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Geometry of slow–fast Hamiltonian systems and Painlevé equations

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Highlights

- Geometric tools to describe slow–fast Hamiltonian systems on smooth manifolds.
- Direct derivation of Painlevé-I equation near a fold point of a slow manifold.
- Direct derivation of Painlevé-II equation near a cusp point of a slow manifold.

Abstract

In the first part of the paper we introduce some geometric tools needed to describe slow–fast Hamiltonian systems on smooth manifolds. We start with a smooth bundle $p : M \to B$ where (M, ω) is a C^∞-smooth presymplectic manifold with a closed constant rank 2-form ω and (B, λ) is a smooth symplectic manifold. The 2-form ω is supposed to be compatible with the structure of the bundle, that is the bundle fibers are symplectic manifolds with respect to the 2-form ω and the distribution on M generated by kernels of ω is transverse to the tangent spaces of the leaves and the dimensions of the kernels and of the leaves are supplementary. This allows one to define a symplectic structure $\Omega_\varepsilon = \omega + \varepsilon^{-1} p^*\lambda$ on M for any positive small ε, where $p^*\lambda$ is the lift of the 2-form λ to M. Given a smooth Hamiltonian H on M one gets a slow–fast Hamiltonian system with respect to Ω_ε. We define a slow manifold SM for this system. Assuming SM is a smooth submanifold, we define a slow Hamiltonian flow on SM. The second part of the paper deals with singularities of the restriction of p to SM. We show that if $\dim M = 4$, $\dim B = 2$ and Hamilton function H is generic, then the behavior of the system near a singularity of fold type is described, to the main order, by the equation Painlevé-I, and if this singularity is a cusp, then the related equation is Painlevé-II.

Keywords: Slow–fast; Hamiltonian; Presymplectic manifold; Singular symplectic; Bundle; Disruption point; Blow-up; Painlevé equations

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1. Introduction

Slow–fast Hamiltonian systems are ubiquitous in the applications in different fields of science. These applications range from astrophysics, plasma physics and ocean hydrodynamics to molecular dynamics. Usually these problems are given in coordinate form, moreover, in the form where a symplectic structure in the phase space is standard (in Darboux coordinates). But there are cases when either the symplectic form is nonstandard or the system under study is of a kind where the corresponding symplectic form has to be found, in particular, when we deal with the system on a manifold.

It is our aim in this paper to present basic geometric tools to describe slow–fast Hamiltonian systems on manifolds, that is in a coordinate-free way. For the non Hamiltonian case this was done by V.I. Arnold [1]. Recall that a customary slow–fast dynamical system is defined by a system of differential equations

$$\varepsilon \dot{x} = f(x, y, \varepsilon), \qquad \dot{y} = g(x, y, \varepsilon), \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^n, \tag{1}$$

depending on a small positive parameter ε (its positivity is needed to fix the direction of increasing time t). It is evident that x-variables in the region of the phase space where $f \neq 0$ change with the speed $\sim 1/\varepsilon$ that is fast. In comparison with them the change of y-variables is slow. Therefore variables x are called fast and y are called slow.

Such system generates two limiting systems whose properties influence the dynamics of the slow–fast system for a small ε. One of the limiting system is called fast or layer system and is derived in the following way. Let us introduce the so-called fast time $\tau = t/\varepsilon$. Then the system acquires the parameter ε in the right hand side of the second equation (due to the differentiation in τ) but looses it in the first equation. Thus, the right hand sides depend on ε in a regular way

$$\frac{dx}{d\tau} = f(x, y, \varepsilon), \qquad \frac{dy}{d\tau} = \varepsilon g(x, y, \varepsilon), \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^n. \tag{2}$$

Setting then $\varepsilon = 0$ we get the system, where y-variables are constants $y = y_0$ and they can be considered as parameters in the equations for x. Sometimes these equations are called layer equations. Because the fast system depends on parameters, it may pass through many bifurcations as parameters y change and this can be useful to find some special motions in the full system for small $\varepsilon > 0$.

The slow equations are derived as follows. Let us formally set $\varepsilon = 0$ in the system (1) and solve the equations $f = 0$ with respect to x (where it is possible). The most natural case when this can be done, is when the matrix f_x is invertible in some domain where solutions for equations $f = 0$ exist. Then by the implicit function theorem one can solve the system $f = 0$. Denote the related branch of solutions as $x = h(y)$ and insert it into the second equation instead of x. Then one gets a system of differential equations for y variables

$$\dot{y} = g(h(y), y, 0),$$

which is called the slow system and the graph of h is called the slow manifold. The idea behind this construction is as follows: for small ε solutions of the system (2) can approach fast to the slow manifold and stay there for a long time, during this time the motions of the full system are described in the first approximation by the slow system.

Now one of the primary problem for slow–fast systems is formulated as follows. Suppose we know something about the dynamics of both (slow and fast) systems, for instance, about some structure in the phase space made up of pieces of fast and slow motions. Can we say anything about the dynamics of the full system for a small positive ε near this structure? There is a vast literature devoted to the study of these systems, see, for instance, some of the references in [7].

This set-up can be generalized to the case of manifolds in a coordinate-free way [1]. Consider a smooth bundle $p : M \to B$ with a fiber F being a smooth manifold and assume a vertical vector field v is given on M. The latter means that any vector $v(x)$ is tangent to the fiber F_b for any $x \in M$ and $b = p(x) \in B$. In other words, each fiber F_b of M is an invariant submanifold for this vector field. Let v_ε be a smooth unfolding of $v = v_0$. Vectors of v_ε need not be tangent to the leaves F_b anymore for $\varepsilon > 0$. Consider the set of zeros for vector field v, that is, one fixes a fiber F_b, then v generates a vector field v^b on this smooth manifold and we consider its zeros (equilibria for this vector field). Let the linearization operator of v^b (along the fiber) at some of the zeros x be a linear operator $Dv^b_x : T_x F_b \to T_x F_b$ acting in invariant linear subspace $V_x = T_x F_b$ of $T_x M$. Suppose the operator has not zero eigenvalues, then the set of zeros is smoothly continued in b for b close to $b = p(x)$. It is a consequence of the implicit function theorem. For this case one gets a local section $z : B \to M,\ p \circ z(b) = b$, which gives a smooth submanifold Z of dimension $\dim B$. One can define a vector field on Z in the following way. Let us represent vector $v_\varepsilon(x)$ in the unique way as $v_\varepsilon(x) = v^1_\varepsilon(x) \oplus v^2_\varepsilon(x)$, a sum of two vectors of which $v^1_\varepsilon(x)$ belongs to V_x and $v^2_\varepsilon(x)$ is in $T_x Z$. Then vector $v^2_\varepsilon(x)$ is of order ε, since v_ε smoothly depends on ε, and it is zero vector as $\varepsilon = 0$. Due to Arnold [1] the vector field on Z given as $(d/d\varepsilon)(v^2_\varepsilon)$ at $\varepsilon = 0$ is called slow vector field, in coordinate form it gives just what was written above.

It is worth remarking that the set in M consisting of zeros for all vertical vector fields (at $\varepsilon = 0$) can be called *slow manifold*. Generically, this set is a smooth submanifold in M but it can be tangent to fibers F_b at some of its points. Genericity here means that it is fulfilled for a residual set of slow–fast vector fields in an appropriate topology. In a neighborhood of the tangency point it is also possible sometimes to define a vector field on Z that can be called a slow vector field, but it is a more complicated problem intimately related with degeneracies of the projection of p at this point (ranks of Dp at these points, etc.) [1].

1.1. Hamiltonian slow–fast systems

Now we turn to Hamiltonian vector fields. It is well known that, in order to define a Hamiltonian vector field in an invariant way, the phase manifold M has to be smooth symplectic: a smooth nondegenerate closed 2-form Ω has to be given on M [3]. For example, the standard way to write a slow–fast Hamiltonian system with a smooth Hamiltonian $H(x, y, u, v, \varepsilon)$ in coordinates is as follows: $(x, y, u, v) = (x_1, \dots, x_n, y_1, \dots, y_n, u_1, \dots, u_m, v_1, \dots, v_m)$ has the form

$$\begin{aligned} \varepsilon \dot{x}_i &= \frac{\partial H}{\partial y_i}, \qquad \varepsilon \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \dots, n, \\ \dot{u}_j &= \frac{\partial H}{\partial v_j}, \qquad \dot{v}_j = -\frac{\partial H}{\partial u_j}, \quad j = 1, \dots, m. \end{aligned} \tag{3}$$

Here the 2-form ω for this system is given by 2-form $\varepsilon dx \wedge dy + du \wedge dv$. It regularly depends on ε and degenerates into 2-form $du \wedge dv$ at $\varepsilon = 0$. Instead, if we introduce the fast time $t/\varepsilon = \tau$, then the transformed system depends on ε regularly but the corresponding symplectic 2-form $dx \wedge dy + \varepsilon^{-1} du \wedge dv$ is singular as $\varepsilon \to +0$.

Let us note that the fast system here is Hamiltonian (evidently). The same is true for the slow system as well. Indeed, if $x = p(u, v),\ y = q(u, v)$ represent solutions of the system

$H_x = 0,\ H_y = 0$ (the set of zeros for the fast systems), then the slow system

$$\dot{u} = H_v, \qquad \dot{v} = -H_u,$$

where the functions p, q are inserted into the right hand sides, is Hamiltonian with the Hamilton function $h(u, v) = H(p(u, v), q(u, v), u, v, 0)$. For a Hamiltonian slow–fast system its slow manifold (if it exists) can be normally hyperbolic giving thus some mechanism for scattering nearby orbits. Also it can be neutrally stable if the slow manifold consists of the elliptic equilibria of the fast systems. Due to the fact that the fast systems depend on parameters (u, v) the types of the equilibria on the slow manifold can vary in parameters.

In order to extend this set-up of slow–fast Hamiltonian systems to vector fields on smooth manifolds one needs to require that a smooth phase manifold M of the system carries a symplectic structure. Moreover this manifold must also have a foliation at $\varepsilon = 0$ into symplectic leaves on which fast Hamiltonian systems are defined.

From this it follows that such Hamiltonian systems can be defined on a manifold with a presymplectic structure, that is on a manifold M with a closed 2-form ω of constant rank at each point $u \in M$. Such manifold carries a distribution of linear subspaces $N_u \subset T_u M$ for $u \in M$ of the kernels of ω in $T_u M$. In order to have at $\varepsilon = 0$ the foliation with symplectic leaves we consider a smooth bundle $p : M \to B$ with a C^∞-smooth connected presymplectic manifold (M, ω) over a C^∞-smooth manifold B. Even in this case the presymplectic structure and the bundle structure can be incompatible, for instance, leaves of the bundle could be not symplectic with respect to the restriction of 2-form ω on a leaf, or the rank of ω could be greater than the dimension of the leaves.

Therefore we shall call the presymplectic form ω *compatible* with the bundle structure, if for any $u \in M$ the restriction of ω on the tangent space $T_u F_u$ to the fiber F_u of the bundle is nondegenerate and kernel N_u of ω at u is transverse to $T_u F_u$. Then fibers F_u are symplectic manifolds with respect to the restriction of ω on F_u. In this case we get a symplectic foliation on M. The dimension of its leaves, due to regularity and connectivity of M, is the integer $2n$ that is the same for all points in M and it is just the rank of the 2-form ω. It turns out that it is possible to define canonically the symplectic structure on the presymplectic total space M of the bundle, if one requires the base B to be a symplectic manifold. In more details this will be explained below, as well as how to define a slow manifold *SM* of a slow–fast Hamiltonian system, the slow and the fast subsystems on M.

