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CLOSED POLYNOMIALS AND SATURATED SUBALGEBRAS OF POLYNOMIAL ALGEBRAS

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The behavior of closed polynomials, i.e., polynomials $f \in \mathbb{k}[x_1, \ldots, x_n] \setminus \mathbb{k}$ such that the subalgebra $\mathbb{k}[f]$ is integrally closed in $\mathbb{k}[x_1, \ldots, x_n]$, is studied under extensions of the ground field. Using some properties of closed polynomials, we prove that, after shifting by constants, every polynomial $f \in \mathbb{k}[x_1, \ldots, x_n] \setminus \mathbb{k}$ can be factorized into a product of irreducible polynomials of the same degree. We consider some types of saturated subalgebras $A \subset \mathbb{k}[x_1, \ldots, x_n]$, i.e., subalgebras such that, for any $f \in A \setminus \mathbb{k}$, a generative polynomial of f is contained in A.

1. Introduction

Recall that a polynomial $f \in \mathbb{k}[x_1, \ldots, x_n] \setminus \mathbb{k}$ is called *closed* if the subalgebra $\mathbb{k}[f]$ is integrally closed in $\mathbb{k}[x_1, \ldots, x_n]$. It turns out that a polynomial f is closed if and only if f is *noncomposite*, i.e., f cannot be represented in the form $f = F(g)$ for some $g \in \mathbb{k}[x_1, \ldots, x_n]$ and $F(t) \in \mathbb{k}[t]$, $\deg(F) > 1$. Since any polynomial in n variables can be obtained from a closed polynomial by taking a polynomial in one variable from it, the problem of studying closed polynomials is of interest. Furthermore, closed polynomials in two variables appear in a natural way as generators of rings of constants of nonzero derivations.

Let us go briefly through the content of the paper. In Sec. 2, we collect numerous characterizations of closed polynomials (Theorem 1). A major part of these characterizations is contained in [1–4], etc., but some results seem to be new. In particular, the implication (i) $\Rightarrow$ (iv) in Theorem 1 over any perfect field and Proposition 1 solve a problem stated in [1] (Sec. 8).

We define a generative polynomial h of a polynomial $f \in \mathbb{k}[x_1, \ldots, x_n] \setminus \mathbb{k}$ as a closed polynomial such that $f = F(h)$ for some $F \in \mathbb{k}[t]$. Clearly, a generative polynomial exists for any f. Moreover, a generative polynomial is unique up to affine transformations (Corollary 1).

The abovementioned results allow us to prove that, over an algebraically closed field $\mathbb{k}$, for any $f \in \mathbb{k}[x_1, \ldots, x_n] \setminus \mathbb{k}$ and all but finitely many $\mu \in \mathbb{k}$, the polynomial $f + \mu$ can be decomposed into a product $f + \mu = \alpha \cdot f_{1\mu} \cdot f_{2\mu} \ldots f_{k\mu}$, $\alpha \in \mathbb{k}^\times$, $k \geq 1$, of irreducible polynomials $f_{i\mu}$ of the same degree d independent of μ and such that $f_{i\mu} - f_{j\mu} \in \mathbb{k}$, $i, j = 1, \ldots, k$ (Corollary 2). This result may be considered as an analog of the fundamental theorem of algebra for polynomials in many variables.

Moreover, the Stein–Lorenzini–Najib inequality (Theorem 2) implies that the number of "exceptional" values of μ is less then $\deg(f)$. The same inequality gives an estimate for the number of irreducible factors in $f + \mu$ for exceptional μ (see Theorem 3).

Section 4 is devoted to saturated $A \subset \mathbb{k}[x_1, \ldots, x_n]$, i.e., subalgebras such that, for any $f \in A \setminus \mathbb{k}$, a generative polynomial of f is contained in A. Clearly, any subalgebra integrally closed in $\mathbb{k}[x_1, \ldots, x_n]$ is saturated. On the other hand, it is known that, for monomial subalgebras, these two conditions are equivalent. In Theorem 4, we characterize subalgebras of invariants $A = \mathbb{k}[x_1, \ldots, x_n]^G$, where G is a finite group acting linearly on

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$\Bbbk[x_1, \dots, x_n]$, with A being saturated. This result provides many examples of saturated homogeneous subalgebras that are not integrally closed in $\Bbbk[x_1, \dots, x_n]$.

