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AN INFORMATIONAL BASIS FOR VOTING RULES

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AN INFORMATIONAL BASIS FOR VOTING RULES[2]

This paper presents a novel combinatorial approach for voting rule analysis. Applying reversal symmetry, we introduce a new class of preference profiles and a new representation (bracelet representation). By applying an impartial, anonymous, and neutral culture model for the case of three alternatives, we obtain precise theoretical values for the number of voting situations for the plurality rule, the run-off rule, the Kemeny rule, the Borda rule, and the scoring rules in the extreme case. From enumerative combinatorics, we obtain an information utilization index for these rules. The main results are obtained for the case of three alternatives.

JEL Classification: D70.

Keywords: ANEC, IANC, plurality, run-off, Kemeny, Borda, scoring rules, reversal symmetry.

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1. Introduction

Social choice theory states that there is no perfect voting rule (Brams, Fishburn 2002). A variety of rules are needed to satisfy different properties and aims. Voting rules are differentiated not only by outcomes, but also by the way of representing preference profiles, aggregation method etc. Many-sided comparisons are applicable for the choice of a relevant procedure. The data requirement is an important aspect of voting rule analysis.

Different voting rules utilize different information from a preference profile. Fishburn (1977) analyzed data requirements for Condorcet social choice functions. The well-known classification introduces three classes: C1 — voting rules that only depend on pairwise majority comparisons, C2 — voting rules that only depend on weighted pairwise majority comparisons, and C3 — voting rules that require more information than weighted pairwise majority comparisons. This classification is not suitable for all (non-Condorcet) voting rules. Almost all voting rules belong to the C3 class.

Fishburn’s classification is a benchmark for different voting rule classification studies. Based on a computational experiment for a particular number of alternatives and agents Eckert et al. (2006), and McCabe-Dansted, Slinko (2006) analyzed the similarity of voting outcomes and the corresponding voting rules. There is no direct relation between Fishburn’s (1977) classification and the similarity of voting result, but there is some correlation. Experimental results are not generalizable for a broader set of parameters.

Data requirements for voting rules are closely related to the computational complexity of social choice problems and mechanisms. Handbook of computational social choice (2016) presents methods and results pertaining to one of Fishburn’s classes.

Zwicker (2016) mentioned that Fishburn’s classification do not represent the relation “need more information”. Intuitively, the plurality rule requires less information than the Borda rule, but according to Fishburn’s classification, the plurality rule belongs to C3, and the Borda rule belongs to C2 (Fischer et al. 2016). The aim of this paper is to quantify the amount of information required by various voting rules and to compare these rules.

Starting from an impartial, anonymous, and neutral culture (IANC) model (Egecioglu 2009; Egecioglu, Giritgil 2013) and enumerative combinatorics of anonymous and neutral equivalence classes of preference profiles (ANECs), we introduce reverse invariant ANECs (RIANECs) and self-symmetric ANECs (SSANECs). These classes of ANECs exploit the reversal symmetry of preference profiles. Note that we do not introduce any probability distribution over preference profiles. We consider the IANC model only as a combinatorial object. For a three-alternative case, we design a multigraph representation of ANECs and a bracelet representation of RIANECs. These objects are well-studied in combinatorics theory.

A voting rule may not allow for all preference profiles to be separately identified. Some preference profiles from different ANECs are indiscernible, e.g., preference profiles with the same structure of the top alternatives under the plurality rule. The class of preference profiles that are indiscernible under a voting rule is called a voting situation.

Voting situations induce a partition of the set of ANECs. A finer partition means that a rule utilizes more information and it is in some sense more complicated. A lower number of voting situations is associated with a simpler computation of the voting outcome. Calculating the number of voting situations, we derive a numerical representation—a “trace” of a voting rule. For a perfectly discernible voting rule, which distinguishes all ANECs, the number of voting situations equals the number of ANECs. The ratio between the number of voting situations and the number of ANECs defines the information utilization index of a voting rule. The new measure is not simulation-based, it has a simple and tractable explicit formula. This measure produces a hierarchy of voting rules according to information utilization. It is a promising alternative to Fishburn's hierarchical classification.

To illustrate the general framework, we analyze five main rules based on different informational bases: the plurality rule, the run-off rule, the Kemeny rule, the Borda rule, and the scoring rules in the extreme case. The main results are obtained for the case of three alternatives. If the number of voters is large enough, then the scoring rules in the extreme case utilize more information than the Kemeny rule, the Kemeny rule utilizes more information than the run-off rule, the run-off rule utilizes more information than the Borda rule, and the Borda rule utilizes more information than the plurality rule. The complexity of the Kemeny rule is confirmed in computational social choice studies. Our framework can be applied to other rules and a higher number of alternatives.

The structure of the paper is as follows: Section 2 describes a mathematical model of IANC and the basic combinatorial theory; Section 3 presents the main result about voting rules; and Section 4 compares voting rules and concludes the paper. Proofs of propositions are given in Appendix 1. Appendix 2 contains a table with the number of ANECs, RIANECs, SSANECs, the number of voting situations, the polynomial degree and the information utilization index for the rules for the 3 alternatives case.

2. Framework

Let a finite set $X = \{1, \dots, m\}$, $m \geq 2$ be the set of alternatives and a finite set $\mathcal{N} = \{1, \dots, n\}$, $n \geq 2$ be the set of agents (voters). Each agent $i \in \mathcal{N}$ has a strict preference P_i over X (linear order). Let $\mathcal{L}(X)$ be the set of all possible linear orders over X. An *n*-tuple of preference orders generates a preference profile $\mathcal{P} = (P_1, \dots, P_n) \in \mathcal{L}(X)^n$.

Within this model, the names of voters (anonymity) and names of alternatives (neutrality) do not matter. An anonymous and neutral equivalence class (ANEC) is a set of preference profiles that can be obtained from each other by permuting preference orders and renaming alternatives. The permutation of preference orders is denoted by $\sigma: \mathcal{N} \to \mathcal{N}$, and the permutation of alternatives is denoted by $\tau: X \to X$. The image of profile $\mathcal{P}$ under permutations σ, τ is denoted by $(\mathcal{P}^\sigma)^\tau$. Preference profiles $\mathcal{P}, \mathcal{P}'$ belong to the same ANEC if and only if there are permutations $\sigma: \mathcal{N} \to \mathcal{N}, \tau: X \to X$, such that $(\mathcal{P}^\sigma)^\tau = \mathcal{P}'$. This relation, which is symmetrical, is denoted as $\mathcal{P} \sim_{ANEC} \mathcal{P}'$. The complementary binary relation is denoted as $\nsim_{ANEC}$. The function $pos(P_i, j) = |\{x \in X | x P_i j\}| + 1$ indicates the position of candidate j in preference profile P_i.

We use the following rounding functions:

$\lfloor x \rfloor$ is rounding down to the nearest integer,

$\lceil x \rceil$ is rounding up to the nearest integer, and

$Round[x]$ is rounding down if the fractional part is less than 0.5 and rounding up if the fractional part is greater than or equal to 0.5.

2.1 ANEC enumeration problem

The ANEC enumeration problem was solved (Egecioglu 2009; Egecioglu, Giritgil 2013). To state the authors' results, we need some notation from their papers. A partition λ of an integer n is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ with $n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Each of the integers $\lambda_i > 0$ is called a part of n. For example, $\lambda = (3,2,2)$ is a partition of $n = 7$ into three parts. It has two parts of size two and one part of size three. If λ is a partition of n, then this is denoted by $\lambda \vdash n$. Each partition of n has a type denoted by the symbol $1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$, which signifies that λ has α_i parts of size i for $1 \leq i \leq n.\ \lambda$ For example, the type of $\lambda = (3,2,2)$ is $1^0 2^2 3^1 4^0 5^0 6^0 7^0$. We can omit the zeros that appear as exponents and write the type of λ as $2^2 3^1$.

