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A multicolour graph as a complete topological invariant for Ω-stable flows without periodic trajectories on surfaces

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Abstract. Studying the dynamics of a flow on surfaces by partitioning the phase space into cells with the same limit behaviour of trajectories within a cell goes back to the classical papers of Andronov, Pontryagin, Leontovich and Maier. The types of cells (the number of which is finite) and how the cells adjoin one another completely determine the topological equivalence class of a flow with finitely many special trajectories. If one trajectory is chosen in every cell of a rough flow without periodic orbits, then the cells are partitioned into so-called triangular regions of the same type. A combinatorial description of such a partition gives rise to the three-colour Oshemkov-Sharko graph, the vertices of which correspond to the triangular regions, and the edges to separatrices connecting them. Oshemkov and Sharko proved that such flows are topologically equivalent if and only if the three-colour graphs of the flows are isomorphic, and described an algorithm of distinguishing three-colour graphs. But their algorithm is not efficient with respect to graph theory. In the present paper, we describe the dynamics of Ω-stable flows without periodic trajectories on surfaces in the language of four-colour graphs, present an efficient algorithm for distinguishing such graphs, and develop a realization of a flow from some abstract graph.

Bibliography: 17 titles.

Keywords: multicolour graph, topological invariant, Ω-stable flow, efficient algorithm.

§ 1. Introduction

A traditional approach to the qualitative study of the dynamics of flows with finitely many special trajectories on surfaces consists in isolating regions on the supporting manifold with predictable behaviour of trajectories — cells. This view of continuous dynamical systems goes back to the classical paper of Andronov and

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Pontryagin [2] of 1937, in which they considered a system of differential equations

$$\dot{x} = v(x), \tag{$*$}$$

where $v(x)$ is a C^1-vector field defined in a disc on the plane whose boundary is a curve without contact, and found a criterion for the roughness of the system ($*$).

In the papers of Leontovich and Maier [8], [9], a more general class of dynamical systems was considered, and their classification was also based on ideas of isolation of a set of special trajectories whose relative disposition (the Leontovich-Maier scheme) completely determines the qualitative structure of the partition of the phase space of the dynamical system into trajectories. The main difficulty in generalizing this result to the case of arbitrary orientable surfaces of positive genus is the possibility of a new type of motion — a nonclosed recurrent trajectory. The absence of such trajectories for rough flows without singularities on a 2-torus was proved by Maier [10] in 1939. In 1971 Peixoto [15] generalized the Leontovich-Maier scheme for structurally stable flows on arbitrary surfaces and obtained a topological classification of such flows, again by analysing all admissible cells for them and by introducing a combinatorial invariant — a directed graph generalizing the Leontovich-Maier scheme.

In 1976 Neumann and O'Brien [12] considered so-called *regular flows* on arbitrary surfaces — flows without nontrivial periodic trajectories, which include the flows described above as a special case. They introduced a complete topological invariant for regular flows — the *orbit complex*, which is the space of orbits of the flow endowed with certain additional information. In 1998 Oshemkov and Sharko [13] introduced a new invariant for structurally stable systems on surfaces — a three-colour graph, and described an algorithm for recognizing isomorphism of such graphs, which, however, is not efficient, that is, its working time is not bounded by some polynomial of the length of definition of the input information. In 2014 Grines, Kapkaeva and Pochinka [4] used three-colour Oshemkov-Sharko graphs to obtain a topological classification of gradient-like diffeomorphisms on surfaces.

In the present paper we consider the class G consisting of Ω-stable flows f^t without periodic trajectories on surfaces S that have at least one saddle point. With every flow of the class under consideration, we associate a four-colour graph, present an efficient algorithm for distinguishing such graphs, and construct a standard representative in every topological equivalence class.

§ 2. Statement of the results

Let f^t be a flow that belongs to the class G consisting of Ω-stable flows f^t without periodic trajectories on surfaces S each of which has at least one saddle point[1]. Recall that a flow f^t is said to be Ω-*stable* if there exists a neighbourhood of it $U(f^t)$ in $C^1(S \times \mathbb{R}, S)$ such that if $f'^t \in U(f^t)$, then the flows f^t and f'^t are topologically equivalent on the nonwandering sets Ω_{f^t} and $\Omega_{f'^t}$, that is, there exists a homeomorphism $h\colon S \to S$ taking the nonwandering trajectories of the flow f^t to the nonwandering trajectories of the flow f'^t preserving the direction of motion

[1]If the flow f^t has no saddle points, then it has exactly two fixed points: a source and a sink, and all such flows are topologically equivalent; therefore we exclude them from the class under consideration.

along the trajectories. It follows from the criterion of Ω-stability (see [16]) that f^t has nonwandering set consisting of finitely many hyperbolic fixed points, and does not have *cycles*, that is, sets of fixed points

$$x_1, \dots, x_k, x_{k+1} = x_1$$

with the property

$$W^s_{x_i} \cap W^u_{x_{i+1}} \neq \varnothing, \qquad i = 1, \dots, k.$$

Here, the flow f^t can be either structurally stable or not, and this is determined by the absence or presence of *connections* — separatrices going from a saddle to a saddle. The condition of Ω-stability implies that the connections of the flow f^t do not form closed curves.

Let $\Omega^0_{f^t}$, $\Omega^1_{f^t}$, $\Omega^2_{f^t}$ denote the sets of all sinks, saddles and sources of the flow f^t, respectively. We set

$$\widetilde{S} = S \setminus \bigl(W^u_{\Omega^0_{f^t} \cup \Omega^1_{f^t}} \cup W^s_{\Omega^1_{f^t} \cup \Omega^2_{f^t}}\bigr).$$

A connected component of the set $\widetilde{S}$ is called a *cell*. Let J_{f^t} denote the set of all cells of the flow f^t. By Lemma 4 (see § 3), the boundary of every cell $J \in J_{f^t}$ contains a unique source α and a unique sink ω, while the whole cell is the union of trajectories going from α to ω. We choose one trajectory θ_J in each cell J and call it a *t-curve*. We set

$$\mathscr{T} = \bigcup_{J \in J_{f^t}} \theta_J, \qquad \overline{S} = \widetilde{S} \setminus \mathscr{T}.$$

A connected component of the set $\overline{S}$ is called a *polygonal region*. Let Δ_{f^t} denote the set of all polygonal regions of the flow f^t. Recall that a *stable* (respectively, *unstable*) *separatrix* of a saddle point σ is defined as a connected component of the set $W^s_\sigma \setminus \{\sigma\}$ (respectively, $W^u_\sigma \setminus \{\sigma\}$). We define *c-curves* to be separatrices connecting saddles (connections), *u-curves* to be unstable saddle separatrices that are not connections, and *s-curves* to be stable saddle separatrices that are not connections. By Lemma 5 (see § 3) the closure of any polygonal region looks as depicted in Figure 1. Any trajectory in Δ goes from α to ω, and the boundary of a polygonal region consists of the closures of saddle separatrices and a t-curve. We consider the boundary of every polygonal region to be oriented correspondingly to the motion along the t-curve from the source to the sink.

We associate with a flow $f^t \in G$ a multicolour graph Γ_{f^t} as follows (Figure 2):

1) the vertices of the graph Γ_{f^t} are in a one-to-one correspondence with the polygonal regions of the set Δ_{f^t};

2) two vertices of the graph are incident to an edge of colour s, t, u, or c if the polygonal regions corresponding to these vertices contain in their closures a common s-, t-, u-, or c-curve, respectively;

3) if there is more than one c-edge going out of some vertex of the graph Γ_{f^t}, then the c-edges are considered to be ordered in accordance with passing the corresponding separatrices when going around the boundary of the corresponding region.

Two multicolour graphs Γ_{f^t} and $\Gamma_{f'^t}$ for flows f^t and f'^t in the class G, respectively, are said to be *isomorphic* if there exists a one-to-one map of the vertices

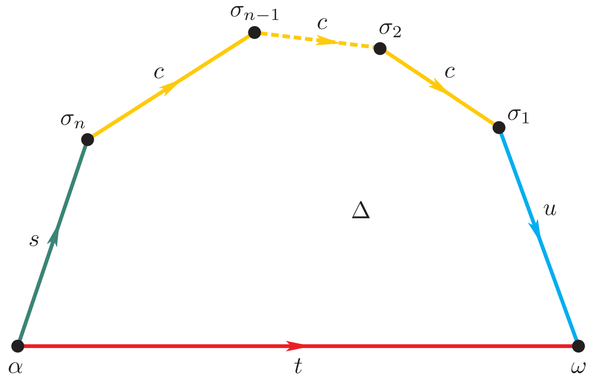

Figure 1. Polygonal region.

and edges of one graph to the vertices and edges of the other graph, respectively, preserving the colours of all edges and the numbering of c-edges.

Theorem 1. *Flows in the class G are topologically equivalent if and only if their multicolour graphs are isomorphic.*

An algorithm for solving the problem of recognition of isomorphism of graphs (in some class of graphs) is customarily considered to be *efficient* if its working time is bounded by some polynomial of the length of definition of the input information. This definition of efficient solubility goes back to Cobham [3]. A standard of intractability is NP-completeness of a problem (see [6]). The complexity status of the problem of recognition of isomorphism of graphs is still unknown, that is, in the class of all graphs, for this problem neither polynomial solubility has been proved, nor NP-completeness. At the same time, the multicolour graphs of flows of class G are not graphs of general form, since they are embeddable in the supporting surface on which the corresponding flows of class G are defined. This fact makes it possible to prove the following theorem.

Theorem 2. *The problem of recognition of isomorphism of the multicolour graphs corresponding to flows in the class G can be solved in polynomial time.*

To solve the realization problem, we consider a simple connected four-colour graph Γ (see the precise definitions in § 4) the edges of which are coloured in the four colours s, u, t, c, and every vertex of which is incident to exactly one edge of each colour s, u, t. There can be any finite (in particular, zero) number n_b of c-edges incident to one vertex b, and they are ordered: $c_1^b,\dots,c_{n_b}^b$ in the case $n_b\geqslant 1$. We call the u-edge and s-edge going out of a vertex b *nominal c-edges* and assign to them the numbers c_0^b and $c_{n_b+1}^b$, respectively. A simple cycle $b_1,(b_1,b_2),b_2,\dots,b_{2k},(b_{2k},b_{2k+1}),b_{2k+1}=b_1$ for $k\in\mathbb{N}$ is called a *c^*-cycle* if

$$(b_{2i-1},b_{2i})=c_m^{b_{2i}},\qquad (b_{2i},b_{2i+1})=c_{m+1}^{b_{2i}}=c_l^{b_{2i+1}},\qquad (b_{2i+1},b_{2i+2})=c_{l-1}^{b_{2i+1}}.$$

A graph Γ is said to be *admissible* if it contains c^*-cycles and every such cycle has length 4. A simple cycle of a graph Γ is called a *tu-cycle* (respectively, *st-cycle*) if all its edges have colour t or u (respectively, t or s).