2. Characterizations of Closed Polynomials

Let $\Bbbk$ be an arbitrary field.

Proposition 1. *Let $f \in \Bbbk[x_1, \dots, x_n] \setminus \Bbbk$ and let $\Bbbk \subset L$ be a separable extension of fields. Then f is closed over $\Bbbk$ if and only if f is closed over L.*

Proof. If $f = F(h)$ over $\Bbbk$, then the same decomposition holds over L.

Now assume that f is closed over $\Bbbk$. Consider an element $g \in L[x_1, \dots, x_n]$ integral over $L[f]$. We prove that $g \in L[f]$. Since the number of nonzero coefficients of g is finite, we may assume that L is a finitely generated extension of $\Bbbk$. Then there exists a finite separable transcendence basis of L over $\Bbbk$, i.e., a finite set $\{\xi_1, \dots, \xi_m\}$ of elements in L that are algebraically independent over $\Bbbk$ and such that L is a finite separable algebraic extension of $L_1 = \Bbbk(\xi_1, \dots \xi_m)$.

Let us show that f is closed over L_1. The subalgebra $\Bbbk[f][\xi_1, \dots, \xi_m]$ is integrally closed in $\Bbbk[x_1, \dots, x_n][\xi_1, \dots, \xi_m]$ [5] (Chap. V.1, Proposition 12). Let T be the set of all nonzero elements of $\Bbbk[\xi_1, \dots, \xi_m]$. Then the localization $T^{-1}\Bbbk[f][\xi_1, \dots, \xi_m]$ is integrally closed in $T^{-1}\Bbbk[x_1, \dots, x_n][\xi_1, \dots \xi_m]$ [5] (Chap. V.1, Proposition 16). This proves that $L_1[f]$ is integrally closed in $L_1[x_1, \dots, x_n]$.

We fix a basis $\{\omega_1, \dots, \omega_k\}$ of L over L_1. With any element $l \in L$ one may associate an L_1-linear operator $M(l)\colon L \to L$, $M(l)(\omega) = l\omega$. Let $\mathrm{tr}(l)$ be the trace of this operator. It is known that there exists a basis $\{\omega_1^\star, \dots, \omega_k^\star\}$ of L over L_1 such that $\mathrm{tr}(\omega_i\omega_j^\star) = \delta_{ij}$ [5] (Chap. V.1.6). Assume that $g = \sum_i \omega_i a_i$, where $a_i \in L_1[x_1, \dots, x_n]$. Any $\omega_j^\star$ is integral over L_1 and, thus, over $L_1[f]$. This shows that $g\omega_j^\star$ is integral over $L_1[f]$. We set $K = L_1(x_1, \dots, x_n)$. The element $g\omega_j^\star$ determines a K-linear map $L \otimes_K K \to L \otimes_K K$, $b \to g\omega_j^\star b$. Since $g\omega_j^\star$ is integral over $L_1[f]$, the trace of this K-linear operator is also integral over $L_1[f]$ [5] (Chap. V.1.6). Note that $\mathrm{tr}(g\omega_j^\star) = \sum_i a_i \,\mathrm{tr}(\omega_i\omega_j^\star)$. On the other hand, the elements $\{\omega_1 \otimes 1, \dots, \omega_k \otimes 1\}$ form a basis of $L \otimes_K K$ over K. Hence, $\mathrm{tr}\,(\omega_i\omega_j^\star) = \delta_{ij}$ and $\mathrm{tr}\,(g\omega_j^\star) = a_j$ is integral over $L_1[f]$. This shows that $a_j \in L_1[f]$ for any j and, thus, $g \in L[f]$.

The proposition is proved.

Let $\mathcal{M}$ be the set of all subalgebras $\Bbbk[f]$, $f \in \Bbbk[x_1, \dots, x_n] \setminus \Bbbk$, partially ordered by inclusion.

In the theorem below, various characterizations of closed polynomials are collected (see [1–4], etc.). A new result here is the implication (i) $\Rightarrow$ (iv).

