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The Gauss–Manin connection on the periodic cyclic homology

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Abstract Let R be the algebra of functions on a smooth affine irreducible curve S over a field k and let $A_\bullet$ be a smooth and proper DG algebra over R. The relative periodic cyclic homology $HP_*(A_\bullet)$ of $A_\bullet$ over R is equipped with the Hodge filtration $\mathcal{F}^\cdot$ and the Gauss–Manin connection ∇ (Getzler, in: Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), Israel mathematics conference proceedings, vol 7, Bar-Ilan University, Ramat Gan, pp 65–78, 1993; Kaledin, in: Algebra, arithmetic, and geometry: in honor of Yu. I. Manin, vol II, pp 23–47, Progress in mathematics, vol 270, Birkhäuser Inc., Boston, 2009) satisfying the Griffiths transversality condition. When k is a perfect field of odd characteristic p, we prove that, if the relative Hochschild homology $HH_m(A_\bullet, A_\bullet)$ vanishes in degrees $|m| \geq p-2$, then a lifting $\tilde{R}$ of R over $W_2(k)$ and a lifting of $A_\bullet$ over $\tilde{R}$ determine the structure of a relative Fontaine–Laffaille module (Faltings, in: Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins University Press, Baltimore, MD, pp 25–80, 1989, §2 (c); Ogus and Vologodsky in PublMath

To Sasha Beilinson on his 60th birthday, with admiration.

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Inst Hautes Études Sci No 106:1–138, 2007 §4.6) on $HP_*(A_\bullet)$. That is, the inverse Cartier transform of the Higgs R-module $(Gr^{\mathcal{F}} HP_*(A_\bullet), Gr^{\mathcal{F}}\nabla)$ is canonically isomorphic to $(HP_*(A_\bullet), \nabla)$. This is non-commutative counterpart of Faltings' result (1989, Th. 6.2) for the de Rham cohomology of a smooth proper scheme over R. Our result amplifies the non-commutative Deligne–Illusie decomposition proven by Kaledin (Algebra, geometry and physics in the 21st century (Kontsevich Festschrift), Progress in mathematics, vol 324. Birkhäuser, pp 99–129, 2017, Th. 5.1). As a corollary, we show that the p-curvature of the Gauss–Manin connection on $HP_*(A_\bullet)$ is nilpotent and, moreover, it can be expressed in terms of the Kodaira–Spencer class $\kappa \in HH^2(A_\bullet, A_\bullet) \otimes_R \Omega^1_R$ [a similar result for the p-curvature of the Gauss–Manin connection on the de Rham cohomology is proven by Katz (Invent Math 18:1–118, 1972)]. As an application of the nilpotency of the p-curvature we prove, using a result from Katz (Inst Hautes Études Sci Publ Math No 39:175–232, 1970), a version of "the local monodromy theorem" of Griffiths–Landman–Grothendieck for the periodic cyclic homology: if $k = \mathbb{C}$, $\overline{S}$ is a smooth compactification of S, then, for any smooth and proper DG algebra $A_\bullet$ over R, the Gauss–Manin connection on the relative periodic cyclic homology $HP_*(A_\bullet)$ has regular singularities, and its monodromy around every point at $\overline{S} - S$ is quasi-unipotent.

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1 Introduction

It is expected that the periodic cyclic homology of a DG algebra over $\mathbb{C}$ (and, more generally, the periodic cyclic homology of a DG category) carries a lot of additional structure similar to the mixed Hodge structure on the de Rham cohomology of algebraic varieties. Whereas a construction of such a structure seems to be out of reach at the moment its counterpart in finite characteristic is much better understood thanks to recent groundbreaking works of Kaledin. In particular, it is proven in [17] that under some assumptions on a DG algebra $A_\bullet$ over a perfect field k of characteristic p, a lifting of $A_\bullet$ over the ring of second Witt vectors $W_2(k)$ specifies the structure of a Fontaine–Laffaille module on the periodic cyclic homology of $A_\bullet$. The purpose of this paper is to develop a relative version of Kaledin's theory for DG algebras over a base k-algebra R incorporating in the picture the Gauss–Manin connection on the relative periodic cyclic homology constructed by Getzler in [13]. Our main result, Theorem 1, asserts that, under some assumptions on $A_\bullet$, the Gauss–Manin connection on its periodic cyclic homology can be recovered from the Hochschild homology of $A_\bullet$ equipped with the action of the Kodaira–Spencer operator as the inverse Cartier transform [29]. As an application, we prove, using the reduction modulo p technique, that, for a smooth and proper DG algebra over a complex punctured disk, the monodromy of the Gauss–Manin connection on its periodic cyclic homology is quasi-unipotent.

1.1 Relative Fontaine–Laffaille modules

Let R be a finitely generated commutative algebra over a perfect field k of *odd* characteristic $p > 2$. Assume that R is smooth over k. Recall from [10, §2(c)] and [29,

§4.6] the notion of *relative Fontaine–Laffaille module*[1] over R. Fix a flat lifting $\tilde{R}$ of R over the ring $W_2(k)$ of second Witt vectors and a lifting $\tilde{F}:\tilde{R}\to\tilde{R}$ of the Frobenius morphism $F:R\to R$. Define the inverse Cartier transform

$$\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}:\mathsf{HIG}(R)\to\mathsf{MIC}(R)$$

to be a functor from the category of Higgs modules i.e., pairs (E,θ), where E is an R-module and $\theta:E\to E\otimes_R\Omega^1_R$ is an R-linear morphism such that the composition $\theta^2:E\to E\otimes_R\Omega^1_R\to E\otimes_R\Omega^2_R$ equals 0,[2] to the category of R-modules with integrable connection. Given a Higgs module (E,θ) we set

$$\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(E,\theta):=(F^*E,\nabla_{can}+C^{-1}_{(\tilde{R},\tilde{F})}(\theta)),$$

where ∇_{can} is the Frobenius pullback connection on F^*E and the map

$$C^{-1}_{(\tilde{R},\tilde{F})}:End_R(E)\otimes\Omega^1_R\to F_*(End_R(F^*E)\otimes_R\Omega^1_R) \tag{1.1}$$

takes $f\otimes\eta$ to $F^*(f)\otimes\frac{1}{p}\tilde{F}^*\tilde{\eta}$, with $\tilde{\eta}\in\Omega^1_{\tilde{R}}$ being a lifting of η. A relative Fontaine–Laffaille module over R consists of a finitely generated R-module M with an integrable connection ∇ and a Hodge filtration

$$0=\mathcal{F}^{l+1}M\subset\mathcal{F}^lM\subset\cdots\subset\mathcal{F}^mM=M$$

satisfying the Griffiths transversality condition, together with isomorphism in $\mathsf{MIC}(R)$:

$$\phi:\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(\mathrm{Gr}^\bullet_{\mathcal{F}}M,\mathrm{Gr}_{\mathcal{F}}\nabla)\xrightarrow{\sim}(M,\nabla).$$

Here

$$\mathrm{Gr}_{\mathcal{F}}\nabla:\mathrm{Gr}^\bullet_{\mathcal{F}}M\to Gr^{\bullet-1}_{\mathcal{F}}M$$

is the "Kodaira–Spencer" Higgs field induced by ∇.[3]

The category $\mathcal{MF}_{[m,l]}(\tilde{R},\tilde{F})$ (where $l\geq m$ are arbitrary integers) of relative Fontaine–Laffaille modules has a number of remarkable properties not obvious from the definition. It is proven by Faltings in [10, Th. 2.1], that $\mathcal{MF}_{[m,l]}(\tilde{R},\tilde{F})$ is abelian, every morphism between Fontaine–Laffaille modules is strictly compatible with the Hodge filtration, and, for every Fontaine–Laffaille module $(M,\nabla,\mathcal{F}^\bullet M,\phi)$, the R-modules M and $Gr_{\mathcal{F}}M$ are flat. Moreover, if $l-m<p$, the category

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[1] In [10], Faltings does not give a name to these objects. In [29], they are called Fontaine modules. The name suggested here is a tribute to [11], where these objects were first introduced in the case when $R=k$.

[2] Equivalently, a Higgs module is a module over the symmetric algebra $S^\bullet T_R$.

[3] In [10], Faltings considers more general objects. In fact, what we call here a relative Fontaine–Laffaille module is the same as a p-torsion object in Faltings' category $\mathcal{MF}^{\nabla}_{[m,l]}(R)$.

$\mathcal{MF}_{[m,l]}(\tilde{R}, \tilde{F}) =: \mathcal{MF}_{[m,l]}(\tilde{R})$ is independent of the choice of the Frobenius lifting.[4] Fontaine–Laffaille modules arise geometrically: it is shown in [10, Th. 6.2] that, for a smooth proper scheme $X \to spec\ R$ of relative dimension less than p, a lifting of X over $\tilde{R}$ specifies a Fontaine–Laffaille module structure on the relative de Rham cohomology $H^{\bullet}_{DR}(X/R)$.

1.2 The Kodaira–Spencer class of a DG algebra

Let $A_\bullet$ be a differential graded algebra over R. Denote by $HH^\bullet(A_\bullet, A_\bullet)$ its Hochschild cohomology and by

$$\kappa \in HH^2(A_\bullet, A_\bullet) \otimes_R \Omega^1_R \tag{1.2}$$

the *Kodaira–Spencer class* of $A_\bullet$. This can be defined as follows. Choose a quasi-isomorphism $A_\bullet \xrightarrow{\sim} B_\bullet$, where $B_\bullet$ is a semi-free DG algebra over R [9, §13.4] and a connection $\nabla' : \bigoplus B_i \to \bigoplus B_i \otimes \Omega^1_R$ on the graded algebra $\bigoplus B_i$ satisfying the Leibniz rule with respect to the multiplication on $\bigoplus B_i$. Then the commutator

$$[\nabla', d] \in \prod Hom_R(B_i, B_{i+1}) \otimes \Omega^1_R \tag{1.3}$$

with the differential d on $B_\bullet$ commutes with d and it is a R-linear derivation of $B_\bullet$ with values in $B_\bullet \otimes \Omega^1_R$ of degree 1. Any such derivation determines a class in

$$HH^2(B_\bullet, B_\bullet) \otimes_R \Omega^1_R \xrightarrow{\sim} HH^2(A_\bullet, A_\bullet) \otimes_R \Omega^1_R.$$

The class corresponding to $[\nabla', d]$ is independent of the choices we made. This is the Kodaira–Spencer class (1.2).[5]

The Kodaira–Spencer class (1.2) acts on the Hochschild homology:

$$e_\kappa : HH_\bullet(A_\bullet, A_\bullet) \to HH_{\bullet-2}(A_\bullet, A_\bullet) \otimes_R \Omega^1_R.$$

The operator e_κ is induced by the action of the Hochschild cohomology algebra on the Hochschild homology (referred to as the "interior product" action).

1.3 The Hodge filtration on the periodic cyclic homology

Denote by $(CH_\bullet(A_\bullet, A_\bullet), b)$ the relative Hochschild chain complex of $A_\bullet$ and by $CP_\bullet(A_\bullet) = (CH_\bullet(A_\bullet, A_\bullet)((u)), b + uB)$ the periodic cyclic complex. The complex $CP_\bullet(A_\bullet)$ is equipped with the Hodge filtration

$$\mathcal{F}^i CP_\bullet(A_\bullet) := (u^i CH_\bullet(A_\bullet, A_\bullet)[[u]], b + uB),$$

[4] Every two liftings $\tilde{R}, \overline{R}$ of R are isomorphic. A choice of such an isomorphism induces an equivalence $\mathcal{MF}_{[m,l]}(\tilde{R}) \xrightarrow{\sim} \mathcal{MF}_{[m,l]}(\overline{R})$. We refer the reader to [29, §4.6] for a construction of the category of Fontaine–Laffaille modules over any smooth scheme X over k equipped with a lifting $\tilde{X}$ over $W_2(k)$.