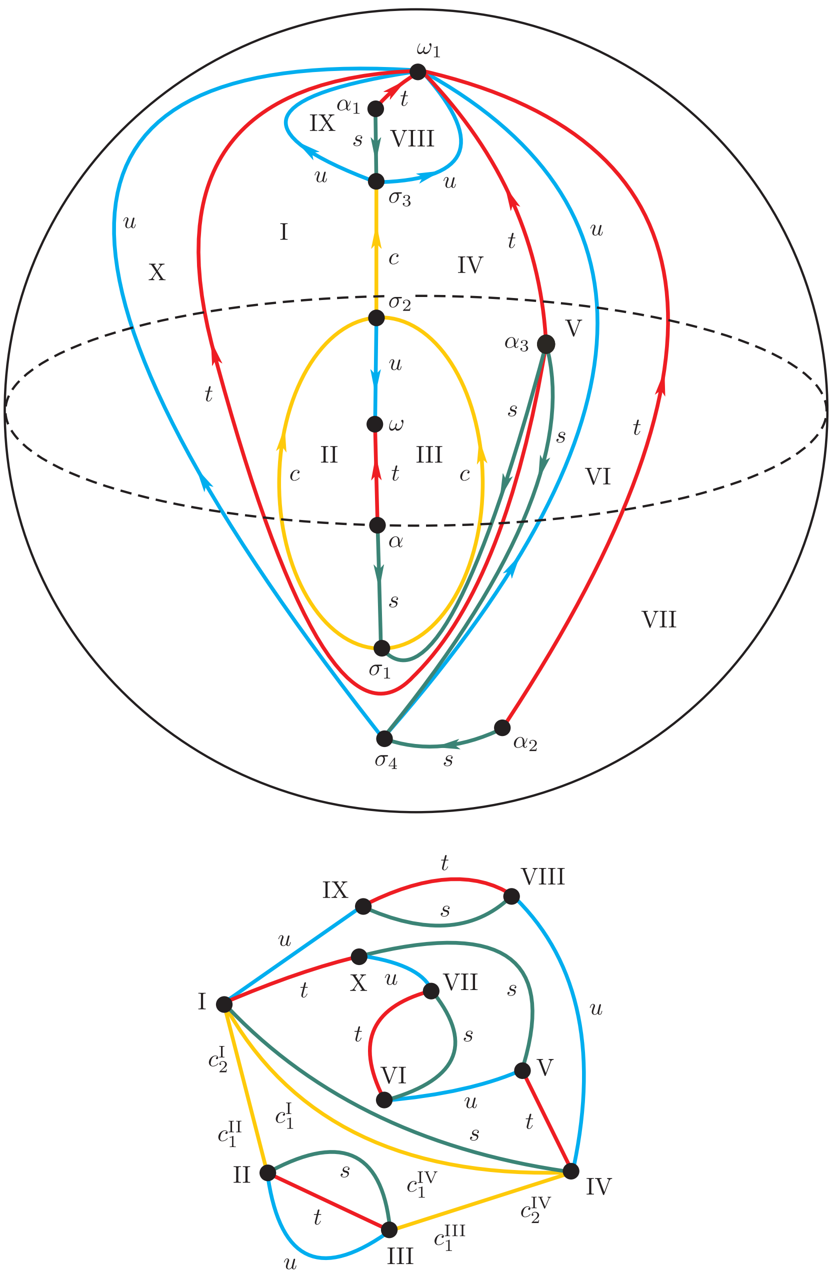

Figure 2. The phase portrait of some flow in the class G (above) and its four-colour graph (below).

Lemma 1. *Let $f^t \in G$. Then the graph Γ_{f^t} is admissible.*

Theorem 3. *For any admissible graph Γ there exists a flow $f^t \in G$ defined on a closed surface S whose graph is isomorphic to the given graph. Furthermore,*

(i) *the Euler characteristic of the surface S is calculated by the formula $\chi(S) = \nu_0 - \nu_1 + \nu_2$, where ν_0, ν_1 and ν_2 are the numbers of all tu-, c^*-, and st-cycles of the graph Γ, respectively*;

(ii) *the surface S is non-orientable if and only if the graph Γ contains at least one cycle of odd length.*

§ 3. Dynamics of a flow $f^t \in G$ and the structure of its polygonal regions

Let f^t be a flow in the class G defined on a closed surface S. In this section we study the dynamics of the flow f^t, which enables us to determine the structure of its polygonal regions. We present the propositions requisite for understanding the dynamics.

Proposition 1 (see [14], Ch. 2, Theorem 5.1, and [17], Ch. 4, Theorem 7.1). *A flow f^t in the class G, in some neighbourhood of a fixed point $p \in \Omega^i_{f^t}$, is topologically equivalent to the linear flow*

$$a^t(x,y) = (2^{-t}x, 2^{-t}y) \quad \text{for } i = 0,$$
$$b^t(x,y) = (2^{-t}x, 2^{t}y) \quad \text{for } i = 1,$$
$$c^t(x,y) = (2^{t}x, 2^{t}y) \quad \text{for } i = 2.$$

Proposition 2 (see [5], Theorem 2.1.1). *Let $f^t \in G$. Then*

1) $S = \bigcup_{p\in\Omega_{f^t}} W^u_p = \bigcup_{p\in\Omega_{f^t}} W^s_p$;

2) W^u_p (*respectively*, W^s_p) *is a smooth submanifold of the manifold S diffeomorphic to $\mathbb{R}^i$* (*respectively*, $\mathbb{R}^{2-i}$) *for any fixed point $p \in \Omega^i_{f^t}$.*

Let p be a fixed point of the flow f^t. Let l^u_p (respectively, l^s_p) denote the unstable (respectively, stable) separatrix of the point p.

Lemma 2. *For any sink ω* (*respectively*, *source α*) *of a flow $f^t \in G$ there exists at least one saddle point σ with unstable* (*respectively*, *stable*) *separatrix l^u_σ* (*respectively*, l^s_σ) *such that* $\mathrm{cl}(l^u_\sigma) \setminus (l^u_\sigma) = \{\sigma, \omega\}$ (*respectively*, $\mathrm{cl}(l^s_\sigma) \setminus (l^s_\sigma) = \{\sigma, \alpha\}$).

Proof. Assuming the opposite for some sink point ω, we obtain by part 1) of Proposition 2 that $\mathrm{cl}(W^s_\omega) = W^s_\omega \cup \bigcup_{i=1}^k \alpha_i$, where α_i, $i = 1, \dots, k$, is a source such that $W^u_{\alpha_i} \cap W^s_\omega \neq \varnothing$. We claim that $W^u_{\alpha_i} \subset \mathrm{cl}(W^s_\omega)$.

Suppose the opposite. Then by part 1) of Proposition 2 there exists a point $p \in \Omega_{f^t}$ different from ω such that $W^s_p \cap W^u_{\alpha_i} \neq \varnothing$. Let x_ω and x_p be points belonging to $W^u_{\alpha_i} \cap W^s_\omega$ and $W^u_{\alpha_i} \cap W^s_p$, respectively. Since the manifold $W^u_{\alpha_i} \setminus \{\alpha_i\}$ is homeomorphic to $\mathbb{R}^2 \setminus \{O\}$ (see part 2) of Proposition 2), there exists a path $c\colon [0,1] \to (W^u_{\alpha_i} \setminus \{\alpha_i\})$ without self-intersections connecting the point $x_\omega = c(0)$ with the point $x_p = c(1)$. Then there exists a value $\tau \in (0,1)$ such that $c(\tau) \notin W^s_\omega$ and $c(t) \in W^s_\omega$ for $t < \tau$. Consequently, there exists a point $r \in \Omega_{f^t}$ such that $r \neq \omega$ and $c(\tau) \in W^s_r$. Furthermore, the point $c(\tau)$ belongs to $\mathrm{cl}(W^s_\omega)$. But if

$c(\tau) \in \mathrm{cl}(W^s_\omega)$, then $c(\tau) = \alpha_{i_0}$ for some $i_0 = 1, \dots, k$; this means that $\alpha_{i_0} \in W^u_{\alpha_i}$, a contradiction to the definition of the unstable manifold of a fixed point.

We have obtained that $W^u_{\alpha_i} \subset W^s_\omega$ for any $i = 1, \dots, k$, and, consequently, the set $\mathrm{cl}(W^s_\omega)$ is open, since it contains each of its points together with some open neighbourhood. Since $\mathrm{cl}(W^s_\omega)$ is simultaneously open and closed, we have $\mathrm{cl}(W^s_\omega) = S$. Then Ω_{f^t} does not contain saddle points, which fact contradicts the definition of the class G.

By passing from f^t to f^{-t}, we can prove the assertion for sources.

The lemma is proved.

Lemma 3. *Let p be a fixed point of a flow $f^t \in G$. Then the following hold.*

(i) *If $p \in \Omega^1_{f^t}$, then*

$$\mathrm{cl}(l^u_p) \setminus (l^u_p \cup \{p\}) = \begin{cases} \{\sigma\} \subset \Omega^1_{f^t} \text{ and } l^u_p = l^s_\sigma, \\ \{\omega\} \subset \Omega^0_{f^t} \text{ and } l^u_p \subset W^s_\omega. \end{cases}$$

(ii) *If $p \in \Omega^2_{f^t}$, then $\mathrm{cl}(l^u_p) \setminus (l^u_p \cup \{p\}) = \bigcup_{\sigma \in \Omega_p} \mathrm{cl}(l^u_\sigma)$, where Ω_p is a non-empty subset of the set $\Omega^1_{f^t}$.*

Proof. Consider case (i): p is a saddle point. Let $x \in \mathrm{cl}(l^u_p)$. By part 1) of Proposition 2, any point l^u_p is a point of W^s_r for some fixed point r. For r there are three possibilities: a) r is a sink; b) r is a saddle; c) r is a source.

a) Consider a sink $r = \omega$ such that $x \in W^s_\omega$. Since ω is a sink and $l^u_p = \mathscr{O}_x$, we have $l^u_p \subset W^s_\omega$. Thus, $\mathrm{cl}(l^u_p) \setminus (l^u_p \cup \{p\}) = \{\omega\}$.

b) Consider a saddle point $r = \sigma$ such that $x \in W^s_\sigma$. In this case, $l^u_p = l^s_\sigma$. Thus, $\mathrm{cl}(l^u_p) \setminus (l^u_p \cup \{p\}) = \{\sigma\}$.

c) Suppose that there exists a source $r = \alpha$ such that $x \in W^s_\alpha$. Since $W^s_\alpha = \{\alpha\}$, we obtain $\alpha \in l^u_p$, which is impossible, since l^u_p consists of wandering points. Consequently, case c) is impossible.

Now consider case (ii): $p = \alpha$ is a source.

It follows from part 1) of Proposition 2 that the set $A = \mathrm{cl}(l^u_\alpha) \setminus (l^u_\alpha \cup \{\alpha\})$ is an f^t-invariant subset of the set $W^u_{\Omega^1_{f^t}} \cup \Omega^0_{f^t}$. Then to prove the assertion it is sufficient to show that

a) if $\sigma \in A$ for some $\sigma \in \Omega^1_{f^t}$, then $l^u_\sigma \subset A$;

b) if $\omega \in A$ for some $\omega \in \Omega^0_{f^t}$, then there exists $\sigma \in \Omega^1_{f^t}$ such that $\omega \in \mathrm{cl}(l^u_\sigma)$ and $l^u_\sigma \subset A$.

In case a), since $\sigma \in A$, there exists a sequence $x_n \in l^u_\alpha$ such that $x_n \to \sigma$ as $n \to +\infty$. Then $\mathscr{O}_{x_n} \subset l^u_\alpha$, and by the equivalence of the flow in a neighbourhood of a hyperbolic saddle point to its linear part (see, for example, [14]), the closure of the set $\bigcup_{n\in\mathbb{N}} \mathscr{O}_{x_n}$ contains l^u_σ.