Theorem 1. *The following conditions on a polynomial $f \in \Bbbk[x_1, \dots, x_n] \setminus \Bbbk$ are equivalent:*

(i) *f is noncomposite;*

(ii) *$\Bbbk[f]$ is a maximal element of $\mathcal{M}$;*

(iii) *f is closed;*

(iv) *$\big(\Bbbk$ is a perfect field$\big)$ $f + \lambda$ is irreducible over $\overline{\Bbbk}$ for all but finitely many $\lambda \in \overline{\Bbbk}$;*

(v) *$\big(\Bbbk$ is a perfect field$\big)$ there exists $\lambda \in \overline{\Bbbk}$ such that $f + \lambda$ is irreducible over $\overline{\Bbbk}$;*

(vi) *$\big(\mathrm{char}\,\Bbbk = 0\big)$ there exists a (finite) family of derivations $\{D_i\}$ of the algebra $\Bbbk[x_1, \dots, x_n]$ such that $\Bbbk[f] = \cap_i \mathrm{Ker}\, D_i$.*

Proof. (i) $\Rightarrow$ (iv). Assume that $\Bbbk = \overline{\Bbbk}$. Consider a morphism $\phi\colon \Bbbk^n \to \Bbbk^1$, $\phi(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$. We should prove that all fibers of this morphism except finitely many are irreducible. But this follows from the first Bertini theorem (see, e.g., [6, p. 139]).

If a perfect field $\Bbbk$ is nonclosed, then Proposition 1 shows that $f \in \Bbbk[x_1, \ldots, x_n]$ is closed over $\Bbbk$, which implies that f is closed over $\overline{\Bbbk}$.

The theorem is proved.

Example 1 [1]**.** If the field $\Bbbk$ is not perfect, then we cannot guarantee that a polynomial f closed over $\Bbbk$ is closed over $\overline{\Bbbk}$ as well. Indeed, let $F = \Bbbk(\eta)$, where $\eta \notin \Bbbk$, $\eta^p \in \Bbbk$. The polynomial $f(x_1, x_2) = x_1^p + \eta^p x_2^p$ is closed over $\Bbbk$. However, one has a decomposition $f = (x_1 + \eta x_2)^p$ over F. The same example works for (i) $\not\Rightarrow$ (iv) in this case.

We are now going to show that a generative polynomial is unique up to affine transformations. Here, we need two auxiliary lemmas.

Lemma 1. *For any $f \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$, the integral closure A of $\Bbbk[f]$ in $\Bbbk[x_1, \ldots, x_n]$ has the form $A = \Bbbk[h]$ for some closed $h \in \Bbbk[x_1, \ldots, x_n]$.*

Proof. Since $\operatorname{tr.deg}_{\Bbbk} Q(A) = 1$, by virtue of the theorem of Gordan (see, e.g., [4, p. 15]) we have $Q(A) = \Bbbk(h)$ for some rational function h. The subfield $Q(A)$ contains nonconstant polynomials. Therefore, by virtue of the theorem of E. Noether (see, e.g., [4, p. 16]), the generator h of the subfield $Q(A)$ can be chosen as a polynomial. Note that $\Bbbk(h) \cap \Bbbk[x_1, \ldots, x_n] = \Bbbk[h]$ because any rational (but polynomial) function of a nonconstant polynomial cannot be a polynomial. Therefore, $A \subseteq \Bbbk[h]$. Since the element h is integral over A and A is integrally closed in $\Bbbk[x_1, \ldots, x_n]$, we have $h \in A$ and $A = \Bbbk[h]$.

The lemma is proved.

Note that, in the case $\operatorname{char} \Bbbk = 0$, this lemma follows immediately from the result of Zaks [7].

Lemma 2. *Let $\Bbbk$ be a field. Polynomials $f, g \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$ are algebraically dependent $\big($over $\Bbbk\big)$ if and only if there exists a closed polynomial $h \in \Bbbk[x_1, \ldots, x_n]$ such that $f, g \in \Bbbk[h]$.*

Proof. Assume that f and g are algebraically dependent. According to the Noether normalization lemma, there exists an element $r \in \Bbbk[f, g]$ such that $\Bbbk[r] \subset \Bbbk[f, g]$ is an integral extension. By virtue of Lemma 1, the integral closure of $\Bbbk[r]$ in $\Bbbk[x_1, \ldots, x_n]$ has the form $\Bbbk[h]$ for some closed polynomial h.