[5] The Kodaira–Spencer class of a A_∞-algebra is defined in [13], §3. Our exposition is inspired by [26].

which induces a Hodge filtration $\mathcal{F}^{\bullet} HP_{\bullet}(A_{\bullet})$ on the periodic cyclic homology and a convergent Hodge-to-de Rham spectral sequence

$$HH_{\bullet}(A_{\bullet}, A_{\bullet})((u)) \Rightarrow HP_{\bullet}(A_{\bullet}). \tag{1.4}$$

The Gauss–Manin connection ∇ on the periodic cyclic homology (we recall its construction in Sect. 3) satisfies the Griffiths transversality condition

$$\nabla : \mathcal{F}^{\bullet} HP_{\bullet}(A_{\bullet}) \to \mathcal{F}^{\bullet -1} HP_{\bullet}(A_{\bullet}) \otimes_R \Omega^1_R.$$

1.4 Statement of the main result

Recall that $A_{\bullet}$ is called homologically proper if $A_{\bullet}$ is perfect as a complex of R-modules. A DG algebra $A_{\bullet}$ is said to be homologically smooth if $A_{\bullet}$ is quasi-isomorphic to a DG algebra $B_{\bullet}$, which is h-flat as a complex of R-modules,[6] and $B_{\bullet}$ is perfect as a DG module over $B_{\bullet} \otimes_R B_{\bullet}{}^{op}$. The following is one of the main results of our paper.

Theorem 1 *Fix the pair* $(\tilde{R}, \tilde{F})$ *as in Sect. 1.1. Let* $A_{\bullet}$ *be a homologically smooth and homologically proper DG algebra over* R *such that*

$$HH_m(A_{\bullet}, A_{\bullet}) = 0, \text{ for every } m \text{ with } |m| \geq p-2. \tag{1.5}$$

Then a lifting[7] *of* $A_{\bullet}$ *over* $\tilde{R}$*, if it exists, specifies an isomorphism*

$$\phi : \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(\mathrm{Gr}^{\bullet}_{\mathcal{F}} HP_{\bullet}(A_{\bullet}), \mathrm{Gr}_{\mathcal{F}} \nabla) \xrightarrow{\sim} (HP_{\bullet}(A_{\bullet}), \nabla) \tag{1.6}$$

giving $(HP_{\bullet}(A_{\bullet}), \nabla, \mathcal{F}^{\bullet} HP_{\bullet}(A_{\bullet}))$ *a Fontaine–Laffaille module structure. In addition, the Hodge-to-de Rham spectral sequence (1.4) degenerates at* E_1 *and induces an isomorphism of Higgs modules*

$$(\mathrm{Gr}^{\bullet}_{\mathcal{F}} HP_{\bullet}(A_{\bullet}), \mathrm{Gr}_{\mathcal{F}} \nabla) \xrightarrow{\sim} (HH_{\bullet}(A_{\bullet}, A_{\bullet})[u, u^{-1}], u^{-1} e_{\kappa}). \tag{1.7}$$

Using (1.7), the isomorphism (1.6) takes the form

$$\phi : (F^* HH_{\bullet}(A_{\bullet}, A_{\bullet})[u, u^{-1}], \nabla_{can} + u^{-1} C^{-1}_{(\tilde{R},\tilde{F})}(e_{\kappa})) \xrightarrow{\sim} (HP_{\bullet}(A_{\bullet}), \nabla), \tag{1.8}$$

where ∇_{can} is the Frobenius pullback connection and $C^{-1}_{(\tilde{R},\tilde{F})}$ is the inverse Cartier operator (1.1).

[6] A complex $B_{\bullet}$ of R-modules is called h-flat if, for any acyclic complex $D_{\bullet}$ of R-modules, the tensor product $B_{\bullet} \otimes_R D_{\bullet}$ is acyclic.

[7] A lifting of $A_{\bullet}$ over $\tilde{R}$ is an h-flat DG algebra $\tilde{A_{\bullet}}$ over $\tilde{R}$ together with a quasi-isomorphism $\tilde{A_{\bullet}} \otimes_{\tilde{R}} R \xrightarrow{\sim} A_{\bullet}$ of DG algebras over R.

Remarks 1.1 (a) If $R = k$ the above result, under slightly different assumptions,[8] is due to Kaledin [17, Th. 5.1].

(b) The construction from Theorem 1 determines a functor from the category of homologically smooth and homologically proper DG algebras over $\tilde{R}$ satisfying (1.5) localized with respect to quasi-isomorphisms to the category of Fontaine–Laffaille modules. We expect, but do not check it in this paper, that this functor extends to the homotopy category of smooth and proper DG categories over $\tilde{R}$ satisfying the analogue of (1.5). When applied to the bounded derived DG category $D^b(Coh(\tilde{X}))$ of coherent sheaves on a smooth proper scheme $\tilde{X}$ over $\tilde{R}$ of relative dimension less than $p-2$, we expect to recover the Fontaine–Laffaille structure on

$$HP_0(D^b(Coh(X)) \xrightarrow{\sim} \bigoplus_i H^{2i}_{DR}(X)(i)$$

$$HP_1(D^b(Coh(X)) \xrightarrow{\sim} \bigoplus_i H^{2i+1}_{DR}(X)(i)$$

constructed by Faltings [10, Th. 6.2]. Here X denotes the scheme over R obtained from $\tilde{X}$ by the base change and $H^*_{DR}(X)(i)$ the Tate twist of the Fontaine–Laffaille structure on the relative de Rham cohomology.

Let us explain some corollaries of Theorem 1. First, under the assumptions of Theorem 1 the Hochschild and cyclic homology of $A_\bullet$ is a locally free R-module. This follows from a general property of Fontaine–Laffaille modules mentioned above.[9] Next, it follows, that under the same assumptions the p-curvature of the Gauss–Manin connection on $HP_\bullet(A_\bullet)$ is nilpotent.[10]. In fact, there is a decreasing filtration,

$$\mathcal{V}_i HP_\bullet(A_\bullet) \subset HP_\bullet(A_\bullet) \tag{1.9}$$

formed by the images under ϕ of

$$u^i F^* HH_\bullet(A_\bullet, A_\bullet)[u^{-1}] \subset F^* HH_\bullet(A_\bullet, A_\bullet)[u, u^{-1}]$$

which is preserved by the connection and such that $\mathrm{Gr}^{\mathcal{V}}_\bullet HP_\bullet(A_\bullet)$ has zero p-curvature:

$$(\mathrm{Gr}^{\mathcal{V}}_\bullet HP_\bullet(A_\bullet), \mathrm{Gr}^{\mathcal{V}} \nabla) \xrightarrow{\sim} (F^* HH_\bullet(A_\bullet, A_\bullet)[u, u^{-1}], \nabla_{can}). \tag{1.10}$$

Moreover, using Theorem 1 we can express the p-curvature of ∇ on $HP_\bullet(A_\bullet)$ in terms of the Kodaira–Spencer operator e_κ: by [29, Th. 2.8], for any Higgs module (E, θ),

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8 Kaledin proves his result assuming, instead of (1.5), vanishing of the *reduced Hochschild cohomology* $\overline{HH}^m(A_\bullet)$ for $m \geq 2p$.

9 After we submitted this paper for publication preprint [27] by Mathew has appeared. Among other results it is proven there, using a different method, the locally freeness of the Hochschild and cyclic homology and the deneration of the Hodge-to-de Rham spectral sequence (1.4) at E_1 under a weaker assumption: the vanishing 1.5 is assumed to hold only for $|m| \geq p$.

10 This suffices for our main application in characteristic 0: Theorem 3 below.

such that the action of S^pT_R on E is trivial, the p-curvature of $\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(E,\theta)$, viewed as a R-linear morphism

$$\psi : F^*E \to F^*E \otimes F^*\Omega^1_R$$

is equal to $-F^*(\theta)$. In particular, under assumption (1.5), the p-curvature of $\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(HH_\bullet(A_\bullet, A_\bullet)[u,u^{-1}], u^{-1}e_\kappa)$, equals $-u^{-1}F^*(e_\kappa)$. As a corollary, we obtain, a version of the Katz formula for the p-curvature of the Gauss–Manin connection on the de Rham cohomology [20, Th. 3.2]: by (1.10) the p-curvature morphism for $HP_\bullet(A_\bullet)$ shifts the filtration $\mathcal{V}_\bullet$:

$$\psi : \mathcal{V}_\bullet HP_\bullet(A_\bullet) \to \mathcal{V}_{\bullet-1}HP_\bullet(A_\bullet) \otimes F^*\Omega^1_R.$$

Thus, ψ induces a morphism

$$\overline{\psi} : \mathrm{Gr}^{\mathcal{V}}_\bullet HP_\bullet(A_\bullet) \to \mathrm{Gr}^{\mathcal{V}}_{\bullet-1} HP_\bullet(A_\bullet) \otimes F^*\Omega^1_R.$$

Our version of the Katz formula asserts the commutativity of the following diagram.

$$\begin{array}{ccc} \mathrm{Gr}^{\mathcal{V}}_i HP_j(A_\bullet) & \xrightarrow{\sim} & F^*HH_{j+2i}(A_\bullet, A_\bullet) \\ \downarrow{\scriptstyle\overline{\psi}} & & \downarrow{\scriptstyle -F^*(e_\kappa)} \\ \mathrm{Gr}^{\mathcal{V}}_{i-1} HP_j(A_\bullet) \otimes F^*\Omega^1_R & \xrightarrow{\sim} & F^*HH_{j+2i-2}(A_\bullet, A_\bullet) \otimes F^*\Omega^1_R. \end{array} \tag{1.11}$$

1.5 The co-periodic cyclic homology, the conjugate filtration, and a generalized Katz p-curvature formula

Though, as explained above, formula (1.11) is an immediate corollary of Theorem 1, a version of the former holds for any DG algebra $A_\bullet$. What makes this generalization possible is the observation that although the morphism (1.8) does depend on the choice of a lifting of $A_\bullet$ over $\tilde{R}$ the induced ∇-invariant filtration (1.9) is canonical: in fact, it coincides with the *conjugate filtration* introduced by Kaledin in [16].[11] However, in general, the conjugate filtration is a filtration on the *co-periodic cyclic homology* $\overline{HP}_\bullet(A_\bullet)$ defined Kaledin in *loc. cit.* to be the homology of the complex

$$\overline{CP}_\bullet(A_\bullet) = (CH_\bullet(A_\bullet, A_\bullet)((u^{-1})), b + uB).$$

For any $A_\bullet$, this comes together with the conjugate filtration $\mathcal{V}_\bullet\overline{CP}_\bullet(A_\bullet)$ satisfying the properties

$$u : \mathcal{V}_\bullet\overline{CP}_\bullet(A_\bullet) \xrightarrow{\sim} \mathcal{V}_{\bullet+1}\overline{CP}_\bullet(A_\bullet)[2],$$
$$\mathrm{Gr}^{\mathcal{V}}\,\overline{CP}_\bullet(A_\bullet) \xrightarrow{\sim} F^*C(A_\bullet, A_\bullet)((u^{-1})).$$

[11] The terminology is borrowed from [19], where the conjugate filtration on the de Rham cohomology in characteristic p was introduced.

This yields a convergent *conjugate spectral sequence*

$$F^* HH_\bullet(A_\bullet, A_\bullet)((u^{-1})) \Rightarrow \overline{HP}_\bullet(A_\bullet), \tag{1.12}$$

whose E_∞ page is $\mathrm{Gr}^{\mathcal{V}}_\bullet \overline{HP}_\bullet(A_\bullet)$. It is shown in [16] that if $A_\bullet$ is smooth and homologically bounded then the morphisms

$$(CH_\bullet(A_\bullet, A_\bullet)[u, u^{-1}], b + uB) \longrightarrow (CH_\bullet(A_\bullet, A_\bullet)((u)), b + uB) \tag{1.13}$$
$$(CH_\bullet(A_\bullet, A_\bullet)[u, u^{-1}], b + uB) \longrightarrow (CH_\bullet(A_\bullet, A_\bullet)((u^{-1})), b + uB) \tag{1.14}$$

are quasi-isomorphisms. In particular, for smooth and homologically bounded DG algebras one has a canonical isomorphism

$$\overline{HP}_\bullet(A_\bullet) \xrightarrow{\sim} HP_\bullet(A_\bullet). \tag{1.15}$$

For an arbitrary DG algebra $A_\bullet$ we introduce in Sect. 3 a Gauss–Manin connection on $\overline{HP}_\bullet(A_\bullet)$. It is compatible with the one on $HP_\bullet(A_\bullet)$ if $A_\bullet$ is smooth and homologically bounded. We show that ∇ preserves the conjugate filtration and the entire conjugate spectral sequence (1.12) is compatible with the connection (where the first page, $F^* HH_\bullet(A_\bullet, A_\bullet)((u^{-1}))$ is endowed with the Frobenius pullback connection). In particular, the p-curvature ψ of the connection on $\overline{HP}_\bullet(A_\bullet)$ is zero on $\mathrm{Gr}^{\mathcal{V}}_\bullet \overline{HP}_\bullet(A_\bullet)$. Hence, ψ induces a morphism

$$\overline{\psi} : \mathrm{Gr}^{\mathcal{V}}_\bullet \overline{HP}_\bullet(A_\bullet) \to \mathrm{Gr}^{\mathcal{V}}_{\bullet-1} \overline{HP}_\bullet(A_\bullet) \otimes F^*\Omega^1_R.$$

In Sect. 3 we prove the following result, which is a generalization of formula (1.11).