In the second part of the paper we consider the local behavior of a slow–fast Hamiltonian system near a point of the tangency of *SM* with a leaf of the bundle p. When $\dim M = 4$ and $\dim B = 2$ then there are generically only two types of such points of tangency, a fold and a cusp, that follows from the theory of singularities of smooth mappings (see, [28,4]). We derive using the blow-up method the Painlevé-I equation for the description in the main approximation of the orbit behavior near a fold point and the Painlevé-II equation for the description in the main approximation of the orbit behavior near a cusp. The scalings for the blow-up procedure are different for the fold and for the cusp. As is known, for a 2-dim slow–fast (dissipative) system the passage near a disruption point is described by the Riccati equation [20,19]. Here we get the Painlevé-I, II equation. Since there is a vast information concerning the behavior of solutions of the Painlevé-I and Painlevé-II equations (see, for instance, [17,6]), the reductions obtained can help in understanding the orbit behavior near the disruption points.

In fact, it is not all the story. If one wants to study the behavior near the fold point, one needs to investigate in detail the blown-up system. As is known, geometrically the blow-up procedure means that we blow up the singular point of the suspended 4-dimensional vector field (i.e. with

$\dot{\varepsilon} = 0$ added) up to the 3-dimensional sphere S^3 and a neighborhood of the singular point becomes a neighborhood of the order r for this sphere. Here $r \geq 0$ is the coordinate in the transverse direction to the sphere which is supposed small. Then one needs to understand the whole picture of the passage of the orbits through a neighborhood of the disruption point. This will be done in a different paper.

2. Presymplectic manifolds and smooth bundles

Let M be a smooth manifold of dimension $2n + k$ and ω be a smooth closed 2-form on M of constant rank $2n$ (here and below we assume C^∞-smoothness). Then the pair (M, ω) is called a *presymplectic manifold* (see, for instance, [14]). Suppose in addition the following assumptions hold

- there is a $2m$-dimensional smooth connected manifold B, $2m = k$, and submersion $p : M \to B$ such that 2-form ω is nondegenerate on fibers of p. In this case we shall call ω compatible with p. Recall that for submersion p its differential $Dp|_u$ has constant rank equal to the dimension of B at any point $u \in M$;
- there is a smooth symplectic 2-form λ on B, i.e. B is a smooth symplectic manifold.

In accordance with the first condition, for each fiber Q of the bundle $\xi = (M, p, B)$ the pair $(Q, \omega|_Q)$ is a symplectic manifold of dimension $2n$.

Consider a point $u \in M$ and let $b = p(u)$. Vectors $X \in T_uM$ that are tangent to the fiber $Q = p^{-1}(b)$ are usually called vertical with respect to the bundle ξ. They form $2n$-dimensional subspace $V_u \subset T_uM$. Denote $N_u = (T_uM)^\perp$ the skew-orthogonal complement to T_uM with respect to the form ω, that is, the space of such $Y \in T_uM$ for which one has $\omega(X, Y) = 0$ for all $X \in T_uM$. Then one has $\dim N_u = \dim T_uM - \text{rank}\omega = 2m = \dim B$ and $\dim T_uM = \dim V_u + \dim B$ due to regularity of $p : M \to B$. Hence one gets $\dim T_uM = \dim V_u + \dim N_u$. On the other hand, since ω is nondegenerate on V_u, we have $V_u \cap N_u = 0$. The two last equalities mean that $T_uM = V_u \oplus N_u$.

The maps $u \to V_u$ and $u \to N_u$ define two smooth distributions V and N on M. Here the distribution N can be interpreted as a connection on the total space M of bundle ξ, therefore vectors $Y \in N_u$ will be called horizontal.

Since rank of ω is constant, each point $u \in M$ possesses a Darboux chart (U, φ), $u \in U$, in which one has $\omega = d\varphi^1 \wedge d\varphi^{n+1} + \cdots + d\varphi^n \wedge d\varphi^{2n}$ [27]. Submanifolds in U given by equations $\varphi^i = c^i$, $i = 1, \ldots, 2n$, for $c^i \in \mathbb{R}$, are integral manifolds for the distribution N. Thus, the connection N is integrable and defines a foliation $\mathcal{F}^N$ on M.

Let $\mathcal{F}^V$ denote the foliation on M formed by the fibers of ξ. Then $\mathcal{F}^V$ and $\mathcal{F}^N$ give a pair of foliations of complementary dimensions $2n$ and $2m$, their leaves intersect transversely. Sometimes such pairs are called bi-foliations.

For any nonzero $\varepsilon \in \mathbb{R}$ we set

$$\Omega_\varepsilon = \omega + \varepsilon^{-1} p^*\lambda. \tag{4}$$

This defines a 2-form Ω_ε on M.

Lemma 1. *For all $\varepsilon \neq 0$ the 2-form Ω_ε is symplectic.*

Proof. The 2-forms ω and λ are closed. Hence one has

$$d\Omega_\varepsilon = d\omega + d(\varepsilon^{-1} p^*\lambda) = d\omega + \varepsilon^{-1} p^* d\lambda = 0.$$

Consider a Darboux chart (U, φ) for ω on M. Then for any point $u \in U$ the determinant of the matrix

$$A = (\omega(\partial_i^\varphi(u), \partial_j^\varphi(u))), \quad i, j = 1, \dots, 2n,$$

is equal to 1 but

$$\omega(\partial_{2n+r}^\varphi(u), \partial_\beta^\varphi(u)) = \omega(\partial_\alpha^\varphi(u), \partial_{2n+s}^\varphi(u)) = 0$$

for all $r, s = 1, \dots, 2m$ and $\alpha, \beta = 1, \dots, 2n+2m$. Since N is the connection on $\xi = (M, p, B)$, the restriction $Dp|_{N_u} : N_u \to T_b B$ is an isomorphism, here $b = p(u)$. Therefore vectors

$$X_{2n+1} = Dp(\partial_{2n+1}^\varphi(u)), \dots, X_{2n+2m} = Dp(\partial_{2n+2m}^\varphi(u))$$

compose a basis of $T_b B$. Thus, matrix

$$C = (\lambda(X_{2n+r}, X_{2n+s})), \quad r, s = 1, \dots, 2m,$$

is non-degenerate.

At last, we remark that with respect to the holonomic (coordinate) basis $\partial_1^\varphi(u), \dots, \partial_{2n+2m}^\varphi(u)$ on the space $T_u M$ the matrix G of the 2-form Ω_ε is

$$G = \begin{pmatrix} A & 0 \\ 0 & \varepsilon^{-1} C \end{pmatrix}, \tag{5}$$

thus $\det G = \det A \varepsilon^{-2m} \det C = \varepsilon^{-2m} \det C \neq 0$. $\square$

Let U be an open set of M. Denote $Q \in \mathcal{F}^V$, $L \in \mathcal{F}^N$ leaves of the foliations $\mathcal{F}^V, \mathcal{F}^N$. Then components of linear connectivity of sets $Q \cap U$ for $Q \in \mathcal{F}^V$ and $L \cap U$ for $L \in \mathcal{F}^N$ generate on U foliations $\mathcal{F}_U^V$ and $\mathcal{F}_U^N$, respectively. Let us choose some point $u_0 \in M$ and denote $b_0 = p(u_0)$. Due to lemma from [18] (v.1, chap.IV, par. 5), there is a chart $(U, \hat{\psi})$ containing u_0 such that $\hat{\psi}(U) = J^{2n+2m}$, $J \subset \mathbb{R}$, and $\hat{\psi} : U \to J^{2n+2m}$ is an isomorphism of the bi-foliation $(\mathcal{F}_U^V, \mathcal{F}_U^N)$ onto the natural bi-foliation of the direct product $J^{2n+2m} = J^{2n} \times J^{2m}$. Without loss of generality one can assume that $\hat{\psi}(u_0) = 0$ and $J = (-1, 1)$. This implies, in particular, that any leaves $Q \in \mathcal{F}_U^V$ and $L \in \mathcal{F}_U^N$ intersect each other at one point only.

If $\hat{q}_0 : J^{2n} \times J^{2m} \to J^{2n}$ is the natural projection, then for each $Q \in \mathcal{F}_U^V$ the composition $\psi_Q = \hat{q}_0 \circ \hat{\psi}|_Q$ is a homeomorphism of Q on J^{2n} and the pair (Q, ψ_Q) is a chart on Q. Denote Q_0 the leaf of the vertical foliation $\mathcal{F}_U^V$ containing the initial point u_0. We also denote as L_v for $v \in U$ that leaf of the horizontal foliation $\mathcal{F}_U^N$ for which $v \in L_v$. Then setting $q(v) = Q_0 \cap L_v$ we construct the map $q : U \to Q_0$ for which one has $\hat{q}_0 \circ \hat{\psi} = \psi_{Q_0} \circ q$. From this formula it follows immediately that $q : U \to Q_0$ is a submersion.

Let now $W = p(U)$, then W is an open set in the base B. For each $v \in U$ we set

$$\psi(v) = (q \times p_U)(v) = (q(v), p(v)) \tag{6}$$

defining a smooth map $\psi : U \to Q_0 \times W$. Since $q : U \to Q_0$ and $p_U : U \to W$ are both submersions and their leaves are transversal then ψ is a regular map of the manifolds of the same dimension $2n + 2m$. For the map ψ there is the inverse map $\psi^{-1} : Q_0 \times W \to U$ defined by the formula $\psi^{-1}(v_0, b) = p_U^{-1}(b) \cap L_{v_0}$. By the implicit function theorem, this map is differentiable. Thus $\psi : U \to Q_0 \times W$ is a diffeomorphism. Moreover, if $q_0 : Q_0 \times W \to Q_0$ and $p_0 : Q_0 \times W \to W$ are the natural projections and $\psi(v) = (v_0, b)$ then due to (6) we have

$q(v) = v_0 = q_0(v_0, b) = q \circ \psi(v)$ and $p_U(v) = p(v) = b = p_0(v_0, b) = p_0 \circ \psi(v)$. Hence the equalities $q = q_0 \circ \psi$ and $p_U = p_0 \circ \psi$ hold.

Consider a Darboux chart (Q_0', η) of the symplectic manifold $(Q_0, \omega|_{Q_0})$ and a Darboux chart (W', θ) of (B, λ) such that $u_0 \in Q_0' \subset Q_0$ and $b_0 \in W' \subset W$.

To simplify the notations let us regard that $Q_0' = Q_0$ and $W' = W$. For a point $v \in U$ we set

$$\varphi(v) = (\eta \circ q(v), \theta \circ p(v)). \tag{7}$$

Lemma 2. *For any $\varepsilon \neq 0$ the pair (U, φ) is a Darboux chart for the symplectic manifold (M, Ω_ε) and the presymplectic manifold (M, ω).*

Proof. By the construction, $\varphi(U) = \eta(Q_0) \times \theta(W)$ is an open subset of $\mathbb{R}^{2n} \times \mathbb{R}^{2m} = \mathbb{R}^{2n+2m}$ and $\varphi = (\eta \times \theta) \circ \psi$, where $\eta \times \theta : Q_0 \times W \to \eta(Q_0) \times \theta(W)$ is the homeomorphism defined by the formula $(\eta \times \theta)(v_0, b) = (\eta(v_0), \theta(b))$. Thus $\varphi : U \to \eta(Q_0) \times \theta(W)$ is a homeomorphism.

Take any point $v \in U$ and two leaves $Q \in \mathcal{F}_U^V$ and $L \in \mathcal{F}_U^N$ through it. Then the restrictions $q|_Q : Q \to Q_0$ and $p|_L : L \to W$ are diffeomorphisms, and what is more, $q|_{Q_0} = \mathrm{id}_{Q_0}$. Let $q(v) = v_0$ and $b = p(v)$.