Conversely, if $f, g \in k[h]$, then these polynomials are obviously algebraically dependent.

The lemma is proved.

Corollary 1. *Let $f \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$. The integral closure of the subalgebra $\Bbbk[f]$ in $\Bbbk[x_1, \ldots, x_n]$ coincides with $\Bbbk[h]$, where h is a generative polynomial of f. In particular, a generative polynomial of f exists and is unique up to affine transformations.*

3. Factorization Theorem

In this section, we assume that the ground field $\Bbbk$ is algebraically closed. Theorem 1 states that, for a closed polynomial $h \in \Bbbk[x_1, \ldots, x_n]$, the polynomial $h + \lambda$ may be reducible only for finitely many $\lambda \in \Bbbk$. Let $E(h)$ denote the set of $\lambda \in \Bbbk$ such that $h + \lambda$ is reducible and let $e(h)$ be the cardinality of this set. Stein's inequality claims that

$$e(h) < \deg h.$$

For any $\lambda \in \Bbbk$, we now consider the decomposition

$$h + \lambda = \prod_{i=1}^{n(\lambda,h)} h_{\lambda,i}^{d_{\lambda,i}},$$

where $h_{\lambda,i}$ is irreducible. A more precise version of Stein's inequality is given in the next theorem.

Theorem 2 [Stein–Lorenzini–Najib inequality]. *Let $h \in \Bbbk[x_1, \ldots, x_n]$ be a closed polynomial. Then*

$$\sum_{\lambda}(n(\lambda, h) - 1) < \min_{\lambda}\left(\sum_{i} \deg(h_{\lambda,i})\right).$$

This inequality has a fairly long history. Stein [8] proved his inequality in characteristic zero for $n = 2$. For any n over $\Bbbk = \mathbb{C}$, this inequality was proved in [9]. In 1993, Lorenzini [10] obtained the inequality as in Theorem 2 in any characteristic, but only for $n = 2$ (see also [11] and [12]). Finally, in [13], the proof for an arbitrary n was reduced to the case $n = 2$.

We now take arbitrary $f \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$ and $\mu \in \Bbbk$ and consider the decomposition

$$f + \mu = \alpha \cdot \prod_{i=1}^{n(\mu,f)} f_{\mu,i}^{d_{\mu,i}},$$

where $\alpha \in \Bbbk^{\times}$ and $f_{\mu,i}$ are irreducible.

Let us state the main result of this section.

Theorem 3. *Let $f \in \Bbbk[x_1, \ldots, x_n] \setminus \Bbbk$. There exists a finite subset $E(f) = \{\mu_1, \ldots, \mu_{e(f)} \mid \mu_i \in \Bbbk\}$ with $e(f) < \deg f$ such that the following assertions are true:*

(1) for any $\mu \notin E(f)$, one has $f + \mu = \alpha \cdot f_{\mu,1} \cdot f_{\mu,2} \ldots f_{\mu,k}$, where all $f_{\mu,i}$ are irreducible and $f_{\mu,i} - f_{\mu,j} \in \Bbbk$;

(2) $f_{\mu,i} - f_{\nu,j} \in \Bbbk^{\times}$ for any $\mu, \nu \notin E(f)$ such that $\nu \neq \mu$; in particular, the degree $d = \deg(f_{\mu,i})$ does not depend on i and μ;

(3) $\deg(f_{\mu,i}) \leq d$ for any $\mu \in \Bbbk$;

(4) $\sum_{\mu}\left(n(\mu, f) - \dfrac{\deg(f)}{d}\right) < \min\limits_{\mu}\left(\sum_{i=1}^{n(\mu,f)} \deg(f_{\mu,i})\right)$.