Theorem 2 *Let $A_\bullet$ be a DG algebra over R and $\kappa \in HH^2(A_\bullet, A_\bullet) \otimes_R \Omega^1_R$ its Kodaira–Spencer class.*

(a) The morphism $u^{-1}F^*(e_\kappa) : F^*HH_\bullet(A_\bullet, A_\bullet)((u^{-1})) \to F^*HH_\bullet(A_\bullet, A_\bullet)((u^{-1})) \otimes F^*\Omega^1_R$ commutes with all the differentials in the conjugate spectral sequence (1.12) inducing a map

$$\mathrm{Gr}^{\mathcal{V}}_\bullet \overline{HP}_\bullet(A_\bullet) \to \mathrm{Gr}^{\mathcal{V}}_{\bullet-1} \overline{HP}_\bullet(A_\bullet) \otimes F^*\Omega^1_R,$$

which we also denote by $u^{-1}F^*(e_\kappa)$. With this notation, we have

$$u^{-1}F^*(e_\kappa) = \overline{\psi}. \tag{1.16}$$

(b) Assume that $HH_m(A_\bullet, A_\bullet) = 0$ for all sufficiently negative m. Then the p-curvature of the Gauss–Manin connection on $\overline{HP}_\bullet(A_\bullet)$ is nilpotent.

Corollary 1.2 *Let $A_\bullet$ be a smooth and proper DG algebra over R and let d be a non-negative integer d such that $HH_m(A_\bullet, A_\bullet) = 0$, for every m with $|m| > d$. Then* *the p-curvature of the Gauss–Manin connection on* $HP_\bullet(A_\bullet)$ *is nilpotent of exponent* $\leq d+1$, *i.e., there exists a filtration*

$$0 = \mathcal{V}_0 HP_\bullet(A_\bullet) \subset \cdots \subset \mathcal{V}_{d+1} HP_\bullet(A_\bullet) = HP_\bullet(A_\bullet)$$

preserved by the connection such that, for every $0 < i \leq d+1$, *the p-curvature of the connection on* $\mathcal{V}_i/\mathcal{V}_{i-1}$ *is 0.*

1.6 An application: the local monodromy theorem

As an application of the nilpotency of the p-curvature we prove, using a result from [19], "the local monodromy theorem" for the periodic cyclic homology in characteristic 0.

Theorem 3 *Let S be a smooth irreducible affine curve over* $\mathbb{C}$, $\overline{S}$ *a smooth compactification of S, and let* $A_\bullet$ *be a smooth and proper DG algebra over* $\mathcal{O}(S)$. *Then the Gauss–Manin connection on the relative periodic cyclic homology* $HP_*(A_\bullet)$ *has regular singularities, and its monodromy around every point at* $\overline{S} - S$ *is quasi-unipotent.*

This result generalizes the Griffiths–Landman–Grothendieck theorem asserting that for a smooth proper scheme X over S the Gauss–Manin connection on the relative de Rham cohomology $H^*_{DR}(X)$ has regular singularities, and its monodromy at infinity is quasi-unipotent. The derivation of Theorem 3 from Corollary 1.2 is essentially due to Katz [19]; we explain the argument in Sect. 4.

1.7 Proofs

Let us outline the proofs of Theorems 1 and 2. Without loss of generality we may assume that $A_\bullet$ is a semi-free DG algebra over R. Let ${A_\bullet}^{\otimes p}$ denote the p-th tensor power of $A_\bullet$ over R. This is a DG algebra equipped with an action of the symmetric group S_p. In particular, it carries an action of the group $\mathbb{Z}/p\mathbb{Z} \xrightarrow{\sim} C_p \subset S_p$ of cyclic permutations. We denote by $T(C_p, {A_\bullet}^{\otimes p})$ the Tate cohomology complex of C_p with coefficients in ${A_\bullet}^{\otimes p}$. The algebra structure on ${A_\bullet}^{\otimes p}$ induces one on the Tate cohomology $\check{H}^\bullet(C_p, {A_\bullet}^{\otimes p})$. Moreover, choosing an appropriate "complete resolution" one can lift the cup product on the cochain level giving $T(C_p, {A_\bullet}^{\otimes p})$ the structure of a DG algebra over R. If $A_\bullet = A$ is an associative algebra then, for $p \neq 2$, one has a canonical isomorphism of algebras

$$\check{H}^*(C_p, A^{\otimes p}) \xrightarrow{\sim} F^*A \otimes \check{H}^*(C_p, \mathbb{F}_p) \xrightarrow{\sim} F^*A[u, u^{-1}, \epsilon],$$

$\deg u = 2$ and $\deg \epsilon = 1$, $\epsilon^2 = 0$. In general, Kaledin defines an increasing filtration

$$\tau^{dec}_{\leq \bullet} T(C_p, {A_\bullet}^{\otimes p}) \subset T(C_p, {A_\bullet}^{\otimes p})$$

making $T(C_p, A_\bullet{}^{\otimes p})$ a filtered DG algebra equipped with a canonical quasi-isomorphism of graded DG algebras

$$\bigoplus_i \mathrm{Gr}_i^\tau\, T(C_p, A_\bullet{}^{\otimes p}) \xrightarrow{\sim} F^* A_\bullet \otimes \bigoplus_i \check{H}^i(C_p, \mathbb{F}_p)[-i]. \tag{1.17}$$

Note that the right-hand side of (1.17) has a canonical connection—the Frobenius pullback connection. A key observation explained in Sect. 2.3 is that there is a canonical connection ∇ on the filtered DG algebra $T(C_p, A_\bullet{}^{\otimes p})$, which induces the Frobenius pullback connection on Gr^τ.

Denote by $T_{[m,l]}(C_p, A_\bullet{}^{\otimes p})$, $(m \leq l)$, the quotient of $\tau^{dec}_{\leq l} T(C_p, A_\bullet{}^{\otimes p})$ by $\tau^{dec}_{\leq m-1} T(C_p, A_\bullet{}^{\otimes p})$. The DG algebra

$$\mathcal{B}(A_\bullet) := T_{[-1,0]}(C_p, A_\bullet{}^{\otimes p}),$$

which is a square-zero extension of $F^* A_\bullet$

$$F^* A_\bullet[1] \xrightarrow{\mu} \mathcal{B}(A_\bullet) \longrightarrow F^* A_\bullet$$

with a compatible connection ∇, admits another description. Let $\hat{R}$ be a flat lifting of $\tilde{R}$ over $W(k)$, $\hat{i}_*$ the functor from the category of DG algebras over R to the category of DG algebras over $\hat{R}$, which carries a DG algebra over R to the same underlying DG ring with the action of $\hat{R}$ induced by the morphism $\hat{R} \to R$, and let $L\hat{i}^*$ be the left adjoint functor, which carries a DG algebra $C_\bullet$ over $\hat{R}$ to the derived tensor product $C_\bullet \overset{L}{\otimes}_{\hat{R}} R$. For any DG algebra $A_\bullet$ over R the composition $L\hat{i}^* \hat{i}_* A_\bullet$ is an algebra over $L\hat{i}^* \hat{i}_* R \xrightarrow{\sim} R[\mu]$, where $\deg \mu = -1$, $\mu^2 = 0$. One can easily check that the functor $L\hat{i}^* \hat{i}_*$ depends on $\tilde{R}$ only (in particular, every automorphism of $\hat{R}$, which restricts to the identity on $\tilde{R}$ acts trivially on $L\hat{i}^* \hat{i}_*$). Similarly, the morphism of crystalline toposes $\mathrm{Cris}(R/k) \to \mathrm{Cris}(R/W(k))$ induces a functor $\hat{i}_{*cris}$ from the category of DG algebras in the category of crystals on $\mathrm{Cris}(R/k)$ (i.e., the category of R-modules with integrable connections) to the category of DG algebras in the category of crystals on $\mathrm{Cris}(R/W(k))$ (i.e., the category of p-adically complete $\hat{R}$-modules with integrable connections) and also its left adjoint functor $L\hat{i}^{*cris}$. These functors extend the functors $\hat{i}_*$, $L\hat{i}^*$ on sheaves of rings.[12]

Theorem 4 *Let $A_\bullet$ be a term-wise flat DG algebra over R.*[13]

(a) There is a canonical quasi-isomorphism of DG algebras with connection

$$(\mathcal{B}(A_\bullet), \nabla) \xrightarrow{\sim} L\hat{i}^{*cris} \hat{i}_{*cris} F^* A_\bullet.$$

[12] Recall that one should think of an algebra with connection on a scheme X over a base ring R as a family of algebras with connection over points of $\mathrm{Spec}(R)$. In terms of this picture, the pushforward and pullback functors i_{*cris}, i^{*cris} are just base change.

[13] Since R has finite homological dimension $A_\bullet$ is also h-flat over R.

(b) *A lifting $(\tilde{R}, \tilde{F})$ of (R, F) over $W_2(k)$ gives rise to a canonical quasi-isomorphism of DG algebras with connection*

$$(\mathcal{B}(A_\bullet), \nabla) \xrightarrow{\sim} \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(L\hat{i}^*\hat{i}_*A_\bullet, \mu\tilde{\kappa}).$$

Here $\tilde{\kappa}$ is the Kodaira–Spencer class of $A_\bullet$ regarded as a derivation of $A_\bullet$ with values in $A_\bullet \otimes \Omega^1_R$ of degree 1 (as defined by formula 1.3), $\mu\tilde{\kappa}$ the induced degree 0 derivation of $L\hat{i}^*\hat{i}_*A_\bullet$ with values in $(L\hat{i}^*\hat{i}_*A_\bullet) \otimes \Omega^1_R$, and $\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}$ is the inverse Cartier transform.

(c) *A lifting of $A_\bullet$ over $\tilde{R}$ gives rise to a canonical quasi-isomorphism of DG algebras with connection*

$$(\mathcal{B}(A_\bullet), \nabla) \xrightarrow{\sim} \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(A_\bullet[\mu], \mu\tilde{\kappa}).$$

Remarks 1.3 (a) If R is a perfect field the above result is due to Kaledin [15, Prop. 6.13].

(b) The first part of the Theorem together with the projection formula gives a canonical isomorphism of DG algebras with connections

$$\hat{i}^{cris}_*\mathcal{B}(A_\bullet) \xrightarrow{\sim} \hat{i}^{cris}_* F^*A_\bullet \oplus \hat{i}^{cris}_* F^*A_\bullet[1],$$

where the right-hand side of the equation is the trivial square-zero extension with the Frobenius pullback connection. However, in general $\mathcal{B}(A_\bullet)$ does not split. For example, from the second part of the Theorem it follows that the p-curvature of ∇ on $\mathcal{B}(A_\bullet)$ equals $-\mu F^*(\mu\tilde{\kappa})$. In particular, it is not zero as long as $\tilde{\kappa}$ is not 0.

Next, we relate the cyclic homology of $\mathcal{B}(A_\bullet)$ together with the connection induced by the one on $\mathcal{B}(A_\bullet)$ with the periodic cyclic homology of $A_\bullet$ with the Gauss–Manin connection. The two-step fitration $F^*A_\bullet[1] \subset \mathcal{B}(A_\bullet)$ gives rise to a filtration $\mathcal{V}_m CC(\mathcal{B}(A_\bullet)) \subset CC(\mathcal{B}(A_\bullet))$, $(m = 0, -1, -2, \cdots)$, on the cyclic complex of $\mathcal{B}(A_\bullet)$.