Let $p_k(n)$ be the number of partitions of n with exactly k parts. It is also the number of partitions of n in which the largest part has size k (Stanley 2012). Let $p_{k,l}(n)$ be the number of partitions with k parts, each of which does not exceed l. It is convenient to use the following formulas:

$$p_{k,l}(n) = \sum_{i=1}^{\left\lfloor \frac{n}{k} \right\rfloor} p_{k-1,min\{l,n-2i\}}(n-i) \tag{1}$$

For $k = 2$ and $n - 1 \geq l \geq n/2$, we have:

$$p_{2,l}(n) = l - \left\lfloor \frac{n-1}{2} \right\rfloor \tag{2}$$

A permutation σ of $[n]$ defines a partition of n where the parts of the partition are the cycle lengths in the cycle decomposition of σ. The cycle type of σ is defined as the type of the resulting partition. For example, $\sigma = (142)(35)(67)$ has cycle type $2^2 3^1$. For any $\lambda \vdash n$ of type $1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$, define a number:

$$z_\lambda = 1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}\alpha_1!\,\alpha_2!\cdots\alpha_n!. \tag{3}$$

The number of permutations of cycle type $1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$ is given by $z_\lambda^{-1}n!$ where λ is the partition of cycle lengths of σ.

Proposition 1. *(Egecioglu 2009) For n voters and m alternatives, the number of ANECs is equal to*

$$\#ANEC(m,n) = \sum_{\lambda \vdash m} z_\lambda^{-1} \binom{\frac{n}{d} + \frac{m!}{d} - 1}{\frac{m!}{d} - 1}, \tag{4}$$

where $d = d(\lambda) = LCM(\lambda)$, z_λ is defined in formula (2) and $\binom{x}{k} = \begin{cases} \frac{x!}{k!(x-k)!} & if\ x\ is\ integer, \\ 0 & otherwise. \end{cases}$

$\#ANEC(m,n)$ is a polynomial in n of degree $m! - 1$. Veselova (2016) compared $\#ANEC(m,n)$ with $\#AEC(m,n)$ and $\#EC(m,n)$ and explored their asymptotic properties. Asymptotically, the IANC, IAC and IC models leads to the same results. IAC and IC models are applicable for the equiprobable generation of preference profiles and simulation studies. In our model we do not make simulations. The neutrality property is needed for defining voting situations. Neutrality also clarifies combinatorial structures, which arise in this paper.

A preference profile $\mathcal{P}'$ is the **reversal of preference profile** $\mathcal{P}$ if $(\forall x \in X, \forall i \in \mathcal{N}$, $pos(P_i, x) = m + 1 - pos(P'_i, x))$. This type of symmetry was studied by Saari (1999), Saari, Barney (2003) and Crisman (2014).

An ANEC is **self-symmetric** if for every $\mathcal{P}$ from the ANEC, the reverse profile $\mathcal{P}'$ belongs to the same ANEC. A pair of ANECs is **reverse symmetric** if for every $\mathcal{P}$ from one ANEC, the reverse profile $\mathcal{P}'$ belongs to the other ANEC.

Considering symmetric ANECs as equivalent, we obtain a set of **reverse invariant ANECs (RIANECs)** and binary relation $\sim_{RIANEC}$, which contains $\sim_{ANEC}$.

2.1. Three alternatives case

This section introduces two new representations of preference profiles. These representations reveal the internal structure of preference profiles and enable us to calculate the number of RIANECs and SSANECs.

Definition 1. Multigraph representation of preference profile. *Having 3 alternatives as vertices of a graph, for each preference order in the profile, we define an arc from the best alternative to the worst alternative.*

Multigraph representation is anonymous and neutral. The renaming of alternatives leads to graph isomorphism. The corresponding multigraph uniquely represents a preference profile. Table 1 contains several examples of the multigraph representation of preference profiles. $\#ANEC(3,n)$ is also the number of multigraphs with 3 nodes and n arcs (it is the A037240 sequence in the on-line encyclopedia of integer sequences, published electronically at http://oeis.org; henceforth OEIS). In the 3 alternatives case, formula (4) leads to:

$$\#ANEC(3,n) = \begin{cases} \frac{1}{6}\binom{n+5}{5} + \frac{1}{16}(n+4)(n+2) + \frac{1}{9}(n+3), \text{if } n \equiv 0 \ (mod\ 6); \\ \frac{1}{6}\binom{n+5}{5}, \text{if } n \equiv (1 \ or\ 5)(mod\ 6); \\ \frac{1}{6}\binom{n+5}{5} + \frac{1}{16}(n+4)(n+2), \text{if } n \equiv (2 \ or\ 4)(mod\ 6); \\ \frac{1}{6}\binom{n+5}{5} + \frac{1}{9}(n+3), \text{if } n \equiv 3 \ (mod\ 6). \end{cases} \tag{5}$$

For $m = 3$, there are six different preference orders presented. Defining residues modulo 6, we numerate preference orders from 0 to 5 in clockwise manner:

$$P_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, P_1 = \begin{pmatrix} a \\ c \\ b \end{pmatrix}, P_2 = \begin{pmatrix} c \\ a \\ b \end{pmatrix}, P_3 = \begin{pmatrix} c \\ b \\ a \end{pmatrix}, P_4 = \begin{pmatrix} b \\ c \\ a \end{pmatrix}, P_5 = \begin{pmatrix} b \\ a \\ c \end{pmatrix}. \tag{6}$$

Each subsequent preference order is obtained from the previous preference order by one pairwise swap of consecutive alternatives. The next nearest preference order is obtained by two swaps. The highest number of swaps is three, which leads to preference order reversal. Putting preference orders on a loop we obtain a circle representation of preference orders, which is presented in Figure 1.

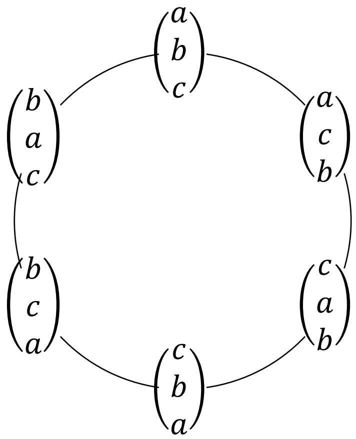


Figure 1. Circle representation of preference orders.

A preference profile is a string of n preference orders, each of 6 possible types P_0, P_1, P_2, P_3, P_4, P_5. By anonymity, the order of preference orders does not matter. Only the number of different preference orders matters. Neutrality links different preference orders in the cycle structure depicted in Figure 1.

Permutating (renaming) one pair of alternatives (one or three swaps in a preference order) leads to a preference order circle turnover. There are 3 possible pairs and 3 axes that divide the circle into halves. Two possibilities of permutating (renaming) three alternatives (two swaps in a preference order) leads to the preference order circle rotating on 2 preference orders in a clockwise or counterclockwise manner. The circle representation of preference orders is applied for the Kemeny rule analysis further in the paper.

Neutrality, anonymity and reverse invariance leads to a simpler representation of preference profiles. Instead of using 6 types of preference orders (letters) in the standard representation of preference profiles, we can use only 2 types of letters. Because we apply the cycle structure of a string, taking all rotations and turnovers as equivalent, we call the new object a bracelet (term is borrowed from combinatorics theory). For demonstration purposes we provide a definition using beads of two colors. It is possible to rewrite definition in terms of string with two types of letters.