The holonomic basis of the tangent space $T_v M$ in the chart (U, φ) will be denoted $\{\partial_\alpha^\varphi(v) | \alpha = 1, \ldots, 2n + 2m\}$, and let $\{\partial_i^\eta(v_0) | i = 1, \ldots, 2n\}$ and $\{\partial_r^\theta(b) | r = 1, \ldots, 2m\}$ be the similar bases of $T_{v_0} Q_0$ and $T_b B$ in charts (Q_0, η) and (W, θ), respectively. Due to (7) we have relations

$$\begin{aligned} &\partial_i^\varphi(v) \in V_v, \quad dq(\partial_i^\varphi(v)) = \partial_i^\varphi(v_0) = \partial_i^\eta(v_0), \qquad \partial_{2n+r}^\varphi(v) \in N_v, \\ &\quad dp(\partial_{2n+r}^\varphi(v)) = \partial_r^\theta(b). \end{aligned} \tag{8}$$

In virtue of (8) we have

$$q^* \Omega_\varepsilon(\partial_i^\varphi(v), \partial_j^\varphi(v)) = \Omega_\varepsilon(\partial_i^\varphi(v_0), \partial_j^\varphi(v_0)) = \omega(\partial_i^\eta(v_0), \partial_j^\eta(v_0)), \tag{9}$$

$$\Omega_\varepsilon(\partial_i^\varphi(v), \partial_{2n+s}^\varphi(v)) = 0, \qquad \Omega_\varepsilon(\partial_{2n+r}^\varphi(v), \partial_{2n+s}^\varphi(v)) = \varepsilon^{-1} \lambda(\partial_r^\theta(b), \partial_s^\theta(b)) \tag{10}$$

for all $i, j = 1, \ldots, 2n$ and $r, s = 1, \ldots, 2m$.

It follows from (9) and (10) that values of Ω_ε take the needed form at all basic vectors from $T_{v_0} M$ and on those basic vectors from $T_v M$ when at least one of its argument is horizontal. Moreover, the form $(q|_Q)^*(\Omega_\varepsilon|_{Q_0})$ takes the canonical form at basic vectors of the space $T_v Q$. To complete the proof it is necessary and sufficient to make sure that the equality $(q|_Q)^*(\Omega_\varepsilon|_{Q_0}) = \Omega_\varepsilon|_Q$ is valid.

To this purpose take a smooth function $h : W \to \mathbb{R}$ and denote X_h its Hamiltonian vector field on the symplectic manifold (W, λ) and let g_h^τ be its Hamiltonian flow on W. We set $H = \varepsilon^{-1} h \circ p|_U$ and define the vector field X_H on U by the formula $\Omega_\varepsilon(Y, X_H) = dH(Y)$. Denote X_h^* the horizontal lift of X_h on U with respect to the connection N. Then for any $v \in U$ and $Y \in T_v U$ we have the identities

$$\begin{aligned} \Omega_\varepsilon(Y, X_H) &= dH(Y) = \varepsilon^{-1} d(h \circ p)(Y) = \varepsilon^{-1} dh(dp(Y)) = \varepsilon^{-1} \lambda(dp(Y), X_h) \\ &= \varepsilon^{-1} \lambda(dp(Y), dp(X_h^*)) = \omega(Y, X_h^*) + \varepsilon^{-1} p^* \lambda(Y, X_h^*) = \Omega_\varepsilon(Y, X_h^*). \end{aligned} \tag{11}$$

This implies $\Omega_\varepsilon(Y, X_H - X_h^*) = 0$ for arbitrary Y and hence $X_H = X_h^*$, since the form Ω_ε is nondegenerate.

Now if $g_h^{*\tau}$ is the horizontal lift of the flow g_h^τ, then its orbits are also orbits of the vector field X_h^*. If g_H^τ is the Hamiltonian flow generated by the field X_H, then we get $g_H^\tau = g_h^{*(\tau)}$ as was said above.

Now let $\theta = (\theta^1, \ldots, \theta^{2m})$ be the coordinate functions on W. Without loss of generality we may assume that $\theta(b_0) = 0$ and $\theta(W)$ is an open cube in $\mathbb{R}^{2m}$. Then Hamiltonian vector fields X_{θ^i} corresponding to functions θ^i generate a local action Φ of group $\mathbb{R}^{2m}$ on W defined by the formula $\Phi(a; t^1, \ldots, t^{2m}) = g_{\theta^{2m}}^{t^{2m}} \circ \cdots \circ g_{\theta^1}^{t^1}(a)$. Due to the choice of W any two its points can be connected by a path composed from coordinate lines. Therefore, a Φ-orbit of any point coincides with W. In particular, if $\theta(b) = x = (x^1, \ldots, x^{2m})$, then setting $t_0^r = -x^{m+r}$ and $t_0^{m+s} = x^s$ for $r, s = 1, \ldots, m$ we get $\Phi(b; t_0^1, \ldots, t_0^{2m}) = b_0 = \theta^{-1}(0, \ldots, 0)$.

Now let $g_{\theta^r}^{*t^r}$ be horizontal lifts of the flows $g_{\theta^r}^{t^r}$, $r = 1, \ldots, 2m$, to the total space U of the bundle $p : U \to W$. The formula

$$\hat{\Phi}(u; t^1, \ldots, t^{2m}) = g_{\theta^{2m}}^{*t^{2m}} \circ \cdots \circ g_{\theta^1}^{*t^1}(u)$$

defines the local action $\hat{\Phi}$ of the group $\mathbb{R}^{2m}$ on U. By the construction one gets $p \circ \hat{\Phi}(u, t^1, \ldots, t^{2m}) = \Phi(p(u), t^1, \ldots, t^{2m})$ that gives $\hat{\Phi}(Q, t_0^1, \ldots, t_0^{2m}) = Q_0$. Since $\hat{\Phi}$ shifts points along the leaves of $\mathcal{F}_U^N$, then $\hat{\Phi}(u, t_0^1, \ldots, t_0^{2m}) = q(u)$ for all $u \in Q$. At last, as was shown above, $g_{\theta^r}^{*t^r}$ are Hamiltonian flows in the symplectic manifold (U, Ω_ε) corresponding to the Hamiltonians $\Theta^r = \varepsilon^{-1}\theta^r \circ p$. Thus, the map $q|_Q$ is the symplectic map between manifolds $(Q, \Omega_\varepsilon|_Q)$ and $(Q_0, \Omega_\varepsilon|_{Q_0})$.

The second assertion follows immediately from the first, since $\omega(X, Y) = \Omega_\varepsilon(X, Y)$ for vertical tangent vectors $X, Y \in T_v M$ and $\omega(X, Y) = 0$ holds, if at least one of the vectors X and Y is horizontal. □

The chart (U, φ) possesses one important feature. In this chart, fibers of the bundle $\xi = (M, p, B)$ are given by equations $\varphi^{2n+r} = c^{2n+r}$, $c^{2n+r} \in \mathbb{R}$, $r = 1, \ldots, 2m$, and local leaves of the horizontal foliation $\mathcal{F}^N$ are given by the equation $\varphi^i = c^i$, $c^i \in \mathbb{R}$, $i = 1, \ldots, 2n$. Hence, the pair (U, φ) can be called a foliated Darboux chart for (M, Ω_ε) and (M, ω).

The separate existence of either a Darboux chart on the symplectic manifold or a Darboux chart on the presymplectic manifold compatible with the bi-foliation structure is obvious. The meaning of Lemma 2 is the fact that for our case in a neighborhood of any point $u_0 \in M$ there exists a chart which possesses both these properties simultaneously.

A vector field X on M is called vertical or horizontal, if the vectors $X(u)$ are such for all points $u \in M$. In the first case X is tangent at each point to leaves of vertical foliation $\mathcal{F}^V$ and in the second case it is tangent to the leaves of the horizontal foliation $\mathcal{F}^N$. Any vector field X on M is decomposed uniquely into a vertical vector field VX and a horizontal vector field NX which satisfy $X = VX + NX$ at every point. The smoothness of X implies the smoothness of both components VX and NX.

Let now $H : M \to \mathbb{R}$ be a smooth function. Then the formula

$$\Omega_\varepsilon(\cdot, X_H) = dH(\cdot), \tag{12}$$

defines the corresponding Hamiltonian vector field X_H on M. At points u of the foliated Darboux chart (U, φ) constructed above (before Lemma 2), this vector field takes the form

$$X_H(u) = a^{ij}\frac{\partial(H \circ \varphi^{-1})}{\partial x^j}(\varphi(u))\partial_i^\varphi(u) + \varepsilon c^{rs}\frac{\partial(H \circ \varphi^{-1})}{\partial x^{2n+s}}(\varphi(u))\partial_{2n+r}^\varphi(u), \tag{13}$$

where a^{ij} and c^{rs} are entries of the constant matrices A^{-1} and C^{-1}.

If in (12) one changes Ω_ε to the presymplectic form ω, then the equation for X_H has infinitely many solutions. But the situation is changed, if one requires that vector fields under consideration are vertical. Namely, the formula

$$\omega(Z, X_H^V) = dH(Z), \tag{14}$$

where Z runs all vertical vector fields on M, defines correctly a vertical vector field X_H^V. It has the following form in the foliated Darboux chart (U, φ)

$$X_H^V(u) = a^{ij} \frac{\partial(H \circ \varphi^{-1})}{\partial x^j}(\varphi(u))\partial_i^{\varphi}(u). \tag{15}$$

It is natural to call the field X_H^V the Hamiltonian vector field corresponding to the function H on the presymplectic manifold (M, ω). In accordance to (13) and (15) one has $X_H^V = VX_H$, i.e. X_H^V is simply the vertical part of X_H.

There is one more treatment of this construction. For any $u \in M$ and $b = p(u)$ let us consider the fiber $Q = p^{-1}(b)$ of the bundle ξ. In accordance to our conditions, the restriction $\omega|_Q$ is a symplectic form on $Q \subset M$. Denote as $h = H|_Q$ the restriction of H on Q and let X_h be the corresponding Hamiltonian vector field to h on $(Q, \omega|_Q)$. Then we conclude that $X_H^V(u) = d(\iota_Q)(X_h(u))$, here $\iota_Q : Q \to M$ is the immersion.

For any two smooth functions F, H on M we define

$$\{H, F\} = dF(X_H), \qquad \{H, F\}^V = dF(X_H^V).$$

This defines Poisson brackets of functions H, F on the symplectic manifold (M, Ω_ε) and the presymplectic manifold (M, ω). On each fiber Q of bundle ξ the equality $\{H, F\}^V = \{h, f\}_Q$ holds where $\{h, f\}_Q$ is the Poisson bracket for restrictions $h = H|_Q$ and $f = F|_Q$ onto the symplectic manifold $(Q, \omega|_Q)$.

Suppose in addition that $h_0, f_0 : B \to \mathbb{R}$ are smooth functions and $\{h_0, f_0\}$ is their Poisson bracket with respect to the symplectic form λ. Then the equality $\{h_0 \circ p, f_0 \circ p\} = \varepsilon\{h_0, f_0\} \circ p$ holds. Thus, at $\varepsilon = 0$ submersion $p : M \to B$ is a Poisson map. Therefore, the bundle $\xi = (M, p, B)$ can be called a Poisson bundle.

3. Symplectic submanifolds transverse to the vertical foliation

Consider now a $2m$-dimensional smooth submanifold $S \subset M$.

Lemma 3. *If S possesses the properties*

- *at each point $v \in S$ the submanifold $S \subset M$ intersects transversely the leaf of the foliation $\mathcal{F}^V$ through v,*
- *S is compact,*

then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the restriction of the 2-form Ω_ε on S is non-degenerate and hence it generates a symplectic form.

Proof. Consider first an arbitrary point $u \in M$ and let $b = p(u)$ be its projection on B. Let (U, φ) be the foliated Darboux chart for a point u of the symplectic manifold (M, Ω_ε) which was constructed before in Lemma 2. When constructing this chart we used a Darboux chart (W, θ) of the manifold (B, λ), a neighborhood Q_0 of the point u in the fiber $p^{-1}(b)$ and the diffeomorphism $\psi : U \to Q_0 \times W$.