Theorem 5 *Let $A_\bullet$ be a term-wise flat DG algebra over R. We have a canonical quasi-isomorphism of filtered complexes with connections*

$$\mathcal{V}_{[-p+2,-1]}CC(\mathcal{B}(A_\bullet))[1] \xrightarrow{\sim} \mathcal{V}_{[-p+2,-1]}\overline{CP}(A_\bullet).$$

Moreover, the multiplication by u^{-1} on the right-hand side corresponds under the above quasi-isomorphism to the multiplication by the class $B\mu$ in the second negative cyclic homology group of the algebra $k[\mu]$.

Let us derive Theorem 1 from Theorems 5 and 4. Since the Cartier transform is a monoidal functor, we have by part 3 of Theorem 5

$$(\mathcal{V}_{[-p+2,-1]}CC(\mathcal{B}(A_\bullet)), \nabla) \xrightarrow{\sim} \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(\mathcal{V}_{[-p+2,-1]}CC(A_\bullet[\mu]), \mu\tilde{\kappa}).$$

We compute the right-hand side using the Künneth formula: with obvious notation we have a quasi-isomorphism of mixed complexes

$$\mathcal{V}_{[-p+2,-1]}C(A_\bullet[\mu]) \xrightarrow{\sim} C(A_\bullet) \otimes \mathcal{V}_{[-p+2,-1]}C(k[\mu]).$$

The Hochschild complex of $k[\mu]$ regarded as a mixed complex is quasi-isomorphic to the divided power algebra:

$$C(k[\mu], k[\mu]) \xrightarrow{\sim} k\langle \mu, B\mu \rangle$$

with zero differential and Connes' operator acting by the formulas: $B((B\mu)^{[m]}) = 0$, $B(\mu(B\mu)^{[m]}) = (m+1)(B\mu)^{[m+1]}$.

It follows that

$$\mathcal{V}_{[-p+2,-1]}CC(A_\bullet[\mu]) \xrightarrow{\sim} \bigoplus_{0 \le m \le p-3} C(A_\bullet) \otimes \mu(B\mu)^{[m]}.$$

Setting $B\mu = u^{-1}$ and using the Cartan formula ([13]; see also Sect. 3 for a review), we find

$$(\mathcal{V}_{[-p+2,-1]}CC(A_\bullet[\mu], \mu\tilde{\kappa})[-1] \xrightarrow{\sim} (C(A_\bullet) \otimes k[u^{-1}]/u^{2-p}, u^{-1}\iota_{\tilde{\kappa}}).$$

Summarizing, we get

$$(\mathcal{V}_{[-p+2,-1]}\overline{CP}(A_\bullet), \nabla) \xrightarrow{\sim} \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(C(A_\bullet) \otimes k[u^{-1}]/u^{2-p}, u^{-1}\iota_{\tilde{\kappa}})[2]$$

This implies the desired result. The derivation of Theorem 2 is similar.

2 The Tate cohomology complex of $A_\bullet^{\otimes p}$

In this section we construct a connection on the Tate complex $T(C_p, A_\bullet{}^{\otimes p})$ and prove Theorem 4.

2.1 The Tate cohomology complex

Let G be a finite group. A complete resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ is an acyclic complex of free $\mathbb{Z}[G]$-modules

$$\longrightarrow \cdots P_{-2} \longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$$

together with an isomorphism of $\mathbb{Z}[G]$-modules

$$\epsilon : \mathbb{Z} \xrightarrow{\sim} \ker(d : P_0 \longrightarrow P_1).$$

One can show that for any two complete resolutions $(P_\bullet, \epsilon)$, $(P_\bullet{}', \epsilon')$ there exists a morphism $f_\bullet : P_\bullet \to P_\bullet{}'$ of complexes of $\mathbb{Z}[G]$-modules such that $f_0 \circ \epsilon = \epsilon'$ and such $f_\bullet$ is unique up to homotopy (in fact, $Hom(P_\bullet, P_\bullet{}')$ in the homotopy category $Ho(\mathbb{Z}[G])$ of complexes of $\mathbb{Z}[G]$-modules is canonically isomorphic to $\mathbb{Z}/r\mathbb{Z}$, where r is the order of $|G|$: to see this observe that $Hom_{Ho(\mathbb{Z}[G])}(M_\bullet, P_\bullet{}') = 0$ if $M_\bullet$ is either a bounded from above complex of free $\mathbb{Z}[G]$-modules or a bounded from below acyclic complex. It follows, using the canonical and "stupid" truncations, that $Hom_{Ho(\mathbb{Z}[G])}(P_\bullet, P_\bullet{}') \simeq Hom_{Ho(\mathbb{Z}[G])}(\mathbb{Z}, P_\bullet{}') \simeq \hat{H}^0(G, \mathbb{Z}) \simeq \mathbb{Z}/r\mathbb{Z}$). Fix a complete resolution $(P_\bullet, \epsilon)$. For a complex $M_\bullet$ of $\mathbb{Z}[G]$-modules we define its Tate cohomology complex $T(G, M_\bullet)$ to be

$$T(G, M_\bullet) := (M_\bullet \otimes_{\mathbb{Z}} P_\bullet)^G.$$

This defines a DG functor $T(G, \cdot)$ from the DG category $C(Mod(\mathbb{Z}[G]))$ of complexes of $\mathbb{Z}[G]$-modules to the DG category of complexes of abelian groups. By construction, $T(G, \cdot)$ commutes with arbitrary direct sums. Also, it easy to check that $T(G, \cdot)$ carries bounded complexes of free $\mathbb{Z}[G]$-modules and bounded acyclic complexes to acyclic complexes.[14] We denote the cohomology groups of $T(G, M_\bullet)$ by $\check{H}^*(G, M_\bullet)$.

Remark 2.1 In [28, §I], Nikolaus and Scholze define the Tate complex of $M_\bullet$ to be the cone of the norm map

$$M_{\bullet hG} \xrightarrow{Nm_G} M_\bullet{}^{hG}.$$

Unless the complex $M_\bullet$ is bounded from below $cone(Nm_G : M_{\bullet hG} \to M_\bullet{}^{hG})$ need not be quasi-isomorphic to $T(G, M_\bullet)$. In fact, for every $M_\bullet$, the cone of the norm map is quasi-isomorphic to a completion of $T(G, M_\bullet)$:

$$cone(Nm_G : M_{\bullet hG} \to M_\bullet{}^{hG}) \xrightarrow{\sim} holim_n T(G, \tau_{>n} M_\bullet),$$

where $M_\bullet \to \tau_{>n} M_\bullet$ is the canonical truncation.

A **multiplicative** complete resolution is a complete resolution $(P_\bullet, \epsilon)$ together with a DG ring structure

$$m : P_\bullet \otimes P_\bullet \to P_\bullet$$

which is compatible with the G-action (i.e., m is a morphism of complexes of $\mathbb{Z}[G]$-modules) such that $\epsilon : \mathbb{Z} \to P_\bullet$ is a morphism of DG rings. Multiplicative complete resolutions exist: e.g., see [6], Chapter 4, §7. From now on $T(G, M_\bullet)$ will denote the Tate complex associated with a fixed multiplicative complete resolution. Then, for any complexes $M_\bullet$, $M_\bullet{}'$ of $\mathbb{Z}[G]$-modules, we get a natural morphism

$$T(G, M_\bullet) \otimes T(G, M_\bullet{}') \to T(G, M_\bullet \otimes_{\mathbb{Z}} M_\bullet{}'),$$

[14] Both statements may fail for unbounded complexes. For example, ϵ induces a quasi-isomorphism $T(G, \mathbb{Z}) \xrightarrow{\sim} T(G, P_\bullet)$. Thus, $T(G, \cdot)$ does not respect arbitrary quasi-isomorphisms.

which induces the cup product on the Tate cohomology groups. In particular, if $M_\bullet$ is a DG ring with an action of G, then the Tate complex $T(G, M_\bullet)$ acquires a DG ring structure.

As explained in [22], for every complete resolution $(P_\bullet, \epsilon)$, one can construct an E_∞ operad $\{\mathcal{C}(i), i \geq 0\}$, $\mathcal{C}(0) = \mathbb{Z}$, in the category of complexes of abelian groups and $G \times S_i$-equivariant morphisms

$$\delta_i : \mathcal{C}(i) \otimes P_\bullet^{\otimes^i} \to P_\bullet,\ \delta_0 = \epsilon$$

which give $(P_\bullet, \epsilon)$ the structure of a G-equivariant algebra over $\{\mathcal{C}(i), i \geq 0\}$. In particular, $T(G, \mathbb{Z})$ has the structure of an E_∞ algebra and, for any G-module (resp. ring with an action of G) $M_\bullet$, the Tate complex $T(G, M_\bullet)$ acquires a module (resp. algebra) structure over $T(G, \mathbb{Z})$. We refer the reader to [28], Appendix S, for another approach to the higher multiplicative structures on the Tate complex based on the interpretation of the latter as the morphism space in a certain symmetric monoidal ∞-category.

2.2 The functor $V \mapsto T(C_p, V^{\otimes^p})$

Let R be a finitely generated smooth commutative algebra over a perfect field k of characteristic $p > 0$. For any complex $V_\bullet$ of flat R-modules the tensor power $V_\bullet^{\otimes^p}$ over R carries an action of the symmetric group S_p. Denote by $\mathbb{Z}/p\mathbb{Z} \simeq C_p \subset S_p$ the subgroup of cyclic permutations. We consider the functor $T(C_p, \cdot^{\otimes p})$

$$V_\bullet \mapsto T(C_p, V_\bullet^{\otimes p})$$

from the category of complexes of flat R-modules to the category of complexes of all R-modules. This functor has a number of remarkable properties not obvious from the definition. First, if $V_\bullet = V$ is supported in cohomological degree 0 and $p > 2$, then, we have a canonical isomorphism of graded modules over $R \otimes \check{H}^*(C_p, \mathbb{F}_p)$

$$\check{H}^*(C_p, V^{\otimes p}) \xrightarrow{\sim} F^*V \otimes_{\mathbb{F}_p} \check{H}^*(C_p, \mathbb{F}_p) \xrightarrow{\sim} F^*V[u, u^{-1}, \epsilon], \tag{2.1}$$

where $\deg u = 2$ and $\deg \epsilon = 1$, $\epsilon^2 = 0$. In [15], §6.2, Kaledin generalized the above isomorphism: for every complex $V_\bullet$ of flat R-modules and any p he defines a canonical increasing filtration

$$\tau^{dec}_{\leq \bullet} T(C_p, V_\bullet^{\otimes p}) \subset T(C_p, V_\bullet^{\otimes p})$$

making $T(C_p, V_\bullet^{\otimes p})$ a filtered DG module over $T(C_p, \mathbb{F}_p)$ (endowed with the canonical filtration) equipped with a canonical quasi-isomorphism of graded DG modules over the graded DG algebra $R \otimes \bigoplus_i \check{H}^i(C_p, \mathbb{F}_p)[-i]$ [15, Lemma 6.5]

$$\bigoplus_i \mathrm{Gr}^\tau_i\, T(C_p, V_\bullet^{\otimes p}) \xrightarrow{\sim} \bigoplus_i (F^*V_\bullet \otimes \check{H}^i(C_p, \mathbb{F}_p)[-i]). \tag{2.2}$$

Namely, consider the (decreasing) stupid filtration on $V_\bullet^{\otimes p} = \dots \to (V_\bullet^{\otimes p})_i \to (V_\bullet^{\otimes p})_{i+1} \to \dots$ rescaled by p:

$$F^i V_\bullet^{\otimes p} = \dots \to 0 \to (V_\bullet^{\otimes p})_{ip} \to (V_\bullet^{\otimes p})_{ip+1} \dots$$

It induces a filtration $F^\bullet$ on $T(C_p, V_\bullet^{\otimes p})$. Now, we apply to the filtered complex $T(C_p, V_\bullet^{\otimes p})$ Deligne's "filtered truncation" construction(§1.3.3 in [7]) and define

$$\tau^{dec}_{\leq n} T(C_p, V_\bullet^{\otimes p})_i = F^{i-n} T(C_p, V_\bullet^{\otimes p})_i \cap d^{-1}(F^{i+1-n} T(C_p, V_\bullet^{\otimes p})_{i+1})$$

We denote by $T_{[n,m]}(C_p, V_\bullet^{\otimes p})$, $(n \leq m)$, the quotient of $\tau^{dec}_{\leq m} T(C_p, V_\bullet^{\otimes p})$ by $\tau^{dec}_{\leq n-1} T(C_p, V_\bullet^{\otimes p})$, and by $\overline{T}(C_p, V_\bullet^{\otimes p})$ the completion of $T(C_p, V_\bullet^{\otimes p})$ with respect to the filtration $\tau^{dec}_{\leq n}$:

$$\overline{T}(C_p, V_\bullet^{\otimes p}) = \varprojlim T(C_p, V_\bullet^{\otimes p})/\tau^{dec}_{\leq n} T(C_p, V_\bullet^{\otimes p}).$$

Formula (2.2) implies the following surprising result.