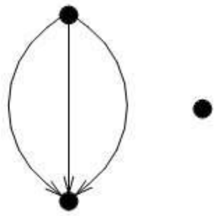
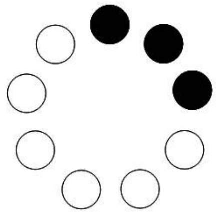
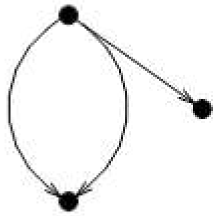
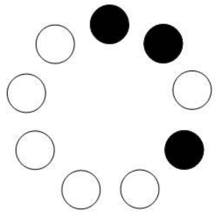
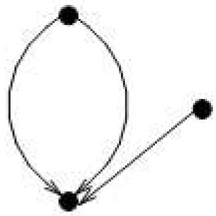
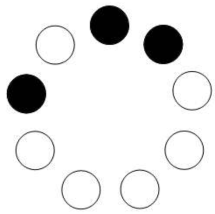
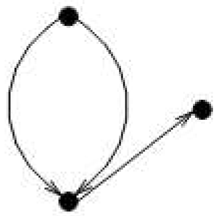
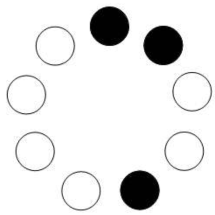
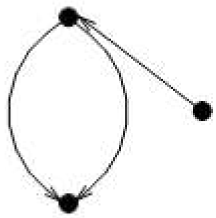
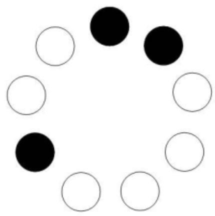
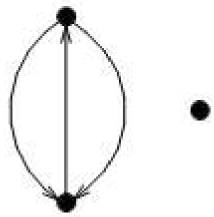
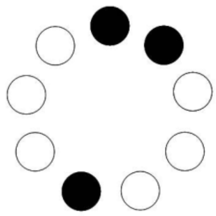
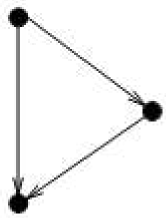
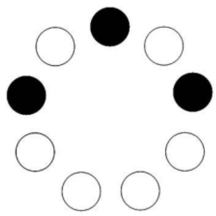
Definition 2. Bracelet representation of preference profiles. *According to the circle representation of preference orders (Formula (6) and Figure 1), we numerate preference orders from 0 to 5 in a clockwise manner. Every preference order in a profile is represented by a black bead. The space between preference orders in the circle representation of preference orders is represented by a white bead. A preference profile is a bracelet with $n+6$ beads, where exactly 6 beads are white. We take n_0 black beads, where n_0 is the number of type 0 preference orders, then one white bead, then n_1 black beads, where n_1 is the number of type 1 preference orders, etc. Adding a white bead between n_5 black beads, where n_5 is the number of type 5 preference orders, and n_0 black beads, where n_0 is the number of type 0 preference orders, we complete the bracelet.*

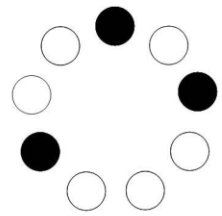
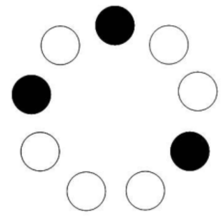
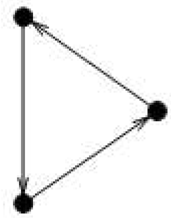
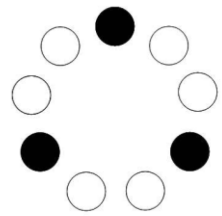
Table 1 contains examples of the bracelet representation of preference profiles. The starting point of the circle representation of preference orders and the numbering (clockwise or counterclockwise) do not matter. In each case, we obtain equivalent bracelets. The bracelet representation of preference profiles is anonymous, neutral and reverse invariant.

Permutating one pair of alternatives (one or three swaps in a preference order) leads to bracelet turnover. Two possibilities of permutating three alternatives (two swaps in a preference order) leads to preference order bracelet rotation. Reversing a preference profile leads to a rotation on 3 preference orders in a clockwise manner.

For $n=m=3$, we have $\#ANEC(3{,}3)=10$. According to Table 1, in the case of 3 alternatives and 3 agents, there are 10 different multigraphs and 7 different bracelets representing 10 ANECs (preference profiles represented different ANECS and numbering of ANECs are borrowed from (Karpov 2017)). Some ANECs have equivalent bracelet representations. These ANECs belong to the same RIANEC. For example, preference profiles $\mathcal{P}_2$ and $\mathcal{P}_3$ are reverse symmetric and bracelets generated by these preference profiles are equivalent.

Table 1. 3 agents and 3 alternatives case. List of ANECs. Multigraph and bracelet representations of preference profiles.

Number	ANEC	Multigraph representation	Bracelet representation
$\mathcal{P}_1$	$a \quad a \quad a$ $b \quad b \quad b$ $c \quad c \quad c$		
$\mathcal{P}_2$	$a \quad a \quad a$ $b \quad b \quad c$ $c \quad c \quad b$		
$\mathcal{P}_3$	$a \quad a \quad b$ $b \quad b \quad a$ $c \quad c \quad c$		
$\mathcal{P}_4$	$a \quad a \quad c$ $b \quad b \quad a$ $c \quad c \quad b$		
$\mathcal{P}_5$	$a \quad a \quad b$ $b \quad b \quad c$ $c \quad c \quad a$		
$\mathcal{P}_6$	$a \quad a \quad c$ $b \quad b \quad b$ $c \quad c \quad a$		
$\mathcal{P}_7$	$a \quad a \quad b$ $b \quad c \quad a$ $c \quad b \quad c$		

$\mathcal{P}_8$	$\begin{matrix} a & a & b \\ b & c & c \\ c & b & a \end{matrix}$		
$\mathcal{P}_9$	$\begin{matrix} a & b & c \\ b & a & a \\ c & c & b \end{matrix}$		
$\mathcal{P}_{10}$	$\begin{matrix} a & b & c \\ b & c & a \\ c & a & b \end{matrix}$		

Every $(n+6)$-bead bracelet (turnover invariant) with 6 white beads corresponds to a RIANEC. Vladimir Shevelev (A005513 in OEIS) proved the formula for the number of such bracelets. Thus, we have proposition 2.

Proposition 2. *(Vladimir Shevelev, A005513 in OEIS). For $m = 3$, the number of reverse invariant ANECs is equal to*

$$\#RIANEC(3,n) = \begin{cases} \frac{1}{12}\binom{n+5}{5} + \frac{1}{96}(n+7)(n+4)(n+2) + \frac{1}{18}(n+6), \text{if } n \equiv 0 \ (mod\ 6); \\ \frac{1}{12}\binom{n+5}{5} + \frac{1}{96}(n+5)(n+3)(n+1), \text{if } n \equiv (1\ or 5)\ (mod\ 6); \\ \frac{1}{12}\binom{n+5}{5} + \frac{1}{96}(n+7)(n+4)(n+2), \text{if } n \equiv (2\ or\ 4)(mod\ 6); \\ \frac{1}{12}\binom{n+5}{5} + \frac{1}{96}(n+5)(n+3)(n+1) + \frac{1}{18}(n+6) - \frac{1}{6}, \text{if } n \equiv 3(mod\ 6). \end{cases} \tag{7}$$

This sequence arises in several enumeration problems. To the best of the author's knowledge, Hoskins and Penfold Street (1982), who studied the geometry of fabrics, were the first with this sequence.

All ANECs are either self-symmetric or have reverse symmetric ANECs. Thus, the number of SSANECs is equal to $2\#RIANEC(3,n) - \#ANEC(3,n)$, which leads to proposition 3.