In case b), if $\omega \in A$, then there exists a sequence $x_n \in l^u_\alpha$ such that $x_n \to \omega$ as $n \to +\infty$. By Lemma 2 there exist finitely many saddle points $\sigma_1, \dots, \sigma_k \in \Omega^1_{f^t}$ such that $\omega \in \mathrm{cl}(l^u_{\sigma_i})$ for $i = 1, \dots, k$. Then among them there exist saddle points $\sigma_{i_1}, \sigma_{i_2}$ (possibly coinciding) such that the sequence x_n is contained in a connected component D of the set $W^s_\omega \setminus (\omega \cup \bigcup_{i=1}^k l^u_{\sigma_i})$. Hence, $D \subset l^u_\alpha$. Thus, $l^u_{\sigma_{i_1}} \subset A$ and $l^u_{\sigma_{i_2}} \subset A$.

The lemma is proved.

An assertion similar to Lemma 3 can be proved for stable separatrices of fixed points of the flow f^t.

Recall that a cell J of the flow f^t was defined to be a connected component of the set $\widetilde{S} = S \setminus (W^u_{\Omega^0_{f^t} \cup \Omega^1_{f^t}} \cup W^s_{\Omega^1_{f^t} \cup \Omega^2_{f^t}})$.

Lemma 4. *Any cell J of the flow f^t contains a unique sink ω and a unique source α in its closure, while the whole cell is the union of trajectories going from α to ω.*

Proof. By Proposition 2,

$$\widetilde{S} = \Big(\bigcup_{\alpha \in \Omega^2_{f^t}} l^u_\alpha\Big) \setminus \Big(\bigcup_{\sigma \in \Omega^1_{f^t}} l^s_\sigma\Big).$$

Then any connected component J of the set $\widetilde{S}$ is a subset of l^u_α for a unique source α. In similar fashion,

$$\widetilde{S} = \Big(\bigcup_{\omega \in \Omega^0_{f^t}} l^s_\omega\Big) \setminus \Big(\bigcup_{\sigma \in \Omega^1_{f^t}} l^u_\sigma\Big).$$

Then any connected component J of the set $\widetilde{S}$ is a subset of l^s_ω for a unique sink ω. Thus,

$$J \subset (W^u_\alpha \cap W^s_\omega),$$

and, consequently, the whole cell is the union of trajectories going from α to ω. The lemma is proved.

Recall that we denoted by J_{f^t} the set of all cells of the flow f^t and chose one trajectory θ_J (a t-curve) in every cell $J \in J_{f^t}$. We also set $\mathscr{T} = \bigcup_{J \subset \widetilde{S}} \theta_J$ and $\overline{S} = \widetilde{S} \setminus \mathscr{T}$ and defined a polygonal region to be a connected component Δ of the set $\overline{S}$. We denoted by Δ_{f^t} the set of all polygonal regions of the flow f^t and defined c-curves to be separatrices connecting saddles (connections), u-curves to be unstable saddle separatrices that are not connections, and s-curves to be stable saddle separatrices that are not connections.

Lemma 5. *Any polygonal region Δ of the flow f^t is homeomorphic to an open disc, and its boundary consists of the closures of one t-curve, one u-curve, one s-curve, and a finite (possibly empty) set of c-curves.*

Proof. By Lemma 4, any cell $J \in J_{f^t}$ is contained in the basin of some source α between two (possibly coinciding) s-curves (see Figure 1). A polygonal region Δ is obtained by removing the t-curve from J. Since W^u_α is homeomorphic to $\mathbb{R}^2$ by Proposition 2, the region Δ is homeomorphic to the sector bounded by two rays going out of the origin in $\mathbb{R}^2$, that is, is homeomorphic to an open disc. By construction, the boundary of the region Δ contains a unique s-curve and a unique t-curve. Since Δ is situated also in the basin of some sink ω, it follows that it is bounded by one u-curve. By part (ii) of Lemma 3 the region Δ is bounded by finitely many c-curves. We obtain that the only possible structure of the boundary of a polygonal region can be the structure depicted in Figure 1, up to the number of c-curves. The lemma is proved.

Figure 3 depicts the phase portrait of some flow in the class G and all its polygonal regions.

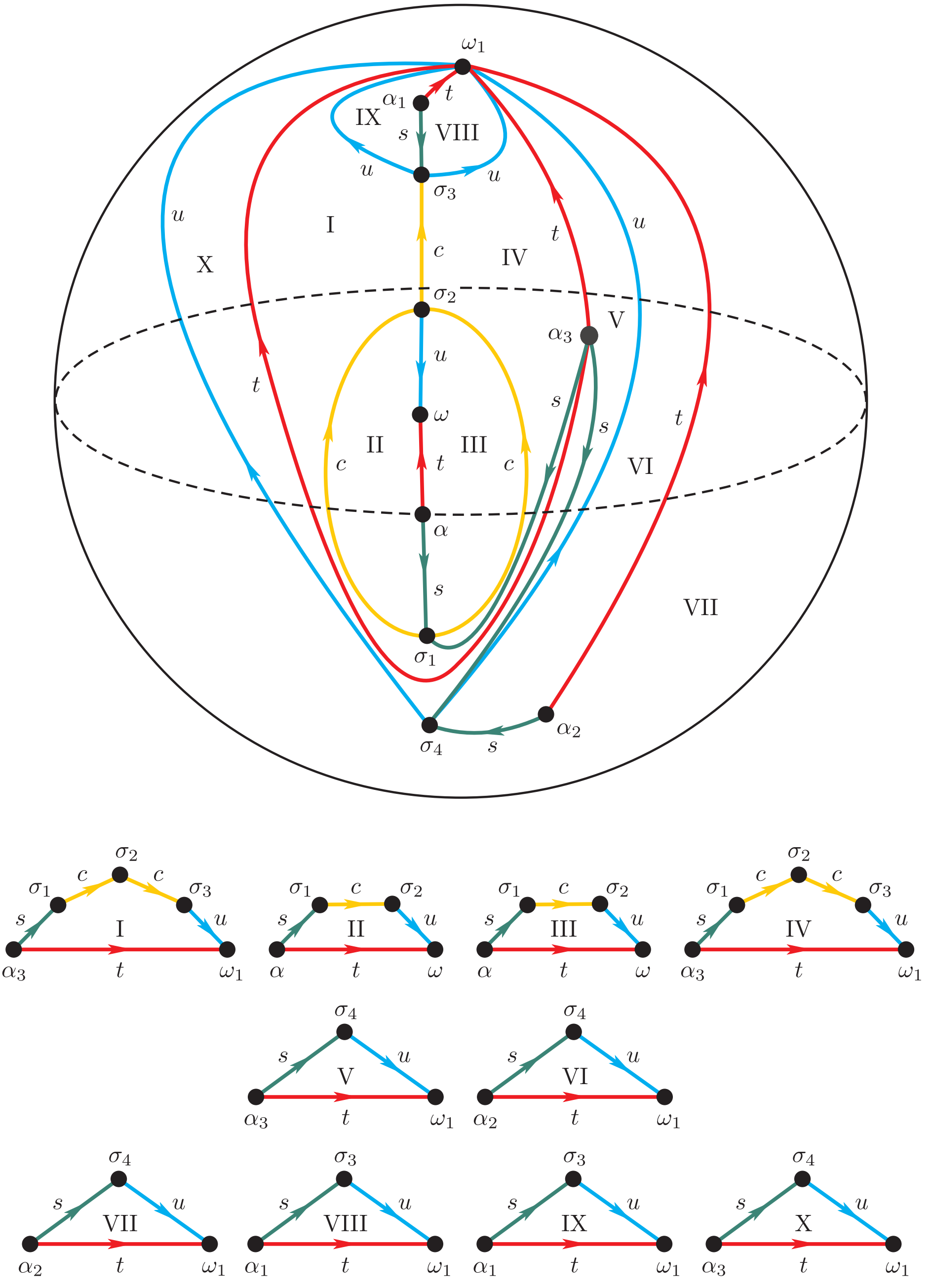

Figure 3. The phase portrait of some flow in the class G (above) and all its polygonal regions (below).

§ 4. Properties of the four-colour graph Γ_{f^t}

In this section, with every flow $f^t \in G$ we associate a four-colour graph Γ_{f^t} and establish the properties of this graph requisite for isolating the set of admissible graphs that are realizable by a flow in the class G.

Recall that a *finite multigraph* Γ is defined to be an ordered pair (B, E) satisfying the following conditions: B is a non-empty set of vertices; E is a multiset of pairs of vertices called edges, while a multiset here means a generalization of the notion of set that allows inclusion of the same element several times.

Henceforth a multigraph is called simply a graph for brevity.

If a graph Γ contains an edge $e = (a, b)$, then each of the vertices a, b is said to be *incident* to the edge e, and the vertices a and b are said to be *connected by the edge* e.

A *path* in a graph is defined to be a finite sequence of its vertices and edges of the form $b_0, (b_0, b_1), b_1, \dots, b_{i-1}, (b_{i-1}, b_i), b_i, \dots, b_{k-1}, (b_{k-1}, b_k), b_k$, $k \geqslant 1$. The number k is called the *length of the path*, it is equal to the number of edges occurring in the path.

A graph is said to be *connected* if any two of its vertices can be connected by a path.

A *cycle* of length $k \in \mathbb{N}$ in a graph is defined to be a finite subset of its vertices and edges of the form $\{b_0, (b_0, b_1), b_1, \dots, b_{i-1}, (b_{i-1}, b_i), b_i, \dots, b_{k-1}, (b_{k-1}, b_0), b_0\}$. A *simple cycle* is defined to be a cycle all of whose vertices and edges are pairwise different.

A graph Γ is said to be *multicolour* if its set of edges is a union of finitely many subsets each of which consists of edges of the same colour.

Let Γ be a simple connected four-colour graph whose edges are coloured in the four colours s, u, t, c, and every vertex of which is incident to exactly one edge of each of the colours s, u, t. Furthermore, there can be any finite (in particular, zero) number n_b of c-edges incident to one vertex b, and they are ordered: $c_1^b, \dots, c_{n_b}^b$ in the case $n_b \geqslant 1$. We call the u-edge and s-edge going out of the vertex b *nominal c-edges* and assign to them the numbers c_0^b and $c_{n_b+1}^b$, respectively. A simple cycle $b_1, (b_1, b_2), b_2, \dots, b_{2k}, (b_{2k}, b_{2k+1}), b_{2k+1} = b_1$ for $k \in \mathbb{N}$ is called a *c^*-cycle* if

$$(b_{2i-1}, b_{2i}) = c_m^{b_{2i}}, \qquad (b_{2i}, b_{2i+1}) = c_{m+1}^{b_{2i}} = c_l^{b_{2i+1}}, \qquad (b_{2i+1}, b_{2i+2}) = c_{l-1}^{b_{2i+1}}.$$

Definition. The graph Γ is said to be *admissible* if it contains c^*-cycles and every such cycle has length 4.

A simple cycle of the graph Γ is called a *tu-cycle* (respectively, *st-cycle*) if all of its edges have colour t or u (respectively, t or s).

In § 3 we proved that the closure of the set of s-, t-, u- and c-curves partitions the surface S into polygonal regions Δ, and denoted by Δ_{f^t} the set of all such regions. The boundary of every polygonal region is considered to be oriented correspondingly to going over the t-curve from the source to the sink.