Lemma 2.2 *(cf.* [15]*, §6,* [24]*, Proposition P*2.2.3*, see also §*III*.1.1 in [28])*
The functors $T_{[n,m]}(C_p, \cdot^{\otimes p})$, $(n \leq m)$, $\overline{T}(C_p, \cdot^{\otimes p})$, *and* $T(C_p, \cdot^{\otimes p})$ *carry an acyclic complex of flat* R*-modules to an acyclic complex. Moreover,* $T_{[n,m]}(C_p, \cdot^{\otimes p})$, $\overline{T}(C_p, \cdot^{\otimes p})$, *and* $T(C_p, \cdot^{\otimes p})$ *are exact: for every diagram of the form*

$$X \xrightarrow{f} Y \xrightarrow{g} Z, \tag{2.3}$$

where X, Y, Z *are complexes of flat* R*-modules with* $g \circ f = 0$ *such that the total complex of the double complex* (2.3) *is acyclic, each of the above 3 functors carries* (2.3) *to a diagram of the same type. That is the total complex of the double complexes* $T_{[n,m]}(C_p, X^{\otimes p}) \to T_{[n,m]}(C_p, Y^{\otimes p}) \to T_{[n,m]}(C_p, Z^{\otimes p})$, $\overline{T}(C_p, X^{\otimes p}) \to \overline{T}(C_p, Y^{\otimes p}) \to \overline{T}(C_p, Z^{\otimes p})$, *and* $T(C_p, X^{\otimes p}) \to T(C_p, Y^{\otimes p}) \to T(C_p, Z^{\otimes p})$ *are acyclic.*

Proof Using formula (2.2), we get a functorial quasi-isomorphism

$$T_{[n,n]}(C_p, V_\bullet^{\otimes p}) \xrightarrow{\sim} F^* V_\bullet[-n].$$

This proves the Lemma for $T_{[n,n]}(C_p, \cdot^{\otimes p})$. The statements on $T_{[m,n]}(C_p, \cdot^{\otimes p})$, $\overline{T}(C_p, \cdot^{\otimes p})$ then follow by dévissage. To prove the Lemma for the Tate complex $T(C_p, \cdot^{\otimes p})$ itself observe that, for a bounded from above complex $V_\bullet$, the completion $\overline{T}(C_p, V_\bullet^{\otimes p})$ is isomorphic to $T(C_p, V^{\otimes p})$. In general, we use that R has finite homological dimension. Denote it by d. Then, for every complex $C_\bullet$ of R-modules such that each C_i is flat over R except, possibly, for C_{i_0}, there exists a complex $C'_\bullet$ of flat R-modules together with a quasi-isomorphism

$$f : C'_\bullet \to C_\bullet$$

such that $f_i : C'_i \to C_i$ is an isomorphism, for every $i \notin [i_0 - d, i_0]$. Applying this assertion to the canonical filtration $\tau_{\leq n} V_\bullet \hookrightarrow V_\bullet$ on an acyclic complex $V_\bullet$ of flat R-modules we conclude that $V_\bullet$ is a filtered colimit of acyclic bounded from above complexes of flat R-modules. Since the functor $T(C_p, \cdot^{\otimes p})$ commutes with filtered colimits of complexes, the first statement of the Lemma follows. The proof of the exactness is similar. □

Remarks 2.3 The proof of the part of Lemma 2.2 concerned $\overline{T}(C_p, \cdot^{\otimes p})$ given here is due to Kaledin. We refer the reader to [24], Proposition 2.2.3, and to §III.1.1 in [28] for different proofs of similar statements and for the historical background.

We do not know whether the assertions on $T(C_p, \cdot^{\otimes p})$ remain valid if one drops the assumption that R has finite homological dimension. However, in [24], Example 2.2.5, Lurie defines a functor which coincides with $T(C_p, \cdot^{\otimes p})$ if R has finite homological dimension and has favorable properties in general.

The filtration $\tau^{dec}_{\leq \bullet} T(C_p, \cdot^{\otimes p})$ is compatible with the cup product in the obvious sense. In particular, if $A_\bullet$ is a termwise flat DG algebra over R then the filtration

$$\tau^{dec}_{\leq \bullet} T(C_p, A_\bullet{}^{\otimes p})$$

defines the structure of a filtered DG algebra on $T(C_p, A_\bullet{}^{\otimes p})$ and on $\overline{T}(C_p, A_\bullet{}^{\otimes p})$.

2.3 Connection on the Tate complex

Denote $spec\, R$ by X. Let $X^{[2]}$ be the first infinitesimal thickening of the diagonal $\Delta \subset X \times X$ and $p_1, p_2 : X^{[2]} \to X$ projections. The following construction is essentially contained in [14], and it does not depend on the fact that X is affine.

Let $A_\bullet$ be a termwise flat DG algebra over R. We will construct a connection on the filtered DG algebra $\overline{T}(C_p, A_\bullet{}^{\otimes p})$ (and, in particular, on $T_{[n,0]}(C_p, A_\bullet{}^{\otimes p})$, $(n \leq 0)$), that is, a quasi-isomorphism of filtered DG algebras

$$\nabla : p_1^* \overline{T}(C_p, A_\bullet{}^{\otimes p}) \cong p_2^* \overline{T}(C_p, A_\bullet{}^{\otimes p})$$

which is, when restricted to Δ is equal to the identity in the category of filtered DG algebras localized with respect to filtered quasi-isomorphisms. The exact sequence of sheaves on $X \times X$

$$0 \to \Omega^1_\Delta \to \mathcal{O}_{X \times X}/I^2_\Delta \to \mathcal{O}_\Delta \to 0$$

induces two exact sequences of complexes

$$0 \to A_\bullet \otimes \Omega^1_X \xrightarrow{\beta} p_{1*} p_2^* A_\bullet \xrightarrow{\alpha} A_\bullet \to 0$$
$$0 \to \overline{T}(C_p, A_\bullet{}^{\otimes p}) \otimes \Omega^1_X \xrightarrow{\beta'} p_{1*} p_2^* \overline{T}(C_p, A_\bullet{}^{\otimes p}) \xrightarrow{\alpha'} \overline{T}(C_p, A_\bullet{}^{\otimes p}) \to 0$$

Giving a connection on $A_{\bullet}$ is equivalent to giving a section $\overline{T}(C_p, A_{\bullet}{}^{\otimes p}) \to p_{1*}p_2^*\overline{T}(C_p, A_{\bullet}{}^{\otimes p})$ of α' in the category of filtered DG algebras localized with respect to filtered quasi-isomorphisms. Let us construct such a section. Denote by

$$0 \subset G^p(p_{1*}p_2^*A_{\bullet})^{\otimes p} \subset G^{p-1}(p_{1*}p_2^*A_{\bullet})^{\otimes p} \subset \cdots \subset G^0(p_{1*}p_2^*A_{\bullet})^{\otimes p} = (p_{1*}p_2^*A_{\bullet})^{\otimes p}$$

the filtration induced by $A_{\bullet} \otimes \Omega^1_X \subset p_{1*}p_2^*A_{\bullet}$:

$$G^i(p_{1*}p_2^*A_{\bullet})^{\otimes p} := \sum_{\sigma \in S_p} \sigma((A_{\bullet} \otimes \Omega^1_X)^{\otimes i} \otimes (p_{1*}p_2^*A_{\bullet})^{\otimes p-i}).$$

Lemma 2.4 *For a term-wise flat DG algebra $A_{\bullet}$ the morphism α induces the following C_p-equivariant isomorphism of DG algebras*

$$G^0(p_{1*}p_2^*A_{\bullet})^{\otimes p}/G^1(p_{1*}p_2^*A_{\bullet})^{\otimes p} \xrightarrow{\sim} A_{\bullet}{}^{\otimes p}$$

and β induces the following C_p-equivariant isomorphism of complexes

$$A_{\bullet}{}^{\otimes p} \otimes_{\mathcal{O}_X} \Omega^1_X \otimes_{\mathbb{Z}} \mathbb{Z}[C_p] \xrightarrow{\sim} G^1(p_{1*}p_2^*A_{\bullet})^{\otimes p}/G^2(p_{1*}p_2^*A_{\bullet})^{\otimes p}$$

The proof is straightforward.

By adjunction, we have a map $m : (p_{1*}p_2^*A_{\bullet})^{\otimes p} \to p_{1*}p_2^*(A_{\bullet}{}^{\otimes p})$. Since $X \to X^{[2]}$ is a square-zero extension, m factors through G^2, so we get the following diagram of complexes of C_p-modules in which the top row is a distinguished triangle

$$\begin{array}{ccccc} A_{\bullet}{}^{\otimes p} \otimes \Omega^1_X \otimes \mathbb{Z}[C_p] & \xrightarrow{i} & G^0/G^2 & \xrightarrow{\pi} & A_{\bullet}{}^{\otimes p} \longrightarrow \\ & & \downarrow m & & \\ & & p_{1*}p_2^*(A_{\bullet}{}^{\otimes p}) & & \end{array} \tag{2.4}$$

The complex of C_p-modules $A_{\bullet}{}^{\otimes p} \otimes \Omega^1_X \otimes \mathbb{Z}[C_p]$ is isomorphic to the tensor product of the complex $A_{\bullet}{}^{\otimes p} \otimes \Omega^1_X$ with *trivial* C_p-action and a free module $\mathbb{Z}[C_p]$. Thus, Tate cohomology complex of this complex is quasi-isomorphic to $A_{\bullet}{}^{\otimes p} \otimes \Omega^1_X \otimes T(C_p, \mathbb{Z}[C_p])$, which is quasi-isomorphic to zero.

Thus, applying the functor $T(C_p, -)$ to (2.4), we get the following diagram of DG algebras in which π is a quasi-isomorphism.

$$\begin{array}{ccc} T(C_p, G^0/G^2) & \xrightarrow{\pi} & T(C_p, A_{\bullet}{}^{\otimes p}) \longrightarrow \\ \downarrow m & & \\ p_{1*}p_2^*T(C_p, A_{\bullet}{}^{\otimes p}) & & \end{array} \tag{2.5}$$

The DG algebra $T(C_p, A_{\bullet}{}^{\otimes p})$ is filtered by $\tau^{dec}_{\leq \bullet}T(C_p, A_{\bullet}{}^{\otimes p})$ and the DG algebra $p_{1*}p_2^*T(C_p, A_{\bullet}{}^{\otimes p})$ is filtered by $p_{1*}p_2^*\tau^{dec}_{\leq \bullet}T(C_p, A_{\bullet}{}^{\otimes p})$. We wish to define a filtration on $T(C_p, G^0/G^2)$ to promote (2.5) to a diagram of filtered DG algebras in which π is a filtered quasi-isomorphism. As in Sect. 2.2 we define the desired filtration $\tau^{dec}_{\leq \bullet} T(C_p, G^0/G^2)$ by applying Deligne's "filtered truncation" construction to filtration on $T(C_p, G^0/G^2)$ induced by the stupid filtration on G^0/G^2 rescaled by p. We leave it to the reader to check that $\tau^{dec}_{\leq \bullet} T(C_p, G^0/G^2)$ has the required properties.

It follows that π induces a quasi-isomorphism of the completed Tate cohomology complexes $\pi : \overline{T}(C_p, G^0/G^2) \xrightarrow{\sim} \overline{T}(C_p, A_\bullet{}^{\otimes p})$. Finally, put $s = m\pi^{-1}$. This is a section of α'. By adjunction, s induces a connection

$$\nabla : p_1^* \overline{T}(C_p, A_\bullet{}^{\otimes p}) \cong p_2^* \overline{T}(C_p, A_\bullet{}^{\otimes p}). \tag{2.6}$$

2.4 The connection on the truncated Tate complex

Denote by $\hat{i}_{cris*}$ and $\hat{i}^{cris*}$ respectively the direct and inverse image functors between the categories of crystals on R over k and over $W(k)$. We will write $\hat{i}_{cris*}$ and $L\hat{i}^{cris*}$ for the corresponding derived functors between the derived categories of crystals. (When there is no risk of ambiguity $L\hat{i}^{cris*}$ will shortened to $\hat{i}^{cris*}$.) By the virtue of Theorem 6.6 from [1,2] we view the category Cris(R/k) of crystals as a full subcategory of the category of R-modules with connection. A key result of this section is the following theorem.