Proposition 3. *For $m = 3$, the number of self-symmetric ANECs is equal to*

$$\#SSANEC(3,n) = \begin{cases} \left\lceil \frac{1}{48}(n+4)^2(n+2) \right\rceil, \text{if } n \text{ is even}; \\ \frac{1}{48}(n+5)(n+3)(n+1), \text{if } n \text{ is odd}. \end{cases} \tag{8}$$

The number of SSANECs is relatively small, with $lim_{n\to\infty}\frac{\#SSANEC(3,n)}{\#ANEC(3,n)}=0$ and $lim_{n\to\infty}\frac{\#RIANEC(3,n)}{\#ANEC(3,n)}=\frac{1}{2}$. Almost all ANECs have their own symmetric ANEC. Appendix 2 contains a table with $\#ANEC(3,n)$, $\#RIANEC(3,n)$ and $\#SSANEC(3,n)$.

3. Voting situations induced by voting rules

Different voting rules utilize different information from preference profiles and distinguish different numbers of voting situations. By a voting situation, we mean a class of ANECs that are indiscernible, having information obtained from a procedure.

For example, for the plurality rule, only the top alternative in each preference order matters. For the case of 3 agents and 3 alternatives, we have only three types of voting situations described by the top alternatives up to renaming alternatives: (a,a,a), (a,a,b), and (a,b,c). From information that is utilized by the plurality rule, we do not distinguish $\mathcal{P}_1$ from $\mathcal{P}_2$. For the case of 3 agents and 3 alternatives, the plurality rule partitions the set of ANECs into three parts: $(\mathcal{P}_1,\mathcal{P}_2)$, $(\mathcal{P}_3,\mathcal{P}_4,\mathcal{P}_5,\mathcal{P}_6,\mathcal{P}_7,\mathcal{P}_8)$, and $(\mathcal{P}_9,\mathcal{P}_{10})$. We have three anonymous and neutral equivalent classes of preference profiles (voting situations) generated by the plurality rule. It is important to note that we do not find the result of a voting rule and do not distinguish preference profiles by the final choice. We consider only anonymous and neutral voting situations.

For the case of 4 agents and 3 alternatives, we have only three types of voting situations described by top alternatives: (a,a,a,a), (a,a,a,b), and (a,a,b,c). The run-off rule utilizes more information than the plurality rule. In this case, we have four voting situations described by the first-best and the second-best alternatives

$$(a,a,a,a),\ (a,a,a,b),\ \begin{pmatrix} a,a,b,c \\ ?,?,?,b \end{pmatrix},\ \begin{pmatrix} a,a,b,c \\ ?,?,?,a \end{pmatrix}.$$

In the last two voting situations, the partition of votes in the second round of the run-off procedure would be different. For the case of 3 agents and 3 alternatives, the run-off rule partitions the set of ANECs into three parts: $(\mathcal{P}_1,\mathcal{P}_2)$, $(\mathcal{P}_3,\mathcal{P}_4,\mathcal{P}_5,\mathcal{P}_6,\mathcal{P}_7,\mathcal{P}_8)$, and $(\mathcal{P}_9,\mathcal{P}_{10})$, which is equivalent to the plurality rule. $\mathcal{P}_9,\mathcal{P}_{10}$ are equivalent because of the equivalence of the partition of votes in the first and second rounds.

$\#Rule(m,n)$ is the number of anonymous and neutral equivalent voting situations induced by a rule. For the case of 3 agents and 3 alternatives, we have $\#Plurality(3,3)=\#Run\_off(3,3)=3$. $\#Rule(m,n)$ does not exceed $\#ANEC(m,n)$. In the case of $\#Rule(m,n)=\#ANEC(m,n)$, we can unambiguously reconstruct the ANEC from a voting situation. In other cases, that reconstruction is impossible.

The information utilization index of a rule is defined as follows:

$$I(Rule,m,n)=\frac{\#Rule(m,n)-1}{\#ANEC(m,n)-1}. \quad (9)$$

The higher the information utilization index, the more information about the preference profile we have from a voting situation. $I(Rule,m,n)=1$ corresponds to the full discernibility power of a voting rule.

If for each preference profile, its reversal belongs to the same voting situation, then a voting rule satisfies the *informational reversal symmetry property*. The Kemeny rule satisfies the informational reversal symmetry property, but the Borda rule does not. Our property differs from the reversal symmetry property (Saari 1999; Saari, Barney 2003; Crisman 2014) in social choice theory, which requires the invariance of the social choice function outcome. Apart from the

Kemeny rule, the Borda rule also satisfies the reversal symmetry property from social choice. For voting rules that satisfy the informational reversal symmetry property, $\#Rule(m,n)$ does not exceed $\#RIANEC(m,n)$. In such cases, it is reasonable to calculate the reverse invariant information utilization index:

$$I^{RI}(Rule,m,n) = \frac{\#Rule(m,n)-1}{\#RIANEC(m,n)-1}. \tag{10}$$

Using $I^{RI}(Rule,m,n)$ instead of $I(Rule,m,n)$ matters only for small n. For large n, we have $\lim_{n\to\infty} I(Rule,m,n) = \lim_{n\to\infty} 2I^{RI}(Rule,m,n)$, but for the considered rules cases, $\lim_{n\to\infty} I(Rule,m,n) = 0$. Although not true for all rules, $I(Rule,m,n)$ has a polynomial form on n for many rules, and it is convenient to distinguish rules by the polynomial degree of $\#Rule(m,n)$ on n. In the case of $m = 3$, there are 6 classes with degrees from 0 to 5.

Strong discernibility is an important property in preference diversity measurement. It has been proven that in some classes of preference diversity indices, there is no function that satisfies strong discernibility (Hashemy, Endriss 2014) and reverse invariant discernibility (Karpov 2017). Some voting rules utilize the same information as preference diversity indices and fail to achieve strong or reverse invariant discernibility.

In the following subsections, the number of voting situations induced by the plurality rule, the run-off rule, the Kemeny rule, the Borda rule, and the scoring rules in extreme cases are calculated.

3.1 The plurality rule

The plurality rule compares alternatives by the number of preference orders where an alternative occupies the top position. The plurality rule utilizes information about the partition of top choices. This partition has from 1 to m parts, which leads to proposition 4.

Proposition 4. *The number of voting situations induced by the plurality rule is equal to*

$$\#Plurality(m,n) = \sum_{i=1}^{m} p_i(n). \tag{11}$$

$\#Plurality(3,n)$ is also the number of multigraphs with 3 nodes and n edges. For each preference order in the profile, we define an edge connecting the worst alternative and the second worst alternative. Because for all $k > n$, we have $p_k(n) = 0$, then for each n, the sequence $\#Plurality(i,n)$ has an upper bound.

Corollary 1. (A001399, OEIS**).** *For $m = 3$, the number of equivalent classes generated by the plurality rule is equal to*

$$\#Plurality(3,n) = round\left[\frac{1}{12}(n+3)^2\right]. \tag{12}$$

Note that ties in rounding in this formula never arise.

The values of $\#Plurality(3,n)$ and the number of voting situations induced by other rules are given in Appendix 2.

3.2 The run-off rule

The run-off rule has two rounds. In the first round, the plurality rule is used, and if there is an absolute majority winner, then the procedure terminates. If not, then the two alternatives with the highest number of votes are promoted to the next round. In the second round, the alternative with the higher number of votes wins.

The run-off rule utilizes information about the partition of voters' top choices and the partition of votes between the two alternatives with the highest number of plurality votes if there is no absolute majority winner in the first round.

For the 3 alternatives and 3 agents case, we have the following voting situations (ANECs and corresponding partitions of the top choices and partition of votes in the second round):

$$\mathcal{P}_1, \mathcal{P}_2\text{: } (3,0,0);$$
$$\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8\text{: } (2,1,0);$$
$$\mathcal{P}_9, \mathcal{P}_{10}\text{: } (1,1,1), (2,1).$$

If the two alternatives with the highest number of votes in the first round have different numbers of votes, then we differentiate situations when there is the same winner in the second round, and when the second winner differs from the first round winner. For example, voting situation with partitions $(11,10,9), (16,14)$ and $(11,10,9), (14,16)$ are different.