The multicolour graph Γ_{f^t} corresponding to a flow $f^t \in G$ is constructed as follows (see Figure 2):

1) the vertices of the graph Γ_{f^t} are in a one-to-one correspondence with the polygonal regions of the flow;

2) two vertices of the graph are incident to an edge of colour s, t, u, or c if the polygonal regions corresponding to these vertices have a common s-, t-, u-, or c-curve, and a one-to-one correspondence is established between this edge and the s-, t-, u-, or c-curve;

3) if there is more than one c-edge going out of some vertex of the graph, the c-edges are numbered in such a way that the numbering corresponds to the order of separatrices when going around the boundary of the corresponding polygonal region.

By construction, the multicolour graphs obtained from different partitions into polygonal regions (depending on the choice of t-curves) are isomorphic.

Let π_{f^t} denote the one-to-one correspondence between the polygonal regions and vertices, as well as between the s-, t-, u-, c-curves and the s-, t-, u-, c-edges of the flow f^t and the graph Γ_{f^t}, respectively.

Proof of Lemma 1. Let us prove that the four-colour graph Γ_{f^t} of the flow f^t is admissible.

Since the flow f^t lies on the closed surface S, and every vertex of the graph corresponds to its polygonal region, it follows that we can construct a graph isomorphic to the given one by simply placing its vertices within the polygonal regions and defining edges to be curves embedded in the surface that connect these vertices and that intersect the corresponding side once (Figure 4). As a graph constructed from the flow f^t, it is obviously isomorphic to the graph Γ_{f^t}. Therefore we can assume without loss of generality that the graph Γ_{f^t} is embedded in the surface S in the manner described above. Since the surface S is connected, the graph Γ_{f^t} is also connected. Since every side of a polygonal region adjoins exactly two different polygonal regions, the graph Γ_{f^t} does not have cycles of length 1, that is, is simple.

Since every point $p \in \Omega_{f^t}$ is adjoined by finitely many polygonal regions separated by coloured curves, the projection π_{f^t} uniquely associates with the point p the cycle of the vertices corresponding to the regions adjoining p and of the coloured edges intersecting the coloured curves going out of p. For example, a saddle is adjoined by exactly four polygonal regions separated by u-, s-, or c-curves. Regarding u- and s-edges as nominal c-edges, we obtain that to every saddle of the flow f^t there corresponds a c^*-cycle of the graph Γ_{f^t}. The converse correspondence also holds: every c^*-cycle can be placed in a neighbourhood of a unique saddle point such that for different c^*-cycles these neighbourhoods are disjoint. Thus, the graph Γ_{f^t} contains c^*-cycles and every such cycle has length 4. Consequently, the graph Γ_{f^t} is admissible.

The lemma is proved.

Proposition 3. *Let $f^t \in G$ and let Γ_{f^t} be the graph of the flow f^t. Then the map π_{f^t} establishes a one-to-one correspondence between the sets $\Omega^0_{f^t}$, $\Omega^1_{f^t}$ and $\Omega^2_{f^t}$ and the sets of tu-, c^*- and st-cycles, respectively.*

Proof. 1. The correspondence by the map π_{f^t} between the set $\Omega^1_{f^t}$ and the set of c^*-cycles follows from the proof of Lemma 1.

2. Every sink point ω of the flow f^t is adjoined by regions that are separated in turns by u- and t-curves contained in the basin W^s_ω. Therefore, the map π_{f^t} associates with the point ω a unique tu-cycle of the graph Γ_{f^t}. The converse correspondence also holds: since the basins of different sinks are separated by c-

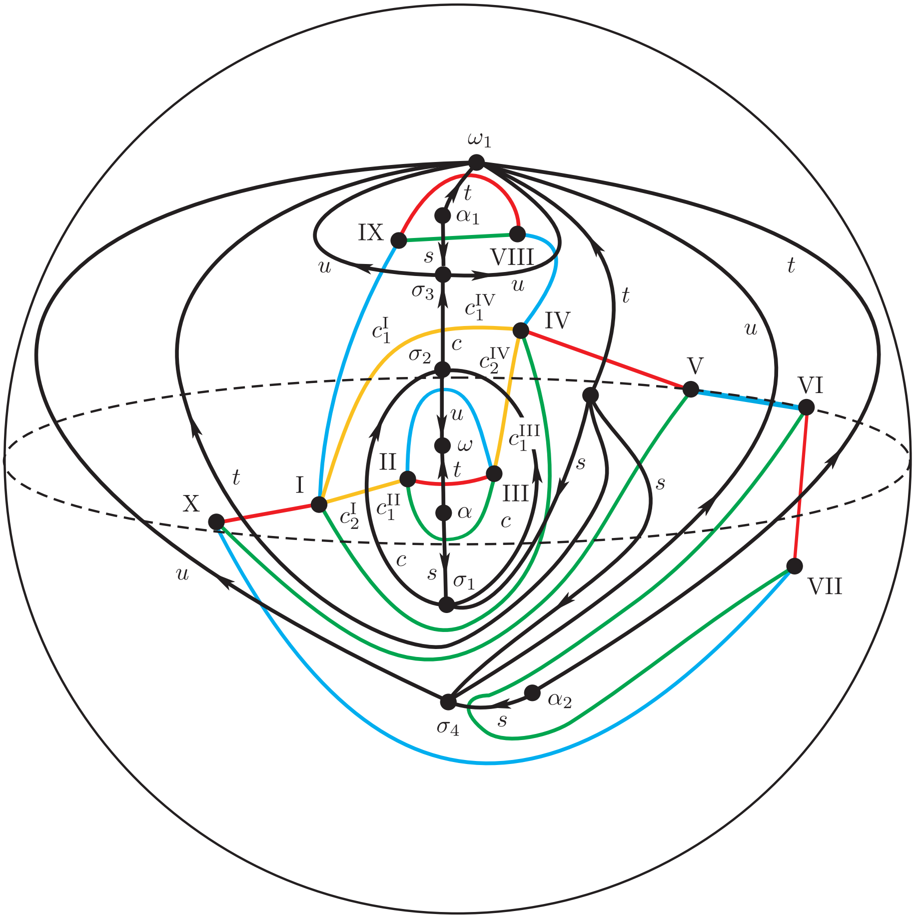

Figure 4. The phase portrait of some flow in the class G and its four-colour graph embedded in the surface S on which this flow is defined.

and s-curves, it follows that every tu-cycle can be placed in the basin of a unique sink. Thus, the map π_{f^t} establishes a one-to-one correspondence between the set $\Omega^0_{f^t}$ and the set of tu-cycles.

3. The correspondence between the set $\Omega^2_{f^t}$ and the set of ts-cycles is established similarly to part 2.

The proposition is proved.

§ 5. Proof of the classification Theorem 1

Let us prove that flows in the class G are topologically equivalent if and only if their multicolour graphs are isomorphic.

Let $f^t \in G$ (respectively, $f'^t \in G$) and let Γ_{f^t} (respectively, $\Gamma_{f'^t}$) be the multicolour graph constructed from the flow f^t (respectively, f'^t).

Necessity. Suppose that the flows f^t and f'^t are topologically equivalent, that is, there exists a homeomorphism $h\colon S \to S$ taking the trajectories of f^t to the trajectories of f'^t. We assume without loss of generality that the set of polygonal regions of the flow f'^t is constructed by using the t-curves $\mathscr{T}' = h(\mathscr{T})$. Then the homeomorphism h takes the polygonal regions of the flow f^t to the polygonal regions of the flow f'^t, and a sought-for isomorphism $\xi\colon \Gamma_{f^t} \to \Gamma_{f'^t}$ is defined by the formula

$$\xi = \pi_{f'^t} h \pi_{f^t}^{-1}.$$

Sufficiency. Suppose that the graphs Γ_{f^t} and $\Gamma_{f'^t}$ of the flows f^t and f'^t are isomorphic under an isomorphism ξ. Consider a polygonal region $\Delta \in \Delta_{f^t}$. Its boundary contains a unique source α, a unique sink ω, and n saddle points $\sigma_1, \sigma_2, \dots, \sigma_n$, $n \in \mathbb{N}$, which we assume to be situated on the boundary in the order of increasing indices in the chosen direction of going around the boundary Δ. Consider a region Δ' for the flow f'^t for which

$$\Delta' = \pi_{f'^t}^{-1} \xi \pi_{f^t}(\Delta).$$

The isomorphism ξ ensures the same number of same-colour edges going out of the vertices of the graphs corresponding to the regions Δ and Δ', which fact implies the existence in the boundary of Δ' of exactly one sink ω', one source α', and n saddles $\sigma'_1, \sigma'_2, \dots, \sigma'_n$ situated in the order of increasing indices in the chosen direction of going around the boundary of Δ'. Since the isomorphism ξ preserves the colours of the edges and the numbering of the c-edges, construction of a homeomorphism $h\colon S \to S$ realizing the topological equivalence of the flows f^t and f'^t reduces to construction of a homeomorphism $h_\Delta\colon \operatorname{cl}(\Delta) \to \operatorname{cl}(\Delta')$ taking the trajectories of the flow f^t contained in $\operatorname{cl}(\Delta)$ to the trajectories of the flow f'^t contained in $\operatorname{cl}(\Delta')$ such that

$$h_\Delta|_{\Delta \cap \widetilde{\Delta}} = h_{\widetilde{\Delta}}|_{\Delta \cap \widetilde{\Delta}}$$

for any polygonal regions $\Delta, \widetilde{\Delta}$ of the flow f^t. We construct the homeomorphism h_Δ step-by-step.

Step 1. First we construct the homeomorphism h_Δ in neighbourhoods of node points. Let

$$u = \{(x,y) \in \mathbb{R}^2\colon x^2 + y^2 < 1\}.$$

Recall that $a^t\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $c^t\colon \mathbb{R}^2 \to \mathbb{R}^2$ are the flows on the plane defined by the formulae $a^t(x,y) = (2^{-t}x, 2^{-t}y)$ and $c^t(x,y) = (2^t x, 2^t y)$ with the point $O(0,0)$ as a sink and a source, respectively. By Proposition 1 there exist neighbourhoods u_ω, u_α ($u_{\omega'}$, $u_{\alpha'}$) of the points ω, α (ω', α'), respectively, such that $f^t|_{u_\omega}$, $f^t|_{u_\alpha}$ ($f'^t|_{u_\omega}$, $f'^t|_{u_\alpha}$) are topologically conjugate with $a^t(x,y)|_u$, $c^t(x,y)|_u$ by some homeomorphisms $h_\omega\colon u_\omega \to u$, $h_\alpha\colon u_\alpha \to u$ ($h_{\omega'}\colon u_{\omega'} \to u$, $h_{\alpha'}\colon u_{\alpha'} \to u$), respectively. We assume without loss of generality that these neighbourhoods are pairwise disjoint.

For $r \in (0,1]$ we set $S_r = \{(x,y) \in \mathbb{R}^2\colon x^2 + y^2 = r\}$ and $S_r^\omega = h_\omega^{-1}(S_r)$, $S_r^\alpha = h_\alpha^{-1}(S_r)$ ($S_r^{\omega'} = h_{\omega'}^{-1}(S_r)$, $S_r^{\alpha'} = h_{\alpha'}^{-1}(S_r)$). Let $\{A\} = S_1^\omega \cap l_{\alpha,\omega}$, $\{A_0\} = S_1^\omega \cap l_{\omega,\sigma_1}$ ($\{A'\} = S_1^{\omega'} \cap l_{\alpha',\omega'}$, $\{A'_0\} = S_1^{\omega'} \cap l_{\omega',\sigma'_1}$) and $\{C\} = S_1^\alpha \cap l_{\alpha,\omega}$, $\{C_0\} = S_1^\alpha \cap l_{\alpha,\sigma_n}$ ($\{C'\} = S_1^{\alpha'} \cap l_{\alpha',\omega'}$, $\{C'_0\} = S_1^{\alpha'} \cap l_{\alpha',\sigma'_n}$).

