Advances in Mathematics 320 (2017) 1135–1157

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Existence of HKT metrics on hypercomplex manifolds of real dimension 8

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ARTICLE INFO

Article history:
Received 3 September 2015
Received in revised form 3 February 2017
Accepted 12 September 2017
Available online xxxx
Communicated by Tristan Rivière

Keywords:
Hypercomplex Geometry
Hyperkähler with torsion metrics

ABSTRACT

A hypercomplex manifold M is a manifold equipped with three complex structures I, J, K satisfying quaternionic relations. Such a manifold admits a canonical torsion-free connection preserving the quaternion action, called the Obata connection. A quaternionic Hermitian metric is a Riemannian metric which is invariant with respect to unitary quaternions. Such a metric is called hyperkähler with torsion (HKT for short) if it is locally obtained as the Hessian of a function averaged with quaternions. An HKT metric is a natural analogue of a Kähler metric on a complex manifold. We push this analogy further, proving a quaternionic analogue of the result of Buchdahl and of Lamari that a compact complex surface M admits a Kähler structure if and only if $b_1(M)$ is even. We show that a hypercomplex manifold M with the Obata holon-

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[1] Misha Verbitsky is partially supported by RFBR grant 10-01-93113-NCNIL-a, RFBR grant 09-01-00242-a, Simons-IUM fellowship, Science Foundation of the SU-HSE award No. 10-09-0015 and AG Laboratory HSE, RF government grant, ag. 11.G34.31.0023. Misha Verbitsky is also partially supported by the Russian Academic Excellence Project '5-100'. Gueo Grantcharov is supported by a grant from the Simons Foundation (#246184).

http://dx.doi.org/10.1016/j.aim.2017.09.020

omy contained in $SL(2,\mathbb{H})$ admits an HKT structure if and only if $H^1\left(\mathcal{O}_{(M,I)}\right)$ is even-dimensional.

1. Introduction

1.1. Hypercomplex manifolds and HKT metrics

A hypercomplex manifold (M, I, J, K) is a manifold M equipped with three integrable almost-complex structures I, J, K satisfying the quaternionic relation $IJ = -JI = K$. Hypercomplex manifolds are the closest quaternionic counterparts of complex manifolds. They have been much studied by physicists since they appeared in the sigma model. However, their mathematical properties still remain a puzzle. One obstacle comes from the fact that compact hypercomplex manifolds are non-Kähler (unless they are hyperkähler [34]). Hypercomplex manifolds appear to be one of the more-studied and better understood classes of non-Kähler manifolds. There are many interesting examples of hypercomplex manifolds and many general theorems, especially about manifolds with trivial canonical bundle or admitting hyperkähler with torsion (HKT for short) metric. For a complete introduction to the HKT geometry, see [12].

HKT metrics play in the hypercomplex geometry the same role as Kähler metrics play in the complex geometry. They admit locally a smooth potential [3]. There is a notion of an "HKT class" (similar to a Kähler class) in a certain finite-dimensional cohomology group, called the Bott–Chern cohomology group (see Subsection 4). Two metrics in the same HKT class differ by a potential, which is a function. When M admits an HKT metric and has a trivial canonical bundle, a version of the Hodge theory has been established [30], giving an $\mathfrak{sl}(2)$-action on the holomorphic cohomology $H^*(\mathcal{O}_{(M,I)})$, and an analogue of the Hodge decomposition and the dd^c-lemma.

Originally, it was conjectured that all hypercomplex manifolds are HKT. The first counterexample to that assertion is due to Fino and Grantcharov [9]; for more examples of non-HKT manifolds, see [4], [28] and [27]. It is therefore important to understand when does a hypercomplex manifold admit an HKT metric. A real 4-dimensional manifold always does, because of the dimension [30]. The present paper builds a new step in this direction in the case when the real dimension is 8. More precisely we provide, for the smaller but important class of so-called $SL(2,\mathbb{H})$-manifolds, a necessary and sufficient condition for an HKT metric to exist.

1.2. An introduction to $SL(n,\mathbb{H})$-geometry

As Obata has shown [23], a hypercomplex manifold (M, I, J, K) admits a necessarily unique torsion-free connection, preserving I, J, K. The converse is also true: if a manifold M equipped with an action of $\mathbb{H}$ on its tangent space admits a torsion-free connection preserving the quaternionic action, it will be hypercomplex. This implies that a hypercomplex structure on a manifold can be defined as a torsion-free connection with the holonomy in $GL(n, \mathbb{H})$. This connection is called the Obata connection.

Connections with a restricted holonomy are one of the central notions in the Riemannian geometry, due to Berger's classification of irreducible holonomy of Riemannian manifolds. However, a similar classification exists for general torsion-free connections [22]. Besides $SL(n, \mathbb{H}) \cdot U(1)$ [17], only three subgroups of $GL(n, \mathbb{H})$ occur in the Merkulov–Schwachhöfer list. In addition to the compact group $Sp(n)$ (which defines the hyperkähler geometry), also $GL(n, \mathbb{H})$ and its commutator $SL(n, \mathbb{H})$ appear, corresponding to hypercomplex manifolds and hypercomplex manifolds with a trivial determinant bundle, respectively. Both of these geometries are interesting, rich in structure and examples, and deserve detailed study.

It is easy to see that (M, I) has a holomorphically trivial canonical bundle, for any $SL(n, \mathbb{H})$-manifold (M, I, J, K) [33]. Using a version of the Hodge theory [30], it was shown that a compact hypercomplex manifold with a trivial canonical bundle has the holonomy in $SL(n, \mathbb{H})$, if it admits an HKT structure [33].

In [4], it was shown that the holonomy of all hypercomplex nilmanifolds lies in $SL(n, \mathbb{H})$. Many working examples of hypercomplex manifolds are in fact nilmanifolds, and by this result they all belong to the class of $SL(n, \mathbb{H})$-manifolds.

The $SL(n, \mathbb{H})$-manifolds were studied in [2] and [35]. On such manifolds, the quaternionic Dolbeault complex is identified with a part of de Rham complex [35], making it possible to write a quaternionic version of the Monge–Ampère equation [2], and to use the quaternionic linear algebra to study positive currents on hyperkähler manifolds [35]. Under this identification, $\mathbb{H}$-positive forms become positive in the usual sense, and ∂, ∂_J-closed or exact forms become $\partial, \overline{\partial}$-closed or exact (see Proposition 14, (4)). This linear-algebraic identification is especially useful in the study of the quaternionic Monge–Ampère equation [2].

1.3. Main result, comparison with Lamari's theorem and structure of the paper

The main result of this paper is the following theorem

Theorem 1. *Let (M, I, J, K) be a compact hypercomplex manifold of real dimension 8 with the Obata holonomy contained in $SL(2, \mathbb{H})$. Then, M is an HKT manifold if and only if $H^1\left(\mathcal{O}_{(M,I)}\right)$ is even-dimensional.*

Actually one of the two implications in Theorem 1 is true in arbitrary dimension, namely: if M is a $SL(n, \mathbb{H})$-manifold that admits an HKT metric then the even-dimensionality of $H^1\left(\mathcal{O}_{(M,I)}\right)$ follows by Hodge theory, as pointed out in the proof of Theorem 1. The main purpose of the present paper is to prove the other implication.

Theorem 1 can be viewed as the quaternionic analogue of the well-known result on compact complex surfaces according to which such a surface is Kähler if and only if the first Betti number is even. This theorem has been known since the mid 1980s, but its proof was based on Kodaira classification of complex surfaces, taking hundreds (if not thousands) of pages and a complicated result of Siu, who proved that all K3 surfaces are Kähler. Buchdahl and Lamari, in two independent papers [6] and [21], gave a direct proof. The proof of Theorem 1 is based on the same arguments as used by Lamari [21] to prove that any complex surface with even b_1 is Kähler. However, in the hypercomplex case, this result is (surprisingly) much easier to prove than in the complex case. Indeed, we are lucky that, for HKT manifolds, the dd^c-lemma (or more precisely, its quaternionic analogue) is the only non-trivial step. In fact, following Lamari's strategy, Theorem 1 will be deduced from the quaternionic dd^c-lemma by employing a suitable extension to the hypercomplex setting of Harvey–Lawson's theory of positive currents: this will be discussed in Section 6 and 7. On the other hand, Sections 3 and 4 are devoted to the tools needed for the proof of the quaternionic dd^c-lemma: quaternionic Gauduchon metrics and quaternionic Aeppli and Bott–Chern cohomology. In Section 5, we prove the quaternionic dd^c-lemma for $(2,0)$-forms (with respect to I) on compact $SL(2,\mathbb{H})$ (not necessarily HKT). In Section 8, we give some applications of Theorem 1.

