorem 3.4 *For given scalar values $a \geqslant 0$ and $\gamma > 0$ system* (3.84) *and* (3.85) *with the transfer function $T(z) \in \mathscr{H}_\infty{}^{q\times m}$ is stable and its a-anisotropic norm is bounded by γ; that is,*

$$|||T|||_a < \gamma$$

if there exist such scalar values $\eta > \gamma^2$, $n \times n$-matrix $\Phi = \Phi^{\mathrm{T}} > 0$ and a random $n \times n$-matrix Y that the following inequalities hold true.

$$\eta - (\mathrm{e}^{-2a} \det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} < \gamma^2, \tag{3.88}$$

$$\begin{bmatrix} -\frac{1}{2}Y - \frac{1}{2}Y^{\mathrm{T}} & YA & YB & \Phi^{\mathrm{T}} - Y^{\mathrm{T}} - \frac{1}{2}Y & 0 \\ A^{\mathrm{T}}Y^{\mathrm{T}} & -\Phi & 0 & A^{\mathrm{T}}Y^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}}Y^{\mathrm{T}} & 0 & -\eta I_m & B^{\mathrm{T}}Y^{\mathrm{T}} & D^{\mathrm{T}} \\ \Phi - Y - \frac{1}{2}Y^{\mathrm{T}} & YA & YB & -Y - Y^{\mathrm{T}} & 0 \\ 0 & C & D & 0 & -I_q \end{bmatrix} < 0. \tag{3.89}$$

Proof Suppose the inequalities (3.88) and (3.89) hold true. Rewrite the expression (3.89) in the form

$$\Xi + \Upsilon^{\mathrm{T}}Y^{\mathrm{T}}\Delta + \Delta^{\mathrm{T}}Y\Upsilon < 0 \tag{3.90}$$

where $\Delta = \begin{bmatrix} I_n & 0 & 0 & I_n \end{bmatrix}$, $\Upsilon = \begin{bmatrix} -\frac{1}{2}I_n & A & B & -I_n \end{bmatrix}$, and a symmetric matrix Ξ is given by

$$\Xi = \begin{bmatrix} 0 & 0 & 0 & \Phi \\ 0 & C^{\mathrm{T}}C - \Phi & C^{\mathrm{T}}D & 0 \\ 0 & D^{\mathrm{T}}C & D^{\mathrm{T}}D - \eta I_m & 0 \\ \Phi & 0 & 0 & 0 \end{bmatrix}.$$

Using the projection lemma [19], we get that inequality (3.90) is solvable for the $n \times n$-matrix Y if and only if

$$M^{\mathrm{T}}\Xi M < 0 \text{ and } N^{\mathrm{T}}\Xi N < 0$$

for

$$M^{\mathrm{T}} = \begin{bmatrix} 0 & I_n & 0 & 0 \\ 0 & 0 & I_m & 0 \\ -I_n & 0 & 0 & I_n \end{bmatrix}, \quad N^{\mathrm{T}} = \begin{bmatrix} I_n & 0 & 0 & -\frac{1}{2}I_n \\ 0 & I_n & 0 & A^{\mathrm{T}} \\ 0 & 0 & I_m & B^{\mathrm{T}} \end{bmatrix}.$$

Also, columns of the matrix N form the basis of the kernel of Υ, and columns of the matrix M form the basis of the kernel of Δ.

Note that

$$N^{\mathrm{T}}\Xi N = \begin{bmatrix} -\Phi & \Phi A & \Phi B \\ A^{\mathrm{T}}\Phi & C^{\mathrm{T}}C & C^{\mathrm{T}}D \\ B^{\mathrm{T}}\Phi & D^{\mathrm{T}}C & D^{\mathrm{T}}D \end{bmatrix} < 0. \tag{3.91}$$

As $\Phi = \Phi^{\mathrm{T}} > 0$, using the Schur complement, we may transform the inequality (3.91) into

$$\begin{bmatrix} C^{\mathrm{T}}C & C^{\mathrm{T}}D \\ D^{\mathrm{T}}C & D^{\mathrm{T}}D \end{bmatrix} - \begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} \Phi(-\Phi)^{-1}\Phi \begin{bmatrix} A & B \end{bmatrix} < 0.$$

Hence,

$$\begin{bmatrix} A^{\mathrm{T}}\Phi A - \Phi + C^{\mathrm{T}}C & A^{\mathrm{T}}\Phi B + C^{\mathrm{T}}D \\ B^{\mathrm{T}}\Phi A + D^{\mathrm{T}}C & B^{\mathrm{T}}\Phi B + D^{\mathrm{T}}D - \eta I_m \end{bmatrix} < 0.$$

Consequently, the conditions of this theorem are equivalent to the conditions of Lemma 3.5, proved in [18].

The theorem is proved. ■

Remark 3.5 In order to avoid product $D^{\mathrm{T}}D$ in inequality (3.88) we introduce a new variable Ψ:

$$\Psi < \eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D. \tag{3.92}$$

Transform (3.92) according to Schur's lemma

$$\Psi - \eta I_m + B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}\left(-I_q\right) D < 0,$$

$$\begin{bmatrix} \Psi - \eta I_m + B^{\mathrm{T}}\Phi B & D^{\mathrm{T}} \\ D & -I_q \end{bmatrix} < 0.$$

Thus the inequality (3.88) can be rewritten as a system of inequalities

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/m} < \gamma^2$$

and

$$\begin{bmatrix} \Psi - \eta I_m + B^{\mathrm{T}}\Phi B & D^{\mathrm{T}} \\ D & -I_q \end{bmatrix} < 0.$$

Novel Bounded Real Lemma for Descriptor Systems

Consider a discrete-time descriptor system (3.41) and (3.42). As it is regular, there exist two transformation matrices $\widetilde{W}$ and $\widetilde{V}$, and the system (3.41) and (3.42) can be rewritten in the equivalent form (2.17)–(2.19). We use the denotations: $E_d = \widetilde{W}E\widetilde{V}$, $A_d = \widetilde{W}A\widetilde{V}$, $B_d = \widetilde{W}B$, $C_d = C\widetilde{V}$, and $D_d = D$.

Now we formulate conditions of anisotropic norm boundedness for the system (3.41) and (3.42).

Theorem 3.5 *For given scalar values $a \geqslant 0$ and the $\gamma > 0$ system (3.41) and (3.42) with a transfer function $P(z) \in \mathscr{H}_\infty{}^{p\times m}$ is admissible and its a-anisotropic norm is bounded by γ; that is,*

$$|\!|\!| P |\!|\!|_a < \gamma$$

if there exist matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $\Psi \in \mathbb{R}^{m\times m}$, scalar values $\eta > \gamma^2$ and $\alpha > 0$ that the following inequalities hold true.

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/m} < \gamma^2, \tag{3.93}$$

$$\begin{bmatrix} \Psi - \eta I_m + B_d^{\mathrm{T}}\Theta B_d & D_d^{\mathrm{T}} \\ D_d & -I_p \end{bmatrix} < 0, \tag{3.94}$$

and

$$\begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_d & L^{\mathrm{T}} - Q^{\mathrm{T}} - \frac{1}{2}Q & 0 \\ A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Pi A_d + A_d^{\mathrm{T}}\Pi^{\mathrm{T}} - \Theta & \Pi B_d & A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & C_d^{\mathrm{T}} + \alpha A_d^{\mathrm{T}}\Pi^{\mathrm{T}}C_d^{\mathrm{T}} \\ B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & B_d^{\mathrm{T}}\Pi^{\mathrm{T}} & -\eta I_m & B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & D_d^{\mathrm{T}} + \alpha B_d^{\mathrm{T}}\Pi^{\mathrm{T}}C_d^{\mathrm{T}} \\ L - Q - \frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_d & -Q - Q^{\mathrm{T}} & 0 \\ 0 & C_d + \alpha C_d \Pi A_d & D_d + \alpha C_d \Pi B_d & 0 & -I_p \end{bmatrix} < 0, \tag{3.95}$$

where $\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}$, $\Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$, $\Gamma = \begin{bmatrix} Q & R \end{bmatrix}$.

Proof Suppose inequalities (3.93)–(3.95) hold true. For the equivalent form (2.17)–(2.19) of system (3.41) and (3.42) it is not difficult to get

$$Z = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & 0 \\ z_{12}^{\mathrm{T}} & -L & z_{23} & 0 & z_{25} & z_{26} \\ z_{13}^{\mathrm{T}} & z_{23}^{\mathrm{T}} & z_{33} & z_{34} & z_{35} & z_{36} \\ z_{14}^{\mathrm{T}} & 0 & z_{34}^{\mathrm{T}} & -\eta I_m & z_{45} & z_{46} \\ z_{15}^{\mathrm{T}} & z_{25}^{\mathrm{T}} & z_{35}^{\mathrm{T}} & z_{45}^{\mathrm{T}} & z_{55} & 0 \\ 0 & z_{26}^{\mathrm{T}} & z_{36}^{\mathrm{T}} & z_{46}^{\mathrm{T}} & 0 & -I_p \end{bmatrix} < 0$$

where

$$z_{11} = -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}},\ z_{12} = QA_{11} + RA_{21},$$

$$z_{13} = QA_{12} + RA_{22},\ z_{14} = QB_1 + RB_2,$$

$$z_{15} = L^{\mathrm{T}} - Q^{\mathrm{T}} - \frac{1}{2}Q,\ z_{23} = A_{21}^{\mathrm{T}}S^{\mathrm{T}},$$

$$z_{25} = A_{11}^{\mathrm{T}}Q^{\mathrm{T}} + A_{21}^{\mathrm{T}}R^{\mathrm{T}},\ z_{26} = C_1^{\mathrm{T}} + \alpha A_{21}^{\mathrm{T}}S^{\mathrm{T}}C_2^{\mathrm{T}}$$

$$z_{33} = SA_{22} + A_{22}^{\mathrm{T}}S^{\mathrm{T}},\ z_{34} = SB_2,$$

$$z_{35} = A_{12}^{\mathrm{T}}Q^{\mathrm{T}} + A_{22}^{\mathrm{T}}R^{\mathrm{T}},\ z_{36} = C_2^{\mathrm{T}} + \alpha A_{22}^{\mathrm{T}}S^{\mathrm{T}}C_2^{\mathrm{T}}$$

$$z_{45} = B_1^{\mathrm{T}}Q^{\mathrm{T}} + B_2^{\mathrm{T}}R^{\mathrm{T}},\ z_{46} = D^{\mathrm{T}} + \alpha B_2^{\mathrm{T}}S^{\mathrm{T}}C_2^{\mathrm{T}},$$

$$z_{55} = -Q - Q^{\mathrm{T}}.$$

As $Z < 0$, we can choose a nonsingular matrix K, such that

$$KZK^{\mathrm{T}} < 0.$$

By setting

$$K = \begin{bmatrix} I_r & 0 & 0 & 0 & 0 & 0 \\ 0 & I_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & I_p \\ 0 & 0 & I_{n-r} & 0 & 0 & 0 \end{bmatrix}$$

we get

$$KZK^{\mathrm{T}} = \begin{bmatrix} z_{11} & z_{12} & z_{14} & z_{15} & 0 & z_{13} \\ z_{12}^{\mathrm{T}} & -L & 0 & z_{25} & z_{26} & z_{23} \\ z_{14}^{\mathrm{T}} & 0 & -\eta I_m & z_{45} & z_{46} & z_{34}^{\mathrm{T}} \\ z_{15}^{\mathrm{T}} & z_{25}^{\mathrm{T}} & z_{45}^{\mathrm{T}} & z_{55} & 0 & z_{35}^{\mathrm{T}} \\ 0 & z_{26}^{\mathrm{T}} & z_{46}^{\mathrm{T}} & 0 & -I_p & z_{36}^{\mathrm{T}} \\ z_{13}^{\mathrm{T}} & z_{23}^{\mathrm{T}} & z_{34} & z_{35} & z_{36} & z_{33} \end{bmatrix} < 0.$$

Consider expression $KZK^{\mathrm{T}} = W + W^{\mathrm{T}}$ where

$$W = \begin{bmatrix} w_{11} & 0 & 0 & 0 & 0 & 0 \\ w_{21} & w_{22} & 0 & w_{24} & w_{25} & w_{26} \\ w_{31} & 0 & w_{33} & w_{34} & w_{35} & w_{36} \\ w_{41} & 0 & 0 & w_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & w_{55} & 0 \\ w_{61} & 0 & 0 & w_{64} & w_{65} & w_{66} \end{bmatrix},$$

$$w_{11} = -\frac{1}{2}Q,\ w_{21} = w_{24} = A_{11}^{\mathrm{T}}Q^{\mathrm{T}} + A_{21}^{\mathrm{T}}R^{\mathrm{T}},$$

$$w_{22} = -\frac{1}{2}L,\ w_{25} = C_1^{\mathrm{T}} + \alpha A_{21}^{\mathrm{T}}S^{\mathrm{T}}C_2^{\mathrm{T}},$$

$$w_{26} = A_{21}^{\mathrm{T}} S^{\mathrm{T}},\ w_{31} = w_{34} = B_1^{\mathrm{T}} Q^{\mathrm{T}} + B_2^{\mathrm{T}} R^{\mathrm{T}},$$

$$w_{33} = -\frac{\eta}{2} I_m,\ w_{35} = D^{\mathrm{T}} + \alpha B_2^{\mathrm{T}} S^{\mathrm{T}} C_2^{\mathrm{T}},$$

$$w_{36} = B_2^{\mathrm{T}} S^{\mathrm{T}},\ w_{41} = L - Q - \frac{1}{2} Q^{\mathrm{T}},$$

$$w_{44} = -Q,\ w_{55} = -\frac{1}{2} I_p,\ w_{65} = C_2^{\mathrm{T}} + \alpha A_{22}^{\mathrm{T}} S^{\mathrm{T}} C_2^{\mathrm{T}},$$

$$w_{61} = w_{64} = A_{12}^{\mathrm{T}} Q^{\mathrm{T}} + A_{22}^{\mathrm{T}} R^{\mathrm{T}},\ w_{66} = A_{22}^{\mathrm{T}} S^{\mathrm{T}}.$$

Thus

$$W + W^{\mathrm{T}} < 0. \tag{3.96}$$

As $z_{33} = A_{22}^{\mathrm{T}} S^{\mathrm{T}} + S A_{22} < 0$, both matrices A_{22} and S are nonsingular. The system (3.41) is causal; it can be transformed into a normal system $\widehat{T}$ of reduced dimension

$$\widehat{x}(k+1) = \widehat{A}\widehat{x}(k) + \widehat{B} w(k), \tag{3.97}$$

$$\widehat{y}(k) = \widehat{C}\widehat{x}(k) + \widehat{D} w(k) \tag{3.98}$$

where $\widehat{x}(k) \in \mathbb{R}^r$,

$$\widehat{A} = A_{11} - A_{12} A_{22}^{-1} A_{21}\ ,\quad \widehat{B} = B_1 - A_{12} A_{22}^{-1} B_2\ ,$$

$$\widehat{C} = C_1 - C_2 A_{22}^{-1} A_{21}\ ,\quad \widehat{D} = D - C_2 A_{22}^{-1} B_2.$$

According to the rank condition, $B_2 = 0$. Hence, the inequality (3.93) coincides with (3.88) for the equivalent system (3.97) and (3.98).

Now show that the matrix $\widehat{A}$ is stable, and $|||\widehat{T}|||_a < \gamma$. As SA_{22} and $A_{22}^{\mathrm{T}} S^{\mathrm{T}}$ are invertible, $A_{22}^{\mathrm{T}} S^{\mathrm{T}} < 0$ and $SA_{22} < 0$; applying Schur's lemma to (3.96), we get

$$\begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}} & Q\widehat{A} & Q\widehat{B} & L^{\mathrm{T}} - Q^{\mathrm{T}} - \frac{1}{2}Q & 0 \\ \widehat{A}^{\mathrm{T}} Q^{\mathrm{T}} & -L & 0 & \widehat{A}^{\mathrm{T}} Q^{\mathrm{T}} & \widehat{C}^{\mathrm{T}} \\ \widehat{B}^{\mathrm{T}} Q^{\mathrm{T}} & 0 & -\eta I_m & \widehat{B}^{\mathrm{T}} Q^{\mathrm{T}} & \widehat{D}^{\mathrm{T}} \\ L - Q - \frac{1}{2}Q^{\mathrm{T}} & Q\widehat{A} & Q\widehat{B} & -Q - Q^{\mathrm{T}} & 0 \\ 0 & \widehat{C} & \widehat{D} & 0 & -I_p \end{bmatrix} < 0. \tag{3.99}$$

According to Theorem 3.4, we have $\rho(\widehat{A}) < 1$ and $|||\widehat{T}|||_a < \gamma$. Thus $|||P|||_a < \gamma$. The theorem is proved. ■

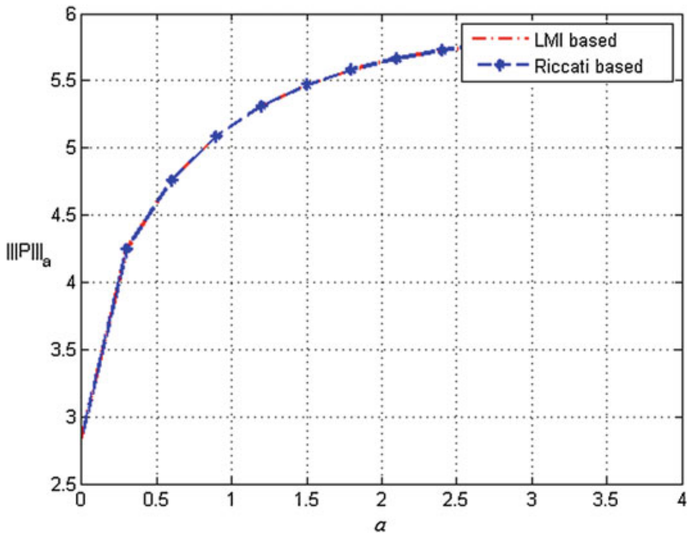

Fig. 3.6 a-Anisotropic norm computation, using novel BRL

Remark 3.6 Denote $\xi = \gamma^2$. While computing the a-anisotropic norm for descriptor systems one should solve the optimization problem: to find $\xi_* = \inf \xi$ on the set $\{L, Q, R, S, \Psi, \eta, \xi\}$, that satisfies the inequalities (3.93)–(3.95). If the minimum value ξ_* is found, the a-anisotropic norm of the system P is calculated as

$$|||P|||_a \approx \sqrt{\xi_*}. \tag{3.100}$$

Here the scalar value $\alpha > 0$ is set.

Example 3.7 Let the matrices of system (3.41) and (3.42) be equal to

$$E = \begin{bmatrix} 3 & 0 & 2 & -5 \\ 0 & 3 & -2 & 2 \\ 2 & 2 & 0 & -2 \\ 2 & -4 & 4 & -6 \end{bmatrix}, \quad A = \begin{bmatrix} 0.7 & -3.25 & -0.7 & 0 \\ 1.8 & 0.4 & -6.4 & 2.6 \\ 1.0 & -1.9 & -5.4 & 2.4 \\ -0.6 & -2.7 & 5.4 & -2.8 \end{bmatrix},$$

$$B = \begin{bmatrix} 3.2 & -3.5 \\ 2.5 & -7.9 \\ 3.8 & -7.6 \\ -1.2 & 8.2 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2 & 0.4 & 0.45 & 0.6 \end{bmatrix}, \quad D = \begin{bmatrix} 0.2 & 1.0 \end{bmatrix}, \text{ and } \alpha = 100.$$

It is easy to check that the system is causal and stable. Results of the a-anisotropic norm computation on basis of the novel bounded real lemma are given in Fig. 3.6. Figure 3.7 depicts the absolute error of anisotropic norm computation compared with

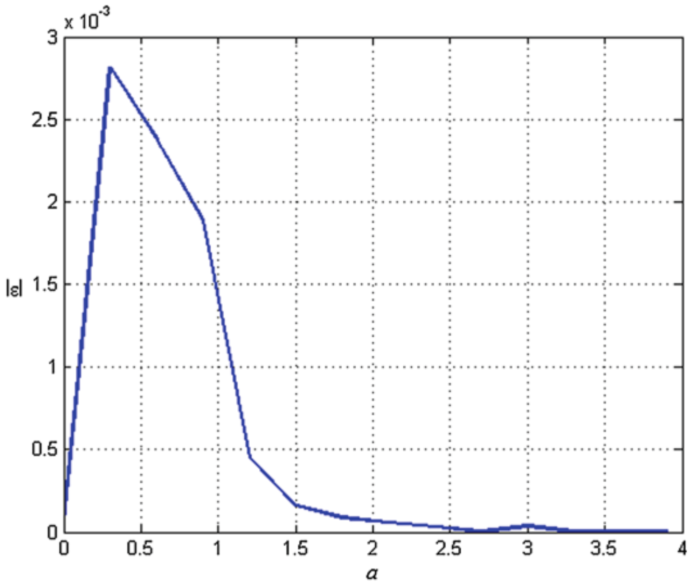

Fig. 3.7 Absolute error of anisotropic norm computation between novel LMI- and Riccati-based algorithms

the results of anisotropic norm calculating based on the Riccati equations method from Sect. 3.3.1. For more information, see [20].

Conclusion

In this chapter, different approaches to anisotropy-based analysis for descriptor systems are introduced. It is shown that the definition of the anisotropic norm of descriptor systems in the frequency domain coincides with the same one for standard state-space systems. However, the conditions and methods of norm computation are different. Different versions of anisotropy-based bounded real lemma are introduced in order to solve control design problems.

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Chapter 4
Optimal Control

4.1 State Feedback Control

Problem Statement

Consider a discrete-time descriptor system P in the form:

$$Ex(k+1) = Ax(k) + B_w w(k) + B_u u(k), \tag{4.1}$$

$$z(k) = Cx(k) + D_w w(k) + D_u u(k) \tag{4.2}$$

where $w(k) \in \mathbb{R}^{m_1}$ and $z(k) \in \mathbb{R}^p$ are the input and output signals, respectively, and $u(k) \in \mathbb{R}^{m_2}$ is the control vector. E, A, B_w, B_u, C, D_w, and D_u are constant real matrices of appropriate dimensions. The system is assumed to be causally controllable and stabilizable. The input signal is a "colored" Gaussian disturbance with known mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$ $(a \geqslant 0)$.

The anisotropy-based control problem for descriptor systems is similar to such a problem for the normal ones [1] and can be formulated as follows.

Problem 4.1 For a given system (4.1) and (4.2) find an admissible state feedback control $u(k) = Kx(k)$ that minimizes the a-anisotropic norm of the closed-loop system:

$$|\!|\!| P_{cl} |\!|\!|_a = \sup_{G(z) \in \mathbf{G}_a} \frac{\|P_{cl} G\|_2}{\|G\|_2} \to \inf_K . \tag{4.3}$$

Physically, a-anisotropic norm minimization for system (4.1) and (4.2) means minimization of the closed-loop impact of an external disturbance $w(k)$ with known spectral color to improve the robust performance of the system.

Recall that $\mathbf{G}_a = \left\{ G \in \mathscr{H}_2{}^{m \times m} : \overline{\mathbf{A}}(G) \leqslant a \right\}$.

Substituting control law $u(k)$ into equations (4.1) and (4.2), we can write the closed-loop system P_{cl} as

A. A. Belov et al., *Control of Discrete-Time Descriptor Systems*, Studies in Systems, Decision and Control 157, https://doi.org/10.1007/978-3-319-78479-3_4

$$Ex(k+1) = (A + B_u K)x(k) + B_w w(k), \tag{4.4}$$
$$z(k) = (C + D_u K)x(k) + D_w w(k). \tag{4.5}$$

Problem Solution

The idea of optimal control Problem 4.1 solution is based on a saddle point condition that can be formulated as follows. For any admissible shaping filter $G \in \mathbf{G}_a$ and any stabilizing control law $K \in \mathbb{K}(P)$ we introduce the following sets.

$$\mathbb{K}_a^\diamond(G) \doteq Arg \min_{K \in \mathbb{K}(P)} \|P_{cl}\|_2, \quad G \in \mathbb{G}_a,$$

$$\mathbf{G}_a^\diamond(K) \doteq Arg \max_{G \in \mathbf{G}_a} \frac{\|P_{cl}\|_2}{\|G\|_2}, \quad K \in \mathbb{K}(P).$$

Set $\mathbb{K}_a^\diamond(G)$ consists of control laws, which are solutions of the weighted $\mathscr{H}_2$-optimization problem. Here input signal $W = GV$ is supposed to be "colored" (correlated). Any control law, given in the form $K \in \mathbb{K}(P)$, minimizes the output dispersion of the input signal $W = GV$. $\mathbb{K}(P)$ is a set of all controllers that make the closed-loop system admissible. The set $\mathbf{G}_a^\diamond(K)$ consists of the shaping filters, which generate Gaussian input disturbances with the worst spectral density of the closed-loop system.