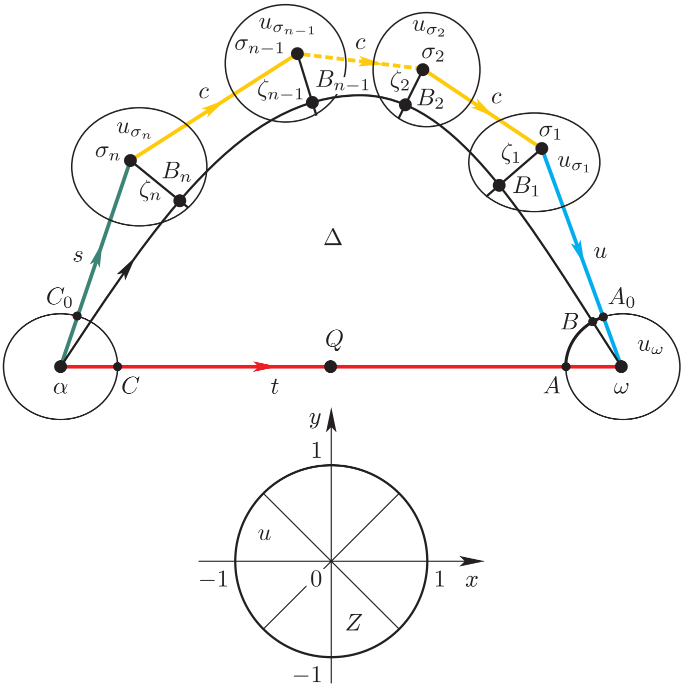

Figure 5. Construction of the map h_Δ in neighbourhoods of node points and the choice of the secants ζ_i.

Throughout what follows we denote by $m_{a,b}$ the closure of a segment of some secant of the trajectories of the flow f^t (f'^t) bounded by points a (a') and b (b'). Note that $m_{a,b} = m_{b,a}$. In particular, let m_{A,A_0} (m_{A',A'_0}) denote the segment that is the intersection $S^\omega_1 \cap \Delta$ ($S^{\omega'}_1 \cap \Delta'$) (Figure 5). Let $x \in m_{A,A_0}$. Let $\mu_{A,A_0} \colon m_{A,A_0} \to [0,1]$ ($\mu_{A',A'_0} \colon m_{A',A'_0} \to [0,1]$) be an arbitrary homeomorphism such that $\mu_{A,A_0}(A) = 0$ ($\mu_{A',A'_0}(A') = 0$). Let

$$h_{m_{A,A_0}} = \mu^{-1}_{A'A'_0}\mu_{A,A_0} \colon m_{A,A_0} \to m_{A',A'_0}.$$

Let $x \in m_{A,A_0}$ ($x' \in m_{A',A'_0}$) and let $\mathscr{O}_x$ ($\mathscr{O}_{x'}$) be the trajectory of the point x (x'). Let $x^\omega \in (\mathrm{cl}(u_\omega) \cap \Delta \setminus \{\omega\})$; then $x^\omega = S^\omega_r \cap \mathscr{O}_x$ for some $r \in (0,1]$ and $x \in m_{A,A_0}$. We construct a homeomorphism $h_{u_\omega} \colon \mathrm{cl}(u_\omega) \cap \Delta \to \mathrm{cl}(u_{\omega'}) \cap \Delta'$ such that $h_{u_\omega}(\omega) = \omega'$ and $h_{u_\omega}(x^\omega) = x'^{\omega'}$, where $x'^{\omega'} = S^{\omega'}_r \cap \mathscr{O}_{h_{m_{A,A_0}}(x)}$. Similarly, for points $x^\alpha \in (\mathrm{cl}(u_\alpha) \cap \Delta \setminus \{\alpha\})$ such that $x^\alpha = S^\alpha_r \cap \mathscr{O}_x$ for some $r \in (0,1]$ and $x \in m_{A,A_0}$, we define a homeomorphism $h_{u_\alpha} \colon \mathrm{cl}(u_\alpha) \cap \Delta \to \mathrm{cl}(u_{\alpha'}) \cap \Delta'$ such that $h_{u_\alpha}(\alpha) = \alpha'$ and $h_{u_\alpha}(x^\alpha) = x'^{\alpha'}$, where $x'^{\alpha'} = S^{\alpha'}_r \cap \mathscr{O}_{h_{m_{A,A_0}}(x)}$.

Step 2. We construct the homeomorphism h_Δ on the boundary of Δ.

Throughout what follows we denote by $l_{a,b}$ the closure of a segment of a trajectory or a saddle separatrix bounded by points a and b, and by $\lambda_{a,b}$ the length of this segment. Note that $l_{a,b} = l_{b,a}$ and $\lambda_{a,b} = \lambda_{b,a}$. For smooth segments $l_{a,b}$, $l_{a',b'}$ of trajectories of the flows f^t, f'^t, we define a *homeomorphism with respect to the arc length* to be the homeomorphism $h_{l_{a,b}}\colon l_{a,b} \to l_{a',b'}$ defined by the following rule for a point $x \in l_{a,b}$:

$$\lambda_{a',h_{l_{a,b}}(x)} = \frac{\lambda_{a,x} \cdot \lambda_{a',b'}}{\lambda_{a,b}}.$$

We use this method to construct the following homeomorphisms: $h_{l_{A,C}}\colon l_{A,C} \to l_{A',C'}$, $h_{l_{A_0,\sigma_1}}\colon l_{A_0,\sigma_1} \to l_{A'_0,\sigma'_1}$, $h_{l_{C_0,\sigma_n}}\colon l_{C_0,\sigma_n} \to l_{C'_0,\sigma'_n}$ and $h_{l_{\sigma_i,\sigma_{i+1}}}\colon l_{\sigma_i,\sigma_{i+1}} \to l_{\sigma'_i,\sigma'_{i+1}}$.

A similar construction on the boundaries of all polygonal regions ensures the validity of the condition $h_\Delta|_{\mathrm{cl}(\Delta)\cap\mathrm{cl}(\widetilde{\Delta})} = h_{\widetilde{\Delta}}|_{\mathrm{cl}(\Delta)\cap\mathrm{cl}(\widetilde{\Delta})}$ for any polygonal regions Δ, $\widetilde{\Delta}$ of the flow f^t.

Step 3. We construct secants connecting saddle points with some interior point of the t-curve.

Let $Q \in \mathrm{int}(l_{A,C})$ and $Q' = h_{l_{A,C}}(Q)$. We define $m_{Q,\sigma_1},\dots,m_{Q,\sigma_n}$ (respectively, $m_{Q',\sigma'_1},\dots,m_{Q',\sigma'_n}$) as follows.

Recall that for $i = 1,\dots,n$ there exists a neighbourhood u_{σ_i} ($u_{\sigma'_i}$) of the saddle point σ_i (σ'_i) such that $f^t|_{u_{\sigma_i}}$ ($f'^t|_{u_{\sigma_i}}$) is topologically conjugate with $b^t|_u$ by some homeomorphism $h_{\sigma_i}\colon u_{\sigma_i} \to u$ ($h_{\sigma'_i}\colon u_{\sigma'_i} \to u$), where $b^t\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $b^t(x,y) = (2^{-t}x, 2^t y)$. Let

$$Z = \{(x,y) \in \mathbb{R}^2 \mid |x| = |y|\} \cap u.$$

The set Z consists of two intervals intersecting at the origin and transversal to the trajectories of the flow b^t. Let $\zeta_i = h^{-1}_{\sigma_i}(Z) \cap \Delta$ (respectively, $\zeta'_i = h^{-1}_{\sigma'_i}(Z) \cap \Delta'$); see Figure 5. We choose some point $B \in m_{A,A_0}$ such that $\mathscr{O}_B \cap \zeta_i \neq \varnothing$ and $\mathscr{O}_{B'} \cap \zeta'_i \neq \varnothing$ for $i = 1,\dots,n$, where $B' = h_{m_{A,A_0}}(B)$ (Figure 6).

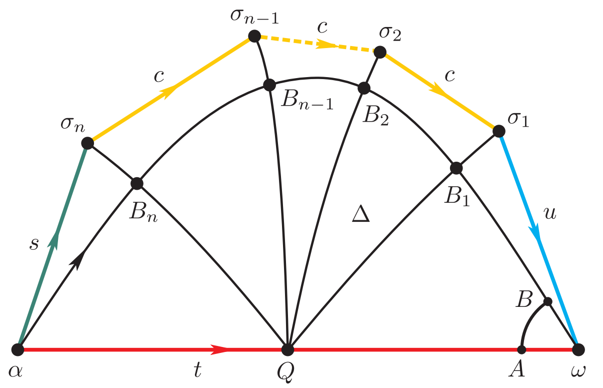

Figure 6. Construction of secants.

Let $\{B_i\} = \mathscr{O}_B \cap \zeta_i$ ($\{B'_i\} = \mathscr{O}_{B'} \cap \zeta'_i$). Let m_{B_i,σ_i} ($m_{B'_i,\sigma'_i}$) denote the subset of ζ_i (ζ'_i) bounded by B_i (B'_i) and σ_i (σ'_i). Let $t_0 \in \mathbb{R}$ ($t'_0 \in \mathbb{R}$) and $t_i \in \mathbb{R}$ ($t'_i \in \mathbb{R}$) be such that $A = f^{t_0}(Q)$ ($A' = f'^{t'_0}(Q')$) and $B = f^{t_i}(B_i)$ ($B' = f'^{t'_i}(B'_i)$). Let

$$m_{B_i,Q} = \left\{ f^{\frac{(-t_i)\cdot\mu_{A,A_0}(x) + (-t_0)(\mu_{A,A_0}(B) - \mu_{A,A_0}(x))}{\mu_{A,A_0}(B)}}(x),\ x \in m_{A,A_0} \right\}.$$

Then $m_{Q,\sigma_i} = m_{Q,B_i} \cup m_{B_i,\sigma_i}$ (see Figure 6). The secants $m_{B'_i,Q'}$ and m_{Q',σ'_i} are defined in similar fashion.

Thus we have obtained the number of secants equal to the number of saddle points of our polygonal regions. They are pairwise disjoint and all have as their endpoints a saddle point and the point Q — an interior point of the t-curve of this region.

Step 4. We now extend the homeomorphism h_Δ to the interior of the region Δ.

Let $x_0 \in m_{A,A_0}$ and $x'_0 = h_{m_{A,A_0}}(x_0)$, let $\mathscr{O}_{x_0}$ be the trajectory of x_0, and $\mathscr{O}_{x'_0}$ the trajectory of x'_0. We set $\{x_i\} = \mathscr{O}_{x_0} \cap m_{Q,\sigma_i}$, $\{x'_i\} = \mathscr{O}_{x'_0} \cap m_{Q',\sigma'_i}$ for $i = 1, \dots, n$, $\{x_{n+1}\} = \mathscr{O}_{x_0} \cap m_{C,C_0}$, $\{x'_{n+1}\} = \mathscr{O}_{x'_0} \cap m_{C',C'_0}$ (Figure 7).