On W, the holonomic vector fields $Y_r = \partial_r^\theta$, $r = 1, \ldots, 2m$ are given. At any point of W matrix $(\lambda(Y_r, Y_s))$, $r, s = 1, \ldots, 2m$, has canonical form. Therefore one has

$$\det(\lambda(Y_r, Y_s)) \equiv 1. \tag{16}$$

In accordance with the first condition of the lemma, for any point $v \in S \cap U$ one has $T_v M = V_v \oplus T_v S$. Consequently, the restriction $Dp|_{T_v S} : T_v S \to T_{p(v)} B$ is an isomorphism. Setting $Y_r^*(v) = (Dp|_{T_v S})^{-1}(Y_r(p(v)))$ for all $v \in S \cap U$ and $r = 1, \ldots, 2m$, we get smooth vector fields $Y_1^*, \ldots, Y_{2m}^*$ on $S \cap U_u$. At any point $v \in S \cap U$ the vectors $Y_1^*(v), \ldots, Y_{2m}^*(v)$ make up a basis of the tangent space $T_v S$.

Let us denote $\Omega^\varepsilon = \varepsilon\, \Omega_\varepsilon$. Then, due to (4) one has

$$\Omega^\varepsilon(Y_r^*, Y_s^*) = \varepsilon\omega(Y_r^*, Y_s^*) + \lambda(Y_r, Y_s)$$

for all $r, s = 1, \ldots, 2m$. Thus for the matrix $D = (\Omega^\varepsilon(Y_r^*, Y_s^*))$ we have

$$\det D = f_{2m}\varepsilon^{2m} + \cdots + f_1\varepsilon + f_0, \tag{17}$$

where for any $t = 0, 1, \ldots, 2m$ the coefficient f_t is the sum of all determinants for which t rows coincide with the corresponding rows of the matrix $(\omega(Y_r^*, Y_s^*))$, but other rows belong to the matrix $(\lambda(Y_r, Y_s))$. It follows from here that $f_t : S \cap U \to \mathbb{R}$ are smooth functions and, due to identity (16), one has

$$f_0 = \det(\lambda(Y_r, Y_s)) \equiv 1. \tag{18}$$

Since B, $p^{-1}(b)$ are manifolds then for points b and u there are neighborhoods W^0 and Q_0^0 whose closures are compact and belong to W and Q_0, respectively. Moreover, the set $U_u^0 = \psi^{-1}(W^0 \times Q_0^0)$ is a neighborhood of the point u and its closure is also compact and belongs to U. In this case functions f_t are bounded on $S \cap U_u^0$. Hence, there is $\varepsilon_u > 0$ such that

$$|f_{2m}\varepsilon^{2m} + \cdots + f_1\varepsilon| < 1 \tag{19}$$

on $S \cap U_u^0$ for any $\varepsilon \in (0, \varepsilon_u)$.

A collection $\mathcal{U} = \{U_u^0 | u \in S\}$ is a covering for the manifold S by open sets. Since S is compact we conclude that $S \subset U_{u_1}^0 \cup \cdots \cup U_{u_l}^0$ for some finite set of points $u_1, \ldots, u_l \in S$. Let us set $\varepsilon_0 = \min\{\varepsilon_{u_1}, \ldots, \varepsilon_{u_l}\}$. Then for any $\varepsilon \in (0, \varepsilon_0)$ inequality (19) is valid on every set from $S \cap U_{u_1}^0, \ldots, S \cap U_{u_l}^0$.

If now v is any point of S, there exists an $i \in \{1, \ldots, l\}$ such that $v \in S \cap U_{u_i}^0$. Here for any $\varepsilon \in (0, \varepsilon_0)$ it follows from (17), (18) and (19) that $\det D \neq 0$. This implies that the 2-form Ω^ε is non-degenerate on the tangent space $T_v S$. Thus, on $T_v S$ the 2-form $\Omega_\varepsilon = \varepsilon^{-1}\Omega^\varepsilon$ is also nondegenerate. □

4. Slow manifold and nearby orbit behavior

Henceforth we assume a bundle $p : M \to B$ is given where M is a C^∞-smooth presymplectic manifold with a constant-rank 2-form ω, B is a smooth symplectic manifold with a symplectic 2-form λ. The ω is supposed to be compatible with p, hence its fibers define a symplectic foliation of M. Suppose a smooth function H on M is given. Thus H generates at $\varepsilon = 0$ for any $b \in B$ a vertical (fast) Hamiltonian vector field X_H^b. The union in b of all zeros for such vertical vector fields for a given function H forms a subset in M which is generically a smooth submanifold SM of dimension $2m = \dim B$ (the genericity here means that it is true for a residual set of functions H). We assume that is the case and shall call SM the *slow manifold* of the vector field X_H. When restricted on SM the corresponding map $p_r : SM \to B$ may be regular or singular at the points of SM. A point $s \in SM$ is called *regular* if $\operatorname{rank} Dp_r(s) = 2m = \dim B$. This implies p_r be a diffeomorphism near s. On the contrary, a point $s \in SM$ is *singular*, if rank Dp_r at s is less than $2m$.

Another characterization of regular and singular points is related to the type of the corresponding equilibria for the fast Hamiltonian system on the symplectic leaf through s. The point s is regular, if the fast vector field has a simple equilibrium for the fast Hamiltonian vector field, i.e. on the corresponding leaf of the symplectic foliation the equilibrium has no zero eigenvalues. For a singular point $s \in SM$ the equilibrium on the symplectic leaf through s is degenerate for the fast Hamiltonian vector field, that is it does have a zero eigenvalue. All this will be seen below in local coordinates, though it is possible to show it in a coordinate-free way.

4.1. A neighborhood of a regular point of SM

The manifold SM near a regular point s can be represented by the implicit function theorem as the graph of a smooth section $z : U \to M$, $p(s) \in U \subset B$, $p \circ z = id_U$. Due to Lemma 3, such compact piece of SM is a symplectic submanifold with respect to the restriction of the 2-form Ω_ε to SM. Hence one can define a slow Hamiltonian vector field on SM generated by the function H. Let X_H be the Hamiltonian vector field on M with respect to the 2-form Ω_ε generated by H.

Denote by H^S the restriction of H to SM and consider a Hamiltonian vector field on SM with the Hamiltonian H^S with respect to the restriction of 2-form Ω_ε to SM. This vector field X^S is of the order ε, hence there is the limit X^S/ε as $\varepsilon \to 0$. This limit vector field is what we call the *slow Hamiltonian vector field* on SM. In the same way one can consider the case when function H on M depends smoothly on a parameter ε.

It is an interesting problem to understand the orbit behavior of the full system (for small $\varepsilon > 0$) within a small neighborhood of a compact piece of regular points in SM. This question is very hard in the general set-up. Nevertheless, there is a rather simple important case to examine, if one assumes the hyperbolicity of this piece of SM. Suppose for a piece of SM each corresponding equilibrium (for $\varepsilon = 0$) is without zero real parts (hyperbolic equilibria in the common terminology, see, for instance, [26,21]). Then this smooth submanifold of the vector field X_H on M is a normally hyperbolic invariant manifold and results of [10,16] are applicable. Namely, for $\varepsilon > 0$ small enough there is a smooth invariant manifold in an $O(\varepsilon)$-neighborhood of that piece of SM. For the full system this invariant manifold is normally hyperbolic and possesses stable and unstable local smooth invariant manifolds. The restriction of X_{H_ε} to this slow manifold can be an arbitrary Hamiltonian system with m degrees of freedom.

One can add to this local picture the structure of the global stable/unstable manifolds of fast systems along with their bifurcations w.r.t. slow variables (parameters of the fast system), then one can say a lot on the behavior of the full system for positive small ε. This behavior is the topics of the averaging theory, theory of adiabatic invariants, etc., see, for instance, [25,22,13].

A much more subtle problem is to understand the local dynamics of the full system near SM when the fast dynamics possesses center equilibria at the points of SM. Nonetheless, one can present some details of this picture when we deal with a real analytic case (manifolds and Hamiltonian). Then results of [11] can be applied. For this case SM was called in [11] an *almost invariant elliptic* slow manifold. At $\varepsilon = 0$ near a piece of the almost elliptic slow manifold one can introduce a coordinate frame where this slow manifold corresponds to the zero section of the bundle $M \to B$. Then the main result of [11] is applicable for the case when the fast system is two dimensional but the slow system can be of any finite dimension and Hamiltonian is analytic in a neighborhood of SM. The result says that the Hamiltonian near SM can be transformed by an analytic transformation to the sum of two functions. One of them contains fast variables (x, y) only in the combination $I = (x^2+y^2)/2$. The second function is exponentially small with respect to the small parameter ε. Thus, up to an exponentially small error, the system has an additional integral I. In particular, if the slow system is also two dimensional, this gives an integrable system up to exponentially small error within a small neighborhood of SM. All this helps a lot when one is interested in the dynamics within this neighborhood, see, for instance, [12]. The case, when fast system has more degrees of freedom and the related equilibria are multi-dimensional elliptic ones, is harder and no results are known to this date. The case of fast equilibria with eigenvalues in the complex plane lying both on the imaginary axis and out of it is even less explored.

4.2. Neighborhood of a singular point of SM

At a singular point $s \in SM$ submanifold SM is tangent to a leaf of the symplectic foliation. More precisely, the rank of Dp_r on the tangent plane T_sSM is smaller than $2m$. This means $KerDp|_{SM} \neq \emptyset$. Let us choose some Darboux coordinate chart (x, y, u, v) near s, then the presymplectic 2-form is written as $\omega = dx \wedge dy$ and the symplectic leaves of the foliation are given as $(u, v) = (u_0, v_0)$. Hamiltonian $H(x, y, u, v)$ is a smooth function of these coordinates, we assume $dH \neq 0$ at s. The Hamiltonian vector field near s with respect to the 2-form ω is written as

$$\dot{x} = H_y, \qquad \dot{y} = -H_x, \qquad \dot{u} = 0, \qquad \dot{v} = 0.$$

The condition for the point s to be on SM (a singular point for the fast vector field) is $H_y(s) = H_x(s) = 0$ and if this point is a singular point for the projection p, then

$$\det \left(\frac{\partial^2 H}{\partial (x, y)^2} \right) \bigg|_s = 0,$$

otherwise the point s in SM would be regular and the system $H_y = 0$, $H_x = 0$ could be resolved with respect to x, y. Thus, the fast vector field at s has a zero eigenvalue. Its eigenspace is invariant w.r.t. the linearization of the fast vector field at s. This is equivalent to the condition that Dp restricted on T_sSM has nonzero kernel and hence its rank is less than $2m$.

The types of degeneracies for the mappings of one smooth manifold to another (for our case it is $p_r : SM \to B$) are studied by the singularity theory of smooth manifolds [28,4,24]. When the dimensions of B and M are large, these degenerations can be very complicated. Keeping this in mind, we consider below only the simplest case of one fast and one slow degree of freedom. For this case both SM and B have dimension two and we have the mapping from one two dimensional smooth manifold to another smooth two dimensional manifold. We need to distinguish singular points of the general type that are possible for such smooth maps. Here the degeneracies can be generically of two types only: folds and cusps, this was done by Whitney [28].

Recall that according to [28], a singular point q of a C^2-smooth map $F : U \to V$ of two open domains in smooth 2-dimensional manifolds is *good*, if the function $J = \det DF$ vanishes at q but its differential dJ is nondegenerate at this point. In a neighborhood of a good singular point q there is a smooth curve of other singular points for F continuing q. Let $\varphi(\tau)$ be a smoothly parameterized curve of singular points through a good singular point q and $\tau = 0$ corresponds to q. We shall use below the notation $A^\top$ for the transpose matrix of any matrix A.