2. Preliminaries

In this section we will present terminology, notions and some results that will be needed later in the paper.

2.1. Hypercomplex manifolds: basic notions

Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_{TM}$. Suppose that I, J, K are integrable almost-complex structures. Then, (M, I, J, K) is called a hypercomplex manifold.

In real dimension 4, compact hypercomplex manifolds have been classified by Boyer [5]. Higher dimensional examples are given for instance in [25] and [20].

Definition 2. Let (M, I, J, K) be a hypercomplex manifold and g a Riemannian metric. We say that g is quaternionic Hermitian if I, J, K are orthogonal with respect to g i.e.

$$g(\cdot,\cdot) = g(I\cdot, I\cdot) = g(J\cdot, J\cdot) = g(K\cdot, K\cdot).$$

Given a quaternionic Hermitian metric g on (M, I, J, K), consider its Hermitian forms (real but not necessarily closed)

$$\omega_I(\cdot,\cdot) := g(\cdot, I\cdot), \quad \omega_J(\cdot,\cdot) := g(\cdot, J\cdot), \quad \omega_K(\cdot,\cdot) := g(\cdot, K\cdot).$$

Then, $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ is of Hodge type (2,0) with respect to I. The quaternionic Hermitian metric g can be easily reconstructed from Ω and J. Indeed, for any $x, y \in T_I^{1,0}(M)$, one has

$$2g(x, \overline{y}) = \Omega(x, J(\overline{y})). \tag{1}$$

If $d\Omega = 0$, one has $d\omega_I = d\omega_J = d\omega_K = 0$, and the manifold (M, I, J, K, g) is hyperkähler.

There are two torsion-free connections that can be naturally introduced on a hypercomplex manifold. The first is the Levi-Civita connection of a quaternionic Hermitian metric. The second is the so-called Obata connection [23] which the unique torsion-free connection ∇ that preserves the hypercomplex structure i.e. $\nabla I = \nabla J = \nabla K = 0$. It follows from the definition that the holonomy of the Obata connection is contained in $GL(n, \mathbb{H})$. For instance, the holonomy of the Obata connection on $SU(3)$ is $GL(2, \mathbb{H})$ [26]. In general, the Obata connection does not preserve the metric and, on the other hand, the Levi-Civita connection does not preserve the hypercomplex structure. An HKT-connection was introduced in order to have both at the same time (on a hypercomplex manifold). Clearly the torsion-free requirement had to be dropped: it was replaced by the weaker requirement that the torsion is a skew-symmetric tensor. Indeed in the original formulation [18] an HKT-connection on a hypercomplex manifold with a quaternionic Hermitian metric (M, I, J, K, g) is a connection ∇ such that $\nabla I = \nabla J = \nabla K = 0, \nabla g = 0$, and the torsion of ∇ is a skew-symmetric tensor. However the equivalent formulation given in [12] is the one that proved more useful for our purposes, in particular in view of the Hodge theory developed in [30].

Definition 3. Let (M, I, J, K) be a hypercomplex manifold, g a quaternionic Hermitian metric and $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ the corresponding $(2,0)$-form. We say that g is HKT if $\partial\Omega = 0$, where ∂ denotes the $(1,0)$-part of the de Rham differential with respect to I. In other words, (M, I, J, K) is called a HKT manifold if $d\Omega \in \Lambda_I^{2,1}(M)$.[2] The form $\Omega \in \Lambda_I^{2,0}(M)$ is called an HKT form on (M, I, J, K).

An important subgroup of $GL(n, \mathbb{H})$ is its commutator $SL(n, \mathbb{H})$. In the standard representation by real matrices, this is the subgroup of matrices with determinant one. As noted in [17], it is also isomorphic to one of the real forms of $SL(2n, \mathbb{C})$ denoted by $SU^*(2n)$ in [15]. In the present note, we focus on manifolds with the holonomy contained in $SL(n, \mathbb{H})$. A hypercomplex manifold of quaternionic dimension n with the holonomy of the Obata connection in $SL(n, \mathbb{H})$ is called an $SL(n, \mathbb{H})$-manifold. An equivalent characterization of $SL(n, \mathbb{H})$-manifolds is obtained through the existence of a certain non-degenerate form of $\Lambda_I^{2n,0}(M)$ [30] (see Definition 4). It can be shown that (M, I) has a holomorphically trivial canonical bundle, for any $SL(n, \mathbb{H})$-manifold [33]. Many nilman-

[2] $\Lambda^*(M)$ denotes the bundle of differential forms, and $\Lambda^*(M) = \oplus_{p,q}\Lambda_I^{p,q}(M)$ its Hodge decomposition, taken with respect to the complex structure I on M.

ifolds (quotients of a nilpotent Lie group by a cocompact lattice) admit hypercomplex structures. In this case, $\mathrm{Hol}(M) \subset SL(n, \mathbb{H})$ [4].

Definition 4. Let (M, I, J, K) be an $SL(n, \mathbb{H})$-manifold. Then there exists a non-degenerate holomorphic section Φ of $\Lambda_I^{2n,0}(M)$. The existence of such form is actually equivalent to $\mathrm{Hol}(M) \subset SL(n, \mathbb{H})$ [33]. Furthermore, we can assume that Φ is real, that is, $J(\Phi) = \overline{\Phi}$, and positive. In particular, $\partial\overline{\Phi} = \partial_J\overline{\Phi} = 0$.

Actually It is often convenient to define an $SL(n, \mathbb{H})$-structure by fixing the quaternionic action and the holomorphic form Φ.

One of the main ingredient of Theorem 1 is to prove that the quaternionic version of the dd^c-lemma, called the "$\partial\partial_J$-lemma", holds on compact $SL(2, \mathbb{H})$-manifold (not necessarily HKT) for the $(2, 0)$-forms with respect to I. We recall that in [30] it is shown that the $\partial\partial_J$-lemma holds on any compact HKT $SL(n, \mathbb{H})$-manifold. More precisely, let (M, I, J, K) be a hypercomplex manifold. We extend

$$J : \Lambda^1(M) \longrightarrow \Lambda^1(M)$$

to $\Lambda^*(M)$ by

$$(J\eta)(x_1, \cdots, x_p) = \eta(J^{-1}x_1, \cdots, J^{-1}x_p).$$

Recall that

$$J(\Lambda_I^{p,q}(M)) = \Lambda_I^{q,p}(M),$$

because I and J anticommute on $\Lambda^1(M)$. Denote by

$$\partial_J : \Lambda_I^{p,q}(M) \longrightarrow \Lambda_I^{p+1,q}(M)$$

the operator $J^{-1} \circ \overline{\partial} \circ J$, where $\overline{\partial} : \Lambda_I^{p,q}(M) \longrightarrow \Lambda_I^{p,q+1}(M)$ is the standard Dolbeault operator on (M, I), that is, the $(0, 1)$-part of the de Rham differential. Since $\overline{\partial}^2 = 0$, we have $\partial_J^2 = 0$. In [30], it is shown that ∂ and ∂_J anticommute:

$$\{\partial_J, \partial\} = 0.$$

The pair of anticommuting differentials ∂, ∂_J is a hypercomplex counterpart to the pair $d, d^c := I^{-1}dI$ of differentials on a complex manifold.

Definition 5. Let (M, I, J, K) be a hypercomplex manifold of real dimension $4n$. We say that the $\partial\partial_J$-lemma holds if every ∂_J-closed, ∂-exact $(p, 0)$-form in $\Lambda_I^{p,0}(M)$ is $\partial\partial_J$-exact, for any $0 \leqslant p \leqslant 2n$. This is equivalent to saying that every ∂-closed, ∂_J-exact is $\partial\partial_J$-exact.

Theorem 6. *The $\partial\partial_J$-lemma holds for any compact $SL(n, \mathbb{H})$-manifold admitting an HKT metric.*

Proof. In [30, Theorem 10.2], it is shown that for any HKT manifold, the Laplacian $\Delta_\partial := \partial\partial^* + \partial^*\partial$ on $\Lambda_I^{p,0}(M) \otimes K_M^{1/2}$ can be written as $\Delta_\partial = \{\partial, \{\partial_J, \Lambda_\Omega\}\}$, where ∂^* is the adjoint of ∂, K_M is the canonical bundle of (M, I), $\{\cdot, \cdot\}$ denotes the anticommutator and Λ_Ω is the contraction by the $(2, 0)$-form Ω (see [30, p. 699]). Then, $\Delta_\partial \eta = \partial\partial_J \Lambda_\Omega \eta$. However, since η is ∂-exact, it is orthogonal to the kernel of Δ_∂, giving $\eta = G\Delta_\partial \eta$, where G is the corresponding Green operator. This gives

$$\eta = G\Delta_\partial \eta = G\partial\partial_J \Lambda_\Omega \eta = \partial\partial_J G\Lambda_\Omega \eta.$$

However, on $SL(n, \mathbb{H})$-manifolds, the canonical bundle is trivial, and this result can be applied to any $\eta \in \Lambda_I^{p,0}(M)$. □

2.2. Quaternionic Dolbeault complex: a definition

It is well-known that any irreducible representation of $SU(2)$ over $\mathbb{C}$ can be obtained as a symmetric power $\mathrm{Sym}^i(V_1)$, where V_1 is a fundamental 2-dimensional representation. We say that a representation W has weight i if it is isomorphic to $\mathrm{Sym}^i(V_1)$. A representation is said to be pure of weight i if all its irreducible components have weight i.