Lemma 4.1 [2] *If the control law K is a saddle point of the mapping $\mathbb{K}_a^\diamond \circ \mathbf{G}_a^\diamond$, then it is the solution to Problem 4.1.*

Hence, the solution is composed of two steps. The first step is to find the worst-case shaping filter $G(z) \in \mathbf{G}_a$ with mean anisotropy level $\overline{\mathbf{A}}(G) \leqslant a$. The second step deals with solving the weighted $\mathscr{H}_2$-control problem.

In order to define the state-space representation of the worst-case shaping filter we give some definitions from dynamical systems theory. Consider a normal system $\mathscr{F}(z)$ in state-space representation

$$x(k+1) = \mathscr{A}x(k) + \mathscr{B}f(k), \tag{4.6}$$
$$y(k) = \mathscr{C}x(k) + \mathscr{D}f(k). \tag{4.7}$$

Here, $x(k) \in \mathbb{R}^n$, $f(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$. Recall that the transfer function of system (4.6) and (4.7) is given by

$$\mathscr{F}(z) = \mathscr{C}(zI - \mathscr{A})^{-1}\mathscr{B} + \mathscr{D}.$$

Definition 4.1 The system $\mathscr{F}(z)$, satisfying the condition $\mathscr{F}^*(z)\Psi\mathscr{F}(z) = \overline{\Psi}$ for a nonzero matrix $\Psi = \Psi^{\mathrm{T}} \in \mathbb{R}^{p \times p}$ and a nonsingular matrix $\overline{\Psi} = \overline{\Psi}^{\mathrm{T}} \in \mathbb{R}^{m \times m}$, is called the *weighted all-pass* system. Here $m \leqslant p$.

The system, satisfying condition $\mathscr{F}^*\mathscr{F} = I_m$, is called the all-pass system. For more information, see [3].

Lemma 4.2 ([3]) *For given matrices Ψ and the $\overline{\Psi}$ system $\mathscr{F} \in \mathscr{H}_\infty{}^{p\times m}$ is the weighted all-pass system if there exists a matrix $\mathscr{R} = \mathscr{R}^{\mathrm{T}}$, which satisfies the following Riccati equation:*

$$\mathscr{R} = \mathscr{A}^{\mathrm{T}}\mathscr{R}\mathscr{A} + \mathscr{C}^{\mathrm{T}}\Psi\mathscr{C}, \tag{4.8}$$

$$0 = \mathscr{B}^{\mathrm{T}}\mathscr{R}\mathscr{A} + \mathscr{D}^{\mathrm{T}}\Psi\mathscr{C}, \tag{4.9}$$

$$\overline{\Psi} = \mathscr{B}^{\mathrm{T}}\mathscr{R}\mathscr{B} + \mathscr{D}^{\mathrm{T}}\Psi\mathscr{D}. \tag{4.10}$$

Now we formulate similar conditions for all-pass descriptor systems

$$\widetilde{E}x(k+1) = \widetilde{A}x(k) + \widetilde{B}w(k), \tag{4.11}$$

$$y(k) = \widetilde{C}x(k) + \widetilde{D}w(k), \tag{4.12}$$

supposing that the following rank assumption takes place,

$$\mathrm{rank}\,(\widetilde{E}) = \mathrm{rank}\,[\,\widetilde{E}\;\;\widetilde{B}\,] = \mathrm{rank}\,[\,\widetilde{E}\;\;\widetilde{C}^{\mathrm{T}}\,]. \tag{4.13}$$

Lemma 4.3 *Admissible system* (4.11) *and* (4.12) *is the all-pass system if there exists a matrix $\widetilde{R} = \widetilde{R}^{\mathrm{T}}$, satisfying the condition $\widetilde{E}^{\mathrm{T}}\widetilde{R}\widetilde{E} \geqslant 0$, such that*

$$\widetilde{B}^{\mathrm{T}}\widetilde{R}\widetilde{B} + \widetilde{D}^{\mathrm{T}}\widetilde{D} = I, \tag{4.14}$$

$$\widetilde{B}^{\mathrm{T}}\widetilde{R}\widetilde{A} + \widetilde{D}^{\mathrm{T}}\widetilde{C} = 0, \tag{4.15}$$

$$\widetilde{A}^{\mathrm{T}}\widetilde{R}\widetilde{A} + \widetilde{C}^{\mathrm{T}}\widetilde{C} - \widetilde{E}^{\mathrm{T}}\widetilde{R}\widetilde{E} = 0. \tag{4.16}$$

Proof For the admissible system there exist two nonsingular matrices $\overline{W}$ and $\overline{V}$ that transform the initial system (4.11) and (4.12) to the form

$$\widetilde{C}\overline{V} = [\,C_1\;C_2\,],\;\; \overline{W}\widetilde{E}\overline{V} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},\; \overline{W}\widetilde{A}\,\overline{V} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix},\; \overline{W}\widetilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Here $r = \mathrm{rank}\,(\widetilde{E})$, $x(k) \in \mathbb{R}^n$. Then rank assumption (4.13) is equivalent to

$$\mathrm{rank}\,\left(\overline{W}\widetilde{E}\overline{V}\right) = \mathrm{rank}\,[\,\overline{W}\widetilde{E}\overline{V},\;\overline{W}\widetilde{B}\,] = \mathrm{rank}\,[\,\overline{W}\widetilde{E}\overline{V},\;\overline{V}^{\mathrm{T}}\widetilde{C}^{\mathrm{T}}\,].$$

It means that

$$\mathrm{rank}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} I & 0 & B_1 \\ 0 & 0 & B_2 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} I & 0 & C_1^{\mathrm{T}} \\ 0 & 0 & C_2^{\mathrm{T}} \end{bmatrix},$$

consequently,

$$B_2 = 0, \quad C_2 = 0. \tag{4.17}$$

Introduce matrix $\widetilde{R}$ in the following way.

$$\widetilde{R} = \overline{W}^{\mathrm{T}} R \overline{W} = \overline{W}^{\mathrm{T}} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \overline{W}. \tag{4.18}$$

Substitute $\widetilde{R}$ from (4.18) into (4.16)

$$\widetilde{A}^{\mathrm{T}} \overline{W}^{\mathrm{T}} R \overline{W} \widetilde{A} + \widetilde{C}^{\mathrm{T}} \widetilde{C} - \widetilde{E}^{\mathrm{T}} \overline{W}^{\mathrm{T}} R \overline{W} \widetilde{E} = 0. \tag{4.19}$$

Left-hand and right-hand multiplying on nonsingular matrices $\overline{V}^{\mathrm{T}}$ and $\overline{V}$ gives

$$\overline{V}^{\mathrm{T}} \widetilde{A}^{\mathrm{T}} \overline{W}^{\mathrm{T}} R \overline{W} \widetilde{A} \overline{V} + \overline{V}^{\mathrm{T}} \widetilde{C}^{\mathrm{T}} \widetilde{C} \overline{V} - \overline{V}^{\mathrm{T}} \widetilde{E}^{\mathrm{T}} \overline{W}^{\mathrm{T}} R \overline{W} \widetilde{E} \overline{V} = 0. \tag{4.20}$$

We can rewrite (4.20) as

$$\begin{aligned} &\begin{bmatrix} A_1^{\mathrm{T}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} - \\ &\qquad - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = 0, \end{aligned} \tag{4.21}$$

or

$$\begin{bmatrix} A_1^{\mathrm{T}} R_{11} A_1 & A_1^{\mathrm{T}} R_{12} \\ R_{21} A_1 & R_{22} \end{bmatrix} + \begin{bmatrix} C_1^{\mathrm{T}} C_1 & C_1^{\mathrm{T}} C_2 \\ C_2^{\mathrm{T}} C_1 & C_2^{\mathrm{T}} C_2 \end{bmatrix} - \begin{bmatrix} R_{11} & 0 \\ 0 & 0 \end{bmatrix} = 0. \tag{4.22}$$

It leads to

$$\begin{aligned} A_1^{\mathrm{T}} R_{11} A_1 + C_1^{\mathrm{T}} C_1 - R_{11} = 0, \quad A_1^{\mathrm{T}} R_{12} + C_1^{\mathrm{T}} C_2 = 0, \\ R_{21} A_1 + C_2^{\mathrm{T}} C_1 = 0, \quad R_{22} + C_2^{\mathrm{T}} C_2 = 0. \end{aligned} \tag{4.23}$$

Now consider equation (4.15): $\widetilde{B}^{\mathrm{T}} \overline{W}^{\mathrm{T}} R \overline{W} \widetilde{A} + \widetilde{D}^{\mathrm{T}} \widetilde{C} = 0$. By right-hand multiplication on $\overline{V}$ we have

$$B_1^{\mathrm{T}} R_{11} A_1 + \widetilde{D}^{\mathrm{T}} C_1 = 0, \tag{4.24}$$

$$B_1^{\mathrm{T}} R_{12} + \widetilde{D}^{\mathrm{T}} C_2 = 0. \tag{4.25}$$

Using the same transformation for (4.14), we have

$$B_1^{\mathrm{T}} R_{11} B_1 + B_1^{\mathrm{T}} R_{12} B_2 + B_2^{\mathrm{T}} R_{22} B_2 + \widetilde{D}^{\mathrm{T}} \widetilde{D} = I. \tag{4.26}$$

Taking into account condition (4.17), equations (4.23) and (4.26) are equivalent to

$$A_1^{\mathrm{T}} R_{11} A_1 + C_1^{\mathrm{T}} C_1 - R_{11} = 0, \tag{4.27}$$
$$B_1^{\mathrm{T}} R_{11} A_1 + \widetilde{D}^{\mathrm{T}} C_1 = 0, \tag{4.28}$$
$$B_1^{\mathrm{T}} R_{11} B_1 + \widetilde{D}^{\mathrm{T}} \widetilde{D} = I. \tag{4.29}$$

Evidently, the initial descriptor system is equivalent to the following normal one.

$$\begin{aligned} x_1(k+1) &= A_1 x_1(k) + B_1 u(k), \\ y(k) &= C_1 x_1(k) + D u(k), \end{aligned}$$

for which conditions (4.23), (4.24), (4.26) coincide with the conditions of Lemma 4.2 if $\Psi = I_p, \overline{\Psi} = I_m$.

The proof is completed. ■

Now we formulate conditions of $\mathscr{H}_2$-norm computation for system (4.11) and (4.12).

Lemma 4.4 *Let the system with state-space representation* (4.11) *and* (4.12) *be admissible. Let the following assumption hold:*

$$\operatorname{rank}(\widetilde{E}) = \operatorname{rank}[\, \widetilde{E} \;\; \widetilde{B} \,]. \tag{4.30}$$

Consider a generalized Lyapunov equation

$$\widetilde{A}\widetilde{\mathscr{G}}\widetilde{A}^{\mathrm{T}} - \widetilde{E}\widetilde{\mathscr{G}}\widetilde{E}^{\mathrm{T}} + \widetilde{B}\widetilde{B}^{\mathrm{T}} = 0. \tag{4.31}$$

Then the $\mathscr{H}_2$-norm of the system can be computed as

$$\|P\|_2 = \operatorname{Tr}(\widetilde{C}\widetilde{\mathscr{G}}\widetilde{C}^{\mathrm{T}} + \widetilde{D}^{\mathrm{T}}\widetilde{D}). \tag{4.32}$$

Proof In [4] it is shown that the solution of the generalized projected Lyapunov equation (2.85) coincides with the solution of the generalized Lyapunov equation (4.31) if $B_1 B_2^{\mathrm{T}} = 0$. Under rank assumption (4.30) $B_2 = 0$ and, therefore, the condition $B_1 B_2^{\mathrm{T}} = 0$ holds. Hence, the solution of the generalized Lyapunov equation (4.31) defines a controllability Gramian of system (4.11) and (4.12). Using the definition of an $\mathscr{H}_2$-norm we get (4.32) that completes the proof. ■

Denote $\widehat{A} = A + B_u K$, $\widehat{C} = C_1 + D_u K$, and suppose that the closed-loop system is admissible. The following theorem gives conditions on the parameters of the worst-case shaping filter with a bounded mean anisotropy level $\overline{\mathbf{A}}(G) \leqslant a$.

Theorem 4.1 *Let system (4.1) and (4.2) be stabilizable and causally controllable. Then for any mean anisotropy level $a \geqslant 0$ there exists a single pair (q, R) where a scalar parameter q satisfies the condition $q \in [0, \;\; \|P_{cl}\|_\infty^{-2})$, and $R = R^{\mathrm{T}}$ with $E^{\mathrm{T}} R E \geqslant 0$ is an admissible solution to the generalized algebraic Riccati equation*

$$
\begin{aligned}
E^{\mathrm{T}} R E &= \widehat{A}^{\mathrm{T}} R \widehat{A} + q \widehat{C}^{\mathrm{T}} \widehat{C} + L^{\mathrm{T}} \Sigma^{-1} L, && (4.33)\\
\Sigma &= (I_{m_1} - q D_w^{\mathrm{T}} D_w - B_w^{\mathrm{T}} R B_w)^{-1}, && (4.34)\\
L &= \Sigma (B_w^{\mathrm{T}} R \widehat{A} + q D_w^{\mathrm{T}} \widehat{C}). && (4.35)
\end{aligned}
$$

Moreover,

$$
-\frac{1}{2} \ln \det \left(\frac{m_1 \Sigma}{\mathrm{Tr}\,(L P_G L^{\mathrm{T}} + \Sigma)} \right) = a \tag{4.36}
$$

where $P_G \in \mathbb{R}^{n \times n}$ is a controllability Gramian for the shaping filter G. It satisfies the projected generalized Lyapunov equation

$$
E P_G E^{\mathrm{T}} = (\widehat{A} + B_w L) P_G (\widehat{A} + B_w L)^{\mathrm{T}} - B_w \Sigma B_w^{\mathrm{T}}. \tag{4.37}
$$

Thus the shaping filter with a state-space representation

$$
G = \left[E,\ \begin{array}{c|c} \widehat{A} + B_w L & B_w \Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array} \right] \tag{4.38}
$$

is the worst-case shaping filter for the closed-loop system. The mean anisotropy level of the signal generated by this filter is equal to a.

Proof Using the definition of an a-anisotropic norm (3.33), we can construct a Lagrange function

$$
\mathfrak{L} = \| P_{cl} G \|_2^2 - \mu \| G \|_2^2 - \lambda \overline{\mathbf{A}}(G). \tag{4.39}
$$

It can be shown [5] that the Lagrange function (4.39) reaches its maximum when

$$
q \Lambda(\omega) - I_{m_1} + \sigma S^{-1}(\omega) = 0 \tag{4.40}
$$

where $\Lambda(\omega) = \widehat{P}_{cl}^{*}(\omega) \widehat{P}_{cl}(\omega)$, $S(\omega) = \widehat{G}^{*}(\omega) \widehat{G}(\omega)$.

Equation (4.40) implies that

$$
S(\omega) = \sigma (I_{m_1} - q \Lambda(\omega))^{-1}
$$

and defines the spectral density of the worst-case input disturbance. Without loss of generality we may put $\sigma = 1$ and rewrite equation (4.40) as

$$
q \widehat{P}_{cl}^{*}(\omega) \widehat{P}_{cl}(\omega) + (\widehat{G}^{*})^{-1}(\omega) \widehat{G}^{-1}(\omega) = I_{m_1}.
$$

Define

$$
\Theta = \begin{bmatrix} \sqrt{q}\, \widehat{P}_{cl}(\omega) \\ \widehat{G}^{-1}(\omega) \end{bmatrix}.
$$

Then the closed-loop system with the worst-case shaping filter must satisfy the following factorization $\Theta^*\Theta = I_{m_1}$. Noting that the shaping filter G is assumed to be invertible we get

$$G^{-1} = \left[E, \begin{array}{c|c} \widehat{A} & B_w \\ \hline -\Sigma^{-1/2}L & \Sigma^{-1/2} \end{array}\right].$$

The state-space representation of Θ is

$$\begin{bmatrix} \sqrt{q}P_{cl} \\ G^{-1} \end{bmatrix} = \left[E, \begin{array}{c|c} \widehat{A} & B_w \\ \hline \Omega & \Delta \end{array}\right]$$

where

$$\Omega = \begin{bmatrix} \sqrt{q}\widehat{C} \\ -\Sigma^{-1/2}L \end{bmatrix}, \ \Delta = \begin{bmatrix} \sqrt{q}D_w \\ \Sigma^{-1/2} \end{bmatrix}.$$

Using Lemma 4.3 and substituting the state-space representation of Θ into (4.14) and (4.16) we get (4.33)–(4.35). Under conditions of Lemma 4.4 the $\mathscr{H}_2$-norm of the shaping filter is defined by the formula

$$\|G\|_2 = \operatorname{Tr}(LP_GL^{\mathrm{T}} + \Sigma). \tag{4.41}$$

Taking into account the definition of mean anisotropy (3.7) and (4.41), we get equation (4.36). The proof is completed. ■

Consider an extended system given as

$$E_*\widehat{x}(k+1) = A_*\widehat{x}(k) + B_{u*}u(k) + B_{v*}v(k), \tag{4.42}$$
$$z(k) = C_*\widehat{x}(k) + D_wv(k) \tag{4.43}$$

where $\widehat{x}(k) \in \mathbb{R}^{2n}$, $z(k) \in \mathbb{R}^p$, and $v(k) \in \mathbb{R}^{m_1}$ is a Gaussian white noise sequence. Here $E_* = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$,

$A_* = \begin{bmatrix} A & B_wL \\ 0 & A + B_wL \end{bmatrix}$, $B_{u*} = \begin{bmatrix} B_u \\ 0 \end{bmatrix}$, $C_* = \begin{bmatrix} C & D_wL \end{bmatrix}$. As the shaping filter G is assumed to be invertible, this problem is equivalent to the standard $\mathscr{H}_2$-optimization problem (see [6] and references therein).

Theorem 4.2 *Let system (4.42) and (4.43) be stabilizable and causally controllable. The optimal state-space control law, which solves the weighted $\mathscr{H}_2$-optimization problem, can be found in the following form:*

$$K = \Gamma_1 + \Gamma_2 \tag{4.44}$$

where Π and $\Gamma = [\,\Gamma_1 \quad \Gamma_2\,]$ are found from the solution of the following generalized algebraic Riccati equation

$$E_*^{\mathrm{T}} T E_* = A_*^{\mathrm{T}} T A_* + C_*^{\mathrm{T}} C_* + \Gamma^{\mathrm{T}} \Pi \Gamma, \tag{4.45}$$

$$\Pi = (B_{u*}^{\mathrm{T}} T B_{u*} + D_w^{\mathrm{T}} D_w), \tag{4.46}$$

$$\Gamma = -\Pi^{-1}(B_{u*}^{\mathrm{T}} T A_* + D_w^{\mathrm{T}} C_*). \tag{4.47}$$

The solution to the optimal control problem includes solving coupled generalized Riccati equations (4.33)–(4.35) and (4.45)–(4.47), the projected generalized Lyapunov equation (4.37), and nonlinear special type equation (4.36).

If $a = 0$, parameters of the worst-case shaping filter $G(z)$ are $L = 0$ and $\Sigma = I_{m_1}$. In this case the solution of the anisotropy-based optimal control problem is equivalent to the solution of the $\mathscr{H}_2$ -optimal control problem.

Example 4.1 Consider the system:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1.15 & -0.3 \\ 0.1 & 0.3 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_w = 0.2, \quad D_u = 0.1.$$

It is easy to see that $\mathrm{rank}\,(E) = \mathrm{rank}\,[\,E \;\; B_w\,] = 1$. The system is causal, but unstable, $\rho(E, A) = 1.25$.

Now we find a state-feedback control $u(k) = Kx(k)$ for the given mean anisotropy level $a = 0.4$, using the techniques from the proven theorems.

The optimal controller is $K^* = \begin{bmatrix} -1.6514 & 0 \end{bmatrix}$. The closed-loop system is admissible. Its spectral radius is $\rho(E, A + B_u K^*) = 0.4014$ and the a-anisotropic norm is $|\!|\!| P_{cl}^{SF} |\!|\!|_a = 0.4978$.

4.2 Output Feedback Control

Problem Statement

The plant is given in the form

$$Ex(k+1) = Ax(k) + B_1 w(k) + B_2 u(k), \tag{4.48}$$

$$z(k) = C_1 x(k) + D_{11} w(k) + D_{12} u(k), \tag{4.49}$$

$$y(k) = C_2 x(k) + D_{21} w(k) + D_{22} u(k), \tag{4.50}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^{m_1}$ is the reference input or disturbance, $u(k) \in \mathbb{R}^{m_2}$ is a control signal, $z \in \mathbb{R}^{p_1}$ is the controllable output, and $y \in \mathbb{R}^{p_2}$ is the measurable output.

We assume that signal $w(k)$ is a stationary Gaussian sequence with bounded mean anisotropy level $a \geqslant 0$. Without loss of generality we suppose that $D_{22} = 0$.

Suppose also that the following assumptions on the system hold.

A1. The system is stabilizable and causally controllable.
A2. The system is detectable and causally observable.
A3. Dimension of the controllable output $z(k)$ is less than the dimension of the input disturbance $w(k)$: $p_1 < m_1$.
A4. Matrix D_{21} has full row rank: rank $(D_{21}) = p_2 \leqslant m_1$.
A5. Matrix D_{12} has full column rank: rank $(D_{12}) = m_2 \leqslant p_1$.

Then the design problem for an anisotropy-based controller can be formulated as follows.

Problem 4.2 For a given mean anisotropy level $a \geqslant 0$ of the input sequence $w(k)$ and for system (4.48) and (4.49), find a dynamical controller K that makes the closed-loop system admissible and minimizes its a-anisotropic norm

$$\sup \frac{\|F_l(P, K)G\|_2}{\|G\|_2};\ G \in \mathbf{G}_a \to \min_K . \tag{4.51}$$

Here $F_l(P, K)$ is a linear fractional transformation (LFT) of the closed-loop system.

Because the initial plant might be noncausal, it is impossible to design an output feedback controller. Therefore, we partition the design procedure into two steps. In the first step, we design a static output feedback control law that makes the closed-loop system causal (causalization). In the second step, we construct an output feedback controller that stabilizes the system and minimizes its a-anisotropic norm.

Problem Solution

Causalization of the System

Under assumption A1, the system is causally controllable; that is, there exists a control law $\widetilde{u}(k) = K_1 y(k)$ such that the pair $(E, A + B_2 K_1 C_2)$ is causal.

Let us consider a procedure for finding coefficients K_1 [7]. Because the system (4.48) and (4.49) is regular, there exist two nonsingular matrices $\widetilde{W}$ and $\widetilde{V}$ such that $\widetilde{W} E \widetilde{V} = \mathrm{diag}(I_r, 0)$ where $r = \mathrm{rank}\,(E)$. A coordinate transformation

$$\widetilde{V}^{-1} x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \tag{4.52}$$

where $x_1(k) \in \mathbb{R}^r$ and $x_2(k) \in \mathbb{R}^{n-r}$, reduces the system (4.48) and (4.49) to SVD equivalent form [8].

Matrices $\widetilde{W}$ and $\widetilde{V}$ are found from SVD decomposition, using the expressions (2.21) and (2.22).

If the initial plant is noncausal, then A_{22} is a singular matrix [9].

Applying coordinate transformation (4.52) to

$$Ex(k+1) = (A + B_2 K_1 C_2) x(k),$$

and left multiplying the result on matrix $\widetilde{W}$, we have

$$\widetilde{W}E\widetilde{V}\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \left(\widetilde{W}A\widetilde{V} + \widetilde{W}B_2K_1C_2\widetilde{V}\right)\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}.$$

The last expression can be rewritten as

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} K_1 \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}\right)\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}.$$

If $\operatorname{rank}(A_{22}) = s < n - r$, we consider the matrix block $(A_{22} + B_{22}K_1C_{22})$. The system is causally controllable if there exists a matrix K_1 such that $\operatorname{rank}(A_{22} + B_{22}K_1C_{22}) = n - r$. Based on SVD decomposition, represent matrix A_{22} as

$$S_{A_{22}}A_{22}U_{A_{22}} = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.53}$$

By left and right multiplication of the expression $(A_{22} + B_{22}K_1C_{22})$ on matrices $S_{A_{22}}$ and $U_{A_{22}}$,we get

$$\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} + \widetilde{B}_{22}K_1\widetilde{C}_{22}, \tag{4.54}$$

where $\widetilde{B}_{22} = S_{A_{22}}B_{22}$ and $\widetilde{C}_{22} = C_{22}U_{A_{22}}$. Then the causalization problem can be represented as a problem of finding a matrix of coefficients K_1 such that matrix (4.54) is nonsingular. For simplicity assume that we need to find a matrix K_1 such that the following equality holds.

$$\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} + \widetilde{B}_{22}K_1\widetilde{C}_{22} = \begin{bmatrix} 2I_s & 0 \\ 0 & I_{n-r-s} \end{bmatrix},$$

or

$$\widetilde{B}_{22}K_1\widetilde{C}_{22} = I_{n-r}.$$

This implies that

$$K_1 = \widetilde{B}_{22}^{+}\widetilde{C}_{22}^{+}, \tag{4.55}$$

where M^{+} is the Moore-Penrose inverse of M.