A good singular point q is called the *fold* point [28], if $dF(\varphi'(0)) \neq (0, 0)^\top$, and it is called the *cusp* point if at q one has $dF(\varphi'(0)) = (0, 0)^\top$ but $d^2(F \circ \varphi)/d\tau^2|_q \neq (0, 0)^\top$. It is worth remarking that if q is a fold, then for any nearby point on the singular curve $\varphi(\tau)$ there is a unique (up to a constant) nonzero vector ξ in the tangent space of the corresponding point such that $DF(\xi) = 0$ (ξ belongs to the kernel of DF). The direction spanned by ξ is transverse to

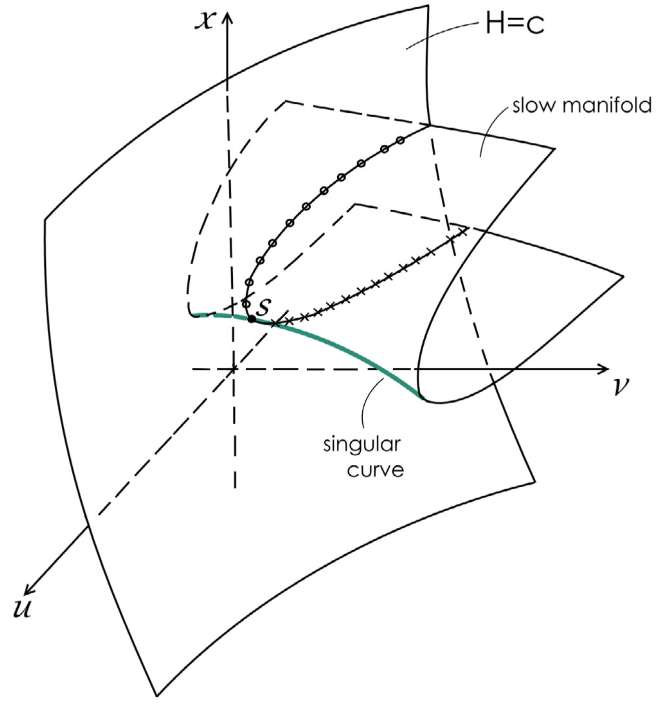

Fig. 1. Slow manifold near fold points.

the tangent direction to the singular curve at the fold point, since $DF(\varphi'(0)) \neq (0, 0)^\top$. These transverse directions form a smooth transverse direction field on the curve.

We shall show below that near a fold point at small $\varepsilon > 0$ the system can be reduced to the case of a family of slowly varying Hamiltonian systems. In such system the Hamiltonians $H(x, y, \varepsilon t, c)$ depend on scalar variables x, y and positive small parameter ε; c is the value of the former autonomous Hamiltonian and also a small parameter. For a fixed c the slow manifold of this system is a slow curve and the fold point corresponds to the point of quadratic tangency of the slow curve with the corresponding two-dimensional symplectic leaf. Here the fast (frozen) Hamiltonian system with one degree of freedom has the equilibrium, corresponding to the tangency point, which is generically a parabolic equilibrium point with the double non-semisimple zero eigenvalue. The local orbit behavior for such fast system does not change as c varies.

The shape of the slow manifold near a fold point in the subspace $y = 0$ when H is written in the form $H = h(x, u, v)+y^2 H_1(x, y, u, v)$ (see Lemma 5) is presented in Fig. 1. The intersection with a level $H = c$ is the slow curve. Small circles on the slow manifold correspond to the elliptic equilibria of the fast systems, small crosses denote the saddle equilibria. The parabolic equilibria lie on the slow curve. The projection of the slow curve on the plane (sometimes it is called the discriminant curve) of slow variables (u, v) is a smooth segment here.

For the case of a cusp we get again a slow–fast Hamiltonian system with two dimensional slow manifold. The fast Hamiltonian system has on the related symplectic leaf an equilibrium of the type of degenerate saddle or degenerate elliptic point (both are of codimension 2). In this case it is also possible to reduce the system to the case of a family of slowly varying nonautonomous

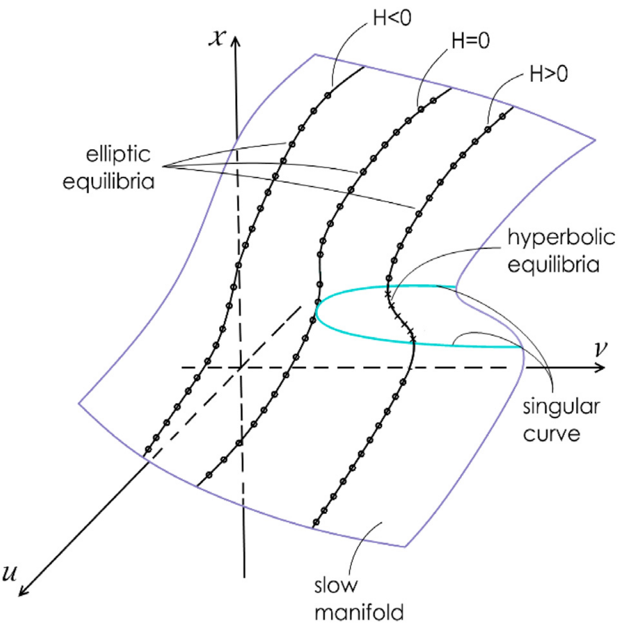

Fig. 2. Slow manifold near a cusp point.

Hamiltonian systems on the corresponding levels $H = c$. But in contrast to the case of a fold, the behavior of these systems near the cusp point s essentially depends on the parameter c.

The shape of the slow manifold near a cusp point in the space $y = 0$ when H is written in the form $H = h(x, u, v) + y^2 H_1(x, y, u, v)$ (see again Lemma 5) is presented in Fig. 2. The intersection with a level $H = c$ is the slow curve. Again small circles on the slow manifold correspond to the elliptic equilibria of the fast systems, small crosses denote the saddle equilibria. The parabolic equilibria lie on the slow curve. The projection of the slow curve on the plane of slow variables (u, v) (the discriminant curve) is a curve like the semi-cubic parabola with a plane cusp at the projection of the cusp point on SM.

Henceforth, we consider only the case of one slow and one fast degree of freedom Hamiltonian systems, that is M will be a smooth 4-dimensional presymplectic manifold with a 2-form ω of rank 2 and B will be a smooth symplectic 2-dimensional manifold with a symplectic 2-form λ, the form ω is compatible with the smooth bundle $p : M \to B$ whose leaves F_b generate the symplectic foliation with respect to ω. The symplectic structure on M is given by the 2-form $\Omega_\varepsilon = \omega + \varepsilon^{-1} p^* \lambda$.

5. Folds for the slow manifold projection

Let a smooth Hamilton function H on M be given. We suppose H is non-degenerate in a neighborhood of a point s where we are working: $dH \neq 0$. Then levels $H = c$ are smooth 3-dimensional disks within this neighborhood. Since the consideration is local, we can work in Darboux coordinates, hence it is supposed that the 2-form Ω_ε is written as $\Omega_\varepsilon = dx \wedge dy + \varepsilon^{-1} du \wedge dv$ with fast variables x, y and slow variables u, v. The corresponding presymplectic manifold is endowed locally with 2-form $\omega = dx \wedge dy$, its symplectic leaves with respect to the bundle map $p : (x, y, u, v) \to (u, v)$ are given by $(u, v) = (u_0, v_0) \in U$, $U \subset B$

is a disk with coordinates (u, v). Without loss of generality we assume that the origin of the coordinate frame $(0, 0, 0, 0)$ is an equilibrium of the fast system on the leaf $(0, 0)$: $H_y = H_x = 0$. The restriction of H to a symplectic leaf is the function $H(x, y, u_0, v_0)$ and the orbit foliation for the fast vector field on this leaf is given by the fast Hamiltonian vector field

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}, \qquad u_0, v_0 \text{ are parameters},$$

depends on parameters (u_0, v_0) and its unfolding, as (u_0, v_0) vary, gives a local piece of the slow manifold *SM* near the point $s = (0, 0, 0, 0)$ in the space (x, y, u, v). Locally near s the slow manifold is indeed a smooth 2-dimensional disk, if rank of the matrix

$$\begin{pmatrix} H_{xy} & H_{yy} & H_{uy} & H_{vy} \\ H_{xx} & H_{yx} & H_{ux} & H_{vx} \end{pmatrix} \tag{20}$$

is 2 at s. We suppose it is the case. Now one can consider the restriction of the projection map p to *SM*: $p_r : SM \to B$. If the inequality $\Delta = H_{xy}^2 - H_{yy}H_{xx} \neq 0$ holds at s, then by the implicit function theorem the set of solutions for the system $H_y = 0,\ H_x = 0$ near s is expressed as $x = f(u, v),\ y = g(u, v)$. Hence locally it is a section of the bundle $p : M \to B$ and Dp_r does not degenerate on this set in some neighborhood of s, i.e. p_r is a diffeomorphism. Thus, the degeneracy happens only if $\Delta(s) = 0$. This equality is equivalent to the condition that the fast Hamiltonian vector field on the corresponding symplectic leaf $F_b,\ b = p(s)$, has at the point s a degenerate equilibrium: it possesses zero eigenvalue (by the Hamiltonian structure, it is double) for the linearization at s. Another characterization of such point is that it is a *singular* point of the mapping p_r: the rank of this mapping at s is less than 2.

Now let us return to the set of points in *SM* near s where p_r degenerates. To be precise, we assume

$$H_{yy}H_{ux} - H_{xy}H_{uy} \neq 0 \tag{21}$$

at s. We can always assume this is the case otherwise one can achieve this by re-ordering slow or fast variables. The unique case when it is impossible and the rank of matrix (20) equals 2 corresponds to $H_{yu}H_{vx} - H_{xu}H_{yv}$ is the only nonzero minor, while the other five are zero. This would indicate a too degenerate case and we do not consider it below. Indeed, the following lemma is valid.

Lemma 4. *Suppose matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

possesses the properties:

- $$\det \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} \neq 0,$$
- *all the other minors of the second order for the matrix A vanish.*

Then the following equalities hold $a_{11} = a_{12} = a_{21} = a_{22} = 0$.

Proof. Suppose the assertion of the lemma is false. Then up to re-enumeration of the rows and the first two columns one may regard $a_{11} \neq 0$. By assumption, all minors of the second order for the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

vanish. This means that its rows are linear dependent. But by assumption the first row is nonzero vector of $\mathbb{R}^3$. Hence, there is $\kappa_1 \in \mathbb{R}$ such that $(a_{21}, a_{22}, a_{23}) = \kappa_1(a_{11}, a_{12}, a_{13})$.

Analogously, if all second order minors of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \end{pmatrix}$$

vanish, then from $a_{11} \neq 0$ follows the existence of a number $\kappa_2 \in \mathbb{R}$ which satisfies the equality $(a_{21}, a_{22}, a_{24}) = \kappa_2(a_{11}, a_{12}, a_{14})$. Then one has $\kappa_1 a_{11} = a_{21} = \kappa_2 a_{11}$, from here equalities $a_{11}(\kappa_1 - \kappa_2) = 0$ and $\kappa_1 = \kappa_2$ follow. Thus, the rows of the matrix A are linear dependent, which contradicts to the first condition of the lemma. □

It follows from the inequality (21) that SM near s is represented as a graph of the mapping $y = f(x, v),\ u = g(x, v)$ (recall that we assume $s = (0, 0, 0, 0)$, then $f(0, 0) = g(0, 0) = 0$). The coordinate representation for the mapping p_r is the following $p_r : (x, v) \to (u = g(x, v), v)$. The derivative of this mapping is the matrix

$$P = Dp_r = \begin{pmatrix} g_x & g_v \\ 0 & 1 \end{pmatrix}, \tag{22}$$

whose rank at $(0, 0)$ is 2, if $g_x(0, 0) \neq 0$ (the point is regular), and it is 1, if $g_x(0, 0) = 0$ (the point is singular). The singular point $(0, 0)$ is good, if $g_x(0, 0) = 0$ and $g_{xx}(0, 0) \neq 0$ or $g_{xv}(0, 0) \neq 0$. The point is a fold if, in addition, one has $P(\xi) \neq (0, 0)^\top$, where ξ is the tangent vector to the singular curve through $(0, 0)$. When $g_{xx}(0, 0) \neq 0$, then the equation $g_x(x, v) = 0$ for the singular points is solved by the implicit function theorem and the singular curve has a representation $(l(v), v),\ l(0) = 0$, so ξ is $(l'(0), 1)^\top$. Thus one has $P(\xi) = (g_v(0, 0), 1)^\top \neq (0, 0)^\top$ and the singular point $(0, 0)$ is indeed the fold.