Remark 7. The Clebsch–Gordan formula [19] claims that the weight satisfies the following: if $i \leqslant j$, then

$$V_i \otimes V_j = \bigoplus_{k=0}^{i} V_{i+j-2k},$$

where $V_i = \mathrm{Sym}^i(V_1)$ denotes the irreducible representation of weight i.

Let $V^k \subset \Lambda^k(M)$ be a maximal $SU(2)$-invariant subspace of weight $< k$. The space V^k is well defined, because it is the sum of all irreducible representations $W \subset \Lambda^k(M)$ of weight $< k$. Since the weight satisfies a certain behavior under tensor products (see Remark 7), $V^* = \bigoplus_k V^k$ is an ideal in $\Lambda^*(M)$.

It is easy to see that the de Rham differential d increases the weight by 1 at most. Therefore, $dV^k \subset V^{k+1}$, and $V^* \subset \Lambda^*(M)$ is a differential ideal in the de Rham DG-algebra $(\Lambda^*(M), d)$.

Definition 8. Denote by $(\Lambda_+^*(M), d_+)$ the quotient algebra $\Lambda^*(M)/V^*$. It is called the quaternionic Dolbeault algebra of M, or the quaternionic Dolbeault complex (qD-algebra or qD-complex for short).

Remark 9. The complex $(\Lambda^*_+(M), d_+)$ was constructed earlier by Capria and Salamon [7] in a different (and much more general) situation, and much studied since then.

2.3. The Hodge decomposition of the quaternionic Dolbeault complex

The Hodge bigrading is compatible with the weight decomposition of $\Lambda^*(M)$, and gives a Hodge decomposition of $\Lambda^*_+(M)$ [30]:

$$\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M).$$

The spaces $\Lambda^{p,q}_{+,I}(M)$ are the weight spaces for a particular choice of a Cartan subalgebra in $\mathfrak{su}(2)$ corresponding to I and sending $\eta \in \Lambda^{p,q}_I(M)$ to $\sqrt{-1}(p-q)\eta$. The $\mathfrak{su}(2)$-action induces an isomorphism of the weight spaces within an irreducible representation. This gives the following result:

Proposition 10. *Let (M, I, J, K) be a hypercomplex manifold and*

$$\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M)$$

the Hodge decomposition of qD-complex defined above. Then, there is a natural isomorphism

$$\Lambda^{p,q}_{+,I}(M) \cong \Lambda^{p+q,0}_I(M). \tag{2}$$

Proof. The above isomorphism is induced by the map

$$\mathcal{R}_{p,q} : \ \Lambda^{p+q,0}_I(M) \longrightarrow \Lambda^{p,q}_I(M)$$

defined by

$$\mathcal{R}_{p,q}(\eta)(x_1, \cdots, x_p, \overline{y}_1, \cdots, \overline{y}_q) := \eta(x_1, \cdots, x_p, J\overline{y}_1, \cdots, J\overline{y}_q),$$

where $x_i, y_i \in T^{1,0}_I(M)$ (see [30] for more details). □

This isomorphism is compatible with a natural algebraic structure on

$$\bigoplus_{p+q=i} \Lambda^{p+q,0}_I(M),$$

and with the Dolbeault differentials, in the following way; Consider the quaternionic Dolbeault complex $(\Lambda^*_+(M), d_+)$ constructed in Subsection 2.2. Using the Hodge bigrading, we can decompose this complex, obtaining a bicomplex

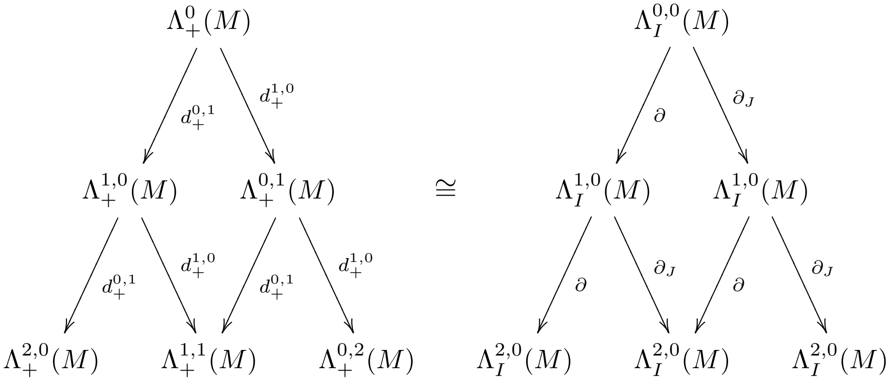
$$\Lambda^{*,*}_{+,I}(M) \xrightarrow{d^{1,0}_{+,I},\, d^{0,1}_{+,I}} \Lambda^{*,*}_{+,I}(M)$$

where $d^{1,0}_{+,I}$, $d^{0,1}_{+,I}$ are the Hodge components of the quaternionic Dolbeault differential d_+, taken with respect to I.

Theorem 11. *[30,32] Under the multiplicative isomorphism*

$$\Lambda^{p,q}_{+,I}(M) \cong \Lambda^{p+q,0}_{I}(M)$$

constructed in Proposition 10, $d^{1,0}_+$ corresponds to ∂ and $d^{0,1}_+$ to ∂_J:



(3)

Moreover, under this isomorphism, $\omega_I \in \Lambda^{1,1}_{+,I}(M)$ corresponds to $\Omega \in \Lambda^{2,0}_I(M)$.

2.4. Positive $(2,0)$*-forms on hypercomplex manifolds*

The notion of positive $(2p,0)$-forms on hypercomplex manifolds (sometimes called q-positive or $\mathbb{H}$-positive) was developed in [29] and [1] (see also [2,35]). For our present purposes, only $(2,0)$-forms are interesting, but everything can be immediately generalized to a general situation.

Let $\eta \in \Lambda^{p,q}_I(M)$ be a differential form. Since I and J anticommute, $J(\eta)$ lies in $\Lambda^{q,p}_I(M)$. Since, $J^2|_{\Lambda^{p,q}_I(M)} = (-1)^{p+q}$, for $p+q$ even, J composed with the conjugation is a complex anti-linear involution of $\Lambda^{p,q}_I(M)$, that is, a real structure on $\Lambda^{p,q}_I(M)$.

Definition 12. A form $\eta \in \Lambda^{2p,0}_I(M)$ is called real if $J(\overline{\eta}) = \eta$. We denote real forms in $\Lambda^{2p,0}_I(M)$ by $\Lambda^{2p,0}_{\mathbb{R}}(M, I)$.

For a real $(2,0)$-form η,

$$\eta\,(x, J(\overline{x}))) = \overline{\eta}\left(J(x), J^2(\overline{x})\right) = \overline{\eta}\,(\overline{x}, J(x))\,,$$

for any $x \in T^{1,0}_I(M)$. For a real form η, we obtain that the scalar $\eta\,(x, J(\overline{x}))$ is always real.

Definition 13. A real $(2,0)$-form η on a hypercomplex manifold is called positive if $\eta\,(x, J(\overline{x})) \geqslant 0$ for every $x \in T^{1,0}_I(M)$, and strictly positive if this inequality is strict, for all $x \neq 0$.

An HKT form $\Omega \in \Lambda^{2,0}_I(M)$ of any HKT structure is strictly positive by (1). Moreover, HKT structures on a hypercomplex manifold are in one-to-one correspondence with closed, strictly positive $(2,0)$-forms. The analogy between Kähler forms and HKT forms can be pushed further: It turns out that any HKT form $\Omega \in \Lambda^{2,0}_I(M)$ has a local potential $\varphi \in C^\infty(M)$, satisfying $\partial\partial_J\varphi = \Omega$ [3,1].