Design of Output Feedback Stabilizing Controller

Consider a control law of the form:

$$u(k) = K_1 y(k) + u_1(k). \tag{4.56}$$

Substituting the expression (4.56) into the equations (4.48) and (4.49), we get

$$
\begin{aligned}
Ex(k+1) &= (A + B_2K_1C_2)x(k) + (B_1 + B_2K_1D_{21})w(k) + B_2u_1(k),\\
z(k) &= (C_1 + D_{12}K_1C_2)x(k) + (D_{11} + D_{12}K_1D_{21})w(k) + D_{12}u_1(k),\\
y(k) &= C_2x(k) + D_{21}w(k).
\end{aligned}
$$

Introduce the following notation: $\overline{A} = A + B_2K_1C_2, \overline{C}_1 = C_1 + D_{12}K_1C_2, \overline{C}_2 = C_2, \overline{B}_1 = B_1 + B_2K_1D_{21}, \overline{B}_2 = B_2$, and $\overline{D}_{11} = D_{11} + D_{12}K_1D_{21}$.

Applying change of variables (4.52) and left-multiplying the equations on matrix $\widetilde{W}$, we get

$$
x_1(k+1) = \overline{A}_{11}x_1(k) + \overline{A}_{12}x_2(k) + \overline{B}_{11}w(k) + \overline{B}_{21}u_1(k), \tag{4.57}
$$

$$
0 = \overline{A}_{21}x_1(k) + \overline{A}_{22}x_2(k) + \overline{B}_{12}w(k) + \overline{B}_{22}u_1(k), \tag{4.58}
$$

$$
z(k) = \overline{C}_{11}x_1(k) + \overline{C}_{12}x_2(k) + \overline{D}_{11}w(k) + D_{12}u_1(k), \tag{4.59}
$$

$$
y(k) = \overline{C}_{21}x_1(k) + \overline{C}_{22}x_2(k) + D_{21}w(k), \tag{4.60}
$$

where

$$
\widetilde{W}\overline{A}\widetilde{V} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix}, \qquad \widetilde{W}\overline{B}_i = \begin{bmatrix} \overline{B}_{i1} \\ \overline{B}_{i2} \end{bmatrix}, \qquad \overline{C}_i\widetilde{V} = \begin{bmatrix} \overline{C}_{i1} & \overline{C}_{i2} \end{bmatrix}, \quad i = 1, 2.
$$

The causalization procedure allows transforming the original system to an equivalent system that contains explicit expressions for the dynamical and algebraic subsystems. Expressing $x_2(k)$ via $x_1(k)$ and substituting it into equations (4.57) and (4.60), we get the normal system:

$$
\begin{aligned}
x_1(k+1) &= \mathscr{A}x_1(k) + \mathscr{B}_1w(k) + \mathscr{B}_2u_1(k),\\
z(k) &= \mathscr{C}_1x_1(k) + \mathscr{D}_{11}w(k) + \mathscr{D}_{12}u_1(k),\\
y(k) &= \mathscr{C}_2x_1(k) + \mathscr{D}_{21}w(k) + \mathscr{D}_{22}u_1(k),
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{A} &= \overline{A}_{11} - \overline{A}_{12}\overline{A}_{22}^{-1}\overline{A}_{21}, & \mathscr{B}_i &= \overline{B}_{i1} - \overline{A}_{12}\overline{A}_{22}^{-1}\overline{B}_{i2},\\
\mathscr{C}_i &= \overline{C}_{i1} - \overline{C}_{i2}\overline{A}_{22}^{-1}\overline{A}_{21}, & \mathscr{D}_{ij} &= D_{ij} - \overline{C}_{i2}\overline{A}_{22}^{-1}\overline{B}_{j2},
\end{aligned}
$$

where $i, j = 1, 2$.

After introducing this causalizing feedback, the original algebraic-difference system can be reduced to a normal one. However, in the general case, the equivalent system does not satisfy standard restrictions imposed on the control plant. Namely, an equivalent state-space system does not necessarily demand that

$$
\mathscr{D}_{22} = 0, \tag{4.61}
$$

To satisfy constraint (4.61) we use the following change of variables [10].

$$
y^{(1)}(k) = y(k) - \mathscr{D}_{22}u_1(k).
$$

This change of variables sets matrix $\mathscr{D}_{22}$ of the control plant to zero by static feedback with the gain $-\mathscr{D}_{22}$ from control $u_1(k)$ to the measured output $y(k)$.

After all the above transformations, the resulting system satisfies standard requirements, thus we can apply the already solved design problem for an anisotropy-based output controller [2]. In this work, we briefly recall the design procedure for solving an anisotropy-based optimal control problem. A more comprehensive description of optimal anisotropy-based controller design can be found in, for example, [11].

The solution of anisotropy-based optimal output feedback control problem consists of several steps.

1. Design of the worst-case shaping filter for the closed-loop system.
2. Design of the full-order observer for the system weighted with the shaping filter.
3. Solving the $\mathscr{H}_2$-optimization problem for the control object weighted with the shaping filter.

The controller's state-space representation has the following form [2].

$$\widetilde{K} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & 0 \end{bmatrix}, \tag{4.62}$$

where

$$\widetilde{A} = \mathscr{A} + \mathscr{B}_1 M + \mathscr{B}_2(\Gamma_1 + \Gamma_2) - \Lambda(\mathscr{C}_2 + \mathscr{D}_{21} M),\ \widetilde{B} = \Lambda,\ \widetilde{C} = \Gamma_1 + \Gamma_2. \tag{4.63}$$

Parameters Γ_1, Γ_2, M, and Λ can be found from the solution of the following equations.

1. Riccati equation with respect to the closed-loop system:

$$\begin{aligned} R &= A_{cl}^{\mathrm{T}} R A_{cl} + q C_{cl}^{\mathrm{T}} C_{cl} + L^{\mathrm{T}} \Sigma^{-1} L, \\ \Sigma &= (I_{m_1} - q \mathscr{D}_{11}^{\mathrm{T}} \mathscr{D}_{11} - B_{cl}^{\mathrm{T}} R B_{cl})^{-1}, \\ L &= \Sigma (B_{cl}^{\mathrm{T}} R A_{cl} + q \mathscr{D}_{11}^{\mathrm{T}} C_{cl}) \end{aligned}$$

where $q \in \mathbb{R}$, $R \in \mathbb{R}^{2n \times 2n}$, and

$$A_{cl} = \begin{bmatrix} \mathscr{A} & \mathscr{B}_2 \widetilde{C} \\ \widetilde{B} \mathscr{C}_2 & \widetilde{A} \end{bmatrix},\ B_{cl} = \begin{bmatrix} \mathscr{B}_1 \\ \widetilde{B} \mathscr{D}_{21} \end{bmatrix},\ C_{cl} = \begin{bmatrix} \mathscr{C}_1 & \mathscr{D}_{12} \widetilde{C} \end{bmatrix}.$$

2. The special type equation

$$a = -\frac{1}{2} \ln \det \left(\frac{m_1 \Sigma}{\mathrm{Tr}\,(L P L^{\mathrm{T}} + \Sigma)} \right).$$

3. Lyapunov equation

$$P = (A_{cl} + B_{cl} L) P (A_{cl} + B_{cl} L)^{\mathrm{T}} + B_{cl} \Sigma B_{cl}^{\mathrm{T}}.$$

4. Riccati equation for observation

$$
\begin{aligned}
S &= (\mathscr{A} + \mathscr{B}_1 L_1) S (\mathscr{A} + \mathscr{B}_1 L_1)^{\mathrm{T}} + \mathscr{B}_1 \Sigma \mathscr{B}_1^{\mathrm{T}} - \Lambda \Theta \Lambda^{\mathrm{T}}, \\
\Theta &= (\mathscr{C}_2 + \mathscr{D}_{21} L_1) S (\mathscr{C}_2 + \mathscr{D}_{21} L_1)^{\mathrm{T}} + \mathscr{D}_{21} \Sigma \mathscr{D}_{21}^{\mathrm{T}}, \\
\Lambda &= \left((\mathscr{A} + \mathscr{B}_1 L_1) S (\mathscr{C}_2 + \mathscr{D}_{21} L_1)^{\mathrm{T}} + \mathscr{B}_1 \Sigma \mathscr{D}_{21}^{\mathrm{T}}\right) \Theta^{-1}
\end{aligned}
$$

where L_1 is an $(m_1 \times n)$ block of the matrix L such that $L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$. Matrix M from (4.63) is defined as $M = L_1 + L_2$.

5. Riccati equation for solving the $\mathscr{H}_2$-optimization problem for the extended control object

$$
\begin{aligned}
T &= A_*^{\mathrm{T}} T A_* + C_*^{\mathrm{T}} C_* - \Gamma^{\mathrm{T}} \Pi \Gamma, \\
\Pi &= B_*^{\mathrm{T}} T B_* + \mathscr{D}_{12}^{\mathrm{T}} \mathscr{D}_{12}, \\
\Gamma &= -\Pi^{-1} (B_*^{\mathrm{T}} T A_* + \mathscr{D}_{12}^{\mathrm{T}} C_*),
\end{aligned}
$$

where $T \in \mathbb{R}^{2n \times 2n}$, $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix}$, and

$$
A_* = \begin{bmatrix} \mathscr{A} & \mathscr{B}_1 M \\ 0 & \mathscr{A} + \mathscr{B}_1 M + \mathscr{B}_2 (\Gamma_1 + \Gamma_2) \end{bmatrix}, \quad B_* = \begin{bmatrix} \mathscr{B}_2 \\ 0 \end{bmatrix} \quad C_* = \begin{bmatrix} \mathscr{C}_1 & \mathscr{D}_{11} M \end{bmatrix}.
$$

Then, finally, the closed-loop system's structure assumes the form depicted in Fig. 4.1. As the figure indicates, the controller has two parts. The first part is required to causalize the backward (algebraic) subsystem; the second one stabilizes the direct (dynamical) subsystem. The coefficient $-\mathscr{D}_{22}$ serves to turn off constraint (4.61).

Taking into account (4.62), the controller's representation is transformed to

$$
\begin{aligned}
\widetilde{x}(k+1) &= \widetilde{A} \widetilde{x}(k) + \widetilde{B} y^{(1)}(k), \\
u_1(k) &= \widetilde{C} \widetilde{x}(k),
\end{aligned}
$$

where $\widetilde{x}(k) \in \mathbb{R}^n$ is the estimator's state. Because $y^{(1)}(k) = y(k) - \mathscr{D}_{22} u_1(k)$, we have

$$
y^{(1)}(k) = y(k) - \mathscr{D}_{22} u_1(k) = y(k) - \mathscr{D}_{22} \widetilde{C} \widetilde{x}(k),
$$

and the controller's parameters for the original observation signal are

$$
K = \begin{bmatrix} \widetilde{A} - \widetilde{B} \mathscr{D}_{22} \widetilde{C} & \widetilde{B} \\ \widetilde{C} & 0 \end{bmatrix}.
$$

Show that this decomposition of the problem into two components lets us solve the original problem where we have to find a single unified controller. Indeed, according

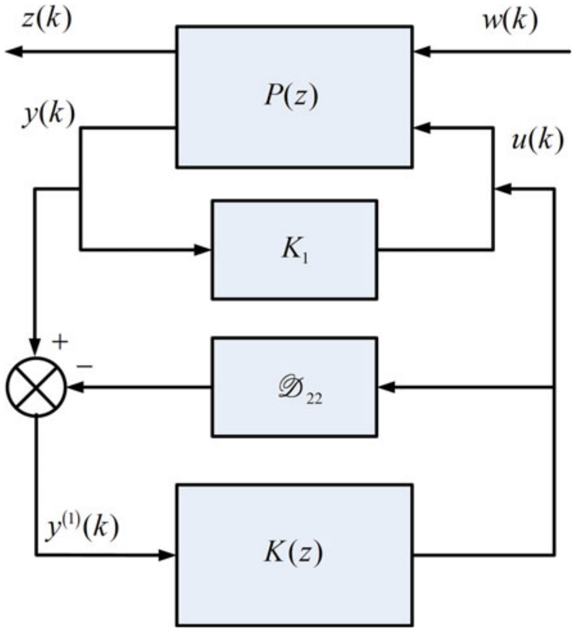

Fig. 4.1 Structural scheme of the closed-loop system

to Fig. 4.1 the controller can be represented as a unified control system with the hierarchical structure consisting of several links. Because the original system is assumed to be regular, it can be represented in the equivalent form (4.53)–(4.55), where matrix A_{22} is singular for noncausal systems. Causal controllability and observability allow us to find a control law $\widetilde{u}(k) = K_1 x(k)$ such that matrix A_{22} becomes nonsingular. Thus, we get an opportunity to express coordinates $x_2(k)$ via $x_1(k)$ and pass to an equivalent input-output operator of lower dimension for which the design procedure is already well known. Because the operator is equivalent, we can solve the original problem for the full-dimension operator by solving the problem for a normal system of lower dimension. Feedback gains that solve the causalization problem for the initial plant are indirectly present in the resulting operator; therefore they are also taken into account in solving the design problem for the optimal output feedback controller.

Example 4.2 Consider a numerical example that demonstrates the method of an anisotropy-based controller design for descriptor systems. Suppose that the control object's parameters are

$$E = \begin{bmatrix} -2 & 0 & 0 & 25 \\ -1 & 0.5 & 0 & 6 \\ 1 & -24 & 0 & 1 \\ -1 & -0.5 & 0 & 20 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & -0.01 & 0.125 & 25.005 \\ -0.9975 & 0.4925 & 0.03 & 5.995 \\ 0.88 & -23.875 & 0 & 0.005 \\ -1.0025 & -0.5025 & 0.1 & 20.015 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.0105 & -0.0093 \\ 0.005 & -0.0011 \\ -0.0015 & -0.0249 \\ 0.004 & -0.0092 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.005 & 0 \\ -0.005 & -0.0025 \\ 0.005 & -0.12 \\ 0.015 & -0.0025 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 \; 0 \; 0 \; 0 \end{bmatrix}, \; D_{11} = \begin{bmatrix} 0 \; 0 \end{bmatrix}, \; D_{12} = \begin{bmatrix} 0 \; 0.1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 1.5 \\ 0 & 0 & 1.3 & 0 \end{bmatrix}, \; D_{21} = \begin{bmatrix} 0.15 & 0 \\ 0 & -0.5 \end{bmatrix}, \; D_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Check whether the system is causal. The system's characteristic polynomial is $\det(zE - A) = 0.0057z^2 - 0.0114z - 0.00567$. The degree of the characteristic polynomial is $\deg\det(zE - A) = 2$, and $\operatorname{rank}(E) = 3$. Thus the system is not causal.

Check that causal controllability and causal observability criteria hold:

$$\operatorname{rank} \begin{bmatrix} E & 0 & 0 \\ A & E & B_2 \end{bmatrix} = 7, \; \operatorname{rank} \begin{bmatrix} E & A \\ 0 & E \\ 0 & C_2 \end{bmatrix} = 7.$$

The system is causally controllable and causally observable; therefore we can design causalizing static output feedback.

The controller's parameters are

$$K_1 = \begin{bmatrix} 0 & -37.74 \\ 0 & 0 \end{bmatrix}.$$

The finite eigenvalues of the system are

$$\lambda(E, A + B_2K_1C_2) = \begin{bmatrix} 1.0001 \; 1.0031 \; 0.9919 \end{bmatrix}.$$

The system is unstable. Its characteristic polynomial is $\det(zE - (A + B_2K_1C_2)) = -55.93z^3 + 167.5z^2 - 167.2z + 55.6$, that has degree equal to 3.

Thus $\deg\det(zE - A) = \operatorname{rank}(E)$, and the system can be transformed into an equivalent normal system.

After transformations, we get a control object with parameters

Table 4.1 Closed-loop system's norm as a function of the level a

a	0	0.05	0.5	1.0	5	∞
$\|\|\|P\|\|\|_a$	0.0398	0.2372	0.5010	0.61995	0.7502	0.7510

$$\mathscr{A} = \begin{bmatrix} 1.0001 & 0 & -0.013 \\ 0 & 0.9945 & -0.159 \\ 0 & -0.0001 & 1.004 \end{bmatrix},$$

$$\mathscr{B}_1 = \begin{bmatrix} -0.015 & -0.0489 \\ 0.0006 & 0.026 \\ 0 & -0.0002 \end{bmatrix}, \quad \mathscr{B}_2 = \begin{bmatrix} -0.033 & -0.0097 \\ 0.0001 & 0.1197 \\ 0 & -0.0002 \end{bmatrix},$$

$$\mathscr{C}_1 = \begin{bmatrix} 0.0021 & -0.0021 & -1.3978 \end{bmatrix}, \quad \mathscr{C}_2 = \begin{bmatrix} -0.0436 & 0.0014 & -1.5488 \\ -0.0008 & 0.0001 & -0.0027 \end{bmatrix},$$

$$\mathscr{D}_{11} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \mathscr{D}_{12} = \begin{bmatrix} 0 & 0.1 \end{bmatrix},$$

$$\mathscr{D}_{21} = \begin{bmatrix} 0.15 & 0 \\ -0.0106 & -0.0029 \end{bmatrix}, \quad \mathscr{D}_{22} = \begin{bmatrix} 0 & 0 \\ 0.0265 & 0 \end{bmatrix}.$$

Design a full-order estimating controller for the resulting equivalent system for different levels of mean anisotropy. The results are summarized in Table 4.1.

The solution of the stabilization problem for the original system under a disturbance on the system's input with mean anisotropy $a = 0.1$ is represented in Fig. 4.3. Controllable outputs for different control laws are given in Fig. 4.2. AC stands for anisotropy-based control.

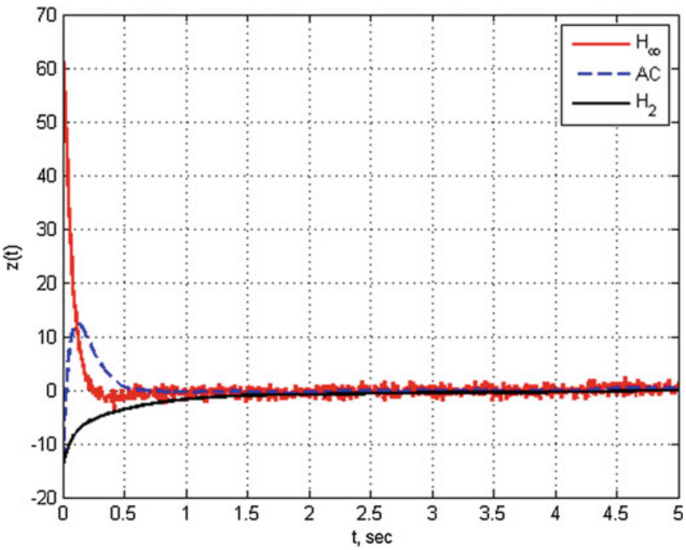

Fig. 4.2 Controllable output of the system

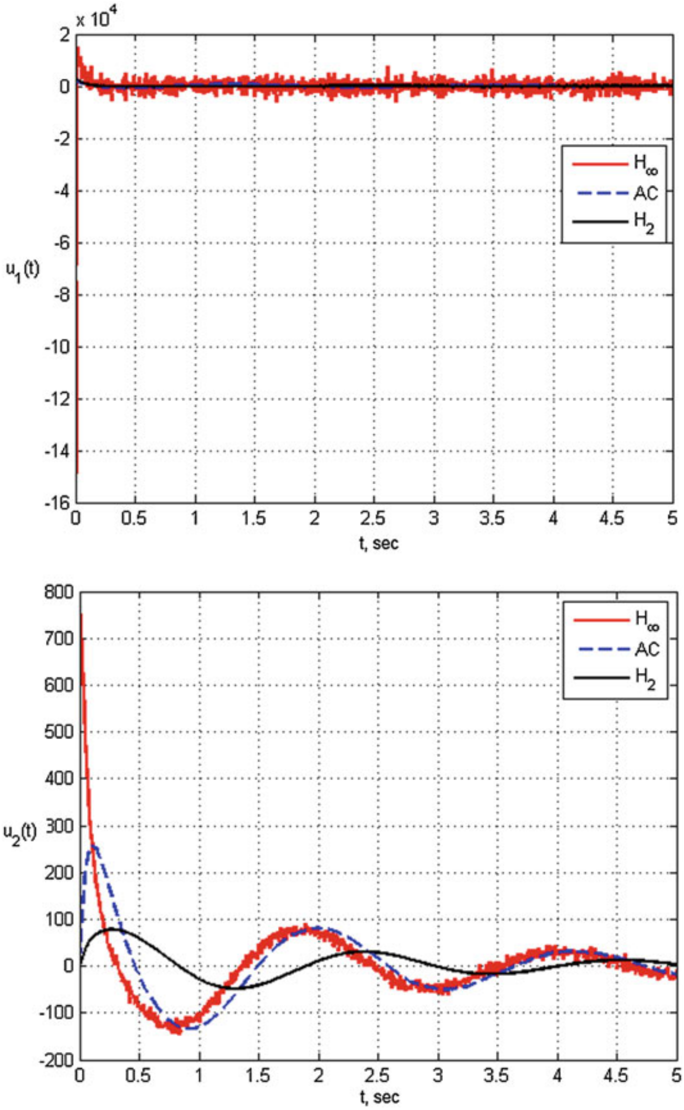

Fig. 4.3 Control signals

Conclusion

In this chapter, a problem of optimal control design is solved. In both state feedback and output feedback cases to solve the problem it's necessary to find parameters of the worst-case shaping filter and to transform the anisotropy-based control problem into a weighted $\mathscr{H}_2$ -optimization problem.

Unlike the analysis problem, the control problem cannot be reduced to its equivalent standard state-space representation due to the presence of noncausal behavior. Thus the solution of the state feedback optimal control problem consists of solving generalized Riccati and Lyapunov equations. In the output feedback case noncausal behavior can be neglected by the causalization procedure. This procedure provides a framework to design a controller of order rank (E) using the equivalent standard state-space model.

Inasmuch as $\|P\|_2/\sqrt{m} \leqslant |||P|||_a \leqslant \|P\|_\infty$, we can tune the sensitivity of the closed-loop system by setting the mean anisotropy level from 0 to $+\infty$. It is shown that the $\mathscr{H}_\infty$ -controller provides fast response speed, but the system, closed by the $\mathscr{H}_\infty$ -controller, is too sensitive to noise. This leads to higher energy losses. The LQG/$\mathscr{H}_2$ -controller provides the optimal attenuation level of white Gaussian noise. However, the $\mathscr{H}_2$ -controller may not satisfy the desirable transient response speed. An anisotropic controller provides both $\mathscr{H}_2$ and $\mathscr{H}_\infty$ advantages. It ensures a good disturbance attenuation level, and the system's operation speed is fast enough.

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Chapter 5
Suboptimal Control

Problem Statement
Consider a discrete-time descriptor system given in a state-space representation as

$$Ex(k+1) = Ax(k) + B_1 w(k) + B_2 u(k), \tag{5.1}$$

$$z(k) = Cx(k) + D_1 w(k) + D_2 u(k), \tag{5.2}$$

$x(k) \in \mathbb{R}^n$ is a state, $w(k) \in \mathbb{R}^{m_1}$ is a random stationary sequence with bounded mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a,\ a \geqslant 0$, $z(k) \in \mathbb{R}^p$ is a controllable output, $u(k) \in \mathbb{R}^{m_2}$ is a control vector, and E, A, B_1, B_2, C, D_1, D_2 are known matrices of appropriate dimensions, rank $(E) = r < n$.

Suppose that

1. System (5.1) is causally controllable and stabilizable.
2. The following rank condition holds true

$$\text{rank}\,(E) = \text{rank}\,[\,E \;\; B_1\,].$$

Problem 5.1 We have to find a state feedback control in the form $u_{SF}(k) = F_2 x(k)$ such that the closed-loop system P_{cl}^{SF}

$$Ex(k+1) = (A + B_2 F_2)x(k) + B_1 w(k), \tag{5.3}$$

$$z(k) = (C + D_2 F_2)x(k) + D_1 w(k) \tag{5.4}$$

is causal and stable; its anisotropic norm $|||P_{cl}^{SF}|||_a$ is bounded by the given value $\gamma > 0$.