Now suppose $g_x(0, 0) = g_{xx}(0, 0) = 0$ but $g_{xv}(0, 0) \neq 0$. Then the singular curve has a representation $(x, r(x)),\ r(0) = 0$, and the vector ξ is $(1, r'(0))^\top$. Because $r(x)$ again solves the equation $g_x(x, v) = 0$, we have the equality

$$r'(0) = -\frac{g_{xx}(0, 0)}{g_{xv}(0, 0)} = 0.$$

It follows from here that $P(\xi) = (0, 0)^\top$. Hence if $g_{xx}(0, 0) = 0$ then singular point $(0, 0)$ is not a fold. In order to verify it is the cusp, one needs to calculate $d^2(g(x, r(x)), r(x))/dx^2$ at the point $(0, 0)$. The calculation gives the vector $(g_v(0, 0)r''(0), r''(0))$, due to the equalities $g_{xx}(0, 0) = 0,\ r'(0) = 0$. Thus, if $r''(0) \neq 0$, the second derivative is not zero vector, and the point is the cusp. For the derivative $r''(0)$ we have

$$r''(0) = -\frac{g_{xxx}(0, 0)}{g_{xv}(0, 0)}. \tag{23}$$

Thus, the conditions for a singular point to be a cusp consist in two equalities and two inequalities

$$g_x(0, 0) = g_{xx}(0, 0) = 0, \qquad g_{xv}(0, 0) \neq 0 \qquad g_{xxx}(0, 0) \neq 0. \tag{24}$$

The mutual position of the singular curve and levels $H = c$ is important. In particular, we need to know when the intersection at s of this curve and the submanifold $H = H(s)$ is transverse and when they are tangent. In the transverse case for all c close enough to $H(s)$ the intersection of this curve with the level $H = c$ will be also transversal and hence, will consist of one point. For the nontransverse case we need to know what will happen for the close levels. In a sense, the transversality condition can be considered as some genericity condition for the chosen function H.

If the point s is the fold point for the map $p_r : SM \to B$, the singular curve on SM near s is expressed in the form $x = l(v)$, $l(0) = 0$. To determine, if this curve intersects transversely the level $H = H(s)$, let us calculate the derivative

$$\frac{d}{dv} H(l(v), f(l(v), v), g(l(v), v), v)|_{v=0} = H_u(0,0,0,0)g_v(0,0) + H_v(0,0,0,0). \quad (25)$$

Taking into account that $f(x, v)$, $g(x, v)$ are solutions of the system $H_y = 0,\ H_x = 0$ near point s, we can calculate $g_v(0, 0)$. This gives that the derivative (25) at the point s does not vanish if

$$H_{xy}[H_u H_{yv} - H_v H_{yu}] - H_{yy}[H_u H_{xv} - H_v H_{xu}] \neq 0. \quad (26)$$

It is hard to calculate this quantity when H is taken in a general form. Therefore we transform H near s to a more tractable form. In order not to care about the smoothness, we assume henceforth all functions are C^∞.

The following assertion is valid.

Lemma 5. *Suppose a smooth function $H(x, y, u, v)$ is given such that the Hamiltonian system*

$$\dot{x} = H_y, \qquad \dot{y} = -H_x$$

depending on two parameters (u, v) has at $(u, v) = (0, 0)$ a degenerate equilibrium $(x, y) = (0, 0)$ with double zero non-semisimple eigenvalue (the related Jordan form is two dimensional). Then there exists a C^∞-smooth transformation $\Phi : (x, y) \to (X, Y)$ smoothly depending on parameters (u, v) and respecting the 2-form $dx \wedge dy$ such that the Hamiltonian $H \circ \Phi$ in the new variables (X, Y) takes the form

$$H(X, Y, u, v) = h(X, u, v) + H_1(X, Y, u, v)Y^2, \quad (27)$$

where $H_1(0, 0, 0, 0) \neq 0$.

Proof. We act as follows. Consider first the Hessian matrix (a_{ij}) of H in variables (x, y) at the point $(x, y) = (0, 0)$ on the leaf $(u, v) = (0, 0)$. Its determinant vanishes but not all its entries are zeros and its rank is 1, since the case is non-semisimple. Then one has either $a_{11} \neq 0$ or $a_{22} \neq 0$ due to symmetry of the Hessian. We assume $a_{22} \neq 0$, this is compatible with the assumption that just the minor $H_{yy}H_{xu} - H_{xy}H_{yu}$ is nonzero. Let us first solve the equation $H_y = 0$ in a neighborhood of the point $(0, 0, 0, 0)$. Due to the implicit function theorem and the assumption $a_{22} = H_{yy}(0, 0, 0, 0) \neq 0$, the equation has a solution $y = f(x, u, v)$, $f(0, 0, 0) = 0$. After the shift transformation $x = X$, $y = Y + f(x, u, v)$ we get the transformed Hamiltonian $\hat{H}(X, Y, u, v)$ of the form

$$\hat{H}(X, Y, u, v) = h(X, u, v) + Y\tilde{H}(X, Y, u, v),$$

where $h(X, u, v) = H(x, f(x, u, v), u, v)$, and

$$\tilde{H}(X, 0, u, v) = H_y(x, f(x, u, v), u, v) \equiv 0.$$

Thus the function $\tilde{H}$ can be also represented as $\tilde{H} = Y\bar{H}$, $\bar{H}(0,0,0,0) = H_{yy}(0,0,0,0)/2 \neq 0$. It is worth noticing that (u, v) are considered here as parameters, therefore the transformation $(x, y) \to (X, Y)$ respects the 2-form: $dx \wedge dy = dX \wedge dY$. □

We now restore the notations (x, y) and assume that H is in the form (27). Then the condition for $(0, 0, 0, 0)$ to be the equilibrium of the fast system leads to the equality $h_x(0, 0, 0) = 0$. The requirement that the equilibrium is degenerate and non-semisimple is $h_{xx}(0, 0, 0) = 0$, due to the inequality $H_1(0, 0, 0, 0) \neq 0$. In this case the requirement that the slow manifold is a smooth solution of the equation $h_x(x, u, v) = 0$ near the point $(0, 0, 0, 0)$ will be assured by one of the inequalities $h_{xu}(0, 0, 0) \neq 0$ or $h_{xv}(0, 0, 0) \neq 0$. One can suppose that the former holds renaming, if necessary, slow variables. This implies that the slow manifold has a representation $y = 0, u = g(x, v)$, where $g(0, 0) = 0$, $g_x(0, 0) = 0$. At last, the point $(0, 0, 0, 0)$ on the slow manifold will be a fold, if $g_{xx}(0, 0) \neq 0$, that is equivalent to the inequality $h_{xxx}(0, 0, 0) \neq 0$.

Now we expand $h(x, u, v)$ in x up to the third order terms

$$\begin{aligned} H(x, y, u, v) &= h(x, u, v) + H_1(x, y, u, v)y^2 \\ &= h_0(u, v) + a_1(u, v)x + a_2(u, v)x^2 + a_3(u, v)x^3 + O(x^4) + H_1 y^2. \end{aligned}$$

Here one has $a_1(0, 0) = 0$, $\partial_u a_1(0, 0) \neq 0$, $a_2(0, 0) = 0$, $a_3(0, 0) \neq 0$. We have some freedom to change parameters (u, v). Using inequality $a_u(0, 0) \neq 0$, we introduce a new parameter $u_1 = a_1(u, v)$. In order to preserve the 2-form $du \wedge dv$ we need to introduce also a new parameter v_1. To that end, we express $u = \hat{a}_1(u_1, v) = R_v$ via a generating function $R(u_1, v)$, where for $|u_1|, |v|$ small enough

$$R(u_1, v) = \int_0^v \hat{a}_1(u_1, z)dz, \qquad \frac{\partial^2 R}{\partial u_1 \partial v} = \frac{\partial \hat{a}_1}{\partial u_1} \neq 0.$$

Then one has $v_1 = R_{u_1}$ and $du \wedge dv = du_1 \wedge dv_1$. After this transformation which does not touch variables x, y we come to the following form of H

$$H(x, y, u_1, v_1) = h_0(u_1, v_1) + u_1 x + \hat{b}(u_1, v_1)x^2 + \hat{c}(u_1, v_1)x^3 + O(x^4) + \hat{H}_1 y^2. \quad (28)$$

In this form we can check the transversality of the singular curve on *SM* and the submanifold $H = h_0(0, 0) = c_0$ at the point $(0, 0, 0, 0)$. Since we have $H^0_{xy} = 0$, $H^0_{yy} \neq 0$ (zeroth upper index means the functions are computed at the point $(0, 0, 0, 0)$), then the inequality (26) casts (we restored the notation u, v again) as

$$H^0_u H^0_{xv} - H^0_v H^0_{xu} \neq 0,$$

that is expressed as follows

$$\frac{\partial h_0}{\partial v}(0, 0) \neq 0. \quad (29)$$

Thus, we come to the conclusion:

if a function H is generic and the singular point s on SM is a fold for the mapping p_r, this is equivalent to the condition that this point on the related symplectic leaf is parabolic and the unfolding of H in parameters (u, v) is generic.

At the next step we want to reduce the dimension of the system near the fold point $s \in SM$ and get a smooth family of nonautonomous Hamiltonian systems in one degree of freedom. This will allow us to describe the principal part of the system near singularity using some rescaling for the system near s. We intend to work in coordinates where H takes the form (27). This form of H was obtained above by the coordinate change respecting the form ω but in order to use it for $\varepsilon > 0$ we need to perform a symplectic transformation with respect to the form Ω_ε such that the transformation reduces to the previous form at $\varepsilon = 0$. This is done in the following lemma.

Lemma 6. *Suppose a smooth function $H(x, y, u, v)$ is given such that the point $(x, y, u, v) = (0, 0, 0, 0)$ is a solution of the system $H_y = 0,\ H_x = 0$ and at this point one has*

$$\Delta = H_{xx}H_{yy} - H_{xy}^2 = 0, \qquad \Delta_1 = H_{yy}H_{xu} - H_{xy}H_{yu} \neq 0.$$

Then there exists a C^∞-smooth transformation $\Phi : (x, y, u, v) \to (X, Y, U, V)$ smoothly depending on the parameter ε and respecting the 2-form $\Omega_\varepsilon = dx \wedge dy + \varepsilon^{-1} du \wedge dv$ such that in the new variables (X, Y, U, V) for all ε small enough the Hamiltonian $H \circ \Phi$ takes the form

$$H(X, Y, U, V, \varepsilon) = h(X, U, V, \varepsilon) + h_1(X, U, V, \varepsilon)Y + H_1(X, Y, U, V, \varepsilon)Y^2, \tag{30}$$

where $h(X, U, V, 0) = h_0(X, U, V)$, $h_1(X, U, V, 0) \equiv 0$ and $H_1(0, 0, 0, 0, 0) \neq 0$.

Proof. We need to extend the transformation $X = x,\ Y = y - f(x, u, v)$ till the symplectic transformation which coincides with that in Lemma 5. This is achieved through a symplectic transformation generated by a generating function $S(x, Y, u, V, \varepsilon) = xY + uV/\varepsilon + S_1(x, u, V, \varepsilon)$ like in [11]

$$X = x, \qquad y = Y + \frac{\partial S_1}{\partial x}, \qquad U = u + \varepsilon\frac{\partial S_1}{\partial V}, \qquad v = V + \varepsilon\frac{\partial S_1}{\partial u}.$$

We take S_1 in the form $S_1(x, u, V) = \int_0^x f(\xi, u, V)$. Then, despite the singular nature of the generation function, the transformation is regular and at the limit $\varepsilon = 0$ we get the previous transformation, since $V = v$ in this case.