2.5. The map $\mathcal{V}_{p,q} :\ \Lambda^{p+q,0}_I(M) \longrightarrow \Lambda^{n+p,n+q}_I(M)$ on $SL(n,\mathbb{H})$-manifolds

Let (M, I, J, K) be an $SL(n,\mathbb{H})$-manifold, and

$$\mathcal{R}_{p,q} :\ \Lambda^{p+q,0}_I(M) \longrightarrow \Lambda^{p,q}_{I,+}(M)$$

the isomorphism induced by the $\mathfrak{su}(2)$-action as in Theorem 11. Consider the projection

$$\Lambda^{p,q}_I(M) \longrightarrow \Lambda^{p,q}_{I,+}(M), \tag{4}$$

and let $R :\ \Lambda^{p,q}_I(M) \longrightarrow \Lambda^{p+q,0}_I(M)$ denotes the composition of (4) and $\mathcal{R}^{-1}_{p,q}$.

Now, on an $SL(n,\mathbb{H})$-manifold (M, I, J, K, Φ), we define the map

$$\mathcal{V}_{p,q} :\ \Lambda^{p+q,0}_I(M) \longrightarrow \Lambda^{n+p,n+q}_I(M)$$

by the relation

$$\mathcal{V}_{p,q}(\eta) \wedge \alpha := \eta \wedge R(\alpha) \wedge \overline{\Phi},$$

for any test form $\alpha \in \Lambda^{n-p,n-q}_I(M)$. The map $\mathcal{V}_{p,p}$ is especially remarkable, because it maps closed, positive $(2p,0)$-forms to closed, positive $(n+p,n+p)$-forms, as the following proposition implies.

Proposition 14. *[35, Proposition 4.2,], [2, Theorem 3.6] Let (M, I, J, K, Φ) be a $SL(n,\mathbb{H})$-manifold, and*

$$\mathcal{V}_{p,q} :\ \Lambda^{p+q,0}_I(M) \longrightarrow \Lambda^{4n-p,4n-q}_I(M)$$

the map defined above. Then

(1) $\mathcal{V}_{p,q}(\eta) = \mathcal{R}_{p,q}(\eta) \wedge \mathcal{V}_{0,0}(1)$.
(2) *The map $\mathcal{V}_{p,q}$ is injective, for all p, q.*
(3) $(\sqrt{-1}\,)^{(n-p)^2}\mathcal{V}_{p,p}(\eta)$ *is real if and only $\eta \in \Lambda^{2p,0}_I(M)$ is real, and weakly positive if and only if η is weakly positive.*

(4) $\mathcal{V}_{p,q}(\partial\eta) = \partial\mathcal{V}_{p-1,q}(\eta)$, *and* $\mathcal{V}_{p,q}(\partial_J\eta) = \overline{\partial}\mathcal{V}_{p,q-1}(\eta)$.
(5) $\mathcal{V}_{0,0}(1) = \lambda\mathcal{R}_{n,n}(\Phi)$, *where* λ *is a positive rational number, depending only on the quaternionic dimension* n.

3. Quaternionic Gauduchon metrics

In order to prove quaternionic version of the $\partial\partial_J$-lemma for $(2,0)$-forms on compact $SL(2,\mathbb{H})$-manifolds, we need to introduce a quaternionic analog of Gauduchon metrics [10]. A Hermitian metric ω on a complex manifold of real dimension $2n$ is called Gauduchon if $\partial\overline{\partial}\omega^{n-1} = 0$. Every Hermitian metric on a compact complex manifold is conformally equivalent to a Gauduchon metric, which is unique in its conformal class, up to a constant multiplier. The Gauduchon metrics is one of the very few instruments available for the study of general non-Kähler manifolds, and probably the most important one.

Definition 15. A quaternionic Hermitian metric g on a hypercomplex manifold (M, I, J, K) of real dimension $4n$ is called quaternionic Gauduchon if $\partial\partial_J\Omega^{n-1} = 0$, where $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ is the corresponding positive (2,0)-form.

Proposition 16. *Let* (M, I, J, K, Φ) *be an* $SL(n,\mathbb{H})$*-manifold, where* Φ *is as in Definition 4, equipped with a quaternionic Hermitian metric* g*, and*

$$|\Phi|^2 := \frac{\Phi \wedge \overline{\Phi}}{(2^{2n}2n!)^{-1}\omega_I^{2n}}.$$

Then, the following conditions are equivalent

- g *is quaternionic Gauduchon.*
- *The Hermitian metric* $|\Phi|^{-1}g$ *is Gauduchon on* (M, I).

Proof. The equivalence follows from

$$\mathcal{V}_{n-1,n-1}(\Omega^{n-1}) = |\Phi|^{-1}\omega_I^{2n-1},$$

proven in [13]. So using Proposition 14, we have that

$$\mathcal{V}_{n,n}(\partial\partial_J\Omega^{n-1}) = \partial\mathcal{V}_{n-1,n}(\partial_J\Omega^{n-1}) = \partial\partial_J\mathcal{V}_{n-1,n-1}(\partial\partial_J\Omega^{n-1}) = \partial\overline{\partial}(|\Phi|^{-1}\omega_I^{2n-1}). \quad \square$$

Corollary 17. *For any* $SL(n,\mathbb{H})$*-manifold* (M, I, J, K, Φ) *equipped with a quaternionic Hermitian metric* g*, there exists a unique (up to a constant multiplier) positive function* μ *such that* μg *is quaternionic Gauduchon.*

Proof. There exists a positive function $\tilde{\mu}$ such that $\tilde{\mu}g$ is a Gauduchon metric on the complex manifold (M, I) so the function μ can be chosen to be $|\Phi|\tilde{\mu}$. $\square$

We are interested in quaternionic Gauduchon metrics because of the following theorem.

Theorem 18. *Let $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Hermitian $SL(n, \mathbb{H})$-manifold of real dimension $4n$. Assume that Ω induces a quaternionic Gauduchon metric. Consider the map*

$$D \ : \ C^\infty(M) \longrightarrow \Lambda^{4n}(M),$$
$$f \mapsto \partial\partial_J f \wedge \Omega^{n-1} \wedge \overline{\Phi}.$$

Then, D induces a bijection between $\{f \in C^\infty(M) \,|\, \int_M f\, \Omega^n \wedge \overline{\Omega}^n = 0\}$ and the space of exact $4n$-forms on M.

Proof. Clearly, D is elliptic, and has index 0, because it has the same symbol as the Riemannian Laplacian, which is self-adjoint. Now, the Hopf maximum principle [11] implies that $\ker D$ is given by constant functions. Therefore, $\operatorname{coker} D$ is a 1-dimensional space. It remains to show that $\operatorname{im} D$ consists of exact $4n$-forms. We have

$$\int_M \partial\partial_J f \wedge \Omega^{n-1} \wedge \overline{\Phi} = -\int_M f \wedge \partial\partial_J(\Omega^{n-1}) \wedge \overline{\Phi} = 0$$

because Ω is quaternionic Gauduchon. Since $H^{4n}(M) = \mathbb{R}$, this implies that all forms in $\operatorname{im} D$ are exact. Converse is also true, because $\operatorname{codim} \operatorname{im} D = 1$. □

4. Quaternionic Aeppli and Bott–Chern cohomology

One of the ingredient to prove Theorem 1 is the quaternionic analog of Bott–Chern and Aeppli cohomologies on complex manifolds. In this section, we introduce the latter cohomologies and prove that they are finite dimensional on compact hypercomplex manifolds. So let (M, I, J, K) be a hypercomplex manifold of real dimension $4n$. Define the quaternionic Bott–Chern cohomology group $H^{p,0}_{BC}(M)$ to be

$$H^{p,0}_{BC}(M) := \frac{\{\eta \in \Lambda^{p,0}_I(M) \,|\, \partial\eta = \partial_J\eta = 0\}}{\partial\partial_J \Lambda^{p-2,0}_I(M)}.$$

Theorem 19. *Let (M, I, J, K) be a compact hypercomplex manifold. Then, the group $H^{p,0}_{BC}(M)$ is finite dimensional.*

Proof. Let g be a quaternionic Hermitian metric on (M, I, J, K). We consider the following operator

$$\Delta_{BC} = \partial^*\partial + \partial_J^*\partial_J + \partial\partial_J\partial_J^*\partial^* + \partial_J^*\partial^*\partial\partial_J + \partial_J^*\partial\partial^*\partial_J + \partial^*\partial_J\partial_J^*\partial,$$

acting on $\Lambda_I^{p,0}(M)$. Here, ∂^* (resp. ∂_J^*) is the adjoint of ∂ (resp. ∂_J) with respect to g. The operator Δ_{BC} is a fourth order self-adjoint elliptic operator. Indeed, for $x \in M$ and $\xi \in \Lambda_{I,x}^{1,0}(M)$, we have the following principal symbols

$$\sigma_\xi(\partial)(\eta) = \xi \wedge \eta, \quad \sigma_\xi(\partial_J)(\eta) = J\overline{\xi} \wedge \eta, \quad \sigma_\xi(\partial^*)(\eta) = -\iota_{(\overline{\xi})^\sharp}\eta, \quad \sigma_\xi(\partial_J^*)(\eta) = -\iota_{(J\xi)^\sharp}\eta.$$