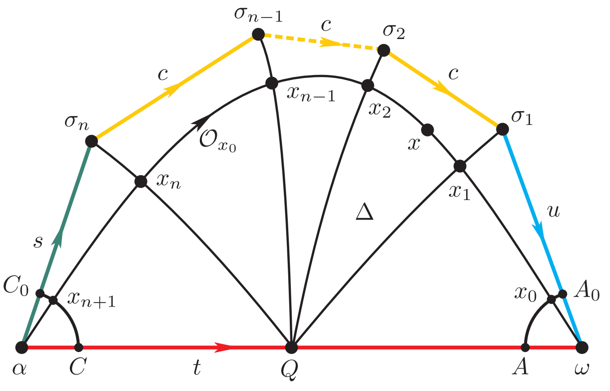

Figure 7. Extension of the homeomorphism to the interior of the region Δ.

We extend the homeomorphism h_Δ to the trajectory $\mathscr{O}_{x_0}$ in such a way that

$$h_\Delta|_{l_{x_i,x_{i+1}}} = h_{l_{x_i,x_{i+1}}} : l_{x_i,x_{i+1}} \to l_{x'_i,x'_{i+1}}.$$

Thus, we have mapped by h_Δ the closure of the polygonal region Δ onto the closure of the polygonal region Δ', which is what finishes the proof of Theorem 1.

§ 6. Proof of Theorem 3

Suppose that Γ is some admissible graph.

I. We construct step-by-step a flow $f^t \in G$ corresponding to the isomorphism class of the graph Γ.

Step 1. Consider some vertex b of the graph Γ. This vertex is incident to n edges, one of which is a t-edge, another is a u-edge, a third one is an s-edge, while the others are c^b_j-edges, $j = 1, \dots, n-3$. We construct on the plane $\mathbb{R}^2$ a regular $2(n-1)$-gon $A_1A_2\dots A_{2(n-1)}$ with centre at the point $O(0,0)$ and with the vertices $A_1(1,0)$ and $A_n(-1,0)$ (Figure 8). Let φ and a denote the central angle and the side of the constructed polygon, respectively. Then

$$\varphi = \frac{\pi}{n-1}, \qquad a = \frac{1}{\sin\varphi}.$$

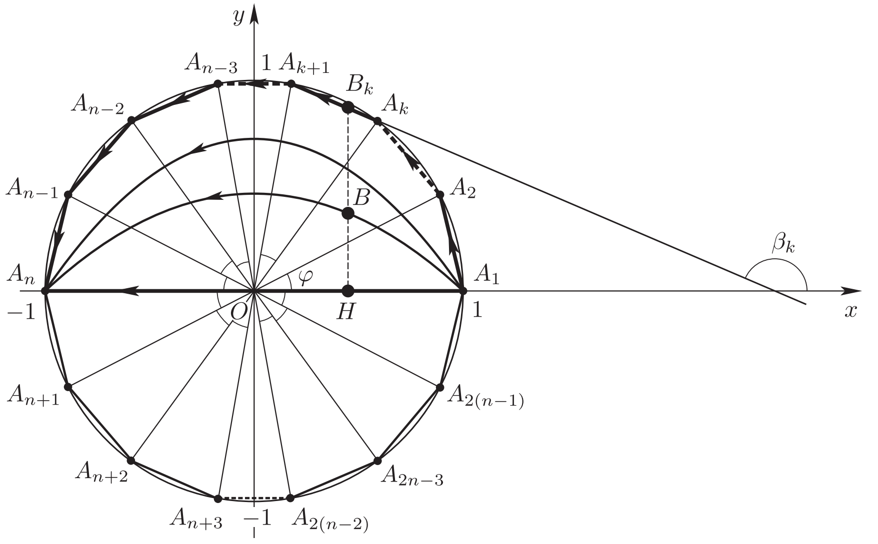

Figure 8. Construction of the vector field v_b.

We obtain that the vertex A_k, $k = 1, \dots, 2(n-1)$, has coordinates $(\cos(k-1)\varphi, \sin(k-1)\varphi)$.

The closure of the half of this polygon situated above the Ox axis is denoted by M_b. By construction, M_b is already an n-gon with the vertices $A_1, A_2, \dots, A_n$, that is, has number of sides equal to the number of edges to which the vertex b is incident. We call the side A_1A_n a t-side, $A_{n-1}A_n$ a u-side or a c_0-side, A_1A_2 an s-side or a c_{n-2}-side, and A_kA_{k+1}, where $k = 2, \dots, n-2$, is called a c_{n-k-1}-side.

Step 2. We construct a vector field v_b on M_b as follows.

First we define a vector field $v_{A_1A_n}$ on the side A_1A_n by the system of differential equations

$$\begin{cases} \dot{y} = 0, \\ \dot{x} = \sin \dfrac{1}{2}\pi(x-1). \end{cases}$$

By construction, A_1, A_n are fixed points, and the flow generated by this vector field is moving from A_1 to A_n. We define the vector field on the other sides of the polygon M_b.

Consider the side A_kA_{k+1}, $k = 1, \dots, n-1$. The straight line passing through the points A_k, A_{k+1} is defined by the equation

$$A_kA_{k+1}\colon \frac{x - \cos(k-1)\varphi}{\cos k\varphi - \cos(k-1)\varphi} = \frac{y - \sin(k-1)\varphi}{\sin k\varphi - \sin(k-1)\varphi},$$

from which the slope angle β_k of the straight line A_kA_{k+1} with respect to the positive direction of the Ox axis is expressed as follows:

$$\beta_k = \operatorname{arctg} \frac{\sin k\varphi - \sin(k-1)\varphi}{\cos k\varphi - \cos(k-1)\varphi}.$$

We now reduce the situation under consideration to the case of the side A_1A_n. For that we establish a one-to-one correspondence t_k between the points of the segments $[\cos k\varphi, \cos(k-1)\varphi]$ and $[-1,1]$ by the formula

$$t_k = 2\frac{x-\cos k\varphi}{\cos(k-1)\varphi - \cos k\varphi} - 1.$$

We set $\gamma_k = \sin\frac{1}{2}\pi(t_k-1)$ and define a vector field $v_{A_kA_{k+1}}$ by the set of systems of differential equations

$$\left[\begin{array}{l}\begin{cases}\beta_k \neq 0,\\ \dot{x} = -\gamma_k\cdot\cos\beta_k\cdot\operatorname{sign}x,\\ \dot{y} = -\gamma_k\cdot\sin\beta_k\cdot\operatorname{sign}x,\end{cases}\\ \begin{cases}\beta_k = 0,\\ \dot{x} = \gamma_k,\\ \dot{y} = 0.\end{cases}\end{array}\right.$$

Step 3. We construct the vector field v_{int} inside the polygon M_b. We choose an arbitrary point B with coordinates (x,y) inside the polygon M_b. Then B belongs to the vertical segment B_kH, where $B_k \in A_kA_{k+1}$ for some $k=1,\dots,n-1$, and H is the projection of B_k onto Ox (see Figure 8). We define the vector field v_{int} by the set of systems of differential equations

$$\left[\begin{array}{l}\begin{cases}\beta_k \neq 0,\\ \dot{x} = \dfrac{B_kB}{B_kH}\sin\dfrac{1}{2}\pi(x-1) - \dfrac{BH}{B_kH}\gamma_k\cdot\cos\beta_k\cdot\operatorname{sign}x,\\ \dot{y} = -\dfrac{BH}{B_kH}\gamma_k\cdot\sin\beta_k\cdot\operatorname{sign}x,\end{cases}\\ \begin{cases}\beta_k = 0,\\ \dot{x} = \dfrac{B_kB}{B_kH}\sin\dfrac{1}{2}\pi(x-1) + \dfrac{BH}{B_kH}\gamma_k,\\ \dot{y} = 0.\end{cases}\end{array}\right.$$

We define the vector field v_b by the system

$$v(x,y) = \begin{cases} v_{A_1A_n}(x,y), & (x,y)\in A_1A_n,\\ v_{A_kA_{k+1}}(x,y), & (x,y)\in A_kA_{k+1}, \quad k=1,\dots,n-1,\\ v_{\mathrm{int}}(x,y), & (x,y)\in \operatorname{int}M_b.\end{cases}$$

Step 4. Let B denote the set of vertices, N the number of vertices, and E the set of edges of the graph Γ. Let η_b be the map associating with a t-, u-, s-, or c_i-edge incident to a vertex b a t-, u-, s-, or c_i-side of the polygon M_b, respectively. Let $\mathscr{M}$ be the disjunct union of the polygons M_b, $b\in B$. We introduce on the set $\mathscr{M}$ the minimal equivalence relation $\sim$ satisfying the following rule: if vertices b_1, b_2 in the set B are incident to an edge e in the set E, then the segments $P_1Q_1 = \eta_{b_1}(e)$ and $P_2Q_2 = \eta_{b_2}(e)$ are identified in such a way that a point $(x_1, y_1) \in P_1Q_1 = [(x_{P_1}, y_{P_1}), (x_{Q_1}, y_{Q_1})]$ is equivalent to the point $(x_2, y_2) \in P_2Q_2 = [(x_{P_2}, y_{P_2}), (x_{Q_2}, y_{Q_2})]$, where

$$\begin{cases} x_2 = x_{P_2} + \dfrac{(x_1 - x_{P_1})(x_{Q_2} - x_{P_2})}{x_{Q_1} - x_{P_1}}, \\ y_2 = y_{P_2} + \dfrac{(y_1 - y_{P_1})(y_{Q_2} - y_{P_2})}{y_{Q_1} - y_{P_1}}. \end{cases}$$

It follows from the properties of an admissible graph that the quotient space $S = \mathscr{M}/\sim$ is a closed topological 2-manifold. Let $q : \mathscr{M} \to S$ denote the natural projection. Note that the vector field has the same length at equivalent points; therefore the projection q induces a continuous vector field on the manifold S, which is denoted by V.

Step 5. We define on S a smooth structure with respect to which the field V is smooth.

We cover the manifold S with finitely many charts (U_z, ψ_z), $z \in S$, where $U_z \subset S$ is an open neighbourhood of a point z, and $\psi_z \colon U_z \to \mathbb{R}^2$ is a homeomorphism onto the images of the following types.

1. Consider on the graph Γ the c^*-cycle

$$\{b_1, c^{b_1}_{j_1} = c^{b_2}_{j_2}, b_2, c^{b_2}_{j_2-1} = c^{b_3}_{j_3-1}, b_3, c^{b_3}_{j_3} = c^{b_4}_{j_4}, b_4, c^{b_4}_{j_4-1} = c^{b_1}_{j_1-1}, b_1\},$$

where the n_i-gon M_{b_i} corresponds to the vertex $b_i \in B$, $i = 1, \dots, 4$, and $\eta_{b_i}(c^{b_i}_{j_i}) = A_{k_i}A_{k_{i+1}}$ for $k_i = n_i - j_i - 1$ (Figure 9).