Proposition 5. *The number of voting situations induced by the run-off rule is equal to*

$$\#Run\_off(m,n) = \left\lceil \frac{n+1}{2} \right\rceil + \sum_{i=3}^{m} \left[\sum_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} p_{i-1}\left(\left\lceil \frac{n}{2} \right\rceil - j \right) + \sum_{j=\left\lceil \frac{n}{i} \right\rceil}^{\left\lfloor \frac{n}{2} \right\rfloor} \left(p_{i-2,j}(n-2j) \left(\left\lceil \frac{n+1}{2} \right\rceil - \mathrm{j} \right) + \sum_{\substack{\lambda \vdash n-j, \\ i-1, j-1}} (n + 1 - \max(\lambda) - \mathrm{j}) \right) \right], \tag{13}$$

where $\sum_{\substack{\lambda \vdash n-j, \\ i-1, j-1}}$ *is the sum over all partitions of* $n-j$ *with* $i-1$ *parts, each part does not exceed* $j-1$, *and* $max(\lambda)$ *is the highest element of partition* λ.

Corollary 2. *For* $m = 3$, *the number of voting situations induced by the run-off rule is equal to*

$$\#Run\_off(3,n) = \begin{cases} \left\lceil \frac{1}{1728}(7n^3 + 192n^2 + 1008n + 1088) \right\rceil, if\ n \equiv (0\ or\ 4)\ (mod\ 12); \\ \left\lceil \frac{1}{1728}(7n^3 + 165n^2 + 729n + 827) \right\rceil, if\ n \equiv (1\ or\ 9)\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 192n^2 + 804n + 1024), if\ n \equiv 2\ (mod\ 12); \\ \left\lceil \frac{1}{1728}(7n^3 + 165n^2 + 837n + 935) \right\rceil, if\ n \equiv (3\ or\ 7)\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 165n^2 + 633n + 475), if\ n \equiv 5\ (mod\ 12); \\ \left\lceil \frac{1}{1728}(7n^3 + 192n^2 + 900n + 1088) \right\rceil, if\ n \equiv (6\ or\ 10)\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 192n^2 + 912n + 1024), if\ n \equiv 8\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 165n^2 + 741n + 583), if\ n \equiv 11\ (mod\ 12). \end{cases} \tag{14}$$

The proof for Proposition 5, Corollary 2 and subsequent propositions are given in Appendix 1. For $\mathrm{m} = 3$, the run-off rule coincides with the alternative vote method (the single transferable vote).

The main difference in the number of voting situations is between even and odd n, but divisibility by 3 and 4 also matters. The function $\#Run\_off(3,n)$ is not monotone on n. We have relatively more voting situations for even n.

For small n, the function $\#Run\_off(3,n)$ is close to $\#Plurality(3,n)$. For example, $\#Run\_off(3,5) = 5$ and $\#Plurality(3,5) = 5$. The reason is that almost all the partitions for top choices lead to an absolute majority winner. Only if the highest part of the partition is greater than or equal to $\left\lceil \frac{n}{3} \right\rceil$ and less than or equal to $\left\lfloor \frac{n}{2} \right\rfloor$ do we have the second round. There are approximately $\frac{n}{6}$ ways to define the highest part of the partition, for which we have the second round.

3.4 The Borda rule

The Borda rule is a scoring rule in which the worst alternative has a score of 0, and the best alternative has a score of $m-1$. The Borda rule utilizes information only about the sum of scores for each alternative.

For the 3 alternatives and 3 agents case, we have the following voting situations (ANECs and corresponding sums of scores vector in decreasing order):

$$\mathcal{P}_1\colon \begin{pmatrix}6\\3\\0\end{pmatrix}; \mathcal{P}_2\colon \begin{pmatrix}6\\2\\1\end{pmatrix}; \mathcal{P}_3\colon \begin{pmatrix}5\\4\\0\end{pmatrix}; \mathcal{P}_4\colon \begin{pmatrix}5\\2\\2\end{pmatrix}; \mathcal{P}_5\colon \begin{pmatrix}4\\4\\1\end{pmatrix}; \mathcal{P}_6, \mathcal{P}_8, \mathcal{P}_9\colon \begin{pmatrix}4\\3\\2\end{pmatrix}; \mathcal{P}_7\colon \begin{pmatrix}5\\3\\1\end{pmatrix}; \mathcal{P}_{10}\colon \begin{pmatrix}3\\3\\3\end{pmatrix}.$$

Voting situations correspond to different partitions of the sum of the scores.

Proposition 6. *For $m = 3$ and $n \geq 2$, the number of voting situations induced by the Borda rule is equal to*

$$\#Borda(3,n) = \left\lceil \frac{1}{2}(n+1)^2 \right\rceil. \quad (15)$$

3.3.The Kemeny rule

The Kemeny rule uses the swap distance between preference orders (the number of pairwise swaps of consecutive alternatives that is needed to transform one order into another). Having the bracelet representation of preference profiles (Figure 1), we obtain a circle of distances from order P_0 (Figure 2). The circles of distances from other preference orders can be obtained by rotating the circle of distances from figure 2.

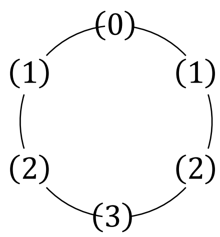


Figure 2. Circle of distances from order P_0.

Permutating one pair of alternatives (one or three swaps in a preference order) leads to the turnover of the circle in Figure 2. Two possibilities of permutating three alternatives (two swaps in the preference order) leads to the circle rotating. Reversing the preference profile leads to the circle rotating on 3 preference orders in a clockwise manner.

The Kemeny rule finds the order with the lowest sum of swap distances from the order to all orders in the preference profile. The Kemeny rule utilizes information only about the sums of swap distances between the preference profile and different preference orders.

For the 3 alternatives and 3 agents case, we have the following voting situations (ANECs and the corresponding sums of distances in the circle form, which is invariant up to rotating and turnover):

$$P_0: -0-3-6-9-6-3-;$$
$$P_2, P_3: -1-2-5-8-7-4-;$$
$$P_4, P_5: -2-3-4-7-6-5-;$$
$$P_6, P_8, P_9: -3-4-5-6-5-4-;$$
$$P_7: -2-3-6-8-6-3-;$$
$$P_{10}: -4-5-4-5-4-5-.$$

Proposition 7. *For $m=3$, the number of voting situations induced by the Kemeny rule is equal to*

$$\#Kemeny(3,n)=\begin{cases}\left\lceil \frac{1}{72}(4n^3+21n^2+54n+56)\right\rceil, \text{if } n \text{ is even;}\\ \left\lceil \frac{1}{72}(4n^3+21n^2+36n+11)\right\rceil, \text{if } n \text{ is odd.}\end{cases} \quad (16)$$

The Kemeny rule is more complicated and has a higher information utilization index than the rules above. Bartholdi, Tovey, and Trick (1989) showed that determining the optimal Kemeny ranking in an election is NP-hard. Hemaspaandra et al. (2005) provided a stronger bound of computational complexity of the Kemeny rule. Zwicker (2018) introduced a generalization of Kemeny's voting rule. Special cases of this generalization include the Borda rule and plurality voting, which are computationally tractable. Zwicker (2018) showed that computational complexity of the Kemeny rule arises from the cyclic part in the fundamental decomposition of a weighted tournament into cyclic and co-cyclic components. This cyclic part is associated with the Condorcet paradox.