Here, ι denotes the inner multiplication and $\sharp$ the dual via the Hermitian metric g. Then, a direct computation shows that

$$\begin{aligned}\sigma_\xi(\Delta_{BC})(\eta) &= (\iota_{\overline{\xi}^\sharp}\xi)(\iota_{(J\xi)^\sharp}J\overline{\xi})\eta + (\iota_{(J\xi)^\sharp}\xi)(\iota_{\overline{\xi}^\sharp}J\overline{\xi})\eta - (\iota_{(J\xi)^\sharp}\xi)\, J\overline{\xi} \wedge \iota_{\overline{\xi}^\sharp}\eta, \\ &\quad - (\iota_{\overline{\xi}^\sharp}J\overline{\xi})\, \xi \wedge \iota_{J\xi^\sharp}\eta, \\ &= |\xi|_g^4\, \eta.\end{aligned}$$

By standard elliptic theory, we obtain the following decomposition

$$\begin{aligned}\Lambda_I^{p,0}(M) &= \mathcal{H}_{\Delta_{BC}} \oplus \operatorname{im} \Delta_{BC}, \\ &= \mathcal{H}_{\Delta_{BC}} \oplus \operatorname{im} \partial\partial_J \oplus (\operatorname{im} \partial^* + \operatorname{im} \partial_J^*),\end{aligned}$$

where $\mathcal{H}_{\Delta_{BC}} = \{\eta \in \Lambda_I^{p,0}(M) \,|\, \partial\eta = \partial_J\eta = \partial_J^*\partial^*\eta = 0\}$ is the (finite dimensional) kernel of Δ_{BC}.

Furthermore, for $\eta \in \Lambda_I^{p,0}(M)$, we write $\eta = \eta_H + \partial\partial_J\rho + \partial^*\alpha + \partial_J^*\beta$, where $\eta_H \in \mathcal{H}_{\Delta_{BC}}$. Then, $\partial\eta = \partial_J\eta = 0$ is equivalent to $\partial^*\alpha + \partial_J^*\beta = 0$. Thus, we deduce

$$\ker \partial|_{\Lambda_I^{p,0}(M)} \cap \ker \partial_J|_{\Lambda_I^{p,0}(M)} = \mathcal{H}_{\Delta_{BC}} \oplus \operatorname{im} \partial\partial_J. \quad \square$$

In a similar way, on a hypercomplex manifold (M, I, J, K), we define the quaternionic Aeppli cohomology group $H_{AE}^{p,0}(M)$ to be

$$H_{AE}^{p,0}(M) := \frac{\{\eta \in \Lambda_I^{p,0}(M) \,|\, \partial\partial_J\eta = 0\}}{\partial\Lambda_I^{p-1,0}(M) + \partial_J\Lambda_I^{p-1,0}(M)}.$$

Theorem 20. *Let (M, I, J, K) be a compact hypercomplex manifold. Then, the group $H_{AE}^{p,0}(M)$ is finite dimensional.*

Proof. Here, we consider the operator

$$\Delta_{AE} = \partial\partial^* + \partial_J\partial_J^* + \partial\partial_J\partial_J^*\partial^* + \partial_J^*\partial^*\partial\partial_J + \partial\partial_J^*\partial_J\partial^* + \partial_J\partial^*\partial\partial_J^*,$$

acting on $\Lambda_I^{p,0}(M)$. The operator Δ_{AE} is a fourth order self-adjoint elliptic operator having the same symbol as Δ_{BC}. We have then

$$\begin{aligned}\Lambda_I^{p,0}(M) &= \mathcal{H}_{\Delta_{AE}} \oplus \operatorname{im} \Delta_{AE}, \\ &= \mathcal{H}_{\Delta_{AE}} \oplus \operatorname{im} \partial_J^*\partial^* \oplus (\operatorname{im} \partial + \operatorname{im} \partial_J),\end{aligned}$$

where $\mathcal{H}_{\Delta_{AE}} = \{\varphi \in \Lambda_I^{p,0}(M) \,|\, \partial^*\eta = \partial_J^*\eta = \partial\partial_J\eta = 0\}$ is the (finite dimensional) kernel of Δ_{AE}. Moreover, if $\eta \in \Lambda_I^{p,0}(M)$ is decomposed as $\eta = \eta_H + \partial_J^*\partial^*\rho + \partial\alpha + \partial_J\beta$, where $\eta_H \in \mathcal{H}_{\Delta_{AE}}$, then $\partial\partial_J\eta = 0$ is equivalent to $\partial_J^*\partial^*\rho = 0$. We obtain that

$$\ker \partial\partial_J|_{\Lambda_I^{p,0}(M)} = \mathcal{H}_{\Delta_{AE}} \oplus (\mathrm{im}\,\partial + \,\mathrm{im}\,\partial_J). \quad \square$$

Remark 21. The groups $H_{BC}^{p,0}(M)$ and $H_{AE}^{2n-p,0}(M)$ are dual when (M, I, J, K, Φ) is a compact $SL(n,\mathbb{H})$-manifold. We consider the pairing on $H_{BC}^{p,0}(M) \times H_{AE}^{2n-p}(M)$ given by

$$([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta \wedge \overline{\Phi}.$$

One can check that this pairing is well defined (recall that $\partial\overline{\Phi} = \partial_J\overline{\Phi} = 0$) and non-degenerate.

5. The quaternionic dd^c-lemma on $SL(2,\mathbb{H})$-manifolds

We have now all the ingredients to prove the $\partial\partial_J$-lemma for $(2,0)$-forms on compact $SL(2,\mathbb{H})$-manifolds (not necessarily HKT). Let $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Gauduchon $SL(n,\mathbb{H})$-manifold. Consider the map

$$\begin{aligned} \deg : H_{AE}^{1,0}(M) &\longrightarrow \mathbb{C} \\ \alpha &\mapsto \int_M \partial\alpha \wedge \Omega^{n-1} \wedge \overline{\Phi}. \end{aligned}$$

Since Ω is quaternionic Gauduchon, $\deg(\alpha)$ is independent from the choice of α in its cohomology class. We call deg the *degree map.*

Consider now the natural map

$$H_{AE}^{1,0}(M) \xrightarrow{\partial} H_{BC}^{2,0}(M).$$

The kernel of this map consists of all cohomology classes $[\alpha]$ such that $\partial\alpha = \partial\partial_J\beta$, for some function β. Hence, the form $\alpha - \partial_J\beta$, cohomological to α in $H_{AE}^{1,0}(M)$, is ∂-closed. We obtain that the kernel of $H_{AE}^{1,0}(M) \xrightarrow{\partial} H_{BC}^{2,0}(M)$ is identified with the space

$$H_\partial^{1,0}(M) := \frac{\{\eta \in \Lambda_I^{1,0}(M) \,|\, \partial\eta = 0\}}{\partial\Lambda_I^{0,0}(M)}.$$

Lemma 22. *Let $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Gauduchon $SL(2,\mathbb{H})$-manifold of real dimension 8. Let $\theta \in \Lambda_I^{1,0}(M)$ be a ∂_J-exact, ∂-closed form. Then, $\theta = 0$.*

Proof. Let $\theta = \partial_J(f)$. Then $\partial\partial_J(f) = \partial\theta = 0$. However, the map $f \longrightarrow \frac{\partial\partial_J(f)\wedge\Omega\wedge\overline{\Phi}}{\Omega^2\wedge\overline{\Phi}}$ is an elliptic operator with vanishing constant term. Hence, any function in its kernel is constant by the Hopf maximum principle [11]. □

It is straightforward to deduce the following from Lemma 22.

Corollary 23. *On a compact $SL(2,\mathbb{H})$-manifold, the natural map*

$$H^{1,0}_{\partial}(M) \longrightarrow H^{1,0}_{AE}(M)$$

is injective.

Theorem 24. *Let $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Gauduchon $SL(2,\mathbb{H})$-manifold of real dimension 8. Then, the sequence*

$$0 \longrightarrow H^{1,0}_{\partial}(M) \longrightarrow H^{1,0}_{AE}(M) \xrightarrow{\deg} \mathbb{C} \tag{5}$$

is exact. Moreover, the space $\ker(\deg)$ is equal to the kernel of the natural map $H^{1,0}_{AE}(M) \xrightarrow{\partial} H^{2,0}_{BC}(M)$.