Theorem 6 *There is a quasi-isomorphism of DG algebras with connection*

$$\mathcal{B}(A_\bullet) := T_{[-1,0]}(C_p, A_\bullet{}^{\otimes p}) \cong L\hat{i}^{cris*} \hat{i}_{cris*} F^* A_\bullet =: \mathcal{B}^{cris}(A_\bullet) \tag{2.7}$$

A proof of Theorem 6 is given in Sect. 2.6.

Under some additional assumptions the DG algebra $\mathcal{B}^{cris}(A_\bullet)$ can be described explicitly. Let $\tilde{R}$ be a flat lifting of R over $W_2(k)$ and $\tilde{F} : \tilde{R} \to \tilde{R}$ a lifting of the Frobenius morphism on $\tilde{R}$. Choose also a lifting $\hat{R}$ of $\tilde{R}$ over $W(k)$. Consider the functors

$$\begin{array}{ll} \hat{i}_* : D(Mod - R) \to D(Mod - \hat{R}) & L\hat{i}^* : D(Mod - \hat{R}) \to D(Mod - R) \\ \tilde{i}_* : D(Mod - R) \to D(Mod - \tilde{R}) & L\tilde{i}^* : D(Mod - \tilde{R}) \to D(Mod - R) \end{array} \tag{2.8}$$

Again, by Theorem 6.6 in [1,2], the categories of crystals on R over $W_2(k)$ and $W(k)$ are equivalent to the categories of respectively $\tilde{R}$- and p-adically complete $\hat{R}$-modules with flat quasi-nilpotent connection. Also, direct and inverse images are compatible with the forgetful functors

$$u_{R/W_2(k)*} : \mathrm{Cris}(R/W_2(k)) \to Mod - \tilde{R}; \; u_{R/W(k)*} : \mathrm{Cris}(R/W(k)) \to Mod - \hat{R}.$$

Hence, as a DG R-algebra $\mathcal{B}^{cris}(A_\bullet)$ is quasi-isomorphic to $L\hat{i}^* \hat{i}_* F^* A_\bullet$.

Theorem 7 *A lifting of $A_\bullet$ to a DG algebra $\tilde{A}_\bullet/\tilde{R}$ gives rise to a quasi-isomorphism of DG algebras with connection*

$$\mathcal{B}^{cris}(A_\bullet) \cong \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(A_\bullet[\mu], \mu\tilde{\kappa}) \tag{2.9}$$

where μ is a generator of degree -1, $\mu^2 = 0$, and $\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}$ is the inverse Cartier transform in the sense of Sect. 1.1.

2.5 Proof of Theorem 7

Replace $\tilde{A}$ by a semi-free resolution (cf. [9] 13.4) over $\tilde{R}$ and $A_\bullet$ by $\tilde{A} \otimes_{\tilde{R}} R$ (the latter will be automatically semi-free over R). Fix a connection ∇' on the free algebra $\bigoplus A_i$. It might not be compatible with the differential—the Kodaira–Spencer class measures this incompatibility: $\tilde{\kappa} = [\nabla', d]$.

Lemma 2.5 *For a free module $\tilde{B}/\tilde{R}$ a connection ∇_0 on $\tilde{i}^*\tilde{B}$ gives rise to a connection on $\tilde{F}^*\tilde{B}$ which reduces to the canonical connection on $F^*\tilde{i}^*\tilde{B}$ under $\tilde{i}^*$.*

Proof Lift ∇_0 to a map of $W_2(k)$-modules $\nabla'_0 : \tilde{B} \to \tilde{B} \otimes \Omega^1_{\tilde{R}/W_2(k)}$. Then define a connection $\widetilde{\nabla}$ on $\tilde{B}$ as the pullback of ∇'_0 under $\tilde{F}$. Namely, for $f \otimes x \in \tilde{R} \otimes_{\tilde{F},\tilde{R}} \tilde{B}$ put

$$\widetilde{\nabla}(f \otimes x) = x \otimes df + f \cdot \tilde{F}^*(\nabla'_0(x)) \tag{2.10}$$

Since ∇'_0 modulo p is a connection, $\widetilde{\nabla}$ is actually a well-defined connection on $\tilde{F}^*\tilde{B}$. It does not depend on the choice of ∇'_0 because for a 1-form $\omega \in \Omega^1_{\tilde{R}/W_2(k)}$ the value of $\tilde{F}^*(\omega)$ depends only on $\tilde{i}^*\omega$ since $\tilde{i}^*\tilde{F}$ is zero on 1-forms. □

Applying the lemma to the underlying $\tilde{R}$-module $\tilde{B} = \bigoplus \tilde{A}_i$ of the given lifting and the connection ∇', we get a connection $\widetilde{\nabla}$. Since $\widetilde{\nabla}$ and $\widetilde{d}$ commute modulo p, we get the following $\tilde{R}$-linear map

$$\frac{[\widetilde{\nabla}, \widetilde{d}]}{p} : \tilde{F}^*\tilde{A}_i \to \tilde{i}_*F^*A_{i+1} \otimes \Omega^1_{\tilde{R}/W_2(k)} \tag{2.11}$$

We are now ready to prove the theorem. Put $\mathcal{F}_\bullet = cone(\tilde{i}_*F^*A_\bullet \overset{p}{\to} \tilde{F}^*\tilde{A}_\bullet)$. This is a complex of $\tilde{R}$-modules with terms

$$\mathcal{F}_i = \tilde{F}^*\tilde{A}_i \oplus \tilde{i}_*F^*A_{i+1}$$

and the differential given by $(x, y) \mapsto (d_{\tilde{A}}x + (-1)^i py, d_A y)$.

Let $r : \mathcal{F}_\bullet \to \tilde{i}_*F^*A_\bullet$ be the morphism which maps $(x, y) \in \mathcal{F}_i$ to the reduction of x modulo p in $\tilde{i}_*F^*A_i$. Note that r is a morphism of complexes because $p \in \tilde{R}$ acts by zero on $\tilde{i}_*F^*A_\bullet$.

Lemma 2.6 *(i) r is a quasi-isomorphism.*

(ii) Considering further $\mathcal{F}_\bullet$ as a complex of $\hat{R}$-modules, the canonical map $L\hat{i}^*\mathcal{F}_\bullet \to \hat{i}^*\mathcal{F}_\bullet$ is a quasi-isomorphism.

Proof (i) It is clear as r is term-wise surjective and its kernel is isomorphic to $cone(\tilde{i}_*F^*A_\bullet \overset{id}{\to} \tilde{i}_*F^*A_\bullet)$ which has zero cohomology.

(ii) Terms of $\mathcal{F}_\bullet$ are not flat over $\hat{R}$ so, a priori, there might be non-zero higher derived functors of $\hat{i}^*$. Let $\bigoplus \mathcal{A}_i$ be a lifting of the graded algebra $\bigoplus \tilde{F}^*\tilde{A}_i$ to a free graded algebra over $\hat{R}$. Pick also a lifting $\delta : \bigoplus \mathcal{A}_i \to \bigoplus \mathcal{A}_i[1]$ of the differential $\widetilde{d}$ (δ is not a differential anymore—its square need not be zero). It enables us to write down the following resolution of $\hat{i}_*A_\bullet$. Put

$$C_i = \mathcal{A}_i \oplus \mathcal{A}_{i+1}; d_C = \begin{pmatrix} \delta & (-1)^i p \\ (-1)^i \frac{\delta^2}{p} & \delta \end{pmatrix} \tag{2.12}$$

Note, that δ^2 is divisible by p because $d^2 = 0$ on $F^*A_\bullet$ and modules $\mathcal{A}_i$ are free over $\hat{R}$. Moreover, δ^2 is divisible by p^2 because $\widetilde{d}^2 = 0$ on $\tilde{A}$. Hence, $\frac{\delta^2}{p}$ is divisible by p, so the reduction maps $C_i \to \mathcal{F}_i$ give a morphism of complexes $\rho : C_\bullet \to \mathcal{F}$. Actually, ρ is a quasi-isomorphism. Indeed, composing it with r we get a term-wise surjective morphism of complexes with kernel given by $K_i = p\mathcal{A}_i \oplus \mathcal{A}_{i+1}$ and the differential restricted from $C_\bullet$. For any $(x, y) \in K_i$ such that $d_C(x, y) = 0$ we have $(x, y) = d_C(0, (-1)^{i-1}\frac{x}{p})$ so $K_\bullet$ is acyclic and $C_\bullet$ is an $\hat{R}$-flat resolution of $\mathcal{F}$. We get a commutative diagram

$$\begin{array}{ccc} L\hat{i}^*C_\bullet & \xrightarrow{\sim} & \hat{i}^*C_\bullet \\ \downarrow & & \downarrow \\ L\hat{i}^*\mathcal{F}_\bullet & \longrightarrow & \hat{i}^*\mathcal{F}_\bullet \end{array}$$

Left vertical arrow is a quasi-isomorphism because $C_\bullet \to \mathcal{F}_\bullet$ is a quasi-isomorphism and the right vertical arrow is an isomorphism because both $C_\bullet, \mathcal{F}_\bullet$ reduce modulo p to the complex $F^*A_\bullet \oplus F^*A_\bullet[1]$. Thus, the lower arrow is a quasi-isomorphism. □

Endow $\mathcal{F}$ with the structure of a DG algebra with connection. As a DG algebra $\mathcal{F}$ is the trivial square-zero extension of $\tilde{F}^*\tilde{A}_\bullet$ by the bimodule $\tilde{i}_*F^*A_\bullet[1]$. Explicitly, the product of $(x, y) \in \tilde{F}^*\tilde{A}_i \oplus \tilde{i}_*F^*A_{i+1}$ and $(x', y') \in \tilde{F}^*\tilde{A}_j \oplus \tilde{i}_*F^*A_{j+1}$ is defined to be $(xx', (-1)^j yx' + (-1)^i xy')$. To see that this algebra structure is compatible with the differential it is enough to check that $D : (x, y) \mapsto ((-1)^i py, 0)$ is a derivation because the diagonal part $(x, y) \mapsto (d_{\tilde{A}}x, d_A y)$ of $d_{\mathcal{F}}$ is a derivation by default. For $(x, y) \in \mathcal{F}_i$, $(x', y') \in \mathcal{F}_j$ we have $D((x, y)(x', y')) = ((-1)^{i+j} p((-1)^j yx' + (-1)^i xy'), 0) = ((-1)^i py, 0)(x', y') + (x, y)((-1)^j py', 0) = D((x, y))(x', y') + (x, y)D((x', y'))$.

Next, define a connection by

$$\nabla_{\mathcal{F}} = \begin{pmatrix} \widetilde{\nabla} & 0 \\ (-1)^i \frac{[\widetilde{\nabla}, \widetilde{d}]}{p} & \tilde{i}_*\nabla^{can} \end{pmatrix} : \tilde{F}^*\tilde{A}_i \oplus \tilde{i}_*F^*A_{i+1} \to (\tilde{F}^*\tilde{A}_i \oplus \tilde{i}_*F^*A_{i+1}) \otimes_{\tilde{R}} \Omega^1_{\tilde{R}/W_2(k)} \tag{2.13}$$

The entry below the diagonal is chosen so that this connection commutes with the differential on the DG algebra. To ensure that this connection respects the algebra structure it is, as above, enough to check that $(x, y) \mapsto (0, (-1)^i \frac{[\widetilde{\nabla}, \widetilde{d}]}{p} x)$ is a derivation which follows from $[\widetilde{\nabla}, \widetilde{d}]$ being a commutator of derivations. Finally, it is clear that our connection is integrable.