3.4 The discernibility potential of the scoring rules

The scoring rules utilize information only about the sum of scores for each alternative. In an extreme case of irrational scores (e.g., for $m=3$ scores $0, 1, \sqrt{2}$), each combination of the scores (e.g., for $m=3$, $n=5$ scores $0, 0, 1, \sqrt{2}, \sqrt{2}$) leads to a unique sum of scores ($1+2\sqrt{2}$ in our example). In this case, we can unambiguously derive the combination of scores from the sum of the score vector. In other words, we can derive the scoring matrix (the scorix in the terminology of Pérez-Fernández and De Baets (2017)). The element at the *i*-th row and *j*-th column of this matrix equals the number of times that the *i*-th candidate is ranked at the *j*-th position

$$a_{ij}=|\{k|pos(P_k,i)=j\}|. \quad (17)$$

The scoring matrix is used not only for scoring rules but also for other voting rules, e.g., for the threshold rule (Aleskerov, Chistyakov, Kalyagin 2010), and the rank-dependent scoring rules (Goldsmith et al 2016). This matrix was investigated in pure mathematics in (MacMahon 1918), where we find 10 scoring matrices for the 3 alternatives and 3 agents case, which corresponds to the voting situations (ANECs and corresponding scoring matrices):

$$\mathcal{P}_1\colon \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}; \mathcal{P}_2\colon \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}; \mathcal{P}_3\colon \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}; \mathcal{P}_4\colon \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}; \mathcal{P}_5\colon \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix};$$
$$\mathcal{P}_6\colon \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}; \mathcal{P}_7\colon \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}; \mathcal{P}_8\colon \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}; \mathcal{P}_9\colon \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}; \mathcal{P}_{10}\colon \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

From the generating function in (A257464 in OEIS) and the author's calculations, we derive Proposition 8.

Proposition 8. *For* $m = 3$*, the number of voting situations induced by the scoring rules in the extreme case is equal to*

$$\#Scoring(3, n) = \begin{cases} \left\lceil \frac{1}{48}(n^4 + 6n^3 + 18n^2 + 36n + 32) \right\rceil, \text{if } n \text{ is even;} \\ \left\lceil \frac{1}{48}(n^4 + 6n^3 + 18n^2 + 18n + 5) \right\rceil, \text{if } n \text{ is odd.} \end{cases} \tag{18}$$

MacMahon (1918) contains series for higher m. In the extreme case, the scoring rule has a higher degree polynomial on n than the other scoring rules considered in this paper (the plurality rule and the Borda rule). For $m = 3$, the simplest example of indiscernibility is:

$$\hat{\mathcal{P}} = \begin{pmatrix} a & a & b & c \\ b & b & c & a \\ c & c & a & b \end{pmatrix}; \tilde{\mathcal{P}} = \begin{pmatrix} a & a & b & c \\ b & c & a & b \\ c & b & c & a \end{pmatrix}.$$

$\hat{\mathcal{P}}$ and $\tilde{\mathcal{P}}$ belong to different ANECs. The last three preference orders are different in different preference profiles, but they have the same scoring matrix. These preference orders represent different versions of the Condorcet cycle. Having the scoring matrix, it is impossible to reconstruct ANEC. For small n, there is a small number of such situations, but for big n the vast majority of ANECs have such indiscernibility.

4. Conclusion

Appendix 2 contains a table with the number of ANECs, the number of voting situations, the polynomial degree and the information utilization index for the above-mentioned rules for the 3 alternatives case.

Only the run-off rule has a non-monotonic function of the number of voting situations. It has a smaller number of voting situations in the case of an odd n. There are two reasons for this phenomenon. First, in the case of an odd n, the absolute majority in the first round arises more frequently. Second, in the second round, the number of partitions for even and odd consecutive numbers are equal. The additional voter in the case of an odd n does not add additional partitions in the second round.

All other rules induce monotonic functions of the number of voting situations. The plurality rule has the lowest number of voting situations. It is the simplest rule, and the information utilization index reflects this. The scoring rules in the extreme case have the highest number of voting situations. For the other rules, the highest information utilization index occurs with the Kemeny rule, which is more complicated in comparison with the other rules.

The number of voting situations induced by the plurality and Borda rules is represented by a polynomial in n of degree 2. At the limit, the number of voting situations induced by the Borda rule is 6 times higher than the number of voting situations induced by the plurality rule. The number of voting situations induced by the run-off and Kemeny rules is represented by a polynomial in n of degree 3. At the limit, the number of voting situations induced by the Kemeny rule is 13.7 times higher than the number of voting situations induced by the run-off rule.

For $n \leq 97$, the Borda rule outreaches the run-off rule for the number of voting situations, but for $n \geq 102$, we have the reverse order. The reason is that for small n, the number of voting situations with the second round is relatively small, and the run-off rule behaves close to the plurality rule, which has a degree 2 polynomial.

Strong discernibility arises only three times in the class of scoring rules: $I(Scoring, 3,2) = I(Scoring, 3,3) = I(Borda, 3,2) = 1$. In other cases, it is impossible to reconstruct the ANEC from the voting situation. The information utilization index decreases relatively rapidly for all of the above-mentioned rules, except for the scoring rule in the extreme case. A general comparison of a higher number of voting rules is the goal for the future research.

Appendix 1.

Proof of proposition 5. We have $1 + p_2(n)$ partitions of top choices with 1 or 2 parts. In this case, there is no second round, or the second round does not add new information. Let i be the number of parts in a partition of top choices; then, $\sum_{j=1}^{\left\lfloor\frac{n}{2}\right\rfloor} p_{i-1}\left(\left\lceil\frac{n}{2}\right\rceil - j\right)$ is the number of partitions with an absolute winner in the first round,

$$\sum_{j=\left\lceil\frac{n}{i}\right\rceil}^{\left\lfloor\frac{n}{2}\right\rfloor}\left(p_{i-2,j}(n-2j)p_{2,n-\mathrm{j}}(n)\right)$$

is the the number of voting situations with the second round and equal number of plurality votes of the two winners in the first round,

$$\sum_{j=\left\lceil\frac{n}{i}\right\rceil}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{\substack{\lambda\vdash n-j,\\ i-1,j-1}}\left(p_{2,n-\max(\lambda)}(n) + p_{2,n-\mathrm{j}}(n) - 1_{n\ is\ even}\right),$$

where $1_{\text{n is even}} = 1$ if n is even and $1_{\text{n is even}} = 0$, if n is odd;
is the number of situations with the second round and unequal number of plurality votes of the two winners in the first round. Thus, we have

$$\#Run\_off(m,n) = \left\lceil\frac{n+1}{2}\right\rceil + \sum_{i=3}^{m}\left[\sum_{j=1}^{\left\lfloor\frac{n}{2}\right\rfloor} p_{i-1}\left(\left\lceil\frac{n}{2}\right\rceil - j\right) + \sum_{j=\left\lceil\frac{n}{i}\right\rceil}^{\left\lfloor\frac{n}{2}\right\rfloor}\left(p_{i-2,j}(n-2j)p_{2,n-\mathrm{j}}(n) + \sum_{\substack{\lambda\vdash n-j,\\ i-1,j-1}}\left(p_{2,n-\max(\lambda)}(n) + p_{2,n-\mathrm{j}}(n) - 1_{n\ is\ even}\right)\right)\right].$$

Substituting formula (2), we obtain the result. ■

Proof of corollary 2. For $m = 3$, formula (13) is as follows

$$\left\lceil\frac{n+1}{2}\right\rceil + \sum_{j=1}^{\left\lfloor\frac{n}{2}\right\rfloor}\left\lfloor\frac{\left\lceil\frac{n}{2}\right\rceil - j}{2}\right\rfloor + \sum_{j=\left\lceil\frac{n}{3}\right\rceil}^{\left\lfloor\frac{n}{2}\right\rfloor}\left(1_{n-2j\geq 1}\left(\left\lceil\frac{n+1}{2}\right\rceil - \mathrm{j}\right) + 1_{j-1\geq\left\lceil\frac{n-j}{2}\right\rceil}\sum_{k=\left\lceil\frac{n-j}{2}\right\rceil}^{j-1}(n+1-\mathrm{j}-k)\right),$$