A. A. Belov et al., *Control of Discrete-Time Descriptor Systems*, Studies in Systems, Decision and Control 157, https://doi.org/10.1007/978-3-319-78479-3_5

5.1 GDARI Approach

In this section, we find conditions in terms of generalized discrete-time algebraic Riccati inequalities (GDARI) in order to design a static state feedback in the form $u_{SF}(k) = F_2 x(k)$ for the system (5.1) and (5.2), which solves Problem 5.1.

The following theorem gives sufficient conditions for solution of this problem.

Theorem 5.1 *For given scalars $a \geqslant 0$ and $\gamma > 0$ the state feedback anisotropy-based control Problem 5.1 is solvable if there exist $\Phi \in \mathbb{R}^{n\times n}$, $\Phi = \Phi^{\mathrm{T}}$,$\Psi \in \mathbb{R}^{m_1\times m_1}$, $\Psi = \Psi^{\mathrm{T}} > 0$, and a positive scalar η satisfying the following conditions:*

$$
\begin{gathered}
E^{\mathrm{T}}\Phi E \geq 0,\\
B_1^{\mathrm{T}}\Phi B_1 + D_1^{\mathrm{T}} D_1 - \eta I_{m_1} < 0,\\
B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}} D_2 > 0,\\
\Psi < \eta I_{m_1} - B_1^{\mathrm{T}}\Phi B_1 - D_1^{\mathrm{T}} D_1,\\
\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/m_1} < \gamma^2,\\
A^{\mathrm{T}}\Phi A - E^{\mathrm{T}}\Phi E + C^{\mathrm{T}}C - (A^{\mathrm{T}}\Phi B + S)(B^{\mathrm{T}}\Phi B + R)^{-1}(B^{\mathrm{T}}\Phi A + S^{\mathrm{T}}) < 0
\end{gathered}
$$

where $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, $S = \begin{bmatrix} C^{\mathrm{T}}D_1 & C^{\mathrm{T}}D_2 \end{bmatrix}$, $R = \begin{bmatrix} D_1^{\mathrm{T}}D_1 - \eta I_{m_1} & D_1^{\mathrm{T}}D_2 \\ D_2^{\mathrm{T}}D_1 & D_2^{\mathrm{T}}D_2 \end{bmatrix}$.

Moreover, $F_2 = -\begin{bmatrix} 0 & I_{m_2} \end{bmatrix}(B^{\mathrm{T}}\Phi B + R)^{-1}(B^{\mathrm{T}}\Phi A + S^{\mathrm{T}})$.

Proof Recall that the closed-loop system is written in the form (5.3) and (5.4). Denote $N = \eta I_{m_1} - B_1^{\mathrm{T}}\Phi B_1 - D_1^{\mathrm{T}} D_1 > 0$. For this system the inequality (3.75) may be rewritten as

$$
\eta - (\mathrm{e}^{-2a}\det(N))^{1/m_1} < \gamma^2.
$$

Consider a matrix $\Psi = \Psi^{\mathrm{T}} > 0$, satisfying the inequality $\Psi < N$.

Thus the following condition holds true:

$$
\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/m_1} < \gamma^2.
$$

Denote

$$
M = -(B^{\mathrm{T}}\Phi B + R) = \begin{bmatrix} N & -B_1^{\mathrm{T}}\Phi B_2 - D_1^{\mathrm{T}}D_2 \\ -B_2^{\mathrm{T}}\Phi B_1 - D_2^{\mathrm{T}}D_1 & -B_2^{\mathrm{T}}\Phi B_2 - D_2^{\mathrm{T}}D_2 \end{bmatrix}.
$$

M is invertible (see [1]) because $N > 0$ and $-B_2^{\mathrm{T}}\Phi B_2 - D_2^{\mathrm{T}}D_2 < 0$.
Let

$$
F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = M^{-1}(B^{\mathrm{T}}\Phi A + S^{\mathrm{T}}),
$$

which can be easily rewritten as

$$
F_1^{\mathrm{T}} N = (A + B_2F_2)^{\mathrm{T}}\Phi B_1 + (C + D_2F_2)^{\mathrm{T}} D_1 \tag{5.5}
$$

Table 5.1 Conditions and results of anisotropy-based control design for different mean anisotropy levels

a	0.1	0.5	0.9
γ	0.050	0.055	0.060
Φ	$\begin{bmatrix} 1.0610 & 0.7604 \\ 0.7604 & 0.4337 \end{bmatrix}$	$\begin{bmatrix} 1.0769 & 0.7714 \\ 0.7714 & 0.4414 \end{bmatrix}$	$\begin{bmatrix} 1.0791 & 0.7730 \\ 0.7730 & 0.4425 \end{bmatrix}$
F_2	[2.5455, –0.8763]	[2.4077, –0.8772]	[2.3883, –0.8773]
$\rho(E, A + B_2F_2)$	0.7403	0.7555	0.7577
$\Vert P_{cl}^{SF} \Vert_a$	0.0420	0.0423	0.0424

and

$$F_1^{\mathrm{T}}(B_1^{\mathrm{T}}\Phi B_2 + D_1^{\mathrm{T}}D_2) = -((A + B_2F_2)^{\mathrm{T}}\Phi B_2 + (C + D_2F_2)^{\mathrm{T}}D_2). \tag{5.6}$$

Using Eqs. (5.5) and (5.6), it is not difficult to show that [2]

$$\begin{aligned}&(A + B_2F_2)^{\mathrm{T}}\Phi(A + B_2F_2) - E^{\mathrm{T}}\Phi E + (C + D_2F_2)^{\mathrm{T}}(C + D_2F_2) + \\ &+ ((A + B_2F_2)^{\mathrm{T}}\Phi B_1 + (C + D_2F_2)^{\mathrm{T}}D_1)N^{-1}(B_1^{\mathrm{T}}\Phi(A + B_2F_2) + D_1^{\mathrm{T}}(C + D_2F_2)) = \\ &= A^{\mathrm{T}}\Phi A - E^{\mathrm{T}}\Phi E + C^{\mathrm{T}}C - (A^{\mathrm{T}}\Phi B + S)(-M)^{-1}(B^{\mathrm{T}}\Phi A + S^{\mathrm{T}}) < 0.\end{aligned} \tag{5.7}$$

After applying Schur's lemma, inequality (5.7) for the closed-loop system gives the same condition as (3.76); therefore by Theorem 3.3 and Remark 3.4, system (5.3) and (5.4) is admissible and the a-anisotropic norm of its transfer function is limited by the given value γ for the set mean anisotropy level a.

The theorem is proved. ■

Example 5.1 Consider the following system:

$$E = \begin{bmatrix} 0.9 & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 0.85 & -0.3 \\ 0.1 & 0.3 \end{bmatrix},\ B_1 = \begin{bmatrix} -0.02 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.35 & 0.09 \end{bmatrix},\ D_1 = [\,0.035\,],\ D_2 = [\,0.1\,].$$

It is easy to check that rank $(E) = \operatorname{rank}\begin{bmatrix} E & B_1 \end{bmatrix} = 1$. The system is causal, but unstable ($\rho(E, A) = 1.0556$).

Now we find a state feedback control $u_{SF}(k) = F_2x(k)$ for the given mean anisotropy level a and scalar value γ, using the technique from Theorem 5.1. Results of control design for different a and γ are represented in Table 5.1.

As we can see, the obtained F_2 guarantees that the anisotropic norm of the closed-loop system is bounded by the given value: $\Vert P_{cl}^{SF} \Vert_a < \gamma$, and system P_{cl}^{SF} is admissible.

5.2 GDARE Approach

In this section, suboptimal anisotropy-based state feedback (SF) and full information (FI) control laws are designed for discrete-time descriptor systems. The obtained conditions are formulated in terms of GDARE.

5.2.1 State Feedback Control

Theorem 5.2 *Let the initial conditions for system* (5.1) *and* (5.2) *satisfy* $Ex(0) = 0$. *Then the closed-loop system* P_{cl}^{SF}, *defined by* (5.3) *and* (5.4) *is admissible, and the inequality* $|||P_{cl}^{SF}|||_a \leqslant \gamma$ *holds true if there exist a matrix* $\Phi = \Phi^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ *and a positive scalar* $\eta > \gamma^2$, *satisfying the conditions:*

$$E^{\mathrm{T}}\Phi E \geqslant 0, \tag{5.8}$$

$$B_1^{\mathrm{T}}\Phi B_1 + D_1^{\mathrm{T}}D_1 - \gamma^2 I_{m_1} < 0, \tag{5.9}$$

$$B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}}D_2 > 0, \tag{5.10}$$

$$-\tfrac{1}{2}\ln(\det((\eta-\gamma^2)(\eta I_{m_1} - B_1^{\mathrm{T}}\Phi B_1 - D_1^{\mathrm{T}}D_1)^{-1})) \geqslant a, \tag{5.11}$$

$$\begin{gathered} E^{\mathrm{T}}\Phi E = A^{\mathrm{T}}\Phi A + C^{\mathrm{T}}C- \\ -(A^{\mathrm{T}}\Phi\overline{B} + S)(\overline{B}^{\mathrm{T}}\Phi\overline{B} + R)^{-1}(\overline{B}^{\mathrm{T}}\Phi A + S^{\mathrm{T}}) \end{gathered} \tag{5.12}$$

where $\overline{B} = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, $S = \begin{bmatrix} C^{\mathrm{T}}D_1 & C^{\mathrm{T}}D_2 \end{bmatrix}$, $R = \begin{bmatrix} D_1^{\mathrm{T}}D_1 - \eta I_{m_1} & D_1^{\mathrm{T}}D_2 \\ D_2^{\mathrm{T}}D_1 & D_2^{\mathrm{T}}D_2 \end{bmatrix}$.
A state feedback control law is defined as

$$F_2 = -(B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}}D_2)^{-1}(B_2^{\mathrm{T}}\Phi A + D_2^{\mathrm{T}}C). \tag{5.13}$$

Proof The proof consists of two steps. In the first step, we prove that the system, closed by a given state feedback $u(k) = F_2x(k)$, is admissible. In the second step anisotropic norm boundedness by positive scalar γ is proved.

Let $M_1 = B_1^{\mathrm{T}}\Phi B_1 + D_1^{\mathrm{T}}D_1 - \eta I_{m_1}$. We get $M_1 < 0$ because $B_1^{\mathrm{T}}\Phi B_1 + D_1^{\mathrm{T}}D_1 - \gamma^2 I_{m_1} < 0$ and $\eta > \gamma^2$.

Denote

$$\begin{aligned} M_2 &= B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}}D_2 > 0, \\ M_3 &= B_1^{\mathrm{T}}\Phi B_1 + D_1^{\mathrm{T}}D_1 - \gamma^2 I_{m_1} < 0, \\ N &= B_1^{\mathrm{T}}\Phi B_2 + D_1^{\mathrm{T}}D_2, \\ A_{cl} &= A + B_2F_2, \ \ C_{cl} = C + D_2F_2. \end{aligned}$$

Note that the matrix

$$M = \overline{B}^{\mathrm{T}} \Phi \overline{B} + R = \begin{bmatrix} M_1 & N \\ N^{\mathrm{T}} & M_2 \end{bmatrix} < 0.$$

Consider an auxiliary variable

$$F = \begin{bmatrix} \overline{F}_1 \\ F_2 \end{bmatrix} = -M^{-1}(\overline{B}^{\mathrm{T}} \Phi A + S^{\mathrm{T}}). \tag{5.14}$$

The expression (5.14) can be rewritten in the form

$$MF = -(\overline{B}^{\mathrm{T}} \Phi A + S^{\mathrm{T}}).$$

Consequently,

$$M_1 \overline{F}_1 + N F_2 = -(B_1^{\mathrm{T}} \Phi A + D_1^{\mathrm{T}} C),$$

$$N^{\mathrm{T}} \overline{F}_1 + M_2 F_2 = -(B_2^{\mathrm{T}} \Phi A + D_2^{\mathrm{T}} C).$$

Without loss of generality one can choose $\overline{F}_1$, satisfying the conditions:

$$M_1 \overline{F}_1 = 0,$$
$$N^{\mathrm{T}} \overline{F}_1 = 0.$$

Hence,

$$(A + B_2 F_2)^{\mathrm{T}} \Phi B_1 + (C + D_2 F_2)^{\mathrm{T}} D_1 = 0. \tag{5.15}$$

Inasmuch as $M < 0$, GDARE (5.12) can be rewritten as

$$E^{\mathrm{T}} \Phi E = A^{\mathrm{T}} \Phi A + \begin{bmatrix} C \\ (-M)^{1/2} F \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} C \\ (-M)^{1/2} F \end{bmatrix}. \tag{5.16}$$

Equation (5.16) is equivalent to the generalized Lyapunov equation (2.91) for E, $A_{cl} = A + B_2 F_2$, and $Q = \begin{bmatrix} C \\ (-M)^{1/2} F \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} C \\ (-M)^{1/2} F \end{bmatrix} > 0$. Hence, the pair (E, A_{cl}) is admissible.

Now we show that $\| P_{cl}^{SF} \|_a \leqslant \gamma$. Introduce a function

$$T(x(k)) \doteq x^{\mathrm{T}}(k) E^{\mathrm{T}} \Phi E x(k) \geqslant 0.$$

Consider an auxiliary function

$$H(x(k), w(k)) \doteq T(x(k+1)) - T(x(k)) + ||z(k)||^2 - \gamma^2 ||w(k)||^2.$$

Show that $H(x(k), w(k)) \leqslant 0$:

$$\begin{aligned}H(x(k),w(k)) = x^{\mathrm{T}}(k+1)E^{\mathrm{T}}\Phi Ex(k+1) - x^{\mathrm{T}}(k)E^{\mathrm{T}}\Phi Ex(k) + \\ +||Cx(k)+D_1w(k)+D_2u(k)||^2 - \gamma^2||w(k)||^2 = \\ = \{\text{substitute } Ex(k+1) \text{ from } (5.1)\} = \\ = (Ax(k)+B_1w(k)+B_2u(k))^{\mathrm{T}}\Phi(Ax(k)+B_1w(k)+B_2u(k)) - \\ -x^{\mathrm{T}}(k)E^{\mathrm{T}}\Phi Ex(k) + ||Cx(k)+D_1w(k)+D_2u(k)||^2 - \gamma^2||w(k)||^2 = \\ = \{u(k) = F_2x(k)\} = \\ = (Ax(k)+B_1w(k)+B_2F_2x(k))^{\mathrm{T}}\Phi(Ax(k)+B_1w(k)+B_2F_2x(k)) - \\ -x^{\mathrm{T}}(k)E^{\mathrm{T}}\Phi Ex(k) + ||Cx(k)+D_1w(k)+D_2F_2x(k)||^2 - \gamma^2||w(k)||^2 = \\ = \{\Theta = (A^{\mathrm{T}}\Phi B_2 + C^{\mathrm{T}}D_2)F_2\} = w^{\mathrm{T}}(k)M_3w(k) + \\ +w^{\mathrm{T}}(k)((A+B_2F_2)^{\mathrm{T}}\Phi B_1 + (C+D_2F_2)^{\mathrm{T}}D_1)^{\mathrm{T}}x(k) + \\ +x^{\mathrm{T}}(k)((A+B_2F_2)^{\mathrm{T}}\Phi B_1 + (C+D_2F_2)^{\mathrm{T}}D_1)w(k) + \\ +x^{\mathrm{T}}(k)(A^{\mathrm{T}}\Phi A + C^{\mathrm{T}}C - E^{\mathrm{T}}\Phi E + F_2^{\mathrm{T}}M_2F_2 + \Theta + \Theta^{\mathrm{T}})x(k) = \\ = \{\text{from (5.13), (5.15) and } A^{\mathrm{T}}\Phi A + C^{\mathrm{T}}C - E^{\mathrm{T}}\Phi E = -F_2^{\mathrm{T}}M_2F_2 \text{ we get}\} = \\ = w^{\mathrm{T}}(k)M_3w(k) - \\ -2x^{\mathrm{T}}(k)((A^{\mathrm{T}}\Phi B_2 + C^{\mathrm{T}}D_2)M_2^{-1}(B_2^{\mathrm{T}}\Phi A + D_2^{\mathrm{T}}C))x(k) \leqslant 0.\end{aligned} \tag{5.17}$$

Summing the expressions $H(x(k), w(k))$, defined by (5.17), from $k = 0$ to $k = \infty$, we get

$$\sum_{k=0}^{\infty} H(x(k), w(k)) = T(x(\infty)) - T(x(0)) + \sum_{k=0}^{\infty}(||z(k)||^2 - \gamma^2||w(k)||^2) \leqslant 0.$$

The closed-loop system is stable thus $T(x(\infty)) = x^{\mathrm{T}}(\infty)E^{\mathrm{T}}\Phi Ex(\infty) = 0$, and as $Ex(0) = 0$, we get $T(x(0)) = x^{\mathrm{T}}(0)E^{\mathrm{T}}\Phi Ex(0) = 0$, then

$$\sum_{k=0}^{\infty}(||z(k)||^2 - \gamma^2||w(k)||^2) \leqslant 0$$

and, hence,

$$\sup_{W} \frac{\sum_{k=0}^{\infty} ||z(k)||^2}{\sum_{k=0}^{\infty} ||w(k)||^2} \leqslant \gamma^2.$$

Consequently, $\sup\limits_{W:\bar{\mathbf{A}}(W)\leqslant a} \frac{\sum_{k=0}^{\infty} ||z(k)||^2}{\sum_{k=0}^{\infty} ||w(k)||^2} \leqslant \gamma^2$, which means $|||P_{cl}^{SF}|||_a^2 \leqslant \gamma^2$. As $\gamma > 0$, we have $|||P_{cl}^{SF}|||_a \leqslant \gamma$.

Using denotations $\eta = q^{-1}$ and $\Phi = q^{-1}\widehat{R}$, it is straightforward to show that GDARE (3.45)–(3.47) for the closed-loop system (5.3) and (5.4) agrees with equation (5.12).

The theorem is proved. ■

Remark 5.1 Consider a limiting case when $a \to +\infty$. Transform the expression (5.11) in the following way.

$$-\ln\left(\det(\eta I_{m_1} - B_1^{\mathrm{T}} \Phi B_1 - D_1^{\mathrm{T}} D_1)^{-1}\right) \geqslant 2a + m_1 \ln(\eta - \gamma^2). \tag{5.18}$$

As $\eta I_{m_1} - B_1^{\mathrm{T}} \Phi B_1 - D_1^{\mathrm{T}} D_1 \leqslant \eta I_{m_1}$, the inequality (5.18) may be rewritten as

$$-\ln\left(\det(\eta^{-1} I_{m_1})\right) \geqslant 2a + m_1 \ln(\eta - \gamma^2).$$

Thus

$$\eta \leqslant \frac{\gamma^2}{1 - \mathrm{e}^{-2a/m_1}}$$

and

$$\gamma^2 < \eta \leqslant \frac{\gamma^2}{1 - \mathrm{e}^{-2a/m_1}}. \tag{5.19}$$

For $a \to +\infty$ from the condition (5.19) we get $\eta \to \gamma^2$, and the inequality (5.11) becomes invalid. Substituting γ^2 instead of η into (5.8)–(5.12), we get the conditions for $\mathscr{H}_\infty$ -control design [3]. Thus $\lim\limits_{a \longrightarrow +\infty} \| P_{cl}^{SF} \|_a = \| P_{cl}^{SF} \|_\infty \leqslant \gamma$.

5.2.2 Full Information Control

In this subsection, we state and solve the anisotropy-based suboptimal full information control problem for discrete-time descriptor systems. Full information means information about the system's state and input disturbance.

Problem 5.2 For system (5.1) and (5.2) and for the known mean anisotropy level a we have to find a full information control law $u_{FI}(k) = F_1 w(k) + F_2 x(k)$ such that the closed-loop system P_{cl}^{FI}

$$Ex(k+1) = (A + B_2 F_2)x(k) + (B_1 + B_2 F_1)w(k), \tag{5.20}$$

$$z(k) = (C + D_2 F_2)x(k) + (D_1 + B_2 F_1)w(k) \tag{5.21}$$

is causal and stable, and its anisotropic norm is bounded by $\gamma > 0$, that is, $\| P_{cl}^{FI} \|_a \leqslant \gamma$.

Sufficient conditions may be formulated as follows.

Theorem 5.3 *Let the initial conditions for system* (5.1) *and* (5.2) *satisfy the constraint*

$$Ex(0) = 0.$$

Then the closed-loop system P_{cl}^{FI} given in the form (5.20) and (5.21) is admissible, and the inequality $|||P_{cl}^{FI}|||_a \leqslant \gamma$ is satisfied if there exist a matrix $\Phi = \Phi^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and a positive scalar value $\eta > \gamma^2$ that satisfy the following conditions:

$$\begin{gathered}
E^{\mathrm{T}}\Phi E \geqslant 0,\\
M_2 = B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}} D_2 > 0,\\
M_3 = B_1^{\mathrm{T}}\Phi B_1 + D_1^{\mathrm{T}} D_1 - \gamma^2 I_{m_1} - N^{\mathrm{T}} M_2^{-1} N < 0,\\
-\tfrac{1}{2}\ln\det\big((\eta-\gamma^2)(-M_1)^{-1}\big) \geqslant a,\\
E^{\mathrm{T}}\Phi E = A^{\mathrm{T}}\Phi A + C^{\mathrm{T}} C -\\
-(A^{\mathrm{T}}\Phi\overline{B} + S)(\overline{B}^{\mathrm{T}}\Phi\overline{B} + R)^{-1}(\overline{B}^{\mathrm{T}}\Phi A + S^{\mathrm{T}})
\end{gathered} \tag{5.22}$$

where

$$M_1 = B_1^{\mathrm{T}}\Phi B_1 + D_1^{\mathrm{T}} D_1 - \eta I_{m_1} - N^{\mathrm{T}} M_2^{-1} N,$$

$$N = B_2^{\mathrm{T}}\Phi B_1 + D_2^{\mathrm{T}} D_1,\ \overline{B} = \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$

$$S = \begin{bmatrix} C^{\mathrm{T}} D_1 & C^{\mathrm{T}} D_2 \end{bmatrix},\ R = \begin{bmatrix} D_1^{\mathrm{T}} D_1 - \eta I_{m_1} & D_1^{\mathrm{T}} D_2 \\ D_2^{\mathrm{T}} D_1 & D_2^{\mathrm{T}} D_2 \end{bmatrix}.$$

Coefficients of the control law can be found as

$$\begin{aligned}
F_1 &= -(B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}} D_2)^{-1}(B_2^{\mathrm{T}}\Phi B_1 + D_2^{\mathrm{T}} D_1),\\
F_2 &= -(B_2^{\mathrm{T}}\Phi B_2 + D_2^{\mathrm{T}} D_2)^{-1}(B_2^{\mathrm{T}}\Phi A + D_2^{\mathrm{T}} C).
\end{aligned} \tag{5.23}$$

Proof The structure of the proof is similar to the proof of the Theorem 5.2 feedback case if we consider an extended state vector $\widetilde{x}(k) = \begin{bmatrix} x(k) & w(k) \end{bmatrix}$. We show how to obtain the special type inequality (5.22).

Using the denotations $\eta = q^{-1}$ and $\Phi = q^{-1}\widehat{R}$, rewrite the inequality (3.44) for the closed-loop system (5.20) and (5.21)

$$\begin{aligned}
-\frac{1}{2}\ln\Big(\det\big((\eta-\gamma^2)(\eta I_{m_1} - (B_1 + B_2F_1)^{\mathrm{T}}\Phi(B_1 + B_2F_1) -\\
-(D_1 + D_2F_1)^{\mathrm{T}}(D_1 + D_2F_1))^{-1}\big)\Big) \geqslant a.
\end{aligned} \tag{5.24}$$

Denote

$$\Psi = \eta I_{m_1} - (B_1 + B_2F_1)^{\mathrm{T}}\Phi(B_1 + B_2F_1) - (D_1 + D_2F_1)^{\mathrm{T}}(D_1 + D_2F_1). \tag{5.25}$$

Substituting F_1 from (5.23) into (5.25), we get

$$\Psi = \eta I_{m_1} - B_1^{\mathrm{T}}\Phi B_1 - D_1^{\mathrm{T}} D_1 + N^{\mathrm{T}} M_2^{-1} N = -M_1.$$

Thus the inequality (5.24) is equal to (5.22).