After the transformation the Hamiltonian casts in the form (30) after the expansion in Y up to the second order terms using the Hadamard lemma

$$H(X, Y, U, V, \varepsilon) = h(X, U, V, \varepsilon) + h_1(X, U, V, \varepsilon)Y + H_1(X, Y, U, V, \varepsilon)Y^2,$$

where $h(X, U, V, 0) = h_0(X, U, V)$ and $h_1(X, U, V, 0) \equiv 0$. $\square$

Now we use the old notations (x, y, u, v). In Darboux coordinates (x, y, u, v) near s the slow–fast Hamiltonian system with Hamiltonian H is written as follows

$$\dot{x} = H_y, \qquad \dot{y} = -H_x, \qquad \dot{u} = \varepsilon H_v, \qquad \dot{v} = -\varepsilon H_u. \tag{31}$$

Without loss of generality, one can assume $H(s) = 0$. Since $H_v(s) \neq 0$ (see (29)), then near s the levels $H = c$ for c close to zero are given as the graphs of the function $v = S(x, y, u, c)$, where $S(0, 0, 0, 0) = 0$, $S_c = 1/H_v \neq 0$. These graphs intersect transversely the singular curve near s, thus the intersection happens at the only point on the corresponding graph. The intersection of SM with a level $H = c$ is a smooth curve (the slow curve on this level) with a unique tangency point with the related leaf $(u, v) = (u_c, v_c)$.

Let us perform the isoenergetic reduction of the system (31) on the level $H = c$, then S is a new (nonautonomous) Hamiltonian and u is the new "time" [1]. After the reduction the system transforms to the system in variables (x, y, u) that reads as follows

$$\varepsilon\frac{dx}{du} = H_y/H_v = -S_y, \qquad \varepsilon\frac{dy}{du} = -H_x/H_v = S_x. \tag{32}$$

If we introduce the fast time setting $du/d\tau = \varepsilon$, we come to the autonomous 3-dimensional system which can be also considered as nonautonomous Hamiltonian one depending on the slow time through $u = \varepsilon\tau + u_0$.

The slow curve for this system is composed by the solutions of the system $S_y = 0,\ S_x = 0$, this system is equivalent to the system $H_y = 0,\ H_x = 0$ where one needs to plug S into H instead of v. Thus we obtain the intersection of SM with the level $H = c$. Because of form (27) for the function H at $\varepsilon = 0$, this curve has a representation in coordinates (x, y, u): $y = 0,\ u - u_c = a(c)(x - x_c)^2 + o((x - x_c)^2),\ a(0) \neq 0$, here $(x_c, 0, u_c, v_c), v_c = S(x_c, 0, u_c, c)$, are coordinates of the trace of the singular curve in the level $H = c$ for c close to $c = 0$. This representation follows from the system $u = g(x, v),\ v = S(x, 0, u, c)$, and by the implicit function theorem, u is expressed via x, since $1 - g_v S_u = (H_u g_v + H_v)/H_v \neq 0$ at the point $(x, y, u, c) = (0, 0, 0, 0)$ (and any point on the singular curve close to it).

The fast system for the reduced system obtained is given by setting $\varepsilon = 0$ in the autonomous 3-dimensional system. Then variable u becomes a parameter $u = u_0$ and u_0 is varied near u_c. On the leaf $u_0 = u_c$ we get a one degree of freedom Hamiltonian system with the equilibrium at $(x_c, 0)$ that has the double zero non-semisimple eigenvalue. Using the form (30) of Hamiltonian, we come to the one-degree-of-freedom system

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{h_1 + 2yH_1 + y^2 H_{1y}}{h_v + yh_{1v} + y^2 H_{1v}}, \\ \frac{dy}{d\tau} &= -\frac{h_x + yh_{1x} + y^2 H_{1x}}{h_v + yh_{1v} + y^2 H_{1v}}. \end{aligned} \tag{33}$$

As follows from Lemma 6, the difference of Hamiltonians $H(x, y, u, v, \varepsilon) - H(x, y, u, v)$ is of the order $O(\varepsilon)$. This implies the relations due to (28):

$$h_1 = O(\varepsilon), \qquad h_x = u + 2\hat{b}x + 3\hat{c}x^2 + O(x^3), \qquad yh_1 = yO(\varepsilon).$$

Returning to the system in the whole phase space M, we can also construct the local phase portrait of the slow system near the singular curve. In this case it is nothing else as the foliation of slow manifold SM near point s into level lines of Hamiltonian H restricted to this manifold. This is given by levels of the function $\hat{h} = H(x, 0, g(x, v), v) = h_0(g(x, v), v) + g(x, v)x + b(g(x, v), v)x + c(g(x, v), v)x^3$. The manifold SM has the line of folds (= the singular curve). This line is projected by the map p on the base B near s as a smooth curve (this curve can be called a *discriminant curve* similar to the theory of implicit differential equations, see [2]). It is clear that the image of SM lies on one side of this curve. Then the curves of the foliation are projected in such a way that they have cusps at the points of the discriminant curve. This picture is the same as for a non Hamiltonian slow–fast system with one fast and two slow variables (see [1]).

5.1. *Rescaling near a fold*

Here we want to find a principal approximation for the system (33) near a fold point. We use a blow-up method like in [9,19]. It is not surprising that we meet the Painlevé-I equation here, like in [15,8], however the derivation is different and more direct.

Let us start with the observation that the one-degree-of-freedom nonautonomous Hamiltonian $S(x, y, u, c)$ can be written in the form (28), if we expand it in (x, y) near the $(x_c, 0)$ where on the related leaf $u = u_c$ the fast system for the reduced system has a parabolic equilibrium

$$S(x, y, u, c) = s_0(u, c) + \alpha(u, c)(x - x_c) + \beta(u, c)(x - x_c)^2 + \gamma(u, c)(x - x_c)^3 + O((x - x_c)^4) + y^2 S_1(x, y, u, c),$$

where $\alpha(u_c, c) = \beta(u_c, c) = 0$, $\alpha_u(u_c, c) \neq 0$, $\gamma(u_c, c) \neq 0$, $S_1(x_c, 0, u_c, c) \neq 0$. Denote $x - x_c = \xi$. This gives the following form of the reduced system for $\varepsilon > 0$ small enough where the Hamiltonian $S(x, y, u, c, \varepsilon)$ is the solution of the equation $H = c$ with H given by (30)

$$\begin{aligned} \varepsilon \frac{d\xi}{du} &= -\frac{\partial S}{\partial y} = -2yS_1 - y^2 \frac{\partial S_1}{\partial y} + O(\varepsilon), \\ \varepsilon \frac{dy}{du} &= \frac{\partial S}{\partial \xi} = \alpha(u, c) + 2\beta(u, c)\xi + 3\gamma(u, c)\xi^2 + O(\xi^3) + y^2 \frac{\partial S_1}{\partial \xi} + yO(\varepsilon). \end{aligned} \tag{34}$$

Now we use the variable $\tau = (u - u_c)/\varepsilon$ and add two more equations $u' = du/d\tau = \varepsilon$ and $\varepsilon' = 0$ to the system. Then the suspended autonomous system will have an equilibrium at the point $(\xi, y, u, \varepsilon) = (0, 0, u_c, 0)$. The linearization of the system at this equilibrium has a matrix that is nothing else as the 4-dimensional Jordan box. To study the solutions to this system near the equilibrium, we, following the idea in [9,19] (see also a close situation in [5]), blow up a neighborhood of this point by means of the coordinate change

$$\xi = r^2 X, \qquad y = r^3 Y, \qquad u = r^4 Z, \qquad \varepsilon = r^5, \quad r \geq 0. \tag{35}$$

Since $\dot{\varepsilon} = 0$ we consider $r = \varepsilon^{1/5}$ as a small parameter. The system in these variables takes the form

$$X' = -r(2YS_1(x_c, 0, u_c, c) + \cdots], \qquad \dot{Y} = r[\alpha_u(u_c, c)Z + 3\gamma(u_c, c)X^2 + \cdots], \qquad \dot{Z} = r.$$

After re-scaling the time $r\tau = T$, denoting $' = d/dT$, setting $r = 0$ we get

$$X' = -2Y, \qquad Y' = \alpha_c Z + 3\gamma_c X^2, \qquad Z' = 1$$

where $\alpha_c = \alpha_u(u_c, c)$, $\gamma_c = \gamma(u_c, c)$. The system obtained is equivalent to the well known Painlevé-I equation [23,15,17]

$$\frac{d^2 X}{dZ^2} + 2\alpha_0 Z + 6\gamma_0 X^2 = 0.$$

By scaling variables this equation can be transformed to the standard form

$$\frac{d^2 W}{dZ^2} = 6W^2 - Z.$$

It is known [15,8,5] that this equation appears when the fast system passes through a parabolic equilibrium. We come to this equation directly using blow-up procedure.

6. Cusp for the slow manifold projection

For the case when s is a cusp for p_r, the related singular curve on SM is tangent to the bundle leaf through s (see above the definition of a cusp and the equality $P(\xi) = 0$ in (22)). Due to the last inequality in (24), this tangency takes place at the only point s, other points on the singular curve near s are folds. Below, without loss of generality, we assume that s

is the origin $(0, 0, 0, 0)$. Recall the coordinate representation of the singular curve is given as $(x, 0, g(x, r(x)), r(x)),\ r(0) = r'(0) = 0$.

The singular curve is also tangent at s to the level $H = H(s)$ (recall that we assume $dH(s) \neq 0$). Indeed, from equalities $H_x(0, 0, 0, 0) = H_y(0, 0, 0, 0) = 0,\ g_x(0, 0) = 0,\ r'(0) = 0$ the tangency follows

$$\frac{d}{dx} H(x, f(x, r(x)), g(x, r(x)), r(x))|_{x=0} = 0.$$

As $\varepsilon = 0$ consider the leaf $F_b,\ b = p(s)$, of the symplectic foliation and the fast Hamiltonian system on this leaf. The system has an equilibrium at s. This equilibrium is degenerate: it has double zero eigenvalue as for a fold, but this equilibrium is even more degenerate than a parabolic one. We want to show that the equilibrium is of co-dimension 2. Indeed, its nonlinear terms satisfy two additional equalities. The partial normal form (27) for such equilibrium in coordinates on the leaf depends on parameters (u, v) and looks as follows

$$H(x, y, u, v) = h_0(u, v) + a_1(u, v)x + \frac{a_2(u, v)}{2}x^2 + \frac{a_3(u, v)}{3}x^3 + \frac{a_4(u, v)}{4}x^4 + O(x^5) + y^2 H_1(x, y, u, v) \quad (36)$$

with $dh_0(0, 0) \neq 0,\ H_1(0, 0, 0, 0) \neq 0,\ a_4(0, 0) \neq 0,\ a_1(0, 0) = 0$. The equalities $a_2(0, 0) = a_3(0, 0) = 0$ are the first condition for the co-dimension 2 here. They follow from the assumption for s to be a cusp of the map $p_r : SM \to B$. Then equality $g_x(0, 0) = 0$ implies $a_2(0, 0) = 0$ and $g_{xx}(0, 0) = 0$ implies $a_3(0, 0) = 0$. The inequality $g_{xxx}(0, 0) \neq 0$ leads to $a_4(0, 0) \neq 0$. It turns out that the sign of $a_4(0, 0)$ plays the essential role, different signs lead to different structures of the fast systems on the neighboring leaves (see Figs. 3–4). In these figures we assume $H_1(0, 0, 0, 0) > 0$.