Calculating sums we obtain

$$\#Run\_off(3,n) = \begin{cases} \frac{1}{1728}(7n^3 + 192n^2 + 1008n + 1728), if\ n \equiv 0\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 165n^2 + 729n + 827), if\ n \equiv 1\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 192n^2 + 804n + 1024), if\ n \equiv 2\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 165n^2 + 837n + 999), if\ n \equiv 3\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 192n^2 + 1008n + 1088), if\ n \equiv 4\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 165n^2 + 633n + 475)\ , if\ n \equiv 5\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 192n^2 + 900n + 1728), if\ n \equiv 6\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 165n^2 + 837n + 935), if\ n \equiv 7\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 192n^2 + 912n + 1024), if\ n \equiv 8\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 165n^2 + 729n + 891), if\ n \equiv 9\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 192n^2 + 900n + 1088), if\ n \equiv 10\ (mod\ 12); \\ \frac{1}{1728}(7n^3 + 165n^2 + 741n + 583), if\ n \equiv 11\ (mod\ 12). \end{cases}$$

Modifying we obtain the result. ■

Proof of proposition 6. For $m=3$, a homogenous preference profile (n preference orders P_0) has vector of rank sums $\begin{pmatrix} 2n \\ n \\ 0 \end{pmatrix}$. Any preference profile has vector of ranks sums either $\begin{pmatrix} 2n \\ n \\ 0 \end{pmatrix}$ or vectors $\alpha, \beta, \gamma, \delta, \varepsilon$, defined here:

$$\alpha = \begin{pmatrix} 2n-x \\ n+x \\ 0 \end{pmatrix}; \beta = \begin{pmatrix} 2n \\ n-x \\ x \end{pmatrix}; \gamma = \begin{pmatrix} 2n-y \\ n-(x-y) \\ x \end{pmatrix}; \delta = \begin{pmatrix} 2n-x \\ n+y \\ x-y \end{pmatrix}; \varepsilon = \begin{pmatrix} 2n-x \\ n \\ x \end{pmatrix}.$$

For all vectors $\alpha, \beta, \gamma, \delta, \varepsilon$, we have that the first component is not less than the second; the second component is not less than the third.

Substituting x preference orders from P_0 to P_5, we obtain type α preference profile. There are $\left\lfloor \frac{n}{2} \right\rfloor$ preference profiles of type α. Similarly, we have $\left\lfloor \frac{n}{2} \right\rfloor$ voting situations of type β.

Substituting $y/2$ preference orders from P_0 to P_3 and $x-y$ preference orders from P_0 to P_1, we obtain type γ voting situations with even y. Substituting $(y-1)/2$ preference orders from P_0 to P_3, $x-y-1$ preference orders from P_0 to P_1, and one preference order from P_0 to P_2, we obtain type γ voting situations with odd y.

We can construct all possible type γ voting situations using these two design methods. There are two natural restrictions on x, y for saving the order of alternatives:

$$x + x - y \leq n,$$
$$y - n \leq x - y.$$

From this we find the number of type γ voting situations (if $n \geq 4$):

for $y \leq x-y$, it is $\sum_{i=2}^{\left\lfloor \frac{n}{2} \right\rfloor} p_2(i) + \sum_{i=\left\lfloor \frac{n}{2} \right\rfloor+1}^{\left\lfloor \frac{2n}{3} \right\rfloor} p_{2,n-i}(i)$;

for $y \geq x-y$, it is $\sum_{i=2}^{\left\lfloor \frac{2n}{3} \right\rfloor} p_2(i) + \sum_{i=\left\lfloor \frac{2n}{3} \right\rfloor+1}^{n-1} \left[p_2(i) - p_{2,2i-n-1}(i)\right]$;

for $y = x-y$, it is $\left\lfloor \frac{\left\lfloor \frac{2n}{3} \right\rfloor}{2} \right\rfloor$.

Thus, the number of type γ voting situations is equal to

$$\sum_{i=2}^{\left\lfloor \frac{n}{2} \right\rfloor} p_2(i) + \sum_{i=\left\lfloor \frac{n}{2} \right\rfloor+1}^{\left\lfloor \frac{2n}{3} \right\rfloor} p_{2,n-i}(i) + \sum_{i=2}^{\left\lfloor \frac{2n}{3} \right\rfloor} p_2(i) + \sum_{i=\left\lfloor \frac{2n}{3} \right\rfloor+1}^{n-1} \left[p_2(i) - p_{2,2i-n-1}(i)\right] - \left\lfloor \frac{\left\lfloor \frac{2n}{3} \right\rfloor}{2} \right\rfloor.$$

It is also the number of type δ voting situations.

Substituting $x/2$ preference orders from P_0 to P_3, we obtain type ε voting situations with even x. Substituting $(x-1)/2$ preference orders from P_0 to P_3, one preference order from P_0 to P_1, and one preference order from P_0 to P_5, we obtain type γ voting situations with odd x.

We can construct all possible type ε voting situations using these two design methods. There is one restriction on x for saving the order of alternatives:

$$x \leq n.$$

If $n \geq 2$, then the number of type ε voting situations is equal to n.

Summing over all types, we have (if $n \geq 4$)

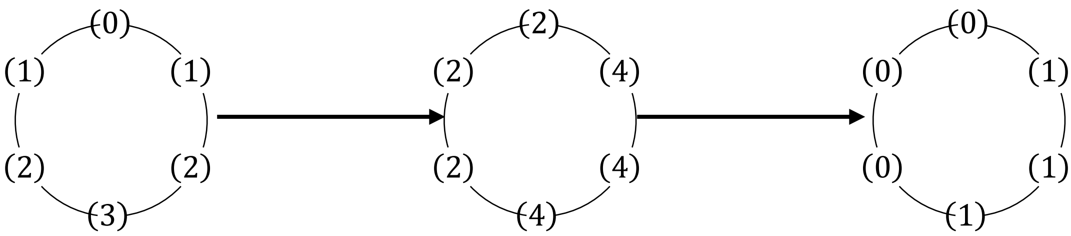
$$1 + 2\left(\left\lfloor\frac{n}{2}\right\rfloor + \sum_{i=2}^{\left\lfloor\frac{n}{2}\right\rfloor} p_2(i) + \sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1}^{\left\lfloor\frac{2n}{3}\right\rfloor} p_{2,n-i}(i) + \sum_{i=2}^{\left\lfloor\frac{2n}{3}\right\rfloor} p_2(i) + \sum_{i=\left\lfloor\frac{2n}{3}\right\rfloor+1}^{n-1}\left[p_2(i) - p_{2,2i-n-1}(i)\right] - \left\lfloor\frac{\left\lfloor\frac{2n}{3}\right\rfloor}{2}\right\rfloor\right) + n.$$

Modifying this, we obtain

$$3n - 1 + \left\lfloor\frac{2n}{3}\right\rfloor^2 - \left\lfloor\frac{2n}{3}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor + 3 - 2n\right) + 2\left(\sum_{i=2}^{\left\lfloor\frac{n}{2}\right\rfloor}\left\lfloor\frac{i}{2}\right\rfloor + \sum_{i=2}^{n-1}\left\lfloor\frac{i}{2}\right\rfloor - \sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1}^{\left\lfloor\frac{2n}{3}\right\rfloor}\left\lfloor\frac{i-1}{2}\right\rfloor + \sum_{i=\left\lfloor\frac{2n}{3}\right\rfloor+1}^{n-1}\left(\left\lfloor\frac{i-1}{2}\right\rfloor\right) - \left\lfloor\frac{\left\lfloor\frac{2n}{3}\right\rfloor}{2}\right\rfloor\right).$$

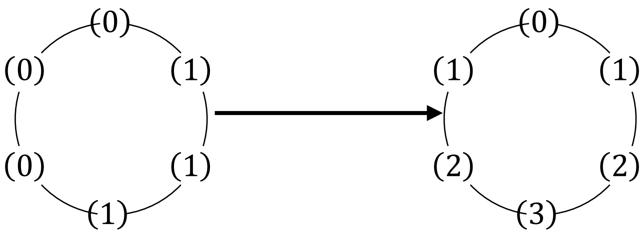
Calculating the sums, we obtain the result, which is also correct for $n = 2$, and $n = 3$. ∎

Proof of proposition 7. Let $f_0(k)\colon \mathbb{N}_0 \to \mathbb{N}_0$ be the sum of distances between preference order P_0 and preference order P_i, where $i \equiv k \ (mod\ 6)$. Let $g_0(k) = f_0(k) + f_0(k+2)$ and $h_0(k) = \big(2g_0(k) - g_0(k+3)\big)/6 = (2f_0(k) + 2f_0(k+2) - f_0(k+3) - f_0(k+5))/6$. This transformation is presented in figure 3a. The first circle is the circle of distances from order P_0 to all six preference orders. After transformation, each preference order is presented by three subsequent ones. This transformation has the inversion presented in figure 3b. We design one-to-one correspondence between $f_0(k)$ and $h_0(k)$.