The theorem is proved. ■

Remark 5.2 In the case of FI control, it is also easy to show that for $a \to +\infty$ the conditions of Theorem 5.3 coincide with the conditions of a $\mathscr{H}_\infty$-based bounded real lemma for the closed-loop system (5.20) and (5.21) [3].

Example 5.2 Consider the following discrete-time descriptor system:

$$E = \begin{bmatrix} 0.9 & 0 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0.85 & -0.3 \\ 0.1 & 0.3 \end{bmatrix}, \; B_1 = \begin{bmatrix} -0.02 \\ 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0.35 & 0.09 \end{bmatrix}, \; D_1 = 0.035, \; D_2 = 0.1.$$

It is easy to check that rank $(E) =$ rank $[\, E \;\; B_1 \,] = 1$. The system is causal, but unstable ($\rho(E, A) = 1.0556$).

Find the state feedback and full information control laws in the forms $u_{SF}(k) = F_2 x(k)$ and $u_{FI}(k) = F_1 w(k) + F_2 x(k)$ for the given mean anisotropy level a and for the known scalar value γ, using the conditions of theorems from this section. Results of suboptimal control law design are given in Tables 5.2, 5.3, and 5.4.

Numerical results show that the FI-control law allows us to obtain less anisotropic norm of the closed-loop system compared to SF-control because of the information about the input disturbance used in the control law design.

Table 5.2 Anisotropy-based suboptimal SF-control design for different mean anisotropy levels

a	0.2	0.5	0.8
γ	0.050	0.055	0.060
$\vert\vert\vert P_{cl}^{SF} \vert\vert\vert_a$	0.0336	0.0422	0.0423
$\rho(E, A + B_2 F_2)$	0.7611	0.7672	0.7631
γ^2	0.0025	0.0030	0.0036
η	0.0042	0.0038	0.0041
Φ	$\begin{bmatrix} 1.0468 & 0.7327 \\ 0.7327 & 0.5129 \end{bmatrix}$	$\begin{bmatrix} 1.0531 & 0.7372 \\ 0.7372 & 0.5160 \end{bmatrix}$	$\begin{bmatrix} 1.0489 & 0.7342 \\ 0.7342 & 0.5139 \end{bmatrix}$
F_2	[2.3498, –0.9]	[2.2956, –0.9]	[2.3319, –0.9]
$E^{\mathrm{T}} \Phi E$	$\begin{bmatrix} 0.8479 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.8530 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.8496 & 0 \\ 0 & 0 \end{bmatrix}$
M_2	0.0205	0.0205	0.0205
M_3	$-8.5629 \cdot 10^{-4}$	-0.0014	-0.0020

Table 5.3 Anisotropy-based suboptimal FI-control design for different mean anisotropy levels

a	0.1	0.3	0.5
γ	0.050	0.080	0.060
$\lVert P_{cl}^{FI} \rVert_a$	0.0162	0.0202	0.0230
$\rho(E, A + B_2F_2)$	0.7247	0.7239	0.7477
γ^2	0.0025	0.0064	0.0036
η	0.0134	0.0141	0.0056
Φ	$\begin{bmatrix} 1.0082 & 0.7057 \\ 0.7057 & 0.4940 \end{bmatrix}$	$\begin{bmatrix} 1.0074 & 0.7052 \\ 0.7052 & 0.4936 \end{bmatrix}$	$\begin{bmatrix} 1.0325 & 0.7228 \\ 0.7228 & 0.5059 \end{bmatrix}$
F_1	–0.2747	–0.2747	–0.2738
F_2	[2.6781, –0.9]	[2.6849, –0.9]	[2.4711, –0.9]
$E^{\mathrm{T}}\Phi E$	$\begin{bmatrix} 0.8167 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.8160 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.8363 & 0 \\ 0 & 0 \end{bmatrix}$
M_2	0.0201	0.0201	0.0203
M_3	−0.0024	−0.0063	−0.0035

Table 5.4 Comparison of SF- and FI-laws for $a = 0.5$ and $\gamma = 0.06$

Case	η	Φ	$\rho(E, A + B_2F_2)$	$\lVert P_{cl} \rVert_a$
FI	0.0056	$\begin{bmatrix} 1.0325 & 0.7228 \\ 0.7228 & 0.5059 \end{bmatrix}$	0.7477	0.0230
SF	0.0047	$\begin{bmatrix} 1.0406 & 0.7284 \\ 0.7284 & 0.5099 \end{bmatrix}$	0.7553	0.0423

5.3 Convex Optimization Technique

In this section, we introduce a solution of the suboptimal anisotropy-based control problem using a strict LMI approach. Here we solve a suboptimal state feedback control design problem for the system (5.1) and (5.2) with $D_2 = 0$.

The system (5.1) is regular, thus there exist two nonsingular matrices, $\widetilde{W}$ and $\widetilde{V}$, that transform the system (5.1) and (5.2) to the equivalent form (2.17)–(2.19). Now we use denotations

$$E_d = \widetilde{W} E \widetilde{V},\ A_d = \widetilde{W} A \widetilde{V},\ B_{1d} = \widetilde{W} B_1,\ B_{2d} = \widetilde{W} B_2,\ C_d = C \widetilde{V},\ D_{1d} = D_1.$$

The following theorem contains sufficient conditions of anisotropic norm boundedness for the closed-loop system; it also gives us the feedback gain, which makes the closed-loop system causal and stable.

To solve the control design problem, apply Theorem 3.5 to the closed-loop system

$$Ex(k+1) = (A + B_2F_2)x(k) + B_1w(k), \tag{5.26}$$
$$z(k) = Cx(k) + D_1w(k). \tag{5.27}$$

Direct implementation of the conditions of Theorem 3.5 to system (5.1) and (5.2), closed by the control law in the form $u = F_2x(k)$, leads to nonlinear terms for which the application of inequality (3.95) as LMI is not possible.

To solve the control problem a better way is to deal with a system dual to (5.26) and (5.27). A state-space representation of a closed-loop dual system is

$$E^{\mathrm{T}}x'(k+1) = (A + B_2F_2)^{\mathrm{T}}x'(k) + C^{\mathrm{T}}w'(k), \tag{5.28}$$
$$z'(k) = B_1^{\mathrm{T}}x'(k) + D_1^{\mathrm{T}}w'(k), \tag{5.29}$$

It is obvious that $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms of the closed-loop system coincide with the same ones of the dual system (5.28) and (5.29). Being a semi-norm, the a-anisotropic norm does not satisfy this property. However, in $p \leqslant m_1$, the design specification is satisfied. To show this fact we recall that the a-anisotropic norm of an admissible system is a convex and monotonic function over a. In addition, when $a = 0$ we get

$$|||P_{cl}|||_0 = \frac{\|P_{cl}\|_2}{\sqrt{m_1}} \leqslant \frac{\|P_{cl}\|_2}{\sqrt{p}} = |||P_{cl}^{dual}|||_0. \tag{5.30}$$

It should be pointed out that $|||P_{cl}|||_a = |||P_{cl}^{dual}|||_a$ when $p = m_1$.

Theorem 5.4 *Consider system* (5.1) *and* (5.2). *Assume that* $\operatorname{rank}(E^{\mathrm{T}}) = \operatorname{rank}\begin{bmatrix} E^{\mathrm{T}} & C^{\mathrm{T}} \end{bmatrix}$ *and* $p \leqslant m_1$. *For a given scalar value* $\gamma > 0$ *and for a known mean anisotropy level of the input disturbance* $a \geqslant 0$ *the closed-loop system* P_{cl}^{SF} *is admissible, and* $|||P_{cl}^{SF}|||_a < \gamma$ *if there exist matrices* $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $Z \in \mathbb{R}^{n\times m_2}$, $\Psi \in \mathbb{R}^{m_1\times m_1}$, *a scalar value* $\eta > \gamma^2$, *such that*

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & \Lambda_{41}^{\mathrm{T}} & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{32}^{\mathrm{T}} & \Lambda_{21} & \Lambda_{52}^{\mathrm{T}} \\ \Lambda_{31} & \Lambda_{32} & -\eta I_p & \Lambda_{31} & \Lambda_{53}^{\mathrm{T}} \\ \Lambda_{41} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & -(Q+Q^{\mathrm{T}}) & 0 \\ 0 & \Lambda_{52} & \Lambda_{53} & 0 & -I_{m_1} \end{bmatrix} < 0, \tag{5.31}$$

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/m_1} < \gamma^2, \tag{5.32}$$

$$\begin{bmatrix} \Psi - \eta I_{m_1} + C_d\Theta C_d^{\mathrm{T}} & D_{1d}^{\mathrm{T}} \\ D_{1d} & -I_p \end{bmatrix} < 0 \tag{5.33}$$

where

$$\begin{aligned}
\Lambda_{11} &= -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}},\ \ \Lambda_{21} = A_d\Gamma^{\mathrm{T}} + B_{2d}Z^{\mathrm{T}}\Omega^{\mathrm{T}},\\
\Lambda_{31} &= C_d\Gamma^{\mathrm{T}},\ \ \Lambda_{41} = L - Q - \frac{1}{2}Q^{\mathrm{T}},\\
\Lambda_{22} &= \Pi A_d^{\mathrm{T}} + A_d\Pi^{\mathrm{T}} + \Phi Z B_{2d}^{\mathrm{T}} + B_{2d}Z^{\mathrm{T}}\Phi^{\mathrm{T}} - \Theta,\\
\Lambda_{52} &= B_{1d}^{\mathrm{T}},\ \ \Lambda_{53} = D_{1d}^{\mathrm{T}},\ \ \Lambda_{32} = C_d\Pi^{\mathrm{T}}.
\end{aligned}$$

In addition, $\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}$, $\Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$, $\Phi = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}$, $\Omega = \begin{bmatrix} I_r & 0 \end{bmatrix}$, $\Gamma = \begin{bmatrix} Q & R \end{bmatrix}$. *A feedback gain is given as*

$$F_2 = Z^{\mathrm{T}} \begin{bmatrix} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}}R^{\mathrm{T}}Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{bmatrix} \widetilde{V}^{-1}. \tag{5.34}$$

Proof We show that the controller, which solves the stated design problem for the singular value decomposition (SVD) canonical form of the system, solves it also for the initial system. The transfer function of the closed-loop system may be written in the form

$$\begin{aligned}
P_{cl}^{SF}(z) = C\widetilde{V}\widetilde{V}^{-1}(zE - A - B_2F_2)^{-1}\widetilde{W}^{-1}\widetilde{W}B_1 + D_1 = \\
= C\widetilde{V}(z\widetilde{W}E\widetilde{V} - \widetilde{W}A\widetilde{V} - \widetilde{W}B_2F_2\widetilde{V})^{-1}\widetilde{W}B_1 + D_1 = \\
= C_d(zE_d - A_d - B_{2d}F_d)^{-1}B_{1d} + D_{1d}
\end{aligned}$$

where $F_d = F_2\widetilde{V}$.

Suppose that inequalities (5.31)–(5.33) hold. Then the $(1, 1)$ entry of (5.31) implies the matrix Q is invertible. We also suppose that the matrix S is invertible. If this condition does not hold, there exists a scalar $\varepsilon \in (0, 1)$ such that the inequality (5.31) holds true for the scalar $\overline{S} = S + \varepsilon I_{n-r}$. Thus we can use $\overline{S}$ instead of S. Replacing Z with $\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}}$ in (5.31), we get conditions of the anisotropy-based bounded real lemma for the system, dual to system (5.1) and (5.2). Thus according to the anisotropy-based bounded real lemma, the closed-loop system (5.26) and (5.27) is admissible, and the a-anisotropic norm of its transfer function is bounded by the given scalar γ.

If the design control problem is solvable, the conditions of Theorem 3.5 hold true for system (5.1) and (5.2). These conditions also hold for the dual system. By the linear change of variables $\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}} = Z$, which implies that $\begin{bmatrix} Q & R \end{bmatrix} F_d^{\mathrm{T}} = \begin{bmatrix} I_r & 0 \end{bmatrix} Z$ and $\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Z$, we get the inequality (5.31).

Moreover, as pointed out before, Q and S are invertible. Thus the feedback gain F_d for the closed-loop system in SVD equivalent form is

Table 5.5 Results of control design for different mean anisotropy levels

a	0	0.1	0.2	0.5	1	2	4.5
$\|\|\| P_{cl}^{SF} \|\|\|_a$	0.2739	0.3266	0.3430	0.3662	0.3796	0.3855	0.3866
$\rho(E, A + B_2F_2)$	0.9999	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998

$$F_d = Z^{\mathrm{T}} \begin{bmatrix} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}} R^{\mathrm{T}} Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{bmatrix}.$$

By inverse change of variables we get F_2 for system (5.3) and (5.4) in the form (5.34).

The theorem is proved. ∎

Remark 5.3 To solve the γ-optimal control problem it is necessary to find $\xi_* = \inf \xi$ on the set $\{L, Q, R, S, Z, \Psi, \eta, \xi\}$, which satisfies inequalities (5.31)–(5.33). Here $\xi = \gamma^2$.

Example 5.3 A numerical example illustrates the computational efficiency of the proposed method. Consider the system with the parameters:

$$E = \begin{bmatrix} 0.3 & 0.5 & 0.1 & 0 & 0.5 \\ 0.7 & 0.8 & 3.3 & 0 & 0.6 \\ 0.6 & 0.8 & 0.3 & 0 & 0.8 \\ 0.7 & 0.5 & 0.9 & 0 & 1 \\ 0.6 & 0.7 & 0.3 & 0 & 0.4 \end{bmatrix}, \quad A = \begin{bmatrix} 0.3 & 0.5001 & 0.1002 & 0.0005 & 0.5006 \\ 0.7 & 0.7941 & 3.2909 & 0.0006 & 0.6002 \\ 0.6 & 0.8 & 0.2999 & 0.0008 & 0.8004 \\ 0.7 & 0.4989 & 0.8978 & 0.001 & 1.0003 \\ 0.6 & 0.7 & 0.2998 & 0.0004 & 0.4013 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.0003 & -0.0002 \\ -0.0058 & 0.0019 \\ 0.0002 & -0.0013 \\ -0.0013 & -0.0015 \\ 0.0001 & 0.0017 \end{bmatrix}, \quad B_2 = 10^{-3} \begin{bmatrix} 0.1 & -0.125 \\ 0.2333 & 0.2 \\ 0.2 & 0.2 \\ 0.2333 & 0.125 \\ 0.2 & 0.175 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}.$$

The system is not causal ($\deg \det(zE - A) = 3$, rank $(E) = 4$), and it is not stable ($\rho(E, A) = 1.000$). Control design results are given in Table 5.5.

5.4 Transient Response Shaping for Closed-Loop Systems

The problem of pole placement for linear descriptor systems is discussed in different works, for example, [4–6]. Pole placement is a well-known technique for shaping desired transient performance. In this section, an anisotropy-based control problem with regional pole placement for discrete-time descriptor systems is considered. The solution to this problem makes it possible to find a state-feedback control law such that the closed-loop system is admissible, its transient response satisfies the desired performance, and anisotropic gain from input disturbance to the controllable output does not exceed the specified level.

Definition 5.1 Consider a region on the complex plane, defined by

$$\mathfrak{D} = \{z \in \mathbb{C} : d + 2b\mathrm{Re}(z) + \mathrm{c}|\mathrm{z}|^2 < 0\}. \tag{5.35}$$

The pair (E, A) is called $\mathfrak{D}$–admissible if it is admissible and its finite eigenvalues lie inside region $\mathfrak{D}$.

For system (5.1) and (5.2) and given scalar numbers a and γ the problem is to find a state-feedback control law

$$u(k) = F_2 x(k), \tag{5.36}$$

such that the closed-loop system with transfer function

$$P_{cl}(z) = C(zE - A - B_2 F_2)^{-1} B_1 + D_1$$

1. is $\mathfrak{D}$-admissible;
2. its a-anisotropic norm satisfies the condition

$$|||P_{cl}|||_a < \gamma.$$

The following lemma, introduced in [4], is useful below.

Lemma 5.1 [4] *Let $\mathfrak{D}$ be a disc centered around the origin and of radius ω; that is, $d = -\omega^2$, $b = 0$, and $c = 1$. The pair (E, A) has g poles inside $\mathfrak{D}$ and $(n - g)$ poles outside $\mathfrak{D}$ if and only if there exist $X = X^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ with g positive, $(n - g)$ negative, and 0 zero eigenvalues satisfying inequality*

$$-\omega^2 EXE^{\mathrm{T}} + AXA^{\mathrm{T}} < 0. \tag{5.37}$$

Theorem 5.5 *Suppose that*

$$\mathrm{rank}\, E = \mathrm{rank}\, [E \;\; B_{1d}].$$

For given scalar values $\gamma > 0$, $0 < \omega < 1$, and $a \geqslant 0$ the a-anisotropic norm of the system is bounded by the value γ; that is, $|||P|||_a < \gamma$, and unforced system P is $\mathfrak{D}$-admissible with radius ω, if there exist matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $X = X^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, and scalar value $\eta > \gamma^2$, satisfying inequalities

$$\eta - (\mathrm{e}^{-2a}\det(\eta I_{m_1} - B_{1d}^{\mathrm{T}}\Theta B_{1d} - D_{1d}^{\mathrm{T}}D_{1d}))^{1/m_1} < \gamma^2, \tag{5.38}$$

$$\begin{bmatrix} \Phi_{11} & \Gamma A_d & \Gamma B_{1d} & \Phi_{41}^{\mathrm{T}} & 0 \\ A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Phi_{22} & \Pi B_{1d} & A^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Phi_{52}^{\mathrm{T}} \\ B_{1d}^{\mathrm{T}}\Gamma^{\mathrm{T}} & B_{1d}^{\mathrm{T}}\Pi^{\mathrm{T}} & -\gamma^2 I_{m_1} & B_{1d}^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Phi_{53}^{\mathrm{T}} \\ \Phi_{41} & \Gamma A_d & \Gamma B_{1d} & -Q - Q^{\mathrm{T}} & 0 \\ 0 & \Phi_{52} & \Phi_{53} & 0 & -I_p \end{bmatrix} < 0, \tag{5.39}$$

$$\begin{bmatrix} -\omega^2 X & 0 \\ 0 & X \end{bmatrix} + \mathrm{sym}\left(\begin{bmatrix} A_d \\ -E_d \end{bmatrix} G\Delta\right) < 0 \tag{5.40}$$

where

$$\Phi_{11} = -\tfrac{1}{2}Q - \tfrac{1}{2}Q^{\mathrm{T}},\ \Phi_{22} = \Pi A_d + A_d^{\mathrm{T}}\Pi^{\mathrm{T}} - \Theta,$$
$$\Phi_{41} = L - Q - \tfrac{1}{2}Q^{\mathrm{T}},\ \Phi_{52} = C_d + \alpha C_d \Pi A_d,$$
$$\Phi_{53} = D_{1d} + \alpha C_d \Pi B_{1d},$$

$$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix},\ \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix},\ \Gamma = \begin{bmatrix} Q & R \end{bmatrix},$$

$$G = \begin{bmatrix} Q^{\mathrm{T}} & 0 \\ R^{\mathrm{T}} & S^{\mathrm{T}} \end{bmatrix}, \tag{5.41}$$

and

$$\Delta = \begin{bmatrix} 0 & 0 & I_r & 0 \\ 0 & I_{n-r} & 0 & 0 \end{bmatrix}. \tag{5.42}$$

A scalar $\alpha > 0$ *is supposed to be sufficiently large.*

Proof The proof of a-anisotropic norm boundedness can be found in Theorem 3.5. Now we need to prove (5.40), which guarantees that all finite eigenvalues lie inside the $\mathfrak{D}$-region.

If the unforced system (5.1) and (5.2) is $\mathfrak{D}$-admissible, then inequality (5.37) holds true for some matrix $\overline{X}$.

Left- and right-multiplying (5.37) on $\overline{W}$ and $\overline{W}^{\mathrm{T}}$, respectively, we get

$$-\omega^2 \overline{W} E \overline{X} E^{\mathrm{T}} \overline{W}^{\mathrm{T}} + \overline{W} A \overline{X} A^{\mathrm{T}} \overline{W}^{\mathrm{T}} < 0. \tag{5.43}$$

Let $\overline{X} = \overline{V} X \overline{V}^{\mathrm{T}}$. It is possible because $\overline{V}$ is nonsingular. Taking into account this notation, inequality (5.43) can be represented as

$$-\omega^2 E_d X E_d^{\mathrm{T}} + A_d X A_d^{\mathrm{T}} < 0. \tag{5.44}$$

Note that the pair (E, A) is admissible. Hence, A_{22} is invertible. Introduce matrices:

$$\overline{\mathscr{W}} = \begin{bmatrix} I_r & -A_{12}A_{22}^{-1} \\ 0 & I_{n-r} \end{bmatrix}, \ \overline{\mathscr{V}} = \begin{bmatrix} I_r & 0 \\ -A_{22}^{-1}A_{21} & A_{22}^{-1} \end{bmatrix},$$

Defining $\widehat{X} = \overline{\mathscr{V}} X \overline{\mathscr{V}}^{\mathrm{T}}$ and by left- and right-multiplying (5.44) on $\overline{\mathscr{W}}$ and $\overline{\mathscr{W}}$, respectively, we get

$$-\omega^2 \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \widehat{X} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \widehat{A} & 0 \\ 0 & I_{n-r} \end{bmatrix} \widehat{X} \begin{bmatrix} \widehat{A}^{\mathrm{T}} & 0 \\ 0 & I_{n-r} \end{bmatrix} < 0 \tag{5.45}$$

with $\widehat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

Let $\widehat{X}$ be divided as $\widehat{X} = \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{12}^{\mathrm{T}} & \widehat{X}_{22} \end{bmatrix}$, $\widehat{X}_{11} \in \mathbb{R}^{r \times r}$.

It follows from (5.45) that $\widehat{X}_{11} > 0$. The expression (5.45) is equivalent to

$$-\omega^2 \widehat{X}_{11} + \widehat{A}\widehat{X}_{11}\widehat{A}^{\mathrm{T}} < 0, \tag{5.46}$$
$$\widehat{X}_{22} < 0. \tag{5.47}$$

We are interested in inequality (5.46). This inequality is strict, hence, there exists a sufficiently small μ such that

$$-\omega^2 \widehat{X}_{11} + \widehat{A}\widehat{X}_{11}\widehat{A}^{\mathrm{T}} + \mu\omega^2 A_{12}A_{12}^{\mathrm{T}} < 0. \tag{5.48}$$

Introduce the matrices:

$$\mathscr{Y} = \begin{bmatrix} 0 & 0 & I_r & 0 \\ 0 & I_{n-r} & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$
$$\mathscr{Z} = \begin{bmatrix} A_{11}^{\mathrm{T}} & A_{21}^{\mathrm{T}} & -I_r & 0 \\ A_{12}^{\mathrm{T}} & I_{n-r} & 0 & 0 \end{bmatrix}.$$

One can check that

$$\mathrm{Ker}\mathscr{Y} = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n-r} \end{bmatrix}^{\mathrm{T}},$$
$$\mathrm{Ker}\mathscr{Z} = \begin{bmatrix} I_r & -A_{12} & \widehat{A} & 0 \\ 0 & 0 & 0 & I_{n-r} \end{bmatrix}^{\mathrm{T}}.$$

Under (5.48) the following inequalities hold true:

$$\begin{cases} \mathrm{Ker}\mathscr{Y}\,\Upsilon\,\mathrm{Ker}\mathscr{Y}^{\mathrm{T}} < 0, \\ \mathrm{Ker}\mathscr{Z}^{\mathrm{T}}\,\Upsilon\,\mathrm{Ker}\mathscr{Z} < 0 \end{cases} \tag{5.49}$$

with

$$\Upsilon = \begin{bmatrix} -\omega^2 \widehat{X}_{11} & -\mu\omega^2 A_{12} & 0 & 0 \\ -\mu\omega^2 A_{12}^{\mathrm{T}} & -\mu\omega^2 I_{n-r} & 0 & 0 \\ 0 & 0 & \widehat{X}_{11} & 0 \\ 0 & 0 & 0 & -\mu I_{n-r} \end{bmatrix}.$$

By the Projection Lemma [7] there exists a matrix $\mathscr{G}$ such that

$$\Upsilon + \operatorname{sym}(\mathscr{Z}^{\mathrm{T}} \mathscr{G} \mathscr{Y}^{\mathrm{T}}) < 0 \tag{5.50}$$

or

$$\begin{bmatrix} -\omega^2 \widehat{X}_{11} & 0 & 0 & 0 \\ 0 & -\mu\omega^2 I_{n-r} & 0 & 0 \\ 0 & 0 & \widehat{X}_{11} & 0 \\ 0 & 0 & 0 & -\mu I_{n-r} \end{bmatrix} + \\ + \operatorname{sym}\left(\mathscr{Z}^{\mathrm{T}}\left(\begin{bmatrix} 0 & 0 \\ 0 & -\mu\omega^2 I_{n-r} \end{bmatrix} + \mathscr{G}\right)\mathscr{Y}^{\mathrm{T}}\right) < 0. \tag{5.51}$$

Denote

$$\overline{G} = \begin{bmatrix} 0 & 0 \\ 0 & -\mu\omega^2 I_{n-r} \end{bmatrix} + \mathscr{G}$$

and

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix} = \begin{bmatrix} -\omega^2 \widehat{X}_{11} & 0 \\ 0 & -\mu\omega^2 I_{n-r} \end{bmatrix}.$$

Then (5.51) can be rewritten as

$$\begin{bmatrix} -\omega^2 X & 0 \\ 0 & X \end{bmatrix} + \mathscr{Z}^{\mathrm{T}} G \mathscr{Y}^{\mathrm{T}} + \mathscr{Y} G \mathscr{Z} < 0. \tag{5.52}$$

Finally, we need to prove that $\overline{G}$ is invertible. If $\overline{G}$ is not invertible, there exists a nonzero vector $c = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$ such that $\overline{G}c = 0$. Let $c_1 \in \mathbb{R}^r$. Then left and right multiplication of (5.52) on $\begin{bmatrix} 0 & c_2 & c_1 & 0 \end{bmatrix}$ and its transpose, respectively, yield $-c_2 X_{22} c_2^{\mathrm{T}} + c_1 X_{11} c_1^{\mathrm{T}} < 0$ which is impossible because $X_{11} > 0$ and $X_{22} < 0$.