Proof. By Corollary 23, the sequence (5) is exact in the first term. It remains to prove that (5) is exact in the second term and to show that $\ker(\deg) = \ker \partial|_{H^{1,0}_{AE}(M)}$.

Let $\alpha \in \ker(\deg)$. By Theorem 18, there exists a function $f \in C^\infty(M)$ such that $(\partial\alpha + \partial\partial_J f) \wedge \Omega \wedge \overline{\Phi} = 0$, equivalently $(\partial\alpha + \partial\partial_J f) \wedge \Omega = 0$. Replacing α by $\alpha + \partial_J f$ in the same cohomology class, we may assume that $\partial\alpha \wedge \Omega = 0$. Since $\partial\alpha$ is primitive, one has

$$\int_M \partial\alpha \wedge \partial_J\alpha \wedge \overline{\Phi} = \|\partial\alpha\|^2$$

by a quaternionic version of Hodge–Riemann relations [31, Theorem 6.3]. However,

$$\|\partial\alpha\|^2 = \int_M \partial\alpha \wedge \partial_J\alpha \wedge \overline{\Phi} = -\int_M \partial\partial_J\alpha \wedge \alpha \wedge \overline{\Phi} = 0.$$

Hence, $\partial\alpha = 0$. This implies that $\ker(\deg) = \ker \partial|_{H^{1,0}_{AE}(M)}$. □

The $\partial\partial_J$-lemma on compact $SL(2,\mathbb{H})$-manifold for even $H^1\left(\mathcal{O}_{(M,I)}\right)$ follows directly from the above theorem. We recall that $H^1\left(\mathcal{O}_{(M,I)}\right)$ is given by the group $\frac{\{\eta\in\Lambda^{0,1}_I(M)\,|\,\overline{\partial}\eta=0\}}{\overline{\partial}\Lambda^{0,0}_I(M)}$.

Theorem 25. *Let (M, I, J, K, Φ) be a compact $SL(2,\mathbb{H})$-manifold with of real dimension 8. Then, the $\partial\partial_J$-lemma holds on $\Lambda^{2,0}_I(M)$ if and only if $H^1\left(\mathcal{O}_{(M,I)}\right)$ is even-dimensional.*

Proof. Clearly, the $\partial\partial_J$-lemma is equivalent to the vanishing of the map

$$\partial: \ H^{1,0}_{AE}(M) \longrightarrow H^{2,0}_{BC}(M),$$

but the kernel of this map is $H^{1,0}_{\partial}(M) = \ker(\deg)$ by Theorem 24. Hence it suffices to show that the degree map vanishes if and only if $\dim H^1\left(\mathcal{O}_{(M,I)}\right)$ is even.

Since J defines the quaternionic structure on $H^{1,0}_{AE}(M)$, this space is even-dimensional. Now, from the exact sequence

$$0 \longrightarrow H^{1,0}_{\partial}(M) \longrightarrow H^{1,0}_{AE}(M) \xrightarrow{\deg} \mathbb{C},$$

we obtain that the degree map vanishes whenever $H^{1,0}_{\partial}(M)$ is even-dimensional. The space $H^{1,0}_{\partial}(M)$ is complex conjugate to $H^1\left(\mathcal{O}_{(M,I)}\right)$. Thus, it has the same dimension. $\square$

6. Currents in HKT-geometry

Another essential tool in the proof of Theorem 1 is the notion of currents. In this section, we study currents on hypercomplex manifolds.

Let (M, I, J, K) be a hypercomplex manifold of real dimension $4n$ equipped with a quaternionic Hermitian metric g. Denote by $\mathcal{D}_{p,q}(M)$ the topological dual to the Fréchet space $\Lambda^{p,q}_I(M)$. An element $T \in \mathcal{D}_{p,q}(M)$ is called a current of bidimension (p,q) and it has a compact support on M. Denote by $\mathcal{D}^{p,q}(M) = \mathcal{D}_{2n-p,2n-q}(M)$.

The complex structure J acts naturally on $\mathcal{D}^{p,q}(M)$ as a map

$$J : \mathcal{D}^{p,q}(M) \longrightarrow \mathcal{D}^{q,p}(M)$$

in the following way

$$(JT)(\varphi) = T(J\varphi),$$

for $T \in \mathcal{D}^{p,q}(M)$ with a compact support. The operators $d, \partial, \overline{\partial}$ are extended in the standard way using the Stokes theorem. For example, $\partial : \mathcal{D}^{p,q}(M) \to \mathcal{D}^{p+1,q}(M)$ is expressed as $\partial T(\varphi) = (-1)^{\deg\varphi} T(\partial\varphi)$, where $\varphi \in \Lambda^{2n-p-1,2n-q}_I(M)$. Similarly, we can define ∂_J on $\mathcal{D}^{p,q}(M)$.

Definition 26. A current $T \in \mathcal{D}^{2p,0}(M)$ is called *real* if $J\overline{T} = T$ and we denote by $\mathcal{D}^{2p,0}_{\mathbb{R}}(M)$ the space of real currents of bidimension $(2n-2p, 2n)$.

The following result is a currents version of the local $\partial\partial_J$-lemma, due to Banos and Swann [3] in the smooth case.

Proposition 27. *Let $T \in \mathcal{D}^{2,0}_{\mathbb{R}}(M)$ be a real ∂-closed current. Then, locally T can be written in the form $T = \partial\partial_J\varphi$, for some real generalized function φ.*

Proof. We use essentially the same arguments as in the proof of the main theorem in [3]. Let T as above. We write $T = T_J + \sqrt{-1}\,T_K$. Since T is real and ∂-closed, a straightforward verification shows that $T_I(\cdot,\cdot) = T_J(\cdot, K\cdot)$ is I-invariant and that $IdT_I = JdT_J = KdT_K$.

Let $Z = M \times S^2$ be the twistor space of M and we consider the current $\eta \in \mathcal{D}^{0,2}(Z)$ given by $\eta = (T_I)^{(0,2)}_{\mathcal{I}}$ i.e. the $(0,2)$-part of T_I with respect to the complex structure $\mathcal{I}_{(p,\overrightarrow{a})} = aI_p + bJ_p + cK_p$, where $(p, \overrightarrow{a}) \in Z$ and $\overrightarrow{a} = (a,b,c) \in \mathbb{R}^3$ with $a^2+b^2+c^2 = 1$. A direct computation shows that $\overline{\partial}_{\mathcal{I}}\eta = 0$. By a 1-pseudo-convexity argument and the $\overline{\partial}$-Poincaré Lemma (for currents), locally $\eta = \overline{\partial}_{\mathcal{I}}(\alpha + \sqrt{-1}\,\mathcal{I}\,\alpha)$ where α is a real current defined locally in M. Hence, the real part of η is given by $\frac{1}{2}(d\alpha - \mathcal{I}d\alpha)$. It follows that $d\alpha$ is a closed I-invariant current. Hence, by the $\partial\overline{\partial}$-Poincaré Lemma (for currents), $T_I = \frac{1}{2}(dd_I\varphi + d_Jd_K\varphi)$ for some real generalized function φ, where $d_I := I^{-1}dI$ etc. By [1], this implies that locally $T = \partial\partial_J\varphi$. □

Using Definition 13, we give the following definition.

Definition 28. On an $SL(n,\mathbb{H})$-manifold (M, I, J, K, Φ), a current $T \in \mathcal{D}^{2n-2,0}_{\mathbb{R}}(M)$ is said to be *positive* if (locally) $T \wedge \alpha \wedge \overline{\Phi}$ is a positive measure for any choice of (local) real strictly positive $(2,0)$-form α.

Definition 29. A generalized function is called *plurisubharmonic* if $\partial\partial_J\varphi$ is a positive (2,0)-current.

We get then the following theorem due to [14].

Theorem 30. *[14, Lemma 3.6] A plurisubharmonic generalized function is subharmonic with respect to any quaternionic Hermitian metric (hence, constant on any compact hypercomplex manifold).*

Now, we consider the group

$$H'^{2,0}_{\mathbb{R}}(M) = \frac{\{T \in \mathcal{D}^{2,0}_{\mathbb{R}}(M) \mid \partial T = 0\}}{\partial\partial_J\mathcal{D}^{0,0}_{\mathbb{R}}(M)}.$$

We want to prove that the group $H'^{2,0}_{\mathbb{R}}(M)$ is isomorphic to the analog one given by forms. Denote by $\mathcal{H}$ the sheaf of real generalized functions satisfying $\partial\partial_J f = 0$. By the proof of Lemma 22, the elements of $\mathcal{H}$ satisfy an elliptic equation. The elliptic regularity implies that all functions in $\mathcal{H}$ are smooth.