Also, quasi-isomorphism r is compatible with connection because $\widetilde{\nabla}$ reduces to ∇^{can} modulo p. In other words, $\hat{i}_{cris*}F^*A_\bullet$ is quasi-isomorphic to $(\mathcal{F}, \nabla_{\mathcal{F}})$. Thus, $T^{cris}(A_\bullet) \cong L\hat{i}^{cris*}((\mathcal{F}, \nabla_{\mathcal{F}}))$. By the virtue of Lemma 2.6, $L\hat{i}^{cris*}(\mathcal{F}, \nabla_{\mathcal{F}})$ is quasi-isomorphic to $(i^*\mathcal{F}, i^*\nabla_{\mathcal{F}})$. The latter complex of R-modules with integrable connection is given by

$$\begin{pmatrix} \nabla^{can} & 0 \\ (-1)^i \frac{[\widetilde{\nabla}, \widetilde{d}]}{p} & \nabla^{can} \end{pmatrix} : F^*A^i \oplus F^*A_{i+1} \to (F^*A_i \oplus F^*A_{i+1}) \otimes \Omega^1_{R/k} \tag{2.14}$$

So, Theorem 7 will follow after we check that

Lemma 2.7

$$\frac{[\widetilde{\nabla}, \widetilde{d}]}{p} = C^{-1}_{\tilde{R}, \tilde{F}}(\tilde{\kappa}) \tag{2.15}$$

Proof By definition $\tilde{\kappa} = [\nabla', d]$. Recall from the Lemma 2.5 that $\tilde{\nabla}_i$ on $\tilde{F}^*\tilde{A}_i$ is given by the formula $\widetilde{\nabla}_i(f \otimes x) = df \otimes x + f \otimes \tilde{F}^*(\nabla'_i(x))$. Hence,

$$\begin{aligned} \frac{[\widetilde{\nabla}, \widetilde{d}]}{p}(f \otimes x) &= \frac{df \otimes \widetilde{d}(\widetilde{x}) + f \otimes \tilde{F}^*(\nabla'_i(dx)) - df \otimes \widetilde{d}(\widetilde{x}) - f \otimes d\tilde{F}^*(\nabla'_i(x))}{p} \\ &= f \otimes \frac{\tilde{F}^*([\nabla'_i, d])}{p} \end{aligned} \tag{2.16}$$

which is exactly the inverse Cartier operator of the Kodaira–Spencer class by the definition (1.1). □

2.6 Proof of Theorem 6

We start with the following result.

Lemma 2.8 *Let $\hat{V}$ be a flat $\hat{R}$-module. For any $n \in \mathbb{Z}$, we have $\hat{H}^{2n-1}(C_p, \hat{V}^{\otimes p}) = 0$ and $\hat{H}^{2n}(C_p, \hat{V}^{\otimes p})$ is canonically isomorphic to $F^*\hat{i}^*\hat{V}$:*

$$\hat{H}^{2n}(C_p, \hat{V}^{\otimes p}) \xrightarrow{\sim} \hat{H}^{2n}(C_p, \hat{i}^*\hat{V}^{\otimes p}) \xrightarrow{\sim} F^*\hat{i}^*\hat{V}. \tag{2.17}$$

Here $\hat{V}^{\otimes p}$ denotes the p-th tensor power of $\hat{V}$ over $\hat{R}$ and C_p acts on $\hat{V}^{\otimes p}$ by cyclic permutations.

Proof The proof is similar to that of the Lemma 6.9 in [16]. We need to prove the vanishing of the odd cohomology groups and that the morphism

$$\hat{H}^{2n}(C_p, \hat{V}^{\otimes p}) \to \hat{H}^{2n}(C_p, \hat{i}^*\hat{V}^{\otimes p})$$

induced by the projection $\hat{V}^{\otimes p} \to \hat{i}^*\hat{V}^{\otimes p}$ is an isomorphism. Since any flat module is a filtered colimit of free modules, it is enough to prove these assertions for free modules. Assume that $\hat{V} = Span_{\hat{R}}(S)$ is freely generated by a set S. Then

$$\hat{V}^{\otimes p} = Span_{\hat{R}}(S^p) = Span_{\hat{R}}(S^p - S) \oplus Span_{\hat{R}}(S),$$

where $S \hookrightarrow S^p$ is the diagonal embedding. We have, that

$$\hat{H}^*(C_p, Span_{\hat{R}}(S^p - S)) = \hat{H}^*(C_p, \hat{i}^* Span_{\hat{R}}(S^p - S)) = 0$$

since the action of C_p on $S^p - S$ is free. On the other hand, the action of C_p on $Span_{\hat{R}}(S)$ is trivial. Hence, the result follows from the fact that $\hat{H}^{2n-1}(C_p, \mathbb{Z}_p) = 0$ and

$$\hat{H}^{2n}(C_p, \mathbb{Z}_p) \xrightarrow{\sim} \hat{H}^{2n}(C_p, \mathbb{F}_p) \xrightarrow{\sim} \mathbb{F}_p.$$

□

Let $\hat{V}_\bullet$ be a complex of flat $\hat{R}$-modules. Using the above lemma Kaledin's construction (reviewed in Sect. 2.2) extends *verbatim* to the present settings yielding a filtration

$$\tau^{dec}_{\leq \bullet} T(C_p, \hat{V}_\bullet^{\otimes p}) \subset T(C_p, \hat{V}_\bullet^{\otimes p})$$

and a canonical quasi-isomorphism of graded DG modules over the graded DG algebra $R \otimes \bigoplus_i \check{H}^i(C_p, \mathbb{Z}_p)[-i] \xrightarrow{\sim} R[u, u^{-1}]$

$$\bigoplus_i \mathrm{Gr}^\tau_i \, T(C_p, \hat{V}_\bullet^{\otimes p}) \xrightarrow{\sim} \bigoplus_i (F^* \hat{i}^* V_\bullet \otimes \check{H}^i(C_p, \mathbb{Z}_p)[-i]). \tag{2.18}$$

Observe a natural quasi-isomorphism:

$$\hat{i}^* T(C_p, \hat{V}_\bullet^{\otimes p}) \xrightarrow{\sim} T(C_p, \hat{i}^* \hat{V}_\bullet^{\otimes p}). \tag{2.19}$$

By construction, morphism (2.19) induces maps

$$\begin{aligned} &\hat{i}^* \tau^{dec}_{\leq \bullet} T(C_p, \hat{V}_\bullet^{\otimes p}) \xrightarrow{\sim} \tau^{dec}_{\leq \bullet} T(C_p, \hat{i}^* \hat{V}_\bullet^{\otimes p}) \\ &\hat{i}^* T_{[-n,0]}(C_p, \hat{V}_\bullet^{\otimes p}) \xrightarrow{\sim} T_{[-n,0]}(C_p, \hat{i}^* \hat{V}_\bullet^{\otimes p}) \end{aligned} \tag{2.20}$$

which are not quasi-isomorphisms in general. For example, $\hat{i}^* T_{[0,0]}(C_p, \hat{R}) = R \oplus R[-1]$, $T_{[0,0]}(C_p, R) = R$. However, we have the following result.

Lemma 2.9 *For every odd n the morphism (2.20) is a quasi-isomorphism.*

Proof Using the exactness of $T_{[-n,0]}(C_p, {}^{\otimes p})$ (Lemma 2.2), periodicity, and divissage it suffices to prove the result for $V_\bullet = \hat{R}$ and $n = -1$. In this case, the statement boils down to the fact that in the diagram

$$\begin{array}{ccc} \hat{R} & \xrightarrow{p} & \hat{R} \\ \downarrow & & \downarrow \\ R & \xrightarrow{0} & R \end{array}$$

the vertical morphism is a quasi-isomorphism of complexes. □

We are ready to prove the Theorem. First, we do this in the special case when $A_\bullet = \hat{i}^* \hat{A}_\bullet$, for some term-wise flat DG algebra $\hat{A}_\bullet$ over $\hat{R}$.

Proposition 2.10 *A lifting $\hat{A}_\bullet$ of $A_\bullet$ over $\hat{R}$ gives rise to a quasi-isomorphism of DG algebras with connections*

$$\mathcal{B}(A_\bullet) = T_{[-1,0]}(C_p, A_\bullet{}^{\otimes p}) \cong \hat{i}^* \hat{i}_* F^* A_\bullet = \mathcal{B}^{cris}(A_\bullet). \tag{2.21}$$

Proof By Lemma 2.9 and (2.18), we have

$$T_{[-1,0]}(C_p, A_\bullet{}^{\otimes p}) \cong \hat{i}^* T_{[-1,0]}(C_p, \hat{A}_\bullet^{\otimes p}) \cong \hat{i}^* T_{[0,0]}(C_p, \hat{A}_\bullet^{\otimes p}) \cong \hat{i}^* \hat{i}_* F^* A_\bullet.$$

The compatibility with connections is straightforward. □

Next, if $A_\bullet$ is arbitrary, apply the proposition to $\hat{i}^* \hat{i}_* A_\bullet$ letting $\hat{A}_\bullet$ be a semi-free resolution of $\hat{i}_* A_\bullet$. We get

$$T_{[-1,0]}(C_p, (\hat{i}^* \hat{i}_* A_\bullet)^{\otimes p}) \cong \hat{i}^* \hat{i}_* F^* \hat{i}^* \hat{i}_* A_\bullet. \tag{2.22}$$

Apply this to $R = k$, $\hat{R} = W(k)$, and $A_\bullet = k$. Then, using a remark from Sect. 2.1, $T_{[-1,0]}(C_p, (\hat{i}^* \hat{i}_* k)^{\otimes p})$ has the structure of an E_∞ algebra, which we denote by Q. Moreover, the projection $T_{[-1,0]}(C_p, (\hat{i}^* \hat{i}_* k)^{\otimes p}) \to T_{[-1,0]}(C_p, k)$ turns $T_{[-1,0]}(C_p, k)$ into an algebra over Q. Next, the quasi-isomorphism (2.22) can be promoted to a quasi-isomorphism of E_∞ algebras $Q \cong \hat{i}^* \hat{i}_* F^* \hat{i}^* \hat{i}_* k$. Therefore, the projection $\hat{i}^* \hat{i}_* F^* \hat{i}^* \hat{i}_* k \to \hat{i}^* \hat{i}_* F^* k$ makes $\hat{i}^* \hat{i}_* F^* k$ into an algebra over Q.

Lemma 2.11 *The Q-algebras $\hat{i}^* \hat{i}_* F^* k$ and $T_{[-1,0]}(C_p, k)$ are quasi-isomorphic.*

Proof There is a natural algebra morphism $s : F^* \hat{i}^* \hat{i}_* k \to \hat{i}^* \hat{i}_* F^* \hat{i}^* \hat{i}_* k$ which splits the projection $\hat{i}^* \hat{i}_* F^* \hat{i}^* \hat{i}_* k \to F^* \hat{i}^* \hat{i}_* k$. Indeed, for any k-algebra D, giving a splitting of the projection $\hat{i}^* \hat{i}_* D \to D$ is equivalent to giving a lifting of D over $W_2(k)$. The algebra $F^* \hat{i}^* \hat{i}_* k \xrightarrow{\sim} \hat{i}^* \hat{i}_* F^* k$[15] has a canonical lifting: $\hat{i}_* F^* k$. We define the desired quasi-isomorphism to be

$$F^* \hat{i}^* \hat{i}_* k \xrightarrow{s} \hat{i}^* \hat{i}_* F^* \hat{i}^* \hat{i}_* k \xrightarrow{\sim} T_{[-1,0]}(C_p, (\hat{i}^* \hat{i}_* k)^{\otimes p}) \to T_{[-1,0]}(C_p, k). \tag{2.23}$$

[15] This isomorphism is given by the lifting of the Frobenius on $W(k)$.