Remark 6.1. Some additional comments to Figs. 3–4 are necessary. In fact, on these figures the parameter plane and the related phase portraits are plotted for the systems in a generic two parameter unfolding of a Hamiltonian system with a singular point of the type of the degenerate center (Fig. 3) or the degenerate saddle (Fig. 4). The points on the bifurcation curve in the parameter plane which are marked with the small circles correspond to the phase portraits marked with the same letters (C^+, C^-), the points marked with numbers 1, 2, 3 stand for the phase portraits in the corresponding bifurcation-free parts of the parameter plane. One more circle marked with the letter k in Fig. 4 corresponds to the one more bifurcation of the formation of a heteroclinic connection between two saddles. The fast Hamiltonian systems on the leaves close to that corresponding to the cusp point depend on the parameters (u, v). Near the equilibrium they have the same phase portraits in dependence on two possible signs of $a_4(0, 0)$.

In order the unfolding in "parameters" (u, v) would be generic, the inequality $\det(D(a_1, a_2)/D(u, v)) \neq 0$ has to be met at $(u, v) = (0, 0)$. The projection of the singular curve on the base B ((u, v)-coordinates) is the cusp-shaped curve which is given up to higher order terms in the parameterized form as follows

$$a_1(u, v) + a_2(u, v)x + a_3(u, v)x^2 + a_4(u, v)x^3 + O(x^4) = 0,$$
$$a_2(u, v) + 2a_3(u, v)x + 3a_4(u, v)x^2 + O(x^3) = 0.$$

To ease again the further calculations we take functions a_1, a_2 as new parameters instead of u, v using the inequality $\det(D(a_1, a_2)/D(u, v)) \neq 0$. Keeping in mind that (u, v) are symplectic

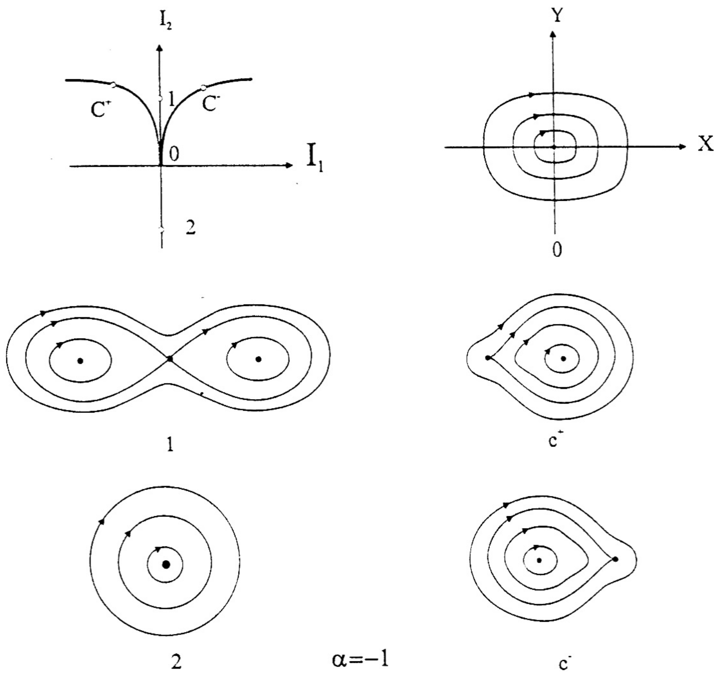

Fig. 3. Unfolding of fast systems near a degenerate center: $\alpha = -\text{sign}\, a_4(0,0) = -1,\ I_1 \sim u,\ I_2 \sim a_2$.

coordinates on B with respect to the 2-form $du \wedge dv$ we make the parameter change via a symplectic transformation. To this end, we assume first this determinant be positive, otherwise we make the redesignation $(a_1, a_2) \to (-a_1, a_2)$. One of partial derivatives of a_1 in u, v at $(0, 0)$ does not vanish, so one can take a_1 as a new parameter $u_1 = a_1(u, v)$. Adding to that some v_1 in order to get a symplectic transformation $(u, v) \to (u_1, v_1)$ we come to the same form of H with respect to the new variables (u_1, v_1) and new coefficients $a_1(u_1, v_1) = u_1, a_2(u_1, v_1)$ which we again denote as (u, v) and $u, a_2(u, v)$. Then we get $\partial a_2/\partial v \neq 0$, since $\det D(a_1, a_2)/D(u, v) \neq 0$. After that we have the Hamiltonian H in the form

$$H(x, y, u, v) = h_0(u, v) + ux + \frac{a_2(u, v)}{2}x^2 + \frac{a_3(u, v)}{3}x^3 + \frac{a_4(u, v)}{4}x^4 + O(x^5) + y^2 H_1(x, y, u, v). \tag{37}$$

At the next step, as in Lemma 6, we transform the initial Hamiltonian near the cusp point to the form ε-close to (37) by a symplectic transformation with respect to the 2-form Ω_ε. The Hamiltonian takes the form (30) but the term h has the form as in (37) with the additional term yh_1, and coefficients in the expansion of h in x differ from those as $\varepsilon = 0$ by terms of the order $O(\varepsilon)$. Recall that the identity $h_1(x, u, v, 0) \equiv 0$ holds.

Now we want to use again the isoenergetic reduction in a neighborhood of the point $s = (0, 0, 0, 0)$ and get a family of nonautonomous Hamiltonian systems in one degree of freedom depending on a parameter c—the value of H. We assume without loss of generality that $H(s) = 0$. Due to the assumption $dH \neq 0$ at s we know that one of the derivatives (or both) $\partial h_0/\partial u,\ \partial h_0/\partial v$ does not vanish.To be definite, we assume that the first derivative is nonzero at

s. One can show that this is not a restriction. This implies that the equation $H = c$ near s can be solved as $u = S(x, y, v, c)$, $S(0, 0, 0, 0) = 0$. The derivative S_c^0 does not vanish, since it is equal to $(\partial h_0/\partial u)^{-1} \neq 0$ at s. Hence, one can represent S as follows $S(x, y, v, c) = S_0(x, y, v) + cS_1(x, y, v, c)$, $S_1(0, 0, 0, 0) \neq 0$.

Thus the system with Hamiltonian $H(x, y, u, v)$ can be written on the level $H = c$ as a nonautonomous Hamiltonian system with one degree of freedom with the new "time" v and the Hamiltonian S. Instead of v we introduce a fast time via $dv/d\tau = \varepsilon$ and add the equation $d\varepsilon/d\tau = 0$. The we get again an equilibrium at (0, 0, 0, 0) for the following system

$$\begin{aligned}
\frac{dx}{d\tau} &= -\frac{h_1 + 2yH_1 + y^2 H_{1y}}{h_{0u} + x + a_{2u}x^2/2 + a_{3u}x^3/3 + a_{4u}x^4/4 + O(x^5) + yh_{1u} + y^2 H_{1u}} = S_y, \\
\frac{dy}{d\tau} &= \frac{u + a_2 x + a_3 x^2 + a_4 x^3 + O(x^4) + yh_{1x} + y^2 H_{1x}}{h_{0u} + x + a_{2u}x^2/2 + a_{3u}x^3/3 + a_{4u}x^4/4 + O(x^5) + yh_{1u} + y^2 H_{1u}} = -S_x, \\
\frac{dv}{d\tau} &= \varepsilon, \qquad \frac{d\varepsilon}{d\tau} = 0.
\end{aligned} \tag{38}$$

To study the system near the equilibrium we perform the blow-up transformation

$$x = rX, \qquad y = r^2 Y, \qquad v = r^2 Z, \qquad u = r^3 C, \qquad \varepsilon = r^3.$$

After writing the system in the new coordinates, scaling the time $r\tau = s$, setting $r = 0$, and denoting the constants $\sigma = 2H_1(0, 0, 0, 0)/(\partial h_0(0, 0)/\partial u)$, $A = C/(\partial h_0(0, 0)/\partial u)$, $\alpha = a_4(0, 0)/(\partial h_0(0, 0)/\partial u)$, $\beta = (\partial a_2(0, 0)/\partial v)/(\partial h_0(0, 0)/\partial u)$, we come to the system

$$\dot{X} = -\sigma Y, \qquad \dot{Y} = A + \beta Z X + \alpha X^3, \qquad \dot{Z} = 1, \tag{39}$$

which is just the Painlevé-II equation. More precisely, the Painlevé-II equation has the following standard form [6]

$$\frac{d^2 w}{dz^2} = 2w^3 + zw + \alpha.$$

Scaling variables and parameters of Eq. (39) reduces it to this standard form.

Thus, we have proved a theorem that gives a connection between an orbit behavior of a slow–fast Hamiltonian system near its disruption point and solutions to the related Painlevé equations. We formulate these results as follows. Let a smooth slow–fast Hamiltonian vector field with a Hamiltonian H be given on a smooth bundle $p : M \to B$ where M is a 4-dimensional presymplectic manifold with rank two 2-form ω compatible with the bundle structure and B is a smooth 2-dimensional symplectic manifold with a symplectic 2-form λ. We endow M by the symplectic structure $\Omega_\varepsilon = \omega + \varepsilon^{-1} p^*\lambda$. Suppose for $\varepsilon = 0$ the set SM of all zeros of the fast vector fields generated by H on the symplectic leaves $F_b,\ b \in B$, is a smooth submanifold in M. A point of tangency of SM with the corresponding symplectic leaf not being a critical point of H is called a disruption point. A disruption point s here is either a fold or a cusp.

Theorem 1. *The slow–fast system near a disruption point s after the isoenergetical reduction can be reduced by some blow-up transformation in the principal approximation to either the Painlevé-I equation, if s is a fold, or to the Painlevé-II equation, if s is a cusp.*

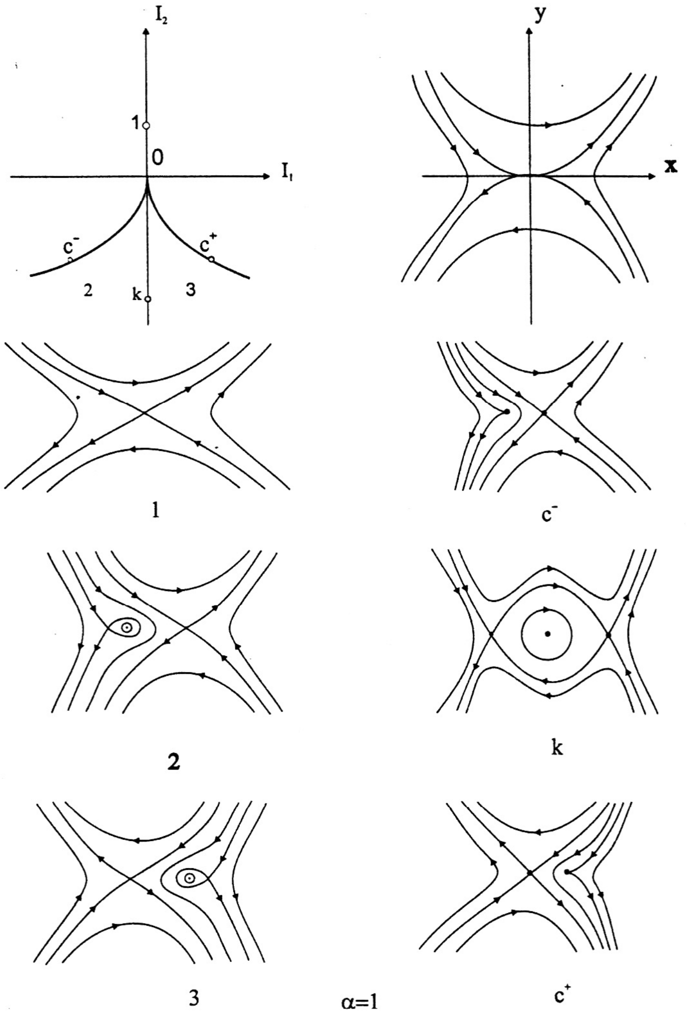

Fig. 4. Unfolding of fast systems near a degenerate saddle: $\alpha = -\text{sign}\, a_4(0,0) = 1,\ I_1 \sim u,\ I_2 \sim a_2$.

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