The sheaf $\mathcal{H}$ admits two resolutions starting by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \Lambda^{0,0}_{\mathbb{R}}(M,I) & \xrightarrow{\partial\partial_J} & \Lambda^{2,0}_{\mathbb{R}}(M,I) & \xrightarrow{\partial} & \Lambda^{3,0}(M,I) \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{D}^{0,0}_{\mathbb{R}}(M) & \xrightarrow{\partial\partial_J} & \mathcal{D}^{2,0}_{\mathbb{R}}(M) & \xrightarrow{\partial} & \mathcal{D}^{3,0}(M), \end{array}$$

where i is the inclusion of forms in the space of currents. We deduce that

$$H'^{2,0}_{\mathbb{R}}(M) \simeq \frac{\{\eta \in \Lambda^{2,0}_{\mathbb{R}}(M,I) \mid \partial\eta = 0\}}{\partial\partial_J \Lambda^{0,0}_{\mathbb{R}}(M,I)}.$$

Moreover, if we consider the group

$$H'^{2n-2,0}_{\mathbb{R}}(M) = \frac{\{T \in \mathcal{D}^{2n-2,0}_{\mathbb{R}}(M) \mid \partial\partial_J T = 0\}}{\{\partial u + \partial_J J^{-1}\overline{u},\ u \in \mathcal{D}^{2n-3,0}(M)\}},$$

then by the same argument in Remark 21, we deduce that $H'^{2n-2,0}_{\mathbb{R}}(M)$ and $H'^{2,0}_{\mathbb{R}}(M)$ are dual when M is a compact $SL(n,\mathbb{H})$-manifold.

7. Harvey–Lawson's theorem in HKT-geometry

We need the following version of Hahn–Banach theorem.

Theorem 31. *[24] Let V be a locally convex topological vector space, $A \subset V$ an open convex subset of V and W a closed subspace of V satisfying $W \cap A = \emptyset$. Then, there is a continuous linear functional θ on V, such that $\theta|_A > 0$ and $\theta|_W = 0$.*

Using the Hahn–Banach Separation Theorem 31, we obtain the following:

Theorem 32. *Let (M, I, J, K, Φ) be an $SL(n,\mathbb{H})$-manifold of real dimension $4n$. Then, M admits no HKT metric if and only if it admits a ∂-exact, real, positive $(2n-2,0)$-current.*

Proof. We apply the Hahn–Banach separation theorem to the space A of strictly positive, real $(2,0)$-forms and W of ∂-closed real (2,0)-forms (with respect to I) to obtain a current $\xi \in \mathcal{D}^{2n-2,0}_{\mathbb{R}}(M)$ which is positive on A (hence, real and positive) and vanishes on W. Such a current exists if and only if $A \cap W = \emptyset$, or equivalently, when M is not an HKT manifold.

We consider now the pairing $\langle \eta, \nu \rangle = \int_M \eta \wedge \nu \wedge \overline{\Phi}$ on $(p,0)$-forms. This pairing is compatible with ∂ and ∂_J and allows one to identify the currents $\mathcal{D}^{2p,0}_{\mathbb{R}}(M)$ with $\Lambda^{2n-2p,0}_{\mathbb{R}}(M,I) \otimes C^\infty(M)^*$, where $C^\infty(M)^*$ denotes generalized functions. This identification is compatible with ∂ and ∂_J, and the cohomology of currents are the same as the cohomology of forms.

Now, since the real current ξ vanishes on the space W of ∂-closed real (2,0)-forms, one has $0 = \langle \xi, \partial\eta \rangle = \langle \partial\xi, \eta \rangle$, for each $\eta \in \Lambda_I^{1,0}(M)$, giving $\partial\xi = 0$. It remains to show that the cohomology class of ξ in $H_\partial^{2n-2,0}(M)$ vanishes. The Serre's duality gives a non-degenerate pairing

$$([\eta], [\nu]) \longrightarrow \int_M \eta \wedge \nu \wedge \overline{\Phi}$$

on cohomology classes in $H_\partial^{*,0}(M)$. Since $\langle [\xi], [\nu] \rangle = 0$ for each ∂-closed real ν, the cohomology class of ξ also vanishes. □

We can finally prove Theorem 1.

Proof of Theorem 1. The even-dimensionality of $H^1(\mathcal{O}_{(M,I)})$ for HKT manifolds with the holonomy contained in $SL(n, \mathbb{H})$ follows from [30, Theorem 10.2 (v)]. Indeed, the Laplacian $\Delta_\partial := \partial\partial^* + \partial^*\partial$ commutes with the action on $\Lambda_I^{p,0}(M)$ given by $\eta \mapsto J(\overline{\eta})$. Hence, if $\eta \in \ker(\Delta_\partial)$, then $\Delta_\partial(J\overline{\eta}) = J\overline{(\Delta_\partial \eta)} = 0$. Finally, we recall that $\dim \ker \Delta_\partial|_{\Lambda_I^{1,0}(M)} = \dim H_\partial^{1,0}(M) = \dim H^1(\mathcal{O}_{(M,I)})$ because $H_\partial^{1,0}(M)$ is conjugate to $H^1(\mathcal{O}_{(M,I)})$.

Conversely, suppose that $H^1(\mathcal{O}_{(M,I)})$ is even-dimensional, but M is not HKT. Then, Theorem 32 implies that there exists a real, positive, ∂-exact current $\xi \in \mathcal{D}_{\mathbb{R}}^{2,0}(M)$. Since ξ is real, we have $\partial_J\xi = J^{-1}\overline{\partial}J\xi = J^{-1}\overline{(\partial\xi)} = 0$. Thus, ξ is ∂_J-closed ∂-exact current. Theorem 25 implies that ξ is $\partial\partial_J$-exact. Hence, $\xi = \partial\partial_J f$, for some generalized function f. Such f is a quaternionic plurisubharmonic function, which has to vanish by Theorem 30. □

8. Examples

The known examples of manifolds with the holonomy in $SL(n, \mathbb{H})$ are either nilmanifolds [4] or obtained via the twist construction of A. Swann [28], which is based on previous examples of D. Joyce. The latter construction provides also simply-connected examples. We describe briefly a simplified version of it.

Let (X, I, J, K, g) be a compact hyperkähler manifold. By definition, an anti-self-dual 2-form on X is a form which is of type $(1,1)$ with respect to I and J and hence with respect to all complex structures of the hypercomplex family. Let $\alpha_1, \ldots, \alpha_{4k}$ be closed 2-forms representing integral cohomology classes on X. Consider the principal T^{4k}-bundle $\pi : M \to X$ with characteristic classes determined by $\alpha_1, \ldots, \alpha_{4k}$. It admits a connection A given by $4k$ 1-forms θ_i such that $d\theta_i = \pi^*(\alpha_i)$. Define an almost-hypercomplex structure on M in the following way: On the horizontal spaces of A, we have the pull-backs of I, J, K and on the vertical spaces, we fix a linear hypercomplex structure of the $4k$-torus. The structures $\mathcal{I}, \mathcal{J}, \mathcal{K}$ on M are extended to act on the cotangent bundle $T^*(M)$ using the following relations:

$$\mathcal{I}(\theta_{4i+1}) = \theta_{4i+2}, \mathcal{I}(\theta_{4i+3}) = \theta_{4i+4}, \qquad \mathcal{J}(\theta_{4i+1}) = \theta_{4i+3}, \mathcal{J}(\theta_{4i+2}) = -\theta_{4i+4},$$
$$\mathcal{I}(\pi^*\alpha) = \pi^*(I\alpha), \mathcal{J}(\pi^*\alpha) = \pi^*(J\alpha),$$

for any 1-form α on X and $i = 0, \dots, k-1$.

It follows from [28] or by direct and easy calculations, that $\mathcal{I}$ is integrable if and only if $\alpha_{4i+1} + \sqrt{-1}\alpha_{4i+2}$ and $\alpha_{4i+3} + \sqrt{-1}\alpha_{4i+4}$ are of type $(2,0)+(1,1)$ with respect to I for every $i = 0, \dots, k-1$. Similarly, $\mathcal{J}$ is integrable if and only if $\alpha_{4i+1} + \sqrt{-1}\alpha_{4i+3}$ and $\alpha_{4i+2} - \sqrt{-1}\alpha_{4i+4}$ are of type $(2,0)+(1,1)$ with respect to J for every $i = 0, \dots, k-1$.

Similarly, one can define a quaternionic Hermitian metric on M from g and a fixed hyperkähler metric on T^{4k} using the splitting of $T(M)$ in horizontal and vertical subspaces. As A. Swann [28] has shown, the structure has a holonomy in $SL(n, \mathbb{H})$ and is HKT when all forms α_i are anti-self-dual (of type $(1,1)$ with respect to all structures).