Proof. Let h be the generative polynomial of f and let $f = F(h)$. Then

$$F(h) + \mu = \alpha \cdot (h + \lambda_{\mu,1}) \ldots (h + \lambda_{\mu,k})$$

for some $\lambda_{\mu,1}, \ldots, \lambda_{\mu,k} \in \Bbbk$. Hence, for any μ such that $\lambda_{\mu,1}, \ldots, \lambda_{\mu,k} \notin E(h)$, we have the decomposition of $f + \mu$ as in (1). Note that $\lambda_{\mu,i} \neq \lambda_{\nu,j}$ for $\mu \neq \nu$. This proves (2) with $d = \deg(h)$ and gives the inequalities

$$e(f) \leq e(h) < \deg(h) \leq \deg(f).$$

Any $f_{\mu,i}$ is a divisor of some $h+\lambda$. This yields (3).
Finally, (4) can be obtained as follows:

$$\sum_{\mu}\left(n(\mu,f)-\frac{\deg(f)}{d}\right)\leq\sum_{\lambda}(n(\lambda,h)-1)<\min_{\lambda}\left(\sum_{i}\deg(h_{\lambda,i})\right)\leq\min_{\mu}\left(\sum_{j}\deg(f_{\mu,j})\right).$$

The theorem is proved.

Remark 1. It follows from the proof of Theorem 3 that

$$E(f)=\big\{-F(-\lambda)\mid\lambda\in E(h)\big\};$$

if f is not closed, then

$$e(f)<\frac{1}{2}\deg(f).$$

Corollary 2. *Let $f\in\Bbbk[x_1,\dots,x_n]\setminus\Bbbk$. Then, for all but finitely many $\mu\in\Bbbk$, the polynomial $f+\mu$ can be decomposed into the product*

$$f+\mu=\alpha\cdot f_{1\mu}\cdot f_{2\mu}\dots f_{k\mu},\qquad\alpha\in\Bbbk^{\times},\quad k\geq 1,$$

of irreducible polynomials $f_{i\mu}$ of the same degree d independent of the number μ and such that $f_{i\mu}-f_{j\mu}\in\Bbbk$, $i,j=1,\dots,k$. The number of exceptional μ's for which such a decomposition does not exist is at most $\deg f-1$.

Example 2. We take

$$f(x_1,x_2)=x_1^2x_2^4-2x_1^2x_2^3+x_1^2x_2^2+2x_1x_2^3-2x_1x_2^2+x_2^2+1.$$

Here, $h=x_1x_2(x_2-1)+x_2$ and $F(t)=t^2+1$. It is easy to check that $E(h)=\{0,-1\}$, and, thus, $E(f)=\{-1,-2\}$. We have the following decompositions:

$$\mu=-1\colon f-1=x_2^2(x_1x_2-x_1+1)^2,$$

$$\mu=-2\colon f-2=(x_2-1)(x_1x_2+1)(x_1x_2(x_2-1)+x_2+1),$$

$$\mu\neq-1,-2\colon f+\mu=\big(x_1x_2(x_2-1)+x_2+\lambda\big)\big(x_1x_2(x_2-1)+x_2-\lambda\big),\quad\lambda^2=-1-\mu.$$

In this case, $\deg(f)=6$, $d=3$,

$$\sum_{\mu}(n(\mu,f)-2)=1,$$

and

$$\min_{\mu}\left(\sum_{i}\deg(f_{\mu,i})\right)=\min\{3,6,6\}=3.$$

4. Saturated Subalgebras and Invariants of Finite Groups

Let $\Bbbk$ be a field.

Definition 1. *A subalgebra $A \subseteq \Bbbk[x_1, \ldots, x_n]$ is said to be saturated if, for any $f \in A \setminus \Bbbk$, the generative polynomial of f is contained in A.*

It is clear that the intersection of a family of saturated subalgebras in $\Bbbk[x_1, \ldots, x_n]$ is again a saturated subalgebra. Therefore, we may define the *saturation* $S(A)$ of a subalgebra A as the minimal saturated subalgebra containing A.

If A is integrally closed in $\Bbbk[x_1, \ldots, x_n]$, then A is saturated. According to Theorem 1, if $A = \Bbbk[f]$, then the converse statement is true. Moreover, the converse is true if A is a monomial subalgebra. In order to prove this, consider the submonoid $P(A)$ in $\mathbb{Z}^n_{\geq 0}$ that consists of multidegrees of all monomials in A. Then the monomials corresponding to elements of the "saturated" semigroup $P'(A) = (\mathbb{Q}_{\geq 0}P(A)) \cap \mathbb{Z}^n_{\geq 0}$ are generative elements of A. On the other hand, it is a basic fact of toric geometry that the monomial subalgebra corresponding to $P'(A)$ is integrally closed in $\Bbbk[x_1, \ldots, x_n]$, [see, e.g., [14] (Sec. 2.1)].