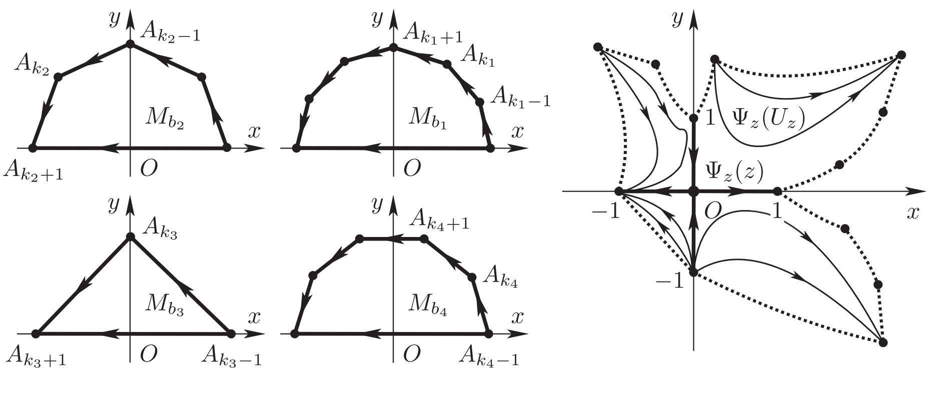

Figure 9. An example of a chart of the first type.

We denote the length of the side $A_{k_i}A_{k_i+1}$, the central angle of the polygon M_{b_i}, and the angle between the vectors $\overrightarrow{A_{k_i}A_{k_i+1}}$ ($\overrightarrow{A_{k_i}A_{k_i-1}}$) and the positive direction of the Ox axis by a_i, φ_i, and $\beta^+_{k_i}$ ($\beta^-_{k_i}$), respectively. Here, the angles $\beta^+_{k_i}$, $\beta^-_{k_i}$ are chosen in such a way that $|\beta^+_{k_i} - \beta^-_{k_i}| < \pi$. We set

$$U_z = \operatorname{int}\biggl(\bigcup_{i=1}^{4} q(M_{b_i})\biggr),$$

$$\psi_z(\varrho) = \mu_i(p_{1,i}((q|_{M_{b_i}})^{-1}(\varrho))) \quad \text{for } \varrho \in q(M_{b_i}), \quad i = 1, \dots, 4,$$

where

$$p_{1,i}(x,y)=\left(\frac{x-\cos(k_i-1)\varphi_i}{a_i},\ \frac{y-\cos(k_i-1)\varphi_i}{a_i}\right),$$
$$\mu_i(x,y)=\mu_i(r\cos\theta,r\sin\theta)=(r\cos\theta_{1,i},r\sin\theta_{1,i}),$$

(r,θ) are polar coordinates, and the function $\theta_{1,i}(\theta)$ is defined by the formula

$$\theta_{1,i}(\theta)=\left(i-2\left(\frac{i}{2}\,(\mathrm{mod}\,1)\right)\right)\cdot\frac{\pi}{2}+(-1)^{i-1}\frac{\pi}{2}\cdot\frac{\theta-\beta^+_{k_i}}{\beta^+_{k_i}-\beta^-_{k_i}}.$$

The function $p_{1,i}(x,y)$ translates the polygon M_{b_i}, placing the vertex A_{k_i} at the origin, and increases the length of the sides $A_{k_i}A_{k_i+1}$ and $A_{k_i-1}A_{k_i}$ to 1. The function $\mu_i(x,y)$ superposes the angle at the vertex A_{k_i} with the ith coordinate angle.

2. Consider on the graph Γ the st-cycle

$$\{b_1,(b_1,b_2),b_2,(b_2,b_3),b_3,\dots,b_{2m-1},(b_{2m-1},b_{2m}),b_{2m},(b_{2m},b_1),b_1\},$$

where the n_i-gon M_{b_i} corresponds to the vertex $b_i\in B$, $i=1,\dots,2m$,
$\eta_{b_{2j-1}}((b_{2j-1},b_{2j}))$ is the side A_1A_2 in the polygon $M_{b_{2j-1}}$,
$\eta_{b_{2j}}((b_{2j-1},b_{2j}))$ is the side A_1A_2 in the polygon $M_{b_{2j}}$,
$\eta_{b_{2j}}((b_{2j},b_{2j+1}))$ is the side $A_1A_{n_{2j}}$ in the polygon $M_{b_{2j}}$,
$\eta_{b_{2j+1}}((b_{2j},b_{2j+1}))$ is the side $A_1A_{n_{2j+1}}$ in the polygon $M_{b_{2j+1}}$ for $j=1,\dots,m$, $n_{2j+1}=n_1$.

Recall that in the polygon M_{b_i} the length of the side A_1A_2 is equal to a_i and the length of the side $A_1A_{n_i}$ is equal to 2. We denote the angle between the vector $\overrightarrow{A_1A_2}$ and the positive direction of the Ox axis by $\beta^+_{1,i}$. We set

$$U_z=\operatorname{int}\left(\bigcup_{i=1}^{2m}q(M_{b_i})\right),$$
$$\psi_z(\varrho)=\nu_i(p_{2,i}(q|_{M_{b_i}})^{-1}(\varrho)))\quad\text{for }\varrho\in q(M_{b_i}),\quad i=1,\dots,2m,$$

where

$$p_{2,i}(x,y)=(x-1,y)$$

and the function

$$\nu_i(x,y)=\nu_i(r\cos\theta,r\sin\theta)=(r_{2,i}(r,\theta)\cdot\cos(\theta_{2,i}(\theta)),\ r_{2,i}(r,\theta)\cdot\sin(\theta_{2,i}(\theta)))$$

is defined by the formulae

$$r_{2,i}(r,\theta)=\frac{r}{2}\cdot\frac{\theta-\beta^+_{1,i}}{\pi-\beta^+_{1,i}}+\frac{r}{a_i}\cdot\frac{\pi-\theta}{\pi-\beta^+_{1,i}},$$
$$\theta_{2,i}(\theta)=\left(i-2\left(\frac{i}{2}\,(\mathrm{mod}\,1)\right)\right)\cdot\frac{\pi}{m}+(-1)^{i-1}\frac{\theta-\beta^+_{1,i}}{\pi-\beta^+_{1,i}}\cdot\frac{\pi}{m}.$$

The function $p_{2,i}(x,y)$ translates the polygon M_{b_i} in such a way that the vertex A_1 is placed at the origin. The functions $\nu_i(x,y)$, $i=1,\dots,2m$, diminish the length of the sides A_1A_2 and $A_1A_{n_i}$ to 1, diminish the magnitude of the angle at the vertex A_1 to π/m, and place the polygons M_{b_i} in such a way that the vertices A_1 are placed at the origin and the angles of these polygons at the vertex A_1 adjoin one another and fill the complete angle, each being situated in the ith place under going around the origin counterclockwise over some circle of radius <1 starting from the positive half-axis Ox, and the sides of the same colour of polygons adjoining each other coincide.

3. Consider on the graph Γ the ut-cycle

$$\{b_1,(b_1,b_2),b_2,(b_2,b_3),b_3,\dots,b_{2m-1},(b_{2m-1},b_{2m}),b_{2m},(b_{2m},b_1),b_1\},$$

where the n_i-gon M_{b_i} corresponds to the vertex $b_i\in B$, $i=1,\dots,2m$,
$\eta_{b_{2j-1}}((b_{2j-1},b_{2j}))$ is the side $A_{n_{2j-1}-1}A_{n_{2j-1}}$ in the polygon $M_{b_{2j-1}}$,
$\eta_{b_{2j}}((b_{2j-1},b_{2j}))$ is the side $A_{n_{2j}-1}A_{n_{2j}}$ in the polygon $M_{b_{2j}}$,
$\eta_{b_{2j}}((b_{2j},b_{2j+1}))$ is the side $A_1A_{n_{2j}}$ in the polygon $M_{b_{2j}}$,
$\eta_{b_{2j+1}}((b_{2j},b_{2j+1}))$ is the side $A_1A_{n_{2j+1}}$ in the polygon $M_{b_{2j+1}}$ for $j=1,\dots,m$, $n_{2j+1}=n_1$.

Recall that in the polygon M_{b_i} the length of the side $A_{n_i-1}A_{n_i}$ is equal to a_i, the length of the side $A_1A_{n_i}$ is equal to 2, and the angle between the vector $\overrightarrow{A_{n_i}A_{n_i-1}}$ and the positive direction of the Ox axis is equal to $\beta^-_{n_i,i}$. We set

$$U_z=\operatorname{int}\biggl(\bigcup_{i=1}^{2m}q(M_{b_i})\biggr),$$
$$\psi_z(\varrho)=\kappa_i(p_{3,i}((q|_{M_{b_i}})^{-1}(\varrho)))\quad\text{for } \varrho\in q(M_{b_i}),\quad i=1,\dots,2m,$$

where

$$p_{3,i}(x,y)=(x+1,y)$$

and the function

$$\kappa_i(x,y)=\kappa_i(r\cos\theta,r\sin\theta)=(r_{3,i}(r,\theta)\cdot\cos(\theta_{3,i}(\theta)),\ r_{3,i}(r,\theta)\cdot\sin(\theta_{3,i}(\theta)))$$

is defined by the formulae

$$r_{3,i}(r,\theta)=\frac{r}{2}\cdot\frac{\beta^-_{n_i}-\theta}{\beta^-_{n_i}}+\frac{r}{a_i}\cdot\frac{\theta}{\beta^-_{n_i}},$$
$$\theta_{3,i}(\theta)=\biggl(i-2\biggl(\frac{i}{2}\ (\mathrm{mod}\,1)\biggr)\biggr)\cdot\frac{\pi}{m}+(-1)^{i-1}\frac{\theta}{\beta^-_{n_i}}\cdot\frac{\pi}{m}.$$

The function $p_{3,i}(x,y)$ translates the polygon M_{b_i} is such a way that the vertex A_{n_i} is placed at the origin. The functions $\nu_i(x,y)$, $i=1,\dots,2m$, change the length of the sides $A_{n_i-1}A_{n_i}$ and $A_1A_{n_i}$ to 1 preserving the continuity of the field, change the magnitude of the angle at the vertex A_{n_i} to π/m, and place the polygons M_{b_i} in such a way that the vertices A_{n_i} are placed at the origin, the angles of these polygons at the vertices A_{n_i} adjoin one another and fill the complete angle being each situated in the ith place under going around the origin counterclockwise over some circle of radius < 1 starting from the positive half-axis Ox, and sides of the same colour of polygons adjoining each other coincide.

For the charts introduced above, the transition maps are compositions of the smooth maps constructed in parts 1–3 and their inverses, which implies that these charts define a smooth structure on the surface S.

II. We now prove parts (i), (ii) of Theorem 3.

(i) We claim that the Euler characteristic of the surface S is calculated by the formula $\chi(S) = \nu_0 - \nu_1 + \nu_2$, where ν_0, ν_1 and ν_2 are the numbers of all *tu*-, c^*- and *st*-cycles of the graph Γ, respectively. It follows from Proposition 3 that the numbers of all sinks, saddles, and sources are equal to ν_0, ν_1 and ν_2, respectively. This implies the assertion that we are proving, since the aforementioned formula is the formula for the sum of indices of singular points of the flow f^t.

(ii) We claim that the surface S is non-orientable if and only if the graph Γ contains at least one cycle of odd length.

The surface S on which we constructed the flow f^t is orientable if and only if all polygonal regions of the flow f^t can be compatibly oriented. An orientation of every polygonal region can be defined by choosing one of the two possible cyclic orders of its fixed points: $\alpha, \sigma_n, \dots, \sigma_1, \omega$ or $\omega, \sigma_1, \dots, \sigma_n, \alpha$, where α is a source, σ_j is a saddle, $j = 1, \dots, n$, and ω is a sink. Suppose that the sign plus is assigned to a polygonal region in the first case, and minus in the second. Clearly, the orientations of two such regions having a common side are compatible if and only if different signs are assigned to them. Since a one-to-one correspondence was established by the map π_{f^t} between the polygonal regions of the flow f^t and the vertices of the graph Γ, the condition of orientability of the surface S can be stated as follows: the surface S is orientable if and only if the signs plus and minus are assigned to the vertices of the graph Γ in such a way that any two of its vertices connected by an edge have different signs. We say that such an arrangement of signs of the vertices of the graph is *regular*.