As a particular case, assume X to be a $K3$ surface. In addition, we suppose the existence on X of 3 closed integral forms inducing a hyperkähler structure and a self-dual integral class defining a principal T^4-bundle M over X with finite fundamental group. After passing to a finite cover, we may assume that M is simply-connected. These forms satisfy the integrability condition above. If $\alpha_2 + \sqrt{-1}\alpha_3$ is a (2,0)-form with respect to I, then $\pi^*(\alpha_2 + \sqrt{-1}\alpha_3) = d(\theta_2 + \sqrt{-1}\theta_3)$ is an exact $(2,0)$-form, which defines a positive current (as in 28). Then, M can not admit any HKT metric; a fact proven by A. Swann using different arguments.

We can also calculate $\dim H^1(\mathcal{O}_{(M,\mathcal{I})}) = h^{0,1}_{\mathcal{I}}(M)$ and apply Theorem 1 to decide on the existence of a HKT structure. One can use the Borel method of doubly graded spectral sequence from [16, Appendix B] to determine $h^{p,q}$. However, in our case, its simpler to use a more direct approach. The vector fields X_1, X_2, X_3, X_4 on M generated by the tori action, which are also dual to θ_i, are hyperholomorphic. So, $\mathcal{L}_{X_i} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{L}_{X_i}$. We can also choose a bundle metric, which is, on the vertical vectors, the flat hyperkähler 4-torus metric, and on the horizontal vectors, a pull-back from the hyperkähler metric from the base $X = K3$. The horizontal and vertical vectors are perpendicular. Such metric is hypercomplex and X_i are Killing vector fields. Since they fix the orientation, the $\mathcal{L}_{X_i}$ commute with the Hodge star operator of this metric. In particular, they also commute with the $\overline{\partial}$-Laplace operator and $\mathcal{L}_{X_i}\alpha$ is a harmonic form for every harmonic α. Since $X_i^{(0,1)}$ is a complex vector field which preserves the structure I and transforms $(0,1)$-form into $(0,1)$-form, for a $\overline{\partial}$-harmonic form α, we have $\mathcal{L}_{X_i^{(0.1)}}\alpha^{(0,1)} = \overline{\partial} f$, where $f = \alpha^{(0,1)}\left(X_i^{(0,1)}\right)$. Moreover, $\mathcal{L}_{X_i^{(0.1)}}\alpha^{(0,1)}$ vanishes because it is harmonic. Since we can use any $(0,1)$-vector field generated by the action and any harmonic $(0,1)$-form, in particular α, we see that the vector fields X_i preserve the harmonic $(0,1)$-forms. Then, any such form has a representation

$$\alpha = A_1(\theta_1 - \sqrt{-1}\theta_2) + A_2(\theta_3 - \sqrt{-1}\theta_4) + \pi^*(\varphi),$$

where A_i are pull-backs of functions on the base and φ is a harmonic form on the base X. Since X is a K3 surface, $\varphi = 0$. Then, from $d\theta_i = \pi^*(\alpha_i)$, we have $\overline{\partial}(\theta_1 - \sqrt{-1}\theta_2) =$

$\alpha_1 - \sqrt{-1}\alpha_2$ if $\alpha_1 - \sqrt{-1}\alpha_2$ is a $(2,0)$-form and 0 if it is $(1,1)$. On the other side, $\overline{\partial}(\theta_3 - \sqrt{-1}\theta_4) = 0$, since the other characteristic classes are $(1,1)$. As a result, we see that $h^{0,1}_{\mathcal{I}}(M) = 2$, if all curvature forms are $(1,1)$ (or instantons), and $h^{0,1}_{\mathcal{I}}(M) = 1$, if one of these forms is of type $(2,0)$. By Theorem 1, in the first case there is a HKT metric and in the second there is none.

In the construction above, we can use a flat 4-torus as a base instead of a $K3$ surface. Then, M is a nilmanifold which corresponds to an example which appeared in [9]. Consider the nilpotent Lie algebra $\mathbb{R} \times \mathfrak{h}_7$, where $\mathfrak{h}_7$ is the algebra of the quaternionic Heisenberg group H_7. It is spanned by the left-invariant vector fields $e_1, \dots, e_8$ and is defined by the following relation on the basis of the dual 1-forms:

$$\begin{aligned}
&de^i = 0, i = 1, \dots, 5,\\
&de^6 = e^1 \wedge e^2 + e^3 \wedge e^4,\\
&de^7 = e^1 \wedge e^3 - e^2 \wedge e^4,\\
&de^8 = e^1 \wedge e^4 + e^2 \wedge e^3.
\end{aligned}$$

On a compact quotient $M = \mathbb{R} \times H_7/\Gamma$, consider the family [9] of complex structures defined via:

$$\begin{aligned}
&I_t(e^1) = \tfrac{t-1}{t} e^2,\ I_t(e^3) = e^4\ J_t(e^5) = \tfrac{1}{t} e^6,\ J_t(e^7) = e^8,\\
&J_t(e^1) = \tfrac{t-1}{t} e^3,\ J_t(e^2) = -e^4,\ J_t(e^5) = \tfrac{1}{t} e^7,\ J_t(e^6) = -e^8,
\end{aligned}$$

for $t \in (0,1)$. Then, for each t, $I_t J_t = -J_t I_t = K_t$ and so it defines a hypercomplex structure on M. Using an averaging argument in [9], it was shown that for $t = \frac{1}{2}$ the structure is HKT and for $t \neq \frac{1}{2}$ there is no HKT metric. Here, we provide a different proof using Theorem 32 and Theorem 1. The manifold M has a projection on $X = T^4$ which makes it a principal bundle, where the fiber and the base are 4-tori. Then, the forms e^1, e^2, e^3, e^4 are pull-backs from forms on the base X and the forms e^5, e^6, e^7, e^8 are connection forms in this bundle. So, up to a constant, the characteristic classes of the bundle are $0, e^1 \wedge e^2 + e^3 \wedge e^4, e^1 \wedge e^3 - e^2 \wedge e^4, e^1 \wedge e^4 + e^2 \wedge e^3$. Now, we note that

$$\begin{aligned}
d(e^7 + \sqrt{-1}e^8) &= (e^1 + \sqrt{-1}e^2) \wedge (e^3 + \sqrt{-1}e^4)\\
&= \frac{2t-1}{2t-2}\left(e^1 + \sqrt{-1}\frac{t-1}{t}e^4\right) \wedge (e^3 + \sqrt{-1}e^4)\\
&\quad - \frac{1}{2t-2}\left(e^1 - \sqrt{-1}\frac{t-1}{t}e^2\right) \wedge (e^3 + \sqrt{-1}e^4).
\end{aligned}$$

So, when $t = \frac{1}{2}$, it is of type $(1,1)$ with respect to $I_{\frac{1}{2}}$. However, for $t \neq \frac{1}{2}$, it is of type $(2,0) + (1,1)$. Moreover, the $(2,0)$-component in this case is $\partial_t(e^7 - \sqrt{-1}e^8) = \frac{2t-1}{2t-2}(e^1 + \sqrt{-1}\frac{t-1}{t}e^4) \wedge (e^3 + \sqrt{-1}e^4)$, which defines a positive (2,0)-current. So, there is no HKT structure if $t \neq \frac{1}{2}$ by Theorem 32. Similarly, we can calculate the Hodge number $h^{0,1}(M, I_t)$ to check its parity. Instead of using the fibration structure, it is easier to use the result of Console and Fino [8] who proved that the Dolbeaut cohomology of a nilmanifold with an invariant complex structure are isomorphic to the $\overline{\partial}$-cohomology of the complex of invariant forms. From the defining equations above, we see that $e^1 + \sqrt{-1}e^2$, $e^3 + \sqrt{-1}e^4$ and $e^5 - \sqrt{-1}e^6$ are nonzero elements of $H^{0,1}(M, I_t)$. Also, $\overline{\partial}_t(e^7 - \sqrt{-1}e^8) = d(e^7 - \sqrt{-1}e^8)|^{(0,2)} = \frac{2t-1}{2t-2}(e^1 - \sqrt{-1}\frac{t-1}{t}e^4) \wedge (e^3 - \sqrt{-1}e^4)$. So, for $t = \frac{1}{2}$, it is non-zero in the cohomology and $h^{0,1}(M, I_{\frac{1}{2}}) = 4$. When $t \neq \frac{1}{2}$, it is not $\overline{\partial}_t$-closed, $h^{0,1}(M, I_t) = 3$ and we can apply Theorem 1.

Acknowledgments

We are grateful to Dan Popovici for interesting discussions about strong Gauduchon metrics. We are very thankful to the referee for his remarks.

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