$f_0(k)$ $g_0(k)$ $h_0(k)$

Figure 3a



$h_0(k)$ $f_0(k) = h_0(k) + h_0(k-1) + h_0(k-2).$

Figure 3b

In the same fashion, we define functions $f_j(k), g_j(k), h_j(k)$, $j = \overline{0,5}$. Instead of summing distances from preference orders $\sum_{j=0}^{5} f_j(k)n_j$, we sum transformed values $\sum_{j=0}^{5} h_j(k)n_j$. We have bisection between these sums. Any sum of transformed values $\sum_{j=0}^{5} h_j(k)n_j$ can be represented by the circle presented in figure 4.

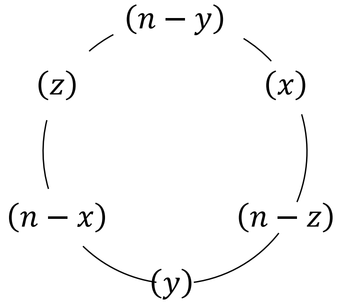


Figure 4

Permutating one pair of alternatives (one or three swaps in a preference order) leads to figure 4 circle turnover. Two possibilities of permutating three alternatives (two swaps in a preference order) lead to the circle rotating. Reversing the preference profile leads to a rotating on 3 preference orders in a clockwise manner. By these operations we can always construct a circle, such that $x+y+z \leq 3n-x-y-z$ and $x \geq y \geq z$. From this definition, we have the following additional restrictions on x, y, z

$$x+y \geq n-z;$$
$$n-y+n-z \geq x;$$
$$z+n-y+n-z+y \geq 2x+2(n-x).$$

From these inequalities, we have $n \leq x+y+z \leq \left\lfloor \frac{3n}{2} \right\rfloor$. We will calculate the number of partitions of $i = \overline{n, \left\lfloor \frac{3n}{2} \right\rfloor}$ with 1, 2, or 3 parts, such that each part does not exceed n (and the special case of $i = \frac{3n}{2}$). A complementary partition has sum $3n-i$.

If $x+y+z = \frac{3n}{2}$, then the partitions of $i = \frac{3n}{2}$ and $3n-i = \frac{3n}{2}$ can be the same. Partitions with $y = \frac{n}{2}$ are symmetric ($n-x=z$, $n-z=x$, $n-y=y$). The number of such partitions is equal to $\frac{n}{2}+1$. All other partitions of $i = \frac{3n}{2}$ have different complement partitions with the same sum. The number of partitions of $\frac{3n}{2}$ and complement partitions of $\frac{3n}{2}$ with 1, 2, or 3 parts, such that each part does not exceed n, is equal to

$$\frac{1}{2}\left(p_{2,n}\left(\frac{3n}{2}\right) + p_{3,n}\left(\frac{3n}{2}\right) + \frac{n}{2} + 1\right) =$$
$$= \frac{1}{2}\left(1 + n - \left\lfloor \frac{n}{4} \right\rfloor \left(\frac{n}{2} - 1\right) + \frac{n^2}{2} + \left\lfloor \frac{n}{4} \right\rfloor^2 - \left\lfloor \frac{3n-2}{4} \right\rfloor - \sum_{j=1}^{\frac{n}{2}} \left\lfloor \frac{3n-2j-2}{4} \right\rfloor\right). \qquad (*)$$

For all other i, we have

$$1 + \sum_{i=n}^{\left\lfloor \frac{3n-1}{2} \right\rfloor} p_{2,n}(i) + \sum_{i=n}^{\left\lfloor \frac{3n-1}{2} \right\rfloor} p_{3,n}(i).$$

For $n \geq 3$, we have

$$1 + \left\lfloor \frac{n}{2} \right\rfloor + \sum_{i=n+1}^{\left\lfloor \frac{3n-1}{2} \right\rfloor} \left(n - \left\lfloor \frac{i-1}{2} \right\rfloor\right) + \sum_{i=n}^{\left\lfloor \frac{3n-1}{2} \right\rfloor} \sum_{j=1}^{\left\lfloor \frac{i}{3} \right\rfloor} \left(\min\{i-2j, n\} - \left\lfloor \frac{i-j-1}{2} \right\rfloor\right). \qquad (**)$$

Summing (*) for even n and (**) for all n, we obtain the result, which is also correct for $n=1$ and $n=2$. ■

Appendix. 2. Number of voting situations, m=3.

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	Polinomial degree
#ANEC	5	10	24	42	83	132	222	335	511	728	1047	1428	1956	2586	3414	4389	5638	7084	8888	5
#RIANEC	4	7	16	26	50	76	126	185	280	392	561	756	1032	1353	1782	2277	2920	3652	4576	5
#SSANEC	3	4	8	10	17	20	30	35	49	56	75	84	108	120	150	165	202	220	264	3
share in #ANEC	0.60	0.40	0.33	0.24	0.20	0.15	0.14	0.10	0.10	0.08	0.07	0.06	0.06	0.05	0.04	0.04	0.04	0.03	0.03	
Plurality	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44	2
I(Rule,3,n)	0.25	0.22	0.13	0.10	0.07	0.05	0.04	0.03	0.03	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.00	
run-off	2	3	5	5	9	10	14	15	21	22	31	31	40	43	55	54	70	72	88	3
I(Rule,3,n)	0.25	0.22	0.17	0.10	0.10	0.07	0.06	0.04	0.04	0.03	0.03	0.02	0.02	0.02	0.02	0.01	0.01	0.01	0.01	
Kemeny	4	6	12	17	28	37	54	69	93	115	148	178	221	261	315	366	433	496	577	3
I(Rule,3,n)	0.75	0.56	0.48	0.39	0.33	0.27	0.24	0.20	0.18	0.16	0.14	0.12	0.11	0.10	0.09	0.08	0.08	0.07	0.06	
I^{IR}(Rule,3,n)	1	0.83	0.73	0.64	0.55	0.48	0.42	0.37	0.33	0.29	0.26	0.23	0.21	0.19	0.18	0.16	0.15	0.14	0.13	
Borda	5	8	13	18	25	32	41	50	61	72	85	98	113	128	145	162	181	200	221	2
I(Rule,3,n)	1	0.78	0.52	0.41	0.29	0.24	0.18	0.15	0.12	0.10	0.08	0.07	0.06	0.05	0.04	0.04	0.03	0.03	0.02	
Scoring	5	10	23	40	73	114	180	262	379	521	712	938	1228	1567	1986	2469	3052	3715	4499	4
I(Rule,3,n)	1	1	0.96	0.95	0.88	0.86	0.81	0.78	0.74	0.72	0.68	0.66	0.63	0.61	0.58	0.56	0.54	0.52	0.51	

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