It is now sufficient to prove that the graph Γ does not have cycles of odd length if and only if there exists a regular arrangement of the signs plus and minus at the vertices of Γ.

The validity of the assertion from right to left is obvious, since it is impossible to regularly arrange the signs plus and minus in a cycle of odd length. We now prove in the other direction: suppose that the graph Γ does not have cycles of odd length. Then it is possible to assign signs regularly to its vertices as follows: we consider some vertex b_0 of the graph Γ and assign the sign plus to it; for any other vertex b_i we consider a path connecting it with the vertex b_0, and if this path has even length, then we assign the sign plus to this vertex, while if it has odd length, then the sign minus. Since by assumption the graph does not have cycles of odd length, this arrangement is independent of the choice of the path and, consequently, the definition is correct.

Theorem 3 is proved.

§ 7. Efficient algorithm for recognition of isomorphism of graphs of flows of the class G

In this section we present a proof of Theorem 2 produced by way of constructing an efficient algorithm for recognition (up to isomorphism) of multicolour graphs of flows in the class G. For this we can assume that the numbers of vertices and edges of these graphs are the same; otherwise they are automatically non-isomorphic. By construction the multicolour graphs of flows in G are not graphs of the general form, since they are embeddable in the supporting surface, on which the corresponding flows of class G are defined. In other words, these graphs can be depicted in such a way that their vertices are points on the surface, and edges are Jordan curves that do not intersect at their interior points. This observation is interesting because of the existence of an efficient algorithm for distinguishing *ordinary graphs* (that is, unlabelled graphs without loops, orientation, or multiple edges) that are embeddable in a given surface; namely, the following fact holds.

Proposition 4 (see [11]). *The problem of recognition of isomorphism of two n-vertex ordinary graphs each of which is embeddable in a surface of genus g can be solved in time $O(n^{O(g)})$.*

Unfortunately, this result cannot be directly applied to recognition of isomorphism of graphs Γ_{f^t} and $\Gamma_{f'^t}$, since these are not ordinary graphs. Nevertheless, the problem of isomorphism of multicolour graphs can be reduced (with a low working time of reduction) to the problem of isomorphism of ordinary graphs embeddable in a surface. For that we need two operations with graphs — k-subpartition of an edge, and (k_1,k_2)-subpartition of an edge.

The operation of *k-subpartition of an edge* (a,b) of a graph consists in deleting this edge from the graph and adding vertices $c_1,\dots,c_k$ and edges $(a,c_1),(c_1,c_2),\dots,$ (c_k,b).

The operation of *(k_1,k_2)-subpartition of an edge* (a,b) of a graph consists in deleting this edge from the graph and adding vertices $c_1,c_2,\dots,c_{k_1}$, v, u, w, $d_1,d_2,\dots,d_{k_2}$ and edges $(a,c_1),(c_1,c_2),\dots,(c_{k_1},v)$, (v,u), (u,w), (v,w), (v,d_1), $(d_1,d_2),\dots,(d_{k_2},b)$.

For a given graph Γ_{f^t}, we construct the corresponding ordinary graph $\Gamma(f^t)$ as follows. In the graph Γ_{f^t} we perform a 1-subpartition of every s-edge, a 2-subpartition of every t-edge, and a 3-subpartition of every u-edge. Let $e=(a,b)$ be an arbitrary c-edge of the graph Γ_{f^t}, and let $\mathrm{num}_a(e)$ and $\mathrm{num}_b(e)$ be the numbers of the edge e in the sets of c-edges incident to the vertices a and b, respectively. We perform a $(\mathrm{num}_a(e),\mathrm{num}_b(e))$-subpartition of the edge e. We perform a similar operation for every c-edge of the graph Γ_{f^t} (Figure 10).

Lemma 6. *The graphs Γ_{f^t} and $\Gamma_{f'^t}$ are isomorphic if and only if the graphs $\Gamma(f^t)$ and $\Gamma(f'^t)$ are isomorphic.*

Proof. Obviously, the graph Γ_{f^t} uniquely defines the graph $\Gamma(f^t)$. We now show that the converse assertion also holds, which will imply the validity of the lemma. Every polygonal region of the set Δ_{f^t} has at least three sides, and therefore every vertex Γ_{f^t} has at least three neighbours in this graph. Obviously, in the graph $\Gamma(f^t)$, none of the vertices of the graph Γ_{f^t} belong to any triangle. Therefore the set of vertices of the graph Γ_{f^t} is formed by those and only those vertices of the

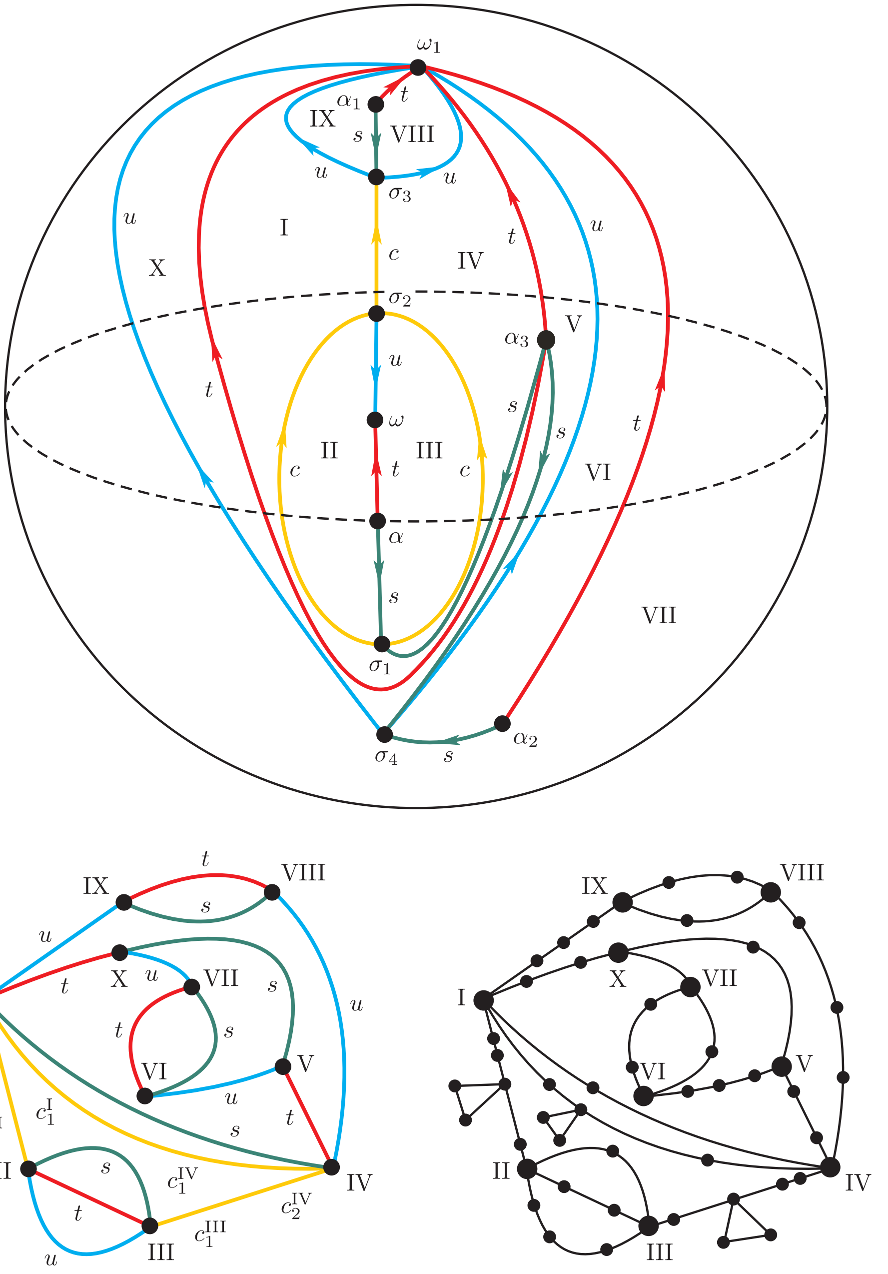

Figure 10. The phase portrait of some flow f^t in the class G (above), its four-colour graph Γ_{f^t} (below on the left), and the corresponding ordinary graph $\Gamma(f^t)$ (below on the right).

graph $\Gamma(f^t)$ that have at least three neighbours and do not belong to triangles. By removing all vertices of the graph Γ_{f^t} from the graph $\Gamma(f^t)$ we obtain a disjunct union of connected subgraphs each of which is either a path, or a path with a triangle 'attached' to some interior vertex. These connected subgraphs are indicators of the presence of edges between the corresponding vertices of the graph Γ_{f^t}. If a subgraph is a path, then its length determines the colour from the set $\{s, t, u\}$ of the corresponding edge of the graph Γ_{f^t}. If a subgraph is a path with an 'attached' triangle, then it corresponds to some c-edge $e = (a, b)$ of the graph Γ_{f^t}. In this subgraph we remove the vertices of the triangle, and obtain two paths, the lengths of which determine the numbers of e in the sets of c-edges incident to the vertices a and b, respectively. Thus, the graph Γ_{f^t} is uniquely reconstructed from the graph $\Gamma(f^t)$.

The lemma is proved.

Let us estimate the number of vertices of the graph $\Gamma(f^t)$ under the assumption that the graph Γ_{f^t} has n vertices and m edges. Obviously, each of the m edges of the graph Γ_{f^t} corresponds to some subgraph of the graph $\Gamma(f^t)$ containing at most $2n + 5$ vertices. Therefore the graph $\Gamma(f^t)$ has at most $(2n + 5)m$ vertices and can be efficiently calculated from the graph Γ_{f^t}. We point out that the graph $\Gamma(f^t)$ is embeddable in the same surface as the graph Γ_{f^t}. Therefore by Lemma 6 we have a polynomial reduction of the problem of recognition of isomorphism of the multicolour graphs of flows in the class G to the problem of recognition of isomorphism of ordinary graphs embedded in a fixed surface.

Recall that by Theorem 3 the surface S is non-orientable if and only if the graph Γ contains a cycle of odd length. By König's theorem [7], an ordinary graph does not contain odd cycles if and only if it is *bipartite*, that is, when the set of its vertices can be partitioned into at most two parts such that there are no edges incident to vertices in the same part. For an ordinary graph with n' vertices and m' edges, being bipartite can be recognized in time $O(n' + m')$ by using breadth-first search (see [1]). Therefore, in order to recognize the orientability of the surface S, we forget about the colours of the edges of the graph Γ and perform a 2-subpartition of each of its edges. Clearly, the resulting graph Γ' is bipartite if and only if the graph Γ does not contain cycles of odd length. The numbers of vertices and edges of the graph Γ' do not exceed the tripled numbers of vertices and edges of the graph Γ, respectively. Therefore the orientability of the surface S can be recognized in linear time in terms of the sum of the numbers of vertices and edges of the graph Γ.

All that has been said above implies the validity of Theorem 2.

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