By choosing $\overline{G} = G$ as in (5.41) and substituting it into (5.52) we get (5.40). Note that $\mathscr{D}$-admissibility is stronger than the admissibility property for $\omega < 1$. Taking into account that (5.39) guarantees admissibility of the system, selection (5.41) does not contradict (5.39). ■

Introduce the following linear change of variables

$$\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F^{\mathrm{T}} = Z. \tag{5.53}$$

Expression (5.53) implies that

$$\begin{bmatrix} Q & R \end{bmatrix} F^{\mathrm{T}} = \begin{bmatrix} I_r & 0 \end{bmatrix} Z \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} F^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Z.$$

Theorem 5.6 *For given scalar values $\gamma > 0$, $0 < \omega < 1$, and mean anisotropy level $a \geqslant 0$ the control design problem is solvable if there exist scalars $\eta > \gamma^2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and matrices $X = X^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $L \in \mathbb{R}^{r\times r}$, $L > 0$, and $Z \in \mathbb{R}^{n\times m_2}$ such that*

$$\eta - (\mathrm{e}^{-2a} \det(\eta I_p - C_d \Theta C_d^{\mathrm{T}} - D_{1d} D_{1d}^{\mathrm{T}}))^{1/p} < \gamma^2, \tag{5.54}$$

$$\begin{bmatrix} -\omega^2 X & 0 \\ 0 & X \end{bmatrix} + \operatorname{sym}\left(\left(\begin{bmatrix} A_d \\ -E_d \end{bmatrix} G + \begin{bmatrix} B_{2d} \\ 0 \end{bmatrix} Z^{\mathrm{T}} \right) \Delta \right) < 0, \tag{5.55}$$

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & \Lambda_{41}^{\mathrm{T}} & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{32}^{\mathrm{T}} & \Lambda_{21} & \Lambda_{52}^{\mathrm{T}} \\ \Lambda_{31} & \Lambda_{32} & -\eta I_p & \Lambda_{31} & \Lambda_{53}^{\mathrm{T}} \\ \Lambda_{41} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & -(Q+Q^{\mathrm{T}}) & 0 \\ 0 & \Lambda_{52} & \Lambda_{53} & 0 & -I_{m_1} \end{bmatrix} < 0 \tag{5.56}$$

with

$$\begin{gathered} \Lambda_{11} = -\tfrac{1}{2} Q - \tfrac{1}{2} Q^{\mathrm{T}},\ \Lambda_{21} = A_d \Gamma^{\mathrm{T}} + B_{2d} Z^{\mathrm{T}} \Omega^{\mathrm{T}}, \\ \Lambda_{31} = C_d \Gamma^{\mathrm{T}},\ \Lambda_{41} = L - Q - \tfrac{1}{2} Q^{\mathrm{T}}, \\ \Lambda_{22} = \Pi A_d^{\mathrm{T}} + A_d \Pi^{\mathrm{T}} + \Phi Z B_{2d}^{\mathrm{T}} + B_{2d} Z^{\mathrm{T}} \Phi - \Theta, \\ \Lambda_{32} = C_d \Pi^{\mathrm{T}},\ \Lambda_{52} = B_{1d}^{\mathrm{T}},\ \Lambda_{53} = D_{1d}^{\mathrm{T}}, \end{gathered}$$

$$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix},\ \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix},\ \Phi = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix},\ \Omega = \begin{bmatrix} I_r & 0 \end{bmatrix},\ \Gamma = \begin{bmatrix} Q & R \end{bmatrix}.$$

The gain matrix can be obtained as

$$F_2 = Z^{\mathrm{T}} \begin{bmatrix} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}} R^{\mathrm{T}} Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{bmatrix} \overline{V}^{-1}. \tag{5.57}$$

Proof Taking into account the linear change of variables (5.53) and substituting it into (5.40) we get (5.55) for the closed-loop system (5.28) and (5.29). By analogy, substitution of (5.53) into (5.39) gives us Λ_{21} and Λ_{22} entries from (5.56), which coincide with the conditions of Theorem 5.5 for unforced system (5.28) and (5.29). Thus according to Theorem 5.5, the closed-loop system (5.1) and (5.2) is $\mathfrak{D}$-admissible, and the a-anisotropic norm of its transfer function is bounded by the given scalar γ.

In addition, as inequality (5.32) holds, the Λ_{11} entry implies matrix Q is invertible. The invertibility of S is guaranteed by (5.55) (see the proof of Theorem 5.5). Therefore the feedback gain F_d for the closed-loop system is defined as

$$F_d = Z^{\mathrm{T}} \begin{bmatrix} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}} R^{\mathrm{T}} Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{bmatrix}.$$

Note that $F_d = F_2 \overline{V}$. The last expression implies F_2 from (5.57). ■

Example 5.4 Consider the system with parameters:

$$E = \begin{bmatrix} 3 & 0 & 2 & -5 \\ 0 & 3 & -2 & 2 \\ 2 & 2 & 0 & -2 \\ 2 & -4 & 4 & -6 \end{bmatrix}, \quad A = \begin{bmatrix} 4.7 & -3.25 & -0.7 & 0 \\ 0.8 & 0.4 & -6.4 & 2.6 \\ 1 & -1.9 & -5.4 & 2.4 \\ -0.6 & -2.7 & 5.4 & -2.8 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 3.2 & -3.5 \\ 2.5 & -7.9 \\ 3.8 & -7.6 \\ -1.2 & 8.2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & -1 \end{bmatrix}, \ D_1 = \begin{bmatrix} 1.2 & 1.3 \end{bmatrix}.$$

The system considered in the example is causal, but not stable. Its finite eigenvalues are $\{\lambda_1, \lambda_2\} = \{1.2523; \quad 0.5994\}$.

The goal is to design a state-feedback control minimizing the a-anisotropic norm of the closed-loop system, such that finite eigenvalues of the closed-loop system lie inside a circle with radius $\omega = 0.5$. We choose the mean anisotropy level $a = 0.2$.

The state-feedback gain is

$$F_2^{(1)} = \begin{bmatrix} -2.9193 & 3.4906 & -3.8471 & 2.3996 \end{bmatrix}.$$

One can check that the closed-loop system is admissible. Its finite eigenvalues are

$$\{\lambda_1^{(1)}, \lambda_2^{(1)}\} = \{-0.4744; \quad 0.4997\}, \quad |||P_{cl}^{(1)}|||_a = 4.4558.$$

Application of the anisotropy-based control design procedure without a pole placement constraint gives us the following result:

$$F_2^{(2)} = \begin{bmatrix} -6.2855 & 5.7999 & 12.8111 & -6.0253 \end{bmatrix}.$$

Finite eigenvalues of the closed-loop system are

$$\{\lambda_1^{(2)}, \lambda_2^{(2)}\} = \{0.1443; \quad 0.6134\}, \quad |||P_{cl}^{(2)}|||_a = 4.1198.$$

A solution of the pole placement problem without the anisotropy-based quality criterion [4] is

$$F_2^{(3)} = \begin{bmatrix} -3.0863 & 4.6807 & -1.8345 & 1.2492 \end{bmatrix}.$$

Finite eigenvalues of the closed-loop system are

$$\{\lambda_1^{(3)}, \lambda_2^{(3)}\} = \{0.0001; \quad 0.4532\}, \quad |\!|\!| P_{cl}^{(3)} |\!|\!|_a = 4.9568.$$

An illustrative example demonstrates an effectiveness of the developed control design procedure. It is shown that using one of the criteria may not satisfy the designer's requirements. Taking into account both criteria we can achieve better performance of the closed-loop system while solving control problems.

Conclusion

In this chapter, the state feedback anisotropy-based suboptimal control problem for LDTI descriptor systems is studied. In this case, a design specification ensures boundedness of the a-anisotropic norm of the closed-loop system. New sufficient conditions for suboptimal control design procedure are established. These results allow developing computationally effective algorithms for anisotropy-based state feedback control.

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Chapter 6
Anisotropy-Based Analysis for LDTI Descriptor Systems with Nonzero-Mean Input Signals

6.1 Mean Anisotropy of the Gaussian Sequence with Nonzero Mean

Anisotropy of the m-dimensional random vector w is introduced in Sect. 3.1.1 as a minimal value of the relative entropy of w with respect to the Gaussian m-dimensional vector with probability density function (PDF)

$$p_{m,\lambda}(x) = (2\pi\lambda)^{-m/2} \exp\left(-\frac{x^T x}{2\lambda}\right), \quad x \in \mathbb{R}^m,$$

and is described by

$$\mathbf{A}(w) = \min_{\lambda>0} \mathbf{E}_f \ln \frac{f(x)}{p_{m,\lambda}(x)} \tag{6.1}$$

where the function f is the PDF of w.

Suppose w is an m-dimensional Gaussian random vector with nonzero mean ν and covariance matrix S, the PDF of which is given by

$$f(x) = \left((2\pi)^m |S|\right)^{-1/2} \mathrm{e}^{-\frac{1}{2}(x-\nu)^T S^{-1}(x-\nu)}, \ x \in \mathbb{R}^m.$$

By definition (6.1) the anisotropy of the random vector w is expressed as

$$\mathbf{A}(w) = -\frac{1}{2} \ln\det\left(\frac{mS}{\operatorname{Tr} S + |\nu|^2}\right).$$

One can show that if $S = \gamma I_m$ and $\nu = 0$, then $\mathbf{A}(w) = 0$. Here γ is a known constant.

Let W be a stationary sequence of random m-dimensional vectors. Mean anisotropy of the stationary ergodic sequence $W = \{w(k)\}_{k\in\mathbb{Z}}$ is defined in [1] by the following expression

A. A. Belov et al., *Control of Discrete-Time Descriptor Systems*, Studies in Systems, Decision and Control 157, https://doi.org/10.1007/978-3-319-78479-3_6

$$\overline{\mathbf{A}}(W) = \lim_{N\to\infty} \frac{\mathbf{A}(W_{0:N-1})}{N}$$

where $W_{0:N-1}$ is an extended vector of the sequence:

$$W_{0:N-1} = \begin{bmatrix} w(0) < \\ \vdots \\ w(N > -1) \end{bmatrix}.$$

Let the sequence $W = \{w(k)\}_{k\in\mathbb{Z}}$ be generated from the Gaussian white noise $V = \{v(k)\}_{k\in\mathbb{Z}}$ by an admissible shaping filter G

$$E_g x(k+1) = A_g x(k) + B_g(v(k) + \mu), \tag{6.2}$$

$$w(k) = C_g x(k) + D_g(v(k) + \mu) \tag{6.3}$$

where $E_g \in \mathbb{R}^{n_1\times n_1}$, $A_g \in \mathbb{R}^{n_1\times n_1}$, $B_g \in \mathbb{R}^{n_1\times m}$, $C_g \in \mathbb{R}^{m\times n_1}$, $D_g \in \mathbb{R}^{m\times m}$. In addition, rank $(E_g) = n < n_1$ and $|\mu| < \infty$. The connection between mean anisotropy $\overline{\mathbf{A}}(W)$ of the sequence W and state-space representation (6.2) and (6.3) of the shaping filter is given by the following theorem [2].

Theorem 6.1 *For a given state-space representation* (6.2) *and* (6.3) *of the shaping filter G mean anisotropy $\overline{\mathbf{A}}(W)$ is determined by*

$$\overline{\mathbf{A}}(W) = -\frac{1}{2}\ln\det\left(\frac{m(\Sigma + \Xi)}{\mathrm{Tr}\,\Sigma + |\mathscr{M}|^2}\right)$$

where Σ and Ξ are connected with solutions of Lyapunov and Riccati equations P and R by formulas

$$\begin{aligned} \Sigma &= \widehat{C} P \widehat{C}^T + \widehat{D}\widehat{D}^T, \\ P &= \widehat{A} P \widehat{A}^T + \widehat{B}\widehat{B}^T, \end{aligned}$$

$$\begin{aligned} \Xi &= \widehat{C} R \widehat{C}^T, \\ R &= \widehat{A} R \widehat{A}^T - \Lambda(\Sigma + \Xi)^{-1}\Lambda^T, \\ \Lambda &= \widehat{B}\widehat{D}^T + \widehat{A}(P + R)\widehat{C}^T \end{aligned}$$

with matrices

$$\widehat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \widehat{B} = B_1 - A_{12}A_{22}^{-1}B_2,$$

$$\widehat{C} = C_1 - C_2A_{22}^{-1}A_{21}, \quad \widehat{D} = D_g - C_2A_{22}^{-1}B_2,$$

connected with matrices A_{ij}, B_i, C_i $(i, j = 1, 2)$ of the SVD equivalent form of the system (6.2) *and* (6.3)*, and vector $\mathscr{M}$ is represented by*

$$\mathscr{M} = (\widehat{D} + \widehat{C}(I_{n\times n} - \widehat{A})^{-1}\widehat{B})\mu.$$

Proof System (6.2) and (6.3) in SVD equivalent form is given by

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1(v(k)+\mu), \tag{6.4}$$
$$0 = A_{21}x_1(k) + A_{22}x_2(k) + B_2(v(k)+\mu), \tag{6.5}$$
$$w(k) = C_1x_1(k) + C_2x_2(k) + D_d(v(k)+\mu) \tag{6.6}$$

where $x_1(k) \in \mathbb{R}^n$, $x_2(k) \in \mathbb{R}^{n_1-n}$. As the system is causal, $\det A_{22} \neq 0$ (see [3]), then

$$x_2(k) = -A_{22}^{-1}(A_{21}x_1(k) + B_2(v(k)+\mu)). \tag{6.7}$$

Substituting $x_2(k)$ into (6.4) and (6.6), one can get

$$x_1(k+1) = \widehat{A}x_1(k) + \widehat{B}(v(k)+\mu), \tag{6.8}$$
$$w(k) = \widehat{C}x_1(k) + \widehat{D}(v(k)+\mu). \tag{6.9}$$

Applying Theorem 1 from [4] to the system (6.8) and (6.9), we finish the proof. ■

Example 6.1 Let the shaping filter (6.2) and (6.3) be formed by the following matrices:

$$E_g = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}, \quad B_g = \begin{bmatrix} 0.03 \\ 0.10 \\ 0.07 \end{bmatrix}, \quad A_g = \begin{bmatrix} 0.7649 & 0.7572 & -0.0581 \\ -0.0424 & 0.2854 & 0.2218 \\ 0.7706 & 0.6003 & 0.7157 \end{bmatrix},$$
$$C_g = \begin{bmatrix} 1 & 2 & 1.5 \end{bmatrix}, \quad D_g = [0.5],$$

and vector $\mu = [0.1]$; $\operatorname{rank}(E_g) = 2$, $m = 1$. The system in SVD equivalent form is defined by matrices

$$\widehat{A} = \begin{bmatrix} 0.7187 & 0.0253 \\ 0.9639 & -0.3064 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} -0.1213 \\ -0.2673 \end{bmatrix},$$
$$\widehat{C} = \begin{bmatrix} -2.2291 & 0.9010 \end{bmatrix}, \quad \widehat{D} = [0.7324].$$

Therefore vector $\mathscr{M} = [0.065]$. Solving Lyapunov and Riccati equations from Theorem 6.1, we obtain

$$\Sigma = [0.5873], \quad \Xi = [-0.0510].$$

Consequently,

$$\overline{\mathbf{A}}(W) = -\frac{1}{2}\ln\det\left(\frac{m(\Sigma+\Xi)}{\operatorname{Tr}\Sigma + |\mathscr{M}|^2}\right) = 0.049.$$

6.2 Anisotropic Norm of Descriptor Systems with Nonzero-Mean Input Signals

Consider an admissible LDTI descriptor system P written in a state-space representation

$$Ex(k+1) = Ax(k) + Bw(k), \tag{6.10}$$
$$z(k) = Cx(k) + Dw(k) \tag{6.11}$$

where $x(k) \in \mathbb{R}^{n_1}$ is the state, and $w(k) \in \mathbb{R}^m$ and $z(k) \in \mathbb{R}^p$ are input and output signals, respectively. $E,\ A,\ B,\ C,\ D$ are constant real matrices of appropriate dimensions. Suppose that matrix E is singular, That is, rank $(E) = n < n_1$. $W = \{w(k)\}_{k\in\mathbb{Z}}$ is the stationary Gaussian sequence of m-dimensional random vectors with a given mean anisotropy level $\overline{\mathbf{A}}(W) = a \geqslant 0$ and known nonzero mean $\mathbf{E}w_\infty = \mathscr{M}$, $|\mathscr{M}| < \infty$.

For a given system P with the input signal W the RMS gain is defined as

$$Q(P, W) = \frac{\|z\|_{\mathscr{P}}}{\|w\|_{\mathscr{P}}} \tag{6.12}$$

where $\|y\|_{\mathscr{P}}$ is the power norm of the signal $\{y(k)\}_{k\in\mathbb{Z}}$.

Let the sequence $\{w(k)\}_{k\in\mathbb{Z}}$ be represented in the form

$$w(k) = C_g x(k) + D_g(v(k) + \mu) \tag{6.13}$$

where $x(k)$ is the state of the system (6.10), and μ is a known vector. Using (6.13), we obtain an admissible filter G

$$Ex(k+1) = (A + BC_g)x(k) + BD_g(v(k) + \mu), \tag{6.14}$$
$$w(k) = C_g x(k) + D_g(v(k) + \mu). \tag{6.15}$$

Power norms of outputs (6.11) and (6.15) are written as

$$\begin{aligned}
\|w\|^2_{\mathscr{P}} &= \lim_{k\to\infty} \left(\mathrm{Tr}\ \mathbf{cov}(w(k)) + |\mathbf{E}w(k)|^2\right) = \|G\|_2^2 + |\mathscr{M}|^2,\\
\|z\|^2_{\mathscr{P}} &= \lim_{k\to\infty} \left(\mathrm{Tr}\ \mathbf{cov}(z(k)) + |\mathbf{E}z(k)|^2\right) = \|PG\|_2^2 + |\mathscr{P}\mathscr{M}|^2
\end{aligned}$$

where

$$\mathscr{P} = P(1) = D + C(E - A)^{-1}B.$$

RMS gain (6.12) for the system with a nonzero-mean input signal is given by the expression:

$$Q(P,W) = Q(P,G) = \sqrt{\frac{\|PG\|_2^2 + |\mathscr{P}\mathscr{M}|^2}{\|G\|_2^2 + |\mathscr{M}|^2}}. \tag{6.16}$$

Finally, the anisotropic norm of the system is defined as [5]

$$|\!|\!| P |\!|\!|_a = \sup_{G:\overline{\mathbf{A}}(G)\leqslant a} Q(P,G). \tag{6.17}$$

The following theorem represents an algorithm of anisotropic norm computation in the frequency domain [2].

Theorem 6.2 *Consider the system defined by* (6.10) *and* (6.11). *Let* W *be a sequence of nonzero-mean* m*-dimensional Gaussian random vectors, generated by an admissible shaping filter* G *in the form* (6.14) *and* (6.15), *with mean anisotropy* $\overline{\mathbf{A}}(W) = a$ *and* $\mathbf{E}w_\infty = \mathscr{M}$. *Then the anisotropic norm of system* (6.10) *and* (6.11) *can be computed in a frequency domain as*

$$|\!|\!| P |\!|\!|_a = \sup_{q\in[0;\|P\|_\infty^{-2})} \{\mathscr{N}(q) \mid \mathscr{A}(q) = a\} \tag{6.18}$$

where

$$\mathscr{A}(q) = \frac{m}{2}\left(\ln\left(\Phi(q) + \frac{1}{m}|\mathscr{M}|^2\right) - \Psi(q)\right),$$

$$\mathscr{N}(q) = \sqrt{\frac{\Phi(q) - 1 + \frac{q}{m}|\mathscr{P}\mathscr{M}|^2}{q\Phi(q) + \frac{q}{m}|\mathscr{M}|^2}},$$

$$\Phi(q) = \frac{1}{2\pi m}\int_{-\pi}^{\pi} \operatorname{Tr} S(q,\omega)d\omega, \tag{6.19}$$

$$\Psi(q) = \frac{1}{2\pi m}\int_{-\pi}^{\pi} \ln\det S(q,\omega)d\omega, \tag{6.20}$$

$$S(q,\omega) = (I_m - q\Lambda(\omega))^{-1}, \quad q \in [0; \|F\|_\infty^2).$$

Here $\Lambda(\omega) = \widehat{P}^*(\omega)\widehat{P}(\omega)$, *and* $\mathscr{P} = P(1)$. *In addition,* $\mathscr{N}(0) = \sqrt{\frac{\|P\|_2^2 + |\mathscr{P}\mathscr{M}|^2}{m+|\mathscr{M}|^2}}$.

Proof Using the definition of mean anisotropy and some notations from Sect. 3.1.1 and the Szegö limit theorem [6], the mean anisotropy of the stationary Gaussian random sequence W may be computed in terms of spectral density $S(\omega)$ and the $\mathscr{H}_2$ -norm of the shaping filter G as

$$\overline{\mathbf{A}}(G) = -\frac{1}{4\pi}\ln\det\frac{m\,\mathbf{cov}(\widetilde{w}_0)}{\|G\|_2^2+|\mathscr{M}|^2} = -\frac{1}{4\pi}\int\limits_{-\pi}^{\pi}\ln\det\frac{m\widehat{G}^*(\omega)\widehat{G}(\omega)}{\|G\|_2^2+|\mathscr{M}|^2}d\omega. \quad (6.21)$$

Using (6.16) and (6.17), we have

$$\|\!|P|\!\|_a^2 = \sup_{G:\overline{\mathbf{A}}(G)\leqslant a}\frac{\|PG\|_2^2+|\mathscr{P}\mathscr{M}|^2}{\|G\|_2^2+|\mathscr{M}|^2} =$$

$$= \sup_{\|G\|_2^2\leqslant 1+|\mathscr{M}|^2}\left\{\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\mathrm{Tr}\,(\Lambda(\omega)S(\omega))d\omega + |\mathscr{P}\mathscr{M}|^2 : \overline{\mathbf{A}}(G)\leqslant a,\ \|G\|_2^2+|\mathscr{M}|^2<\gamma\right\}$$

where $\gamma > 1$ is a positive real constant.