We now come from monomial to homogeneous saturated subalgebras. The degree of monomials

$$\deg(\alpha x_1^{i_1} \ldots x_n^{i_n}) = i_1 + \ldots + i_n$$

defines a $\mathbb{Z}_{\geq 0}$-grading on the polynomial algebra $\Bbbk[x_1, \ldots, x_n]$. Recall that a subalgebra $A \subset \Bbbk[x_1, \ldots, x_n]$ is called *homogeneous* if, for any element $a \in A$, all its homogeneous components belong to A.

Consider a subgroup $G \subset GL_n(\Bbbk)$. The linear action $G\colon \Bbbk[x_1, \ldots, x_n] \to \Bbbk[x_1, \ldots, x_n]$ determines the homogeneous subalgebra $\Bbbk[x_1, \ldots, x_n]^G$ of G-invariant polynomials.

Theorem 4. *Let $G \subseteq GL_n(\Bbbk)$ be a finite subgroup. The subalgebra $A = \Bbbk[x_1, \ldots, x_n]^G$ is saturated in $\Bbbk[x_1, \ldots, x_n]$ if and only if G admits no nontrivial homomorphisms $G \to \Bbbk^\times$.*

Proof. Assume that there is a nontrivial homomorphism $\phi\colon G \to \Bbbk^\times$. Let G_ϕ be the kernel of ϕ and let $G^\phi = G/G_\phi$. Then G^ϕ is a finite cyclic group of some order k, and it may be identified with a subgroup of $\Bbbk^\times$.

Lemma 3. *Let H be a cyclic subgroup of order k in $\Bbbk^\times$. Then any finite-dimensional $\big($over $\Bbbk\big)$ H-module W is a direct sum of one-dimensional submodules.*

Proof. The polynomial $X^k - 1$ annihilates the linear operator P in $GL(W)$ corresponding to a generator of H. By assumption, $X^k - 1$ is a product of k nonproportional linear factors in $\Bbbk[X]$. This shows that the operator P is diagonalizable.

Lemma 4. *Let $H \subset G$ be a proper subgroup. Then $\Bbbk[x_1, \ldots, x_n]^H \neq \Bbbk[x_1, \ldots, x_n]^G$.*

Proof. Let K be a field and let G be a finite group of its automorphisms. By virtue of the Artin theorem [15] (Sec. 2.1, Theorem 1.8), $K^G \subset K$ is a Galois extension and $[K\colon K^G] = |G|$. This implies that $\Bbbk(x_1, \ldots, x_n)^H \neq \Bbbk(x_1, \ldots, x_n)^G$. The implication

$$\frac{f}{h} \in \Bbbk(x_1, \ldots, x_n)^G \implies \frac{f\prod_{g\in G, g\neq e} g\cdot f}{h\prod_{g\in G, g\neq e} g\cdot f} \in \Bbbk(x_1, \ldots, x_n)^G$$

shows that $\Bbbk(x_1, \ldots, x_n)^G$ $\big($respectively, $\Bbbk(x_1, \ldots, x_n)^H\big)$ is the quotient field of $\Bbbk[x_1, \ldots, x_n]^G$ $\big($respectively, $\Bbbk[x_1, \ldots, x_n]^H\big)$, and, thus, $\Bbbk[x_1, \ldots, x_n]^H \neq \Bbbk[x_1, \ldots, x_n]^G$.

The lemma is proved.

We may now take a finite-dimensional G-submodule $W \subset \Bbbk[x_1,\dots,x_n]^{G_\phi}$ that is not contained in $\Bbbk[x_1,\dots,x_n]^G$. Then W is a G^ϕ-module. According to Lemma 3, one can find a G^ϕ-eigenvector $h \in W$, $h \notin \Bbbk[x_1,\dots,x_n]^G$. Then $h^k \in \Bbbk[x_1,\dots,x_n]^G$ and $\Bbbk[x_1,\dots,x_n]^G$ is not saturated.