Construct a Lagrange function as

$$\mathscr{L} = \|PG\|_2^2 + |\mathscr{P}\mathscr{M}|^2 - \alpha_1(\|G\|_2^2+|\mathscr{M}|^2) - \alpha_2\overline{\mathbf{A}}(G).$$

Using the definitions of the $\mathscr{H}_2$ -norm and anisotropic norm of descriptor systems, we get

$$\mathscr{L} = \int\limits_{-\pi}^{\pi}\left(\mathrm{Tr}\,(\Lambda(\omega)S(\omega)-\alpha_1 S(\omega)) + \frac{1}{2}\alpha_2\ln\det S(\omega)\right)d\omega + \\ +|\mathscr{P}\mathscr{M}|^2 - \alpha_1|\mathscr{M}|^2. \quad (6.22)$$

Find an extremum point of function (6.22) from the condition

$$\frac{\partial\mathscr{L}}{\partial S(\omega)} = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\mathrm{Tr}\,\left(\Lambda(\omega)-\alpha_1 I_m + \frac{\alpha_2}{2}S^{-1}(\omega)\right)d\omega = 0.$$

Hence,

$$\Lambda(\omega) - \alpha_1 I_m + \frac{\alpha_2}{2}S^{-1}(\omega) = 0. \quad (6.23)$$

Let $q = \frac{1}{\alpha_1}$, $\sigma = \frac{\alpha_2}{2\alpha_1}$, then equation (6.23) can be rewritten as

$$q\Lambda(\omega) - I_m + \sigma S^{-1}(\omega) = 0.$$

The expression

$$S(\omega) = S(q,\omega) = G^*(\omega)G(\omega) = \sigma(I_m - q\Lambda(\omega))^{-1} \quad (6.24)$$

defines the spectral density for the worst case of the input disturbance. Without loss of generality we consider $\sigma = 1$ [5]. Substituting (6.24) into (6.21) and using notations (6.19) and (6.20), we obtain

$$\overline{\mathbf{A}}(G) = \frac{m}{2}\left(\ln\left(\Phi(q) + \frac{1}{m}|\mathscr{M}|^2\right) - \Psi(q)\right) = \mathscr{A}(q).$$

Finally, substituting spectral density $S(q, \omega)$ into (6.17), we have

$$Q^2(P, G) = \frac{m\frac{\Phi(q)-1}{q} + |\mathscr{P}\mathscr{M}|^2}{m\Phi(q) + |\mathscr{M}|^2} = \mathscr{N}^2(q).$$

Function $\Phi(q)$ satisfies the following properties.
$\lim\limits_{q\to 0+0} \Phi(q) = 1$, and $\lim\limits_{q\to 0+0} \frac{\Phi(q)-1}{q} = \frac{1}{m}\|P\|_2^2$. Thus $\mathscr{N}(0) = \sqrt{\frac{\|P\|_2^2+|\mathscr{P}\mathscr{M}|^2}{m+|\mathscr{M}|^2}}$.
The theorem is proved. ■

Example 6.2 Let system P be described by

$$E = \begin{bmatrix} 0.9 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.7 & -0.3 \\ 0.1 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} -0.02 \\ 0.07 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.50 & 0.09 \end{bmatrix}, \quad D = [0.035].$$

The transfer function of the system is

$$P(z) = \frac{0.235}{9z - 8} + 0.014.$$

The spectral density of the system is

$$\Lambda(\omega) = \frac{0.031(1 + \cos\omega)}{-144\cos\omega + 145}.$$

The spectral density of the worst-case shaping filter is

$$S(q, \omega) = \frac{144\cos\omega + 145}{(-144 - 0.031q)\cos\omega + 145 - 0.031q}.$$

Figures 6.1 and 6.2 show $\mathscr{A}(q)$ and $\mathscr{N}(q)$ plots for different values of $\mathscr{M}$, respectively.

For large values of $\mathscr{M}$ functions, $\mathscr{A}(q)$ and $\mathscr{N}(q)$ lose their monotony (see Fig. 6.3). The set $\{\mathscr{N}(q)\ |_{\mathscr{A}(q)=a}\}$ can be empty or contain several values of $\mathscr{N}(q)$. Therefore, the anisotropic norm is defined as a supremum function by (6.18).

When $a = 0.34$, the anisotropic norm of the system is equal to $|||P|||_a = 0.1841$ for $\mathscr{M} = 0$, $|||P|||_a = 0.1823$ for $\mathscr{M} = 1$ and cannot be computed for $\mathscr{M} = \{2, 3\}$ (see Figs. 6.4 and 6.5).

Remark 6.1 In the general case functions $\mathscr{A}(q)$ and $\mathscr{N}(q)$ are not monotonic [4], but if the condition $\|G\|_2^2 + |\mathscr{M}|^2 = 1$ is satisfied, they become monotone increasing and get the form

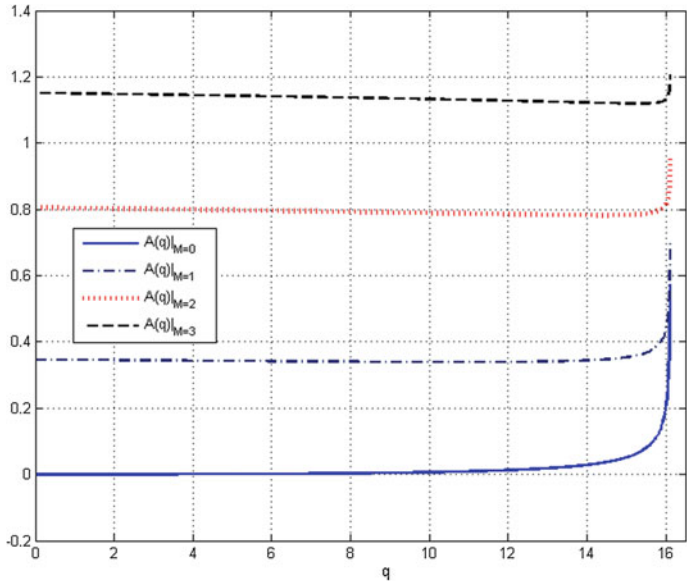

Fig. 6.1 $\mathscr{A}(q)$ for different values of $\mathscr{M}$

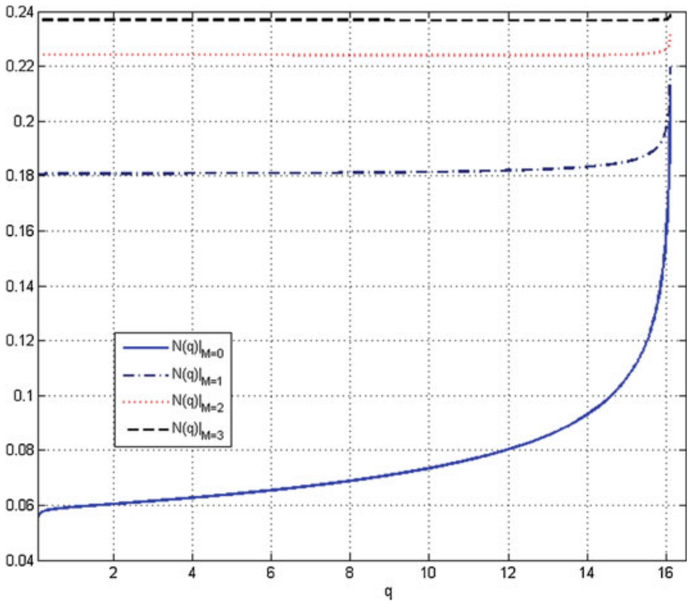

Fig. 6.2 $\mathscr{N}(q)$ for different values of $\mathscr{M}$

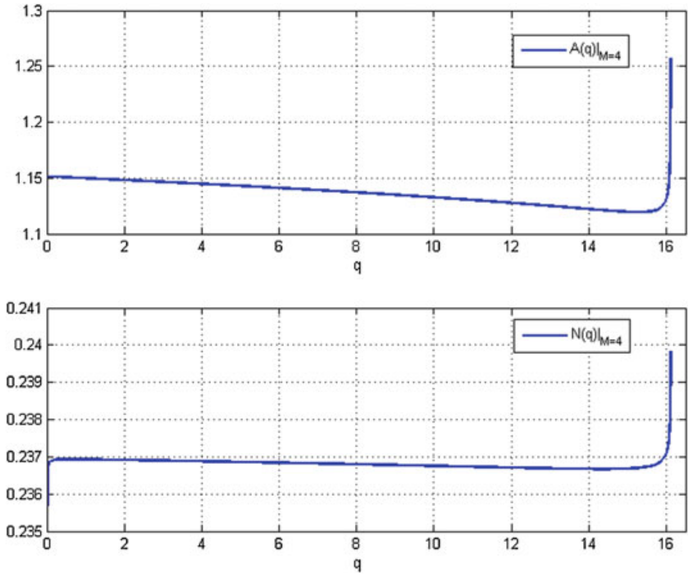

Fig. 6.3 $\mathscr{A}(q)$ and $\mathscr{N}(q)$ for $\mathscr{M} = 3$

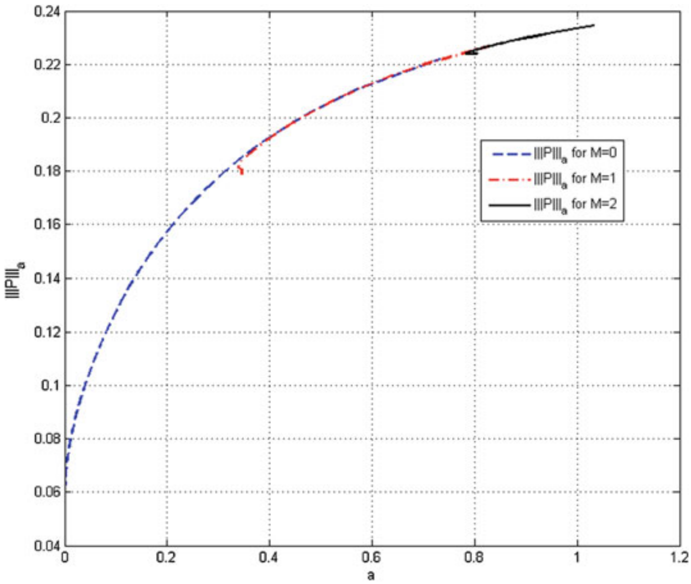

Fig. 6.4 $\mathscr{N}(\mathscr{A}(q))$ for different $\mathscr{M}$

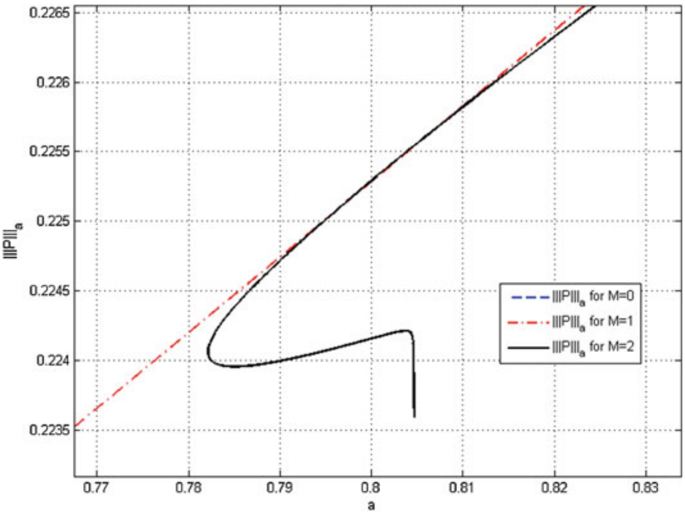

Fig. 6.5 $\mathscr{N}(\mathscr{A}(q))$ plots for different $\mathscr{M}$ (magnified)

$$\mathscr{A}(q) = \overline{\mathbf{A}}(W) = \frac{m}{2}\left(\ln\left(\frac{\Phi(q)}{1-|\mathscr{M}|^2}\right) - \Psi(q)\right), \tag{6.25}$$

$$\mathscr{N}(q) = Q\,(P, W) = \sqrt{\frac{\Phi(q)-1}{q\Phi(q)}(1-|\mathscr{M}|^2) + |\mathscr{P}\mathscr{M}|^2}. \tag{6.26}$$

Moreover, the constraint $\mathscr{A}(q) \leqslant a$ gives a convex set now.

Conclusion

In this chapter, we provide an anisotropy-based analysis problem subject to nonzero-mean random input signals. It is shown that for a nonzero mean of random input signals the anisotropic norm function loses monotonicity. This may result in nonuniqueness of the solution of equation $\mathscr{N}(\mathscr{A}(q)) = a$. However, some limitations on a value of mathematical expectation $\mathscr{M}$ of input random signals can lead to monotonic behavior of both $\mathscr{N}(q)$ and $\mathscr{A}(q)$.

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Chapter 7
Robust Anisotropy-Based Control

7.1 Problem Statement

Consider the following discrete-time descriptor system:

$$Ex(k+1) = A_\Delta x(k) + B_{\Delta 1} w(k) + B_2 u(k), \tag{7.1}$$

$$y(k) = C_\Delta x(k) + D_{\Delta 1} w(k) \tag{7.2}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^{m_1}$ is a random stationary sequence with bounded mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$, $y(k) \in \mathbb{R}^p$ is the output, and $u(k) \in \mathbb{R}^{m_2}$ is the control input. The matrix E is singular; that is, $\operatorname{rank}(E) = r < n$. $A_\Delta = A + M_A \Delta N_A$, $B_{\Delta 1} = B_1 + M_B \Delta N_B$, $C_\Delta = C + M_C \Delta N_C$, and $D_{\Delta 1} = D_1 + M_D \Delta N_D$.

Matrix $\Delta \in \mathbb{R}^{s \times s}$ is unknown norm-bounded, that is, $\|\Delta\|_2 \leqslant 1$. Note that $\|\Delta\|_2 := \overline{\sigma}(\Delta) \leqslant 1$ if and only if $\Delta^{\mathrm{T}} \Delta \leqslant I_s$.

Introduce notations

$$\begin{aligned} A_d = \widetilde{W} A \widetilde{V} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_{1d} = \widetilde{W} B_1 = \begin{bmatrix} B_1^1 \\ B_1^2 \end{bmatrix}, \quad B_{2d} = \widetilde{W} B_2, \\ C_d = C \widetilde{V} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad D_{1d} = D_1. \end{aligned} \tag{7.3}$$

$$M_A^d = \widetilde{W} M_A,\ N_A^d = N_A \widetilde{V},\ M_B^d = \widetilde{W} M_B = \begin{bmatrix} M_{B1}^d \\ M_{B2}^d \end{bmatrix}, N_B^d = N_B,\ M_C^d = M_C,$$
$$N_C^d = N_C \widetilde{V} = \begin{bmatrix} N_{C1}^d & N_{C2}^d \end{bmatrix},$$

A. A. Belov et al., *Control of Discrete-Time Descriptor Systems*, Studies in Systems, Decision and Control 157, https://doi.org/10.1007/978-3-319-78479-3_7

Suppose that

$$\operatorname{rank}\left(E^{\mathrm{T}}\right) = \operatorname{rank}\left[E^{\mathrm{T}}, C^{\mathrm{T}}, N_C^{\mathrm{T}}\right], \tag{7.4}$$

$$\operatorname{rank}(E) = \operatorname{rank}\left[E, B_1, M_B\right]. \tag{7.5}$$

Matrices $\widetilde{W}$ and $\widetilde{V}$ are found from the SVD of matrix E.
We consider two problems.

- Anisotropy-based analysis of system (7.1) and (7.2). This problem is solved in Sect. 7.2;
- State feedback anisotropy-based control design for (7.1) and (7.2). The solution of this problem is given in Sect. 7.3.

7.2 Anisotropy-Based Analysis for Uncertain Descriptor Systems

In an anisotropy-based analysis problem control input is assumed to be zero; that is, $B_2 = 0$. Output $y(k)$ is considered as a measurable output. System (7.1) and (7.2) is supposed to be admissible for all Δ from the given set. Its transfer function is given by $P_\Delta(z) = C_\Delta(zE - A_\Delta)^{-1}B_{\Delta 1} + D_{\Delta 1}$.

For known values $a \geqslant 0$ and $\gamma > 0$ the problem is to find the conditions which allow us to check that the inequality

$$|||P_\Delta|||_a < \gamma$$

holds true.

To solve the anisotropy-based analysis problem for uncertain systems, we use the following Petersen's lemma.

Lemma 7.1 [1] *Let matrices $M \in \mathbb{R}^{n\times p}$ and $N \in \mathbb{R}^{q\times n}$ be nonzero, and $G = G^{\mathrm{T}} \in \mathbb{R}^{n\times n}$. The inequality*

$$G + M\Delta N + N^{\mathrm{T}}\Delta^{\mathrm{T}}M^{\mathrm{T}} \leqslant 0 \tag{7.6}$$

is true for all $\Delta \in \mathbb{R}^{p\times q}$: $\|\Delta\|_2 \leqslant 1$ if there exists a scalar value $\varepsilon > 0$ such that

$$G + \varepsilon M M^{\mathrm{T}} + \frac{1}{\varepsilon}N^{\mathrm{T}}N \leqslant 0. \tag{7.7}$$

Theorem 7.1 [2] *For given scalars $a \geqslant 0$ and $\gamma > 0$ system (7.1) and (7.2) is admissible and its a-anisotropic norm $|||P_\Delta|||_a < \gamma$ if there exist scalars $\eta > \gamma^2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and matrices $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $\Psi \in \mathbb{R}^{m_1\times m_1}$, $L \in \mathbb{R}^{r\times r}$, $L > 0$, $\Upsilon \in \mathbb{R}^{r\times r}$, and $\Upsilon > 0$, such that*

$$\varUpsilon L = I_r, \tag{7.8}$$

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/m_1} < \gamma^2, \tag{7.9}$$

$$\begin{bmatrix} \mho + \varepsilon_1 N_1^{\mathrm{T}} N_1 & M_1 \\ M_1^{\mathrm{T}} & -\varepsilon_1 I_{2s} \end{bmatrix} < 0, \tag{7.10}$$

$$\begin{bmatrix} \Sigma + \varepsilon_2 N_2^{\mathrm{T}} N_2 & M_2 \\ M_2^{\mathrm{T}} & -\varepsilon_2 I_{4s} \end{bmatrix} < 0. \tag{7.11}$$

Here

$$\mho = \begin{bmatrix} \Psi - \eta I_{m_1} & D_{1d}^{\mathrm{T}} & \left(B_1^1\right)^{\mathrm{T}} \\ D_{1d} & -I_p & 0 \\ B_1^1 & 0 & -\varUpsilon \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0 & 0 \\ M_D & 0 \\ 0 & M_{B1}^d \end{bmatrix}, \; N_1 = \begin{bmatrix} N_D & 0 & 0 \\ N_B^d & 0 & 0 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_{1d} & L^{\mathrm{T}} - Q^{\mathrm{T}} - \frac{1}{2}Q & 0 \\ A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Pi A_d + A_d^{\mathrm{T}}\Pi^{\mathrm{T}} - \Theta & \Pi B_{1d} & A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & C_d^{\mathrm{T}} \\ B_{1d}^{\mathrm{T}}\Gamma^{\mathrm{T}} & B_{1d}^{\mathrm{T}}\Pi^{\mathrm{T}} & -\eta I_{m_1} & B_{1d}^{\mathrm{T}}\Gamma^{\mathrm{T}} & D_{1d}^{\mathrm{T}} \\ L - Q - \frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_{1d} & -Q - Q^{\mathrm{T}} & 0 \\ 0 & C_d & D_{1d} & 0 & -I_p \end{bmatrix}, \tag{7.12}$$

$$M_2 = \begin{bmatrix} \Gamma M_A^d & \Gamma M_B^d & 0 & 0 \\ \Pi M_A^d & \Pi M_B^d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Gamma M_A^d & \Gamma M_B^d & 0 & 0 \\ 0 & 0 & M_C^d & M_D \end{bmatrix}, \; N_2 = \begin{bmatrix} 0 & N_A^d & 0 & 0 & 0 \\ 0 & 0 & N_B^d & 0 & 0 \\ 0 & N_C^d & 0 & 0 & 0 \\ 0 & 0 & N_D & 0 & 0 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}, \; \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}, \; \Gamma = \begin{bmatrix} Q & R \end{bmatrix}.$$

Proof Under assumptions (7.4) and (7.5) $B_1^2 = 0$ and $C_2 = 0$. It is easy to check that in (3.95) $\alpha C_d \Pi A_d = 0$ and $\alpha C_d \Pi B_{1d} = 0$. Consider inequality (3.94) from Theorem 3.5. Taking into account $B_1^2 = 0$, transform the expression $B_{1d}^{\mathrm{T}}\Theta B_{1d} = \begin{bmatrix} \left(B_1^1\right)^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1^1 \\ 0 \end{bmatrix} = \left(B_1^1\right)^{\mathrm{T}} L B_1^1 > 0$. Thus the inequality (3.94) is equal to

$$\begin{bmatrix} \Psi - \eta I_{m_1} + \left(B_1^1\right)^{\mathrm{T}} L B_1^1 & D_{1d}^{\mathrm{T}} \\ D_{1d} & -I_p \end{bmatrix} < 0,$$

using Schur's lemma and denoting $\Upsilon = L^{-1}$, we have

$$\begin{bmatrix} \Psi - \eta I_{m_1} & D_{1d}^{\mathrm{T}} & \left(B_1^1\right)^{\mathrm{T}} \\ D_{1d} & -I_p & 0 \\ B_1^1 & 0 & -\Upsilon \end{bmatrix} < 0. \tag{7.13}$$

Now we write the inequality of the form (7.13) for system (7.1) and (7.2) with norm-bounded uncertainties:

$$\begin{bmatrix} \Psi - \eta I_{m_1} & (D_{1d} + M_D \Delta N_D)^{\mathrm{T}} & (B_1^1 + M_{B1}^d \Delta N_B^d)^{\mathrm{T}} \\ D_{1d} + M_D \Delta N_D & -I_p & 0 \\ B_1^1 + M_{B1}^d \Delta N_B^d & 0 & -\Upsilon \end{bmatrix} < 0 \tag{7.14}$$

or

$$\mho + \mathrm{sym}\,(M_1 \Delta N_1) < 0. \tag{7.15}$$

Using the conditions of Schur's and Petersen's lemmas, we can rewrite inequality (7.15) as (7.10). Now we transform expression (3.95) for system (7.1) and (7.2)

$$\Sigma + \mathrm{sym}\,(M_2 \Delta N_2) < 0. \tag{7.16}$$

Applying the same lemmas to inequality (7.16), we get

$$\Sigma + \frac{1}{\varepsilon_2} M_2 M_2^{\mathrm{T}} + \varepsilon_2 N_2^{\mathrm{T}} N_2 < 0,$$

$$\Sigma + \varepsilon_2 N_2^{\mathrm{T}} N_2 - M_2(-\varepsilon_2 I)^{-1} M_2^{\mathrm{T}} < 0,$$

$$\begin{bmatrix} \Sigma + \varepsilon_2 N_2^{\mathrm{T}} N_2 & M_2 \\ M_2^{\mathrm{T}} & -\varepsilon_2 I \end{bmatrix} < 0.$$

The last inequality coincides with (7.11). Expression (7.9) is equal to (3.93). Consequently, conditions of Theorem 3.5 hold true for system (7.1) and (7.2); it means that its anisotropic norm is bounded by a positive scalar value, that $|||P_\Delta|||_a < \gamma$. ∎

Remark 7.1 The mutually inverse matrices search procedure can be found, for example, in [3].

Remark 7.2 If $M_B = 0$ and $N_B = 0$, then conditions of Theorem 7.1 become simpler:

$$\eta - (\mathrm{e}^{-2a} \det(\Psi))^{1/m_1} < \gamma^2,$$

$$\begin{bmatrix} \mho + \varepsilon_1 N_1^{\mathrm{T}} N_1 & M_1 \\ M_1^{\mathrm{T}} & -\varepsilon_1 I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \Sigma + \varepsilon_2 N_2^{\mathrm{T}} N_2 & M_2 \\ M_2^{\mathrm{T}} & -\varepsilon_2 I \end{bmatrix} < 0.$$

Here

$$\mho = \begin{bmatrix} \Psi - \eta I_{m_1} + \left(B_1^1\right)^{\mathrm{T}} L B_1^1 & D_{1d}^{\mathrm{T}} \\ D_{1d} & -I_p \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0 \\ M_D \end{bmatrix}, \; N_1 = \begin{bmatrix} N_D & 0 \end{bmatrix}.$$

In this case, the algorithm of mutually inverse matrices computation in order to find Υ is no longer required.

The procedure of a-anisotropic norm calculation of an uncertain descriptor system (7.1) and (7.2) is based on conditions of Theorem 7.1 and can be formulated as follows. Introduce the notation $\xi = \gamma^2$. Thus the a-anisotropic norm calculation problem is to find

$$\xi_* = \inf \xi$$

on the set

$$\{\eta,\ \xi,\ L,\ \Psi,\ \Upsilon,\ Q,\ R,\ S,\ \varepsilon_1,\ \varepsilon_2\}$$

that satisfies (7.8)–(7.11). If the minimum value ξ_* is found, then the a-anisotropic norm of system $P_\Delta(z)$ can be approximately calculated as

$$|\!|\!| P_\Delta |\!|\!|_a \approx \sqrt{\xi_*}. \tag{7.17}$$

Example 7.1 Consider a descriptor system with parameters:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix},\; A = \begin{bmatrix} -0.25 & 0 & 0 \\ -0.5 & 0.5 & 2 \\ 0.13 & -0.18 & -0.66 \end{bmatrix},\; B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.2 & 0.1 \end{bmatrix},\; C = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.01 & -0.01 \end{bmatrix},\; M_A = \begin{bmatrix} 0.1 & -0.1 & 0.05 \end{bmatrix}^{\mathrm{T}} \text{ and } N_A = \begin{bmatrix} 0 & 0.1 & 0.1 \end{bmatrix}.$$

Uncertainty Δ is scalar, therefore $\Delta \in [-1,\ 1]$. It is easy to check that the system is admissible for all values of Δ. The lower and upper values of the a-anisotropic norm of uncertain systems are shown in Fig. 7.4. The dashed line in Fig. 7.4 displays the result of γ-minimization using conditions of Theorem 7.1. The estimation error is shown in Fig. 7.5.

Example 7.2 We consider a model of a hydraulic tank system with three tanks represented in Fig. 7.3 [4]. A linearized discrete-time state-space model in descriptor form is given by

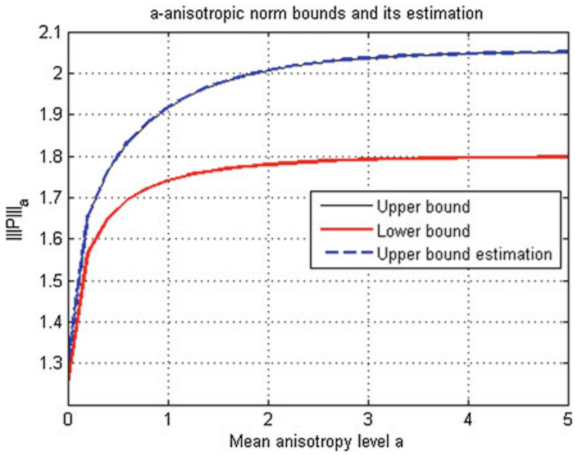

Fig. 7.1 a-Anisotropic norm of uncertain system and its estimation

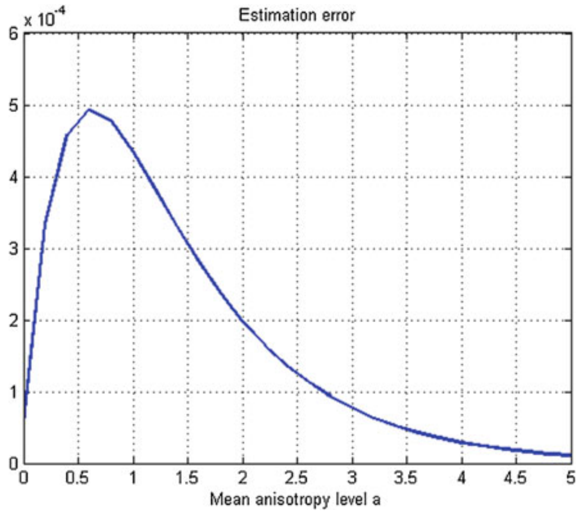

Fig. 7.2 Error of a-anisotropic norm estimation

$$Eq(k+1) = Aq(k) + B_u u(k) + B_\xi \xi(k), \tag{7.18}$$

$$y(k) = Cq(k) + 0.3\eta(k) \tag{7.19}$$

where $q(k)$ is a vector consisting of volumes in the tanks, $u(k)$ is a pump flow, $\xi(k)$ is a plant noise, and $\eta(k)$ is a measurement noise. Matrices in state-space representation (7.18) and (7.19) are given by

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.9692 & 0 & 0 \\ 0.0095 & 0.9867 & 0 \\ 1 & 2.3328 & 1 \end{bmatrix},$$

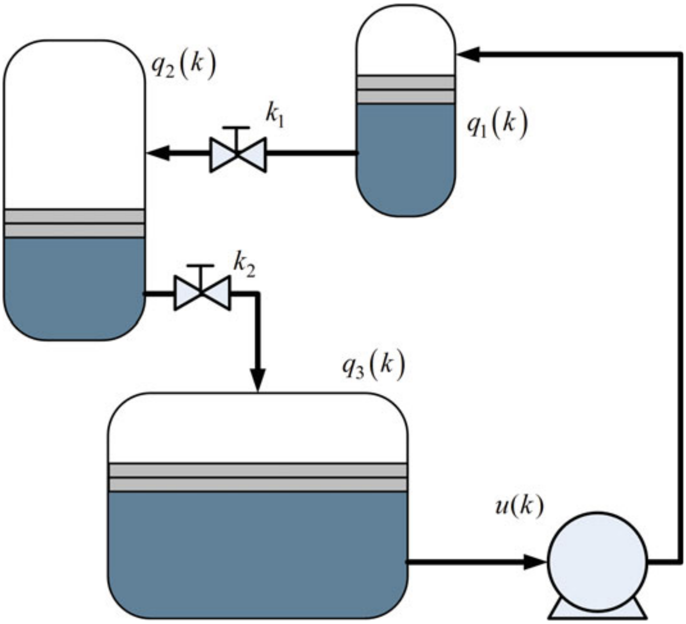

Fig. 7.3 Hydraulic tank system

$$B_u = \begin{bmatrix} 0.056 \\ 0.003 \\ 0 \end{bmatrix}, \ B_\xi = \begin{bmatrix} 0.02 \\ 0.01 \\ 0 \end{bmatrix}, \ \ C = \begin{bmatrix} 0 \ 1 \ 0 \end{bmatrix},$$

Define $w = [\xi \ \ \eta]^{\mathrm{T}}$. Then $B_1 = \begin{bmatrix} 0.02 & 0 \\ 0.01 & 0 \\ 0 & 0 \end{bmatrix}$, $D_1 = \begin{bmatrix} 0 \ 0.3 \end{bmatrix}$. In addition,

$$M_A = \begin{bmatrix} 0.1 \ -0.1 \ 0.3 \end{bmatrix}^{\mathrm{T}} \text{ and } N_A = \begin{bmatrix} 0.2 \ 0.1 \ 0.1 \end{bmatrix}.$$

One can check that the system is admissible for all $\Delta \in [-1; \ 1]$; its worst-case generalized spectral radius is $\rho(E, A) < 1$.

The exact upper bound of the a-anisotropic norm of uncertain systems for different mean anisotropy levels a is shown in Fig. 7.4. The dashed line in Fig. 7.4 displays the result of γ-minimization using the conditions of Theorem 7.1. The estimation error is shown in Fig. 7.5.

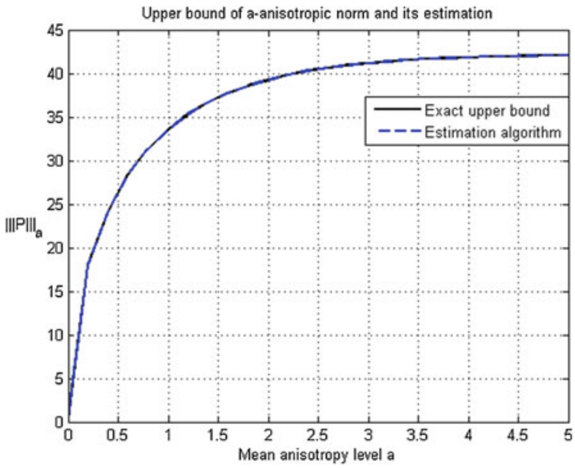

Fig. 7.4 a-Anisotropic norm of uncertain hydraulic tank system and its estimation

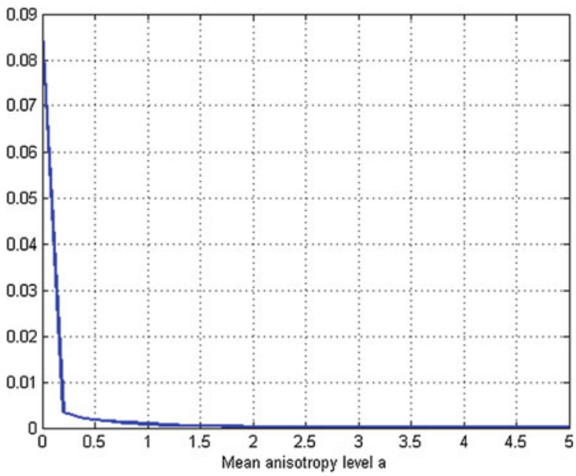

Fig. 7.5 Error of a-anisotropic norm estimation

7.3 State-Space Anisotropy-Based Robust Control Design for Uncertain Descriptor Systems

In the state-space anisotropy-based robust control design problem the system is supposed to be noncausal and unstable; $y(k)$ stands for controllable output. The problem is to find a feedback gain $u(k) = Fx(k)$ such that the closed-loop system with a transfer function

$$P_{\Delta}^{cl}(z) = C_{\Delta}(zE - (A_{\Delta} + B_2F))^{-1}B_{\Delta 1} + D_{\Delta 1}$$

is admissible and

$$|\!|\!| P_\Delta^{cl} |\!|\!|_a < \gamma$$

for all Δ from the given set.

Assume that

1. System (7.1) is causally controllable.
2. System (7.1) is stabilizable.
3. Mean anisotropy of the input disturbance is bounded: $\overline{\mathbf{A}}(W) \leqslant a$ (a is a known value).
4. A scalar value $\gamma > 0$ is given.
5. $p \leqslant m_1$.

Definitions of causal controllability and stabilizability can be found in Sect. 2.6. The following theorem defines the control design procedure.

Theorem 7.2 *For a given scalar $\gamma > 0$ and a known mean anisotropy level a ($\overline{\mathbf{A}}(W) \leqslant a$) the control design problem is solvable if there exist scalars $\eta > \gamma^2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and matrices $Q \in \mathbb{R}^{r \times r}$, $R \in \mathbb{R}^{r \times (n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $\Psi \in \mathbb{R}^{p \times p}$, $L \in \mathbb{R}^{r \times r}$, $L > 0$, $\Upsilon \in \mathbb{R}^{r \times r}$, $\Upsilon > 0$, and $Z \in \mathbb{R}^{n \times m_2}$ such that*

$$\Upsilon L = I_r \tag{7.20}$$

$$\eta - (\mathrm{e}^{-2a} \det(\Psi))^{1/p} < \gamma^2, \tag{7.21}$$

$$\begin{bmatrix} \mho + \varepsilon_1 M_1^{\mathrm{T}} M_1 & N_1 \\ N_1^{\mathrm{T}} & -\varepsilon_1 I_{2s} \end{bmatrix} < 0, \tag{7.22}$$

$$\begin{bmatrix} \Lambda + \varepsilon_2 M_2^{\mathrm{T}} M_2 & N_2 \\ N_2^{\mathrm{T}} & -\varepsilon_2 I_{4s} \end{bmatrix} < 0 \tag{7.23}$$

where

$$\mho = \begin{bmatrix} \Psi - \eta I_p & D_{1d} & C_1 \\ D_{1d}^{\mathrm{T}} & -I_{m_1} & 0 \\ C_1^{\mathrm{T}} & 0 & -\Upsilon \end{bmatrix},$$

$$M_1 = \begin{bmatrix} (M_D)^{\mathrm{T}} & 0 & 0 \\ \left(M_C^d\right)^{\mathrm{T}} & 0 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 0 \\ (N_D)^{\mathrm{T}} & 0 \\ 0 & \left(N_{C1}^d\right)^{\mathrm{T}} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & (M_A^d)^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & 0 & (M_C^d)^{\mathrm{T}} & 0 & 0 \\ 0 & (M_B^d)^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & 0 & (M_D)^{\mathrm{T}} & 0 & 0 \end{bmatrix}, \tag{7.24}$$

$$N_2 = \begin{bmatrix} \Gamma(N_A^d)^{\mathrm{T}} & \Gamma(N_C^d)^{\mathrm{T}} & 0 & 0 \\ \Pi(N_A^d)^{\mathrm{T}} & \Pi(N_C^d)^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Gamma(N_A^d)^{\mathrm{T}} & \Gamma(N_C^d)^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & (N_B^d)^{\mathrm{T}} & (N_D)^{\mathrm{T}} \end{bmatrix}, \tag{7.25}$$

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & \Lambda_{41}^{\mathrm{T}} & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{32}^{\mathrm{T}} & \Lambda_{21} & \Lambda_{52}^{\mathrm{T}} \\ \Lambda_{31} & \Lambda_{32} & -\eta I_p & \Lambda_{31} & \Lambda_{53}^{\mathrm{T}} \\ \Lambda_{41} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & -(Q+Q^{\mathrm{T}}) & 0 \\ 0 & \Lambda_{52} & \Lambda_{53} & 0 & -I_{m_1} \end{bmatrix}, \tag{7.26}$$

$$\Lambda_{11} = -\tfrac{1}{2}Q - \tfrac{1}{2}Q^{\mathrm{T}},\ \Lambda_{21} = A_d\Gamma^{\mathrm{T}} + B_{2d}Z^{\mathrm{T}}\Omega^{\mathrm{T}},$$
$$\Lambda_{31} = C_d\Gamma^{\mathrm{T}},\ \Lambda_{41} = L - Q - \tfrac{1}{2}Q^{\mathrm{T}},$$
$$\Lambda_{22} = \Pi A_d^{\mathrm{T}} + A_d\Pi^{\mathrm{T}} + \Phi Z B_{2d}^{\mathrm{T}} + B_{2d}Z^{\mathrm{T}}\Phi^{\mathrm{T}} - \Theta,$$
$$\Lambda_{32} = C_d\Pi^{\mathrm{T}},\ \Lambda_{52} = B_{1d}^{\mathrm{T}},\ \Lambda_{53} = D_{1d}^{\mathrm{T}}.$$
$$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix},\ \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix},\ \Phi = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix},$$
$$\Omega = \begin{bmatrix} I_r & 0 \end{bmatrix},\ \Gamma = \begin{bmatrix} Q & R \end{bmatrix}.$$

The gain matrix can be obtained as

$$F = Z^{\mathrm{T}} \begin{bmatrix} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}}R^{\mathrm{T}}Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{bmatrix} \widetilde{V}^{-1}. \tag{7.27}$$

Proof Show that controller (7.27) is a solution of the anisotropy-based control problem for initial system (7.1) and (7.2). Indeed,

$$\begin{aligned} P_{cl}(z) &= C\widetilde{V}\widetilde{V}^{-1}(zE - A - B_2F)^{-1}\widetilde{W}^{-1}\widetilde{W}B_1 + D_1 = \\ &= C\widetilde{V}(z\widetilde{W}E\widetilde{V} - \widetilde{W}A\widetilde{V} - \widetilde{W}B_uF\widetilde{V})^{-1}\widetilde{W}B_1 + D_1 = \\ &= C_d(zE_d - A_d - B_{2d}F_d)^{-1}B_{1d} + D_{1d}, \end{aligned}$$

where $F_d = F\widetilde{V}$.

Introduce the following linear change of variables

$$\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}} = Z.$$

It implies that $\begin{bmatrix} Q & R \end{bmatrix} F_d^{\mathrm{T}} = \begin{bmatrix} I_r & 0 \end{bmatrix} Z$ and $\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Z$. Substituting the last expression into (7.12) we get Λ_{21} and Λ_{22} entries from (7.26), which coincide with the conditions of Theorem 7.1 for the system, dual to the system (7.1) and (7.2). Therefore according to Theorem 7.1, the closed-loop system (7.1) and (7.2) is admissible, and the a-anisotropic norm of its transfer function is bounded by the given scalar γ.

Because inequality (7.23) holds, the (1,1) entry implies matrix Q is invertible. We also suppose that matrix S is invertible. If it does not hold, there exists a scalar $\varepsilon \in (0, 1)$, such that inequality (7.23) holds true for matrix $\overline{S} = S + \varepsilon I_{n-r}$. Thus we can use $\overline{S}$ instead of S.

As pointed out before, Q and S are invertible. Therefore the feedback gain F_d for the closed-loop system is defined as $F_d = Z^{\mathrm{T}} \begin{bmatrix} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}} R^{\mathrm{T}} Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{bmatrix}$. Note that $F_d = F\widetilde{V}$. By the inverse change of variables we get F from (7.27).

This completes the proof. ■

Example 7.3 Consider the system:

$$E = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0.5 \\ 1 & 2 & 0 & -1 \\ -2 & 3 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2.02 & 0.99 & 3 & 3 \\ 0.02 & 0.97 & 0.5 & 1 \\ 1.04 & 1.95 & -1 & -1 \\ -1.94 & 2.89 & 0 & 0.5 \end{bmatrix},$$

$$B_u = \begin{bmatrix} 5.9 \\ 2.3 \\ 1.15 \\ 4.05 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.05 & 0.1 \\ -0.2 & 0 \\ 0 & 0.33 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D_w = \begin{bmatrix} 0.03 & 0.07 \\ 0.09 & 0.12 \end{bmatrix},$$

Matrix A is assumed to be uncertain with $\Delta \in [-1;\ 1]$ and

$$M_A = \begin{bmatrix} 0.2 & 0.3 & 0.5 & 0.1 \end{bmatrix}^{\mathrm{T}}, \ N_A = \begin{bmatrix} 1 & 1 & 1 & 0.3 \end{bmatrix}.$$

The system is neither causal nor stable. It is easy to check that all assumptions **A1**–**A5** hold. The generalized spectral radius of the nominal system is $\rho(E, A) = 1.054$.

We consider the mean anisotropy level $a = 0.2$. The design objective is to find the minimal value of γ for which conditions of Theorem 7.2 hold.

Minimization of γ gives $\overline{\gamma}_{\min} = 9.0370$. The controller's parameters are

$$F_{\mathrm{rob}} = \begin{bmatrix} -0.2460 & -0.1821 & 0.6837 & 0.0172 \end{bmatrix}.$$

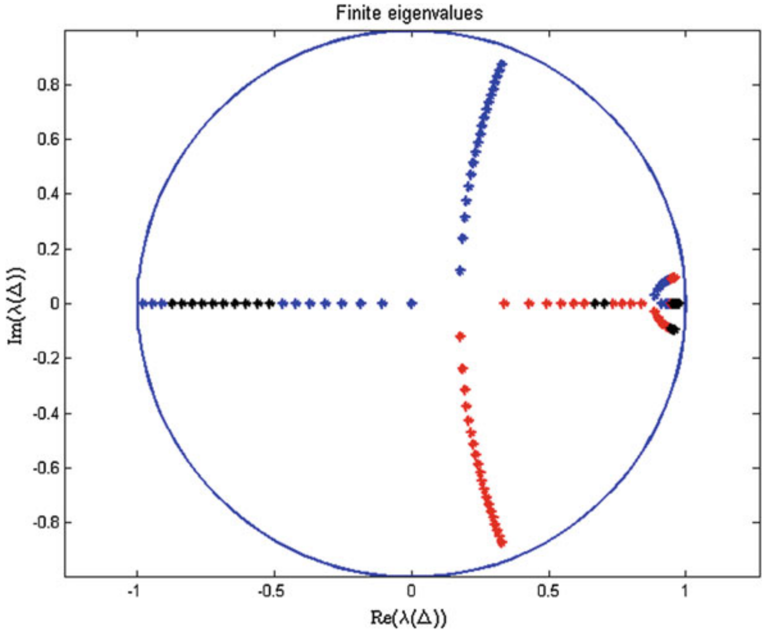

Fig. 7.6 Finite eigenvalues of the closed-loop uncertain system with control gain F_{rob}

The lower and upper bounds of $|||P_{\Delta}^{cl}|||_a$ are $\underline{\gamma} = 1.5419$ and $\overline{\gamma} = 6.0066$, respectively. Hence, the design objective is satisfied. Arrangement of finite eigenvalues of the closed-loop system for all uncertainties is depicted in Fig. 7.6. It is obvious that the closed-loop system is stable for all $\Delta \in [-1;\;\; 1]$.

In order to compare the system performance, we minimized γ for the nominal system (i.e., without uncertainties) and substituted obtained feedback gain into the uncertain system.

$$F_{\mathrm{nom}} = \begin{bmatrix} -0.1942 & -0.1109 & 0.3419 & 0 \end{bmatrix}.$$

The uncertain system, closed by feedback gain $u(k) = F_{\mathrm{nom}} x(k)$ loses stability. The arrangement of finite eigenvalues of the closed-loop system for all uncertainties is depicted in Fig. 7.7.

Conclusion

In this chapter, the state feedback control design problem for discrete-time descriptor systems with norm-bounded uncertainties in the presence of colored noise is examined. It has been shown that the above problem can be solved via the matrix inequality approach involving no parameter uncertainties. Thus the derived result can be applied to design anisotropy-based controllers with a guaranteed robust performance for descriptor systems.

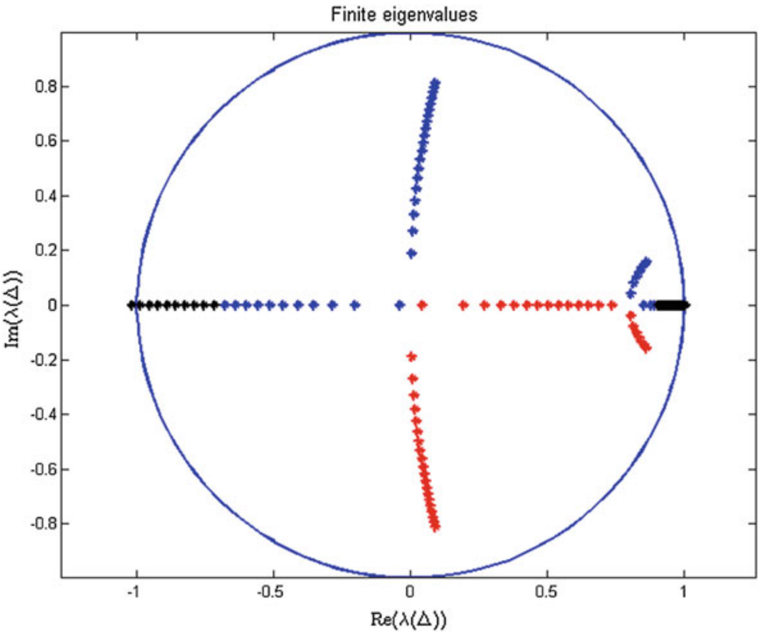

Fig. 7.7 Finite eigenvalues of the closed-loop uncertain system with control gain F_{nom}

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Conclusion

In conclusion, note that stochastic anisotropy-based robust control theory was developed for systems described by linear difference equations in the mid 1990s. The basic concepts of anisotropy-based control theory are anisotropy of the random vector, mean anisotropy of the random sequence, and anisotropic norm of the system.

The concepts of anisotropy of the random vector, mean anisotropy of the sequence of random vectors, and anisotropic norm of linear stationary systems were introduced in [1] in 1994. Anisotropy of the random vector is defined as minimal relative entropy (Kullback-Leibler information divergence) between the probability density functions of the random vector and the Gaussian signal with zero mean and scalar covariance matrix.

Mean anisotropy is defined as the limit of the ratio of anisotropy of the vector, composed of n random vectors, to the number n, when n goes to infinity. Mean anisotropy characterizes "spectral color" of the input sequence, or its difference from the Gaussian white noise that has zero "spectral color."

The induced $\mathscr{H}_2$ -norm of the system with random input signals with limited mean anisotropy is called the anisotropic norm of the stationary system. To distinguish the anisotropic norm from other norms, the authors of the anisotropy-based theory introduced the notation for the anisotropic norm of a matrix transfer function $|||P|||_a$ corresponding to a given level of mean anisotropy of the input sequence. In [2], a homotopy method for solving the anisotropy-based analysis problem is introduced. The anisotropic norm of the system lies between the $\mathscr{H}_2$ -norm of the system scaled over the square root of the McMillan control object degree and the $\mathscr{H}_\infty$ -norm of the system.

Both these values are limiting cases of the anisotropic norm (when the mean anisotropy is equal to 0 and tends to ∞, respectively). Therefore, anisotropy-based control methods, in some sense, generalize methods of $\mathscr{H}_2$ - and $\mathscr{H}_\infty$ -optimal control design.

Thus, by the end of the 1990s the basis of stochastic (anisotropy-based) robust control theory had been established. This theory has been successfully developed in subsequent years. Generalizing methods for many well-known problems of stochastic control in the case of unknown probabilistic characteristics of input signals and

A. A. Belov et al., *Control of Discrete-Time Descriptor Systems*, Studies
in Systems, Decision and Control 157, https://doi.org/10.1007/978-3-319-78479-3

different kinds of parametric disturbances were created. This monograph represents an extension of the stochastic anisotropy-based theory of robust control to discrete-time descriptor systems.

Important results were obtained while developing a suboptimal anisotropy-based theory of stochastic robust control. Much attention in the present work was paid to the development of effective computational methods of anisotropy-based controller design.

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