Conversely, assume that any homomorphism $\chi\colon G \to \Bbbk$ is trivial. If h is a generative element of a polynomial $f \in \Bbbk[x_1,\dots,x_n]^G$, then, for any $g \in G$, the element $g \cdot h$ is also a generative element of f. By virtue of Corollary 1, the generative element is unique up to affine transformation. Without loss of generality, we can assume that the constant term of h is zero. Then the element $g \cdot h$ obviously has the zero constant term, and by virtue of Corollary 1 this element is proportional to h for any $g \in G$. Thus, G acts on the line $\langle h \rangle$ via some character. But any character of G is trivial, whence $h \in \Bbbk[x_1,\dots,x_n]^G$, and $\Bbbk[x_1,\dots,x_n]^G$ is saturated.

The theorem is proved.

Remark 2. Since all coefficients of the polynomial

$$F_f(T) = \prod_{g \in G} (T - g \cdot f)$$

are in $\Bbbk[x_1,\dots,x_n]^G$, any element $f \in \Bbbk[x_1,\dots,x_n]$ is integral over $\Bbbk[x_1,\dots,x_n]^G$. Thus, Theorem 4 provides many saturated homogeneous subalgebras that are not integrally closed in $\Bbbk[x_1,\dots,x_n]$.

Corollary 3. *Assume that $\Bbbk$ is algebraically closed and $\operatorname{char}\Bbbk = 0$. Then the following assertions are true:*

(1) the subalgebra $\Bbbk[x_1,\dots,x_n]^G$ is saturated in $\Bbbk[x_1,\dots,x_n]$ if and only if G coincides with its commutant;

(2) the saturation of $\Bbbk[x_1,\dots,x_n]^G$ is $\Bbbk[x_1\dots,x_n]$ if and only if G is solvable.

Example 3. In general, the saturation $S(A)$ is not generated by generative elements of elements of A. Indeed, take any field $\Bbbk$ that contains a primitive root of unit of degree six. Let $G = S_3$ be the permutation group acting naturally on $\Bbbk[x_1,x_2,x_3]$ and let $A_3 \subset S_3$ be the alternating subgroup. The proof of Theorem 4 shows that any generative element of an S_3-invariant is an S_3-semiinvariant and thus belongs to $\Bbbk[x_1,x_2,x_3]^{A_3}$. On the other hand, $S(\Bbbk[x_1,x_2,x_3]^{S_3}) = \Bbbk[x_1,x_2,x_3]$.

Example 4. It follows from Theorem 4 that the property of a subalgebra to be saturated is not preserved under field extensions. Let us give an explicit example of this effect.

Let $\Bbbk = \mathbb{R}$ and let G be the cyclic group of order three acting on $\mathbb{R}^2$ by rotations. We begin with the calculation of generators of the algebra of invariants $\mathbb{R}[x,y]^G$. Consider the complex polynomial algebra $\mathbb{C}[x,y] = \mathbb{R}[x,y] \oplus \mathrm{i}\mathbb{R}[x,y]$ with natural G-action. Then

$$\mathbb{C}[x,y]^G = \mathbb{R}[x,y]^G \oplus \mathrm{i}\mathbb{R}[x,y]^G.$$

We set $z = x + \mathrm{i}y$, $\overline{z} = x - \mathrm{i}y$. It is clear that $\mathbb{C}[x,y] = \mathbb{C}[z,\overline{z}]$, and G acts on z and $\overline{z}$ as $z \to \epsilon z$ and $\overline{z} \to \overline{\epsilon z}$, where $\epsilon^3 = 1$. This yields $\mathbb{C}[z,\overline{z}]^G = \mathbb{C}[f_1,f_2,f_3]$, where $f_1 = z^3$, $f_2 = \overline{z}^3$, and $f_3 = z\overline{z}$. Finally,

$$\mathbb{R}[x,y]^G = \mathbb{R}\left[\mathrm{Re}(f_i),\, \mathrm{Im}(f_i);\, i = 1,2,3\right] = \mathbb{R}\left[x^3 - 3xy^2,\, y^3 - 3x^2y,\, x^2 + y^2\right].$$

By virtue of Theorem 4, the subalgebra $\mathbb{R}[x,y]^G$ is saturated in $\mathbb{R}[x,y]$. On the other hand, the subalgebra $\mathbb{C}\left[x^3 - 3xy^2, y^3 - 3x^2y, x^2 + y^2\right]$ contains $x^3 - 3xy^2 + \mathrm{i}(y^3 - 3x^2y) = (x - \mathrm{i}y)^3$.

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