Let us check that the composition above is indeed a quasi-isomorphism. Clearly, this is an isomorphism on H^0. Assume that is 0 on H^{-1}. Then the same map would be 0 on H^{-1} for k replaced by any k-vector space V. In the other words, the following extension of polynomial functors $Vect_k \to Vect_k$ is split

$$0 \to F^*V \to (V^{\otimes p})_{C_p} \to (V^{\otimes p})^{C_p} \to F^*V \to 0.$$

This extension is equivalent to a similar one with C_p replaced by the symmetric group S_p

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^*V & \longrightarrow & (V^{\otimes p})_{C_p} & \xrightarrow{N_{C_p}} & (V^{\otimes p})^{C_p} & \longrightarrow & F^*V & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle av_p} & & \| & & \\ 0 & \longrightarrow & F^*V & \longrightarrow & (V^{\otimes p})_{S_p} & \xrightarrow{N_{S_p}} & (V^{\otimes p})^{S_p} & \longrightarrow & F^*V & \longrightarrow & 0 \end{array}$$

Here π_p is the projection and av_p is the averaging over left cosets of $C_p \subset S_p$ that is $av_p(x) = \frac{1}{(p-1)!}\sum_{gC_p \in S_p/C_p} g(x)$(note that this does not depend on the choice of representatives of cosets). However, it follows from Corollary 4.7 ($r = j = 1$) and Lemma 4.12 in [12] that the latter extension is non-split. Hence, the composition (2.23) is a quasi-isomorphism. □

Using the lemma we can complete the proof of the theorem.

$$\begin{aligned} T_{[-1,0]}(C_p, A_\bullet{}^{\otimes p}) &\xrightarrow{\sim} T_{[-1,0]}(C_p, (\hat{i}^*\hat{i}_*A_\bullet)^{\otimes p}) \otimes_Q T_{[-1,0]}(C_p, k) \\ &\xrightarrow{\sim} \hat{i}^*\hat{i}_*F^*\hat{i}^*\hat{i}_*A_\bullet \otimes_Q \hat{i}^*\hat{i}_*F^*k \xrightarrow{\sim} \hat{i}^*\hat{i}_*F^*A_\bullet. \end{aligned}$$

3 The Gauss–Manin connection on the (co-)periodic cyclic homology

In this section we review Getzler's and Kaledin's constructions of the Gauss–Manin connection, check that the two constructions agree, show that the Gauss–Manin connection preserves the conjugate filtration, and prove Theorem 5.

3.1 Getzler's construction

Let R be a smooth finitely generated commutative algebra over a field k, and let $A_\bullet$ be a semi-free differential graded algebra over R [9, §13.4]. Denote by $(CH_\bullet(A_\bullet, A_\bullet), b)$ the relative Hochschild chain complex of $A_\bullet$ over R[16] and by $CP_\bullet(A_\bullet) = (CH_\bullet(A_\bullet, A_\bullet)((u)), b+uB)$ the periodic cyclic complex. Getzler defined in [13] a connection on $CP_\bullet(A_\bullet)$

$$\nabla : CP_\bullet(A_\bullet) \to CP_\bullet(A_\bullet) \otimes_R \Omega^1_R.$$

[16] Here "relative over R" means that all the tensor products in the standard complex are taken over R.

His construction can be explained as follows: choose a connection $\nabla' : \bigoplus A_i \to \bigoplus A_i \otimes \Omega^1_R$ on the graded algebra $\bigoplus A_i$ satisfying the Leibnitz rule with respect to the multiplication on $\bigoplus A_i$. Then the commutator

$$\tilde{\kappa} = [\nabla', d] \in \prod Hom_R(A_i, A_{i+1}) \otimes \Omega^1_R$$

with the differential d on $A_\bullet$ commutes with d and it is a R-linear derivation of $A_\bullet$ (with values in $A_\bullet \otimes \Omega^1_R$) of degree 1.[17] As a derivation, $\tilde{\kappa}$ acts on $CH_\bullet(A_\bullet, A_\bullet)$ by the Lie derivative

$$\mathcal{L}_{\tilde{\kappa}} : CH_\bullet(A_\bullet, A_\bullet) \to CH_\bullet(A_\bullet, A_\bullet) \otimes \Omega^1_R[1], \quad [\mathcal{L}_{\tilde{\kappa}}, B] = 0$$

and the "interior product" operator

$$e_{\tilde{\kappa}} : CH_\bullet(A_\bullet, A_\bullet) \to CH_\bullet(A_\bullet, A_\bullet) \otimes \Omega^1_R[2].$$

The operators $\mathcal{L}_{\tilde{\kappa}}, e_{\tilde{\kappa}}, B$ satisfy the Cartan formula up to homotopy: there is a canonical operator

$$E_{\tilde{\kappa}} : CH_\bullet(A_\bullet, A_\bullet) \to CH_\bullet(A_\bullet, A_\bullet) \otimes \Omega^1_R, \quad [E_{\tilde{\kappa}}, B] = 0$$

such that $[e_{\tilde{\kappa}}, B] = \mathcal{L}_{\tilde{\kappa}} - [E_{\tilde{\kappa}}, b]$ [25, §4.1.8]. One defines

$$\nabla := \nabla' - u^{-1}\iota_{\tilde{\kappa}}, \tag{3.1}$$

where the first summand is the connection on $\bigoplus CP_i(A_\bullet)$ induced the connection ∇' on $\bigoplus A_i$ and $\iota_{\tilde{\kappa}} : \bigoplus CP_i(A_\bullet) \to \bigoplus CP_i(A_\bullet) \otimes \Omega^1_R$ is an $R((u))$ linear map given by the formula $\iota_{\tilde{\kappa}} = e_{\tilde{\kappa}} + uE_{\tilde{\kappa}}$. By construction, ∇ commutes with $b + uB$. Thus, it induces a connection on $CP_\bullet(A_\bullet)$. Getzler showed that up to homotopy ∇ does not depend on the choice of ∇'.[18] He also proved that the induced connection

[17] Denote by $Der^\bullet_R(A_\bullet)$ the DG Lie algebra of R-linear derivations of $A_\bullet$: $Der^i_R(A_\bullet)$ is the R-module of R-linear derivations of the graded algebra $\bigoplus A_i$; the differential on $Der^\bullet_R(A_\bullet)$ is given by the commutator with d. The cohomology class $\kappa \in H^1(Der^\bullet_R(A_\bullet)) \otimes \Omega^1_R$ of $\tilde{\kappa}$ does not depend on the choice of ∇'. (Indeed, any two connections differ by an element of $Der^0_R(A_\bullet)$.) Recall that the Hochschild cochain complex of $A_\bullet$ is quasi-isomorphic to the cone of the map $A_\bullet \to Der^\bullet_R(A_\bullet)$ which takes an element of A_i to the corresponding inner derivation. We refer to the image $\overline{\kappa}$ of κ under the induced morphism $H^1(Der^\bullet_R(A_\bullet)) \otimes \Omega^1_R \to HH^2(A_\bullet, A_\bullet) \otimes \Omega^1_R$ as the Kodaira–Spencer class of $A_\bullet$. In [13], Getzler chooses local coordinates $x_1, \cdots x_n$ on spec$R$. His notation (*loc. cit.*, §3) for $\overline{\kappa}$ coupled with $\frac{d}{dx_i}$ is $\mathcal{A}_i$.

[18] One can rephrase the above construction to make this fact obvious: let $Der^\bullet_k(R \to A_\bullet)$ be the DG Lie algebra of k-linear derivations which take the subalgebra $R \subset A_0$ to itself. Then $Der^\bullet_R(A_\bullet)$ is a Lie ideal in $Der^\bullet_k(R \to A_\bullet)$. Denote by $\widetilde{Der_k}(R)$ the cone of the morphism $Der^\bullet_R(A_\bullet) \to Der^\bullet_k(R \to A_\bullet)$. The restriction morphism $\widetilde{Der_k}(R) \to Der_k(R)$ a homotopy equivalence of DG Lie algebras: a choice of ∇' as above yields a homotopy inverse map. Next, we have a canonical morphism of complexes $\widetilde{Der_k}(R) \otimes_R CP_\bullet(A_\bullet) \to CP_\bullet(A_\bullet)$ given by the formulas $\theta \otimes c \mapsto u^{-1}\iota_\theta(c)$, for $\theta \in Der^\bullet_R(A_\bullet)$, and $\zeta \otimes c \mapsto \mathcal{L}_\zeta(c)$, for $\zeta \in Der^\bullet_R(R \to A_\bullet)$. This yields a morphism $Der_k(R) \otimes_R CP_\bullet(A_\bullet) \to CP_\bullet(A_\bullet)$ well defined up to homotopy.

on $HP_\bullet(A_\bullet)$ is flat. However, we do not know how to make ∇ on $CP_\bullet(A_\bullet)$ flat up to coherent homotopies.[19]

By construction, the connection ∇ satisfies the Griffiths transversality property with respect to the Hodge filtration $\mathcal{F}^i CP_\bullet(A_\bullet) := (u^i CH_\bullet(A_\bullet, A_\bullet)[[u]], b + uB)$:

$$\nabla : \mathcal{F}^i CP_\bullet(A_\bullet) \to \mathcal{F}^{i-1} CP_\bullet(A_\bullet) \otimes_R \Omega^1_R.$$

Thus, ∇ induces a degree one R-linear morphism of graded complexes

$$Gr^{\mathcal{F}}\nabla : Gr^{\mathcal{F}} CP_\bullet \to Gr^{\mathcal{F}} CP_\bullet \otimes_R \Omega^1_R.$$

Abusing terminology, we refer to $Gr^{\mathcal{F}}\nabla$ as the Kodaira–Spencer operator. Under the identification $Gr^{\mathcal{F}} CP_\bullet = (CH_\bullet(A_\bullet, A_\bullet)((u)), b)$ the Kodaira–Spencer operator is given by the formula

$$Gr^{\mathcal{F}}\nabla = u^{-1} e_{\tilde{\kappa}}.$$

3.2 Kaledin's definition

Following [14, §3], we extend the argument from Sect. 2.3 to give another definition of the Gauss–Manin connection which will be used in our proofs. Consider a two-term filtration on $p_{1*}p_2^*A_\bullet$ given as $I^0 = p_{1*}p_2^*A_\bullet$, $I^1 = A_\bullet \otimes \Omega^1_X$, $I^2 = 0$. Note that $I^0/I^1 = A_\bullet$. Taking tensor powers of the filtered complex $p_{1*}p_2^*A_\bullet$, we obtain a filtration on the cyclic object $(p_{1*}p_2^*A_\bullet)^\#$. This gives rise to a filtration I^i on the periodic cyclic complex of $p_{1*}p_2^*A_\bullet$ such that $I^0 CP_\bullet(p_{1*}p_2^*A_\bullet)/I^1 CP_\bullet(p_{1*}p_2^*A_\bullet) = CP_\bullet(A_\bullet)$. So we get a diagram with the upper row being a distinguished triangle

$$\begin{array}{ccccc} I^1/I^2 & \xrightarrow{i} & I^0/I^2 & \xrightarrow{\pi} & CP_\bullet(A_\bullet) \longrightarrow \\ & & \downarrow m & & \\ & & p_{1*}p_2^* CP_\bullet(A_\bullet) & & \end{array}$$

Lemma 3.1 *$I^1 CP_\bullet(p_{1*}p_2^*A_\bullet)/I^2 CP_\bullet(p_{1*}p_2^*A_\bullet)$ is contractible.*

Proof The cyclic object $I^1(p_{1*}p_2^*A_\bullet)^\#/I^2(p_{1*}p_2^*A_\bullet)^\#$ is isomorphic to the tensor product of $A^\# \otimes_k \Omega^1$ and a cyclic object Q, which does not depend on R and $A_\bullet$. In fact, Q is the cyclic object representable by $[1] \in \Lambda$: $Q([n])$ is the k-vector space spanned by the set $Mor_\Lambda([1], [n])$. In particular, the cyclic homology of Q is k. by the Künneth formula [18] it follows that action of the periodicity morphism u on the cyclic complex of $I^1(p_{1*}p_2^*A_\bullet)^\#/I^2(p_{1*}p_2^*A_\bullet)^\#$ is homotopic to 0.[20] Thus, periodic cyclic complex of $I^1(p_{1*}p_2^*A_\bullet)^\#/I^2(p_{1*}p_2^*A_\bullet)^\#$ is contractible. □

[19] The problem is that, in general, the canonical morphism $\widetilde{Der_k(R)} \otimes_R CP_\bullet(A_\bullet) \to CP_\bullet(A_\bullet)$ is not a Lie algebra action.

[20] Here is another interpretation of Q. Consider the cyclic object corresponding to the algebra $k[\epsilon]$ of dual numbers. Introduce a grading on $k[\epsilon]$ so that ϵ has degree 1. Then Q is precisely degree 1 summand in

Hence, π is a quasi-isomorphism, and the connection is defined as $\nabla = m\pi^{-1} : CP_\bullet(A_\bullet) \to p_{1*}p_2^*CP_\bullet(A_\bullet)$

Proposition 3.2 *Kaledin's connection is equal to Getzler's connection as a morphism* $CP_\bullet(A_\bullet) \to p_{1*}p_2^*CP_\bullet(A_\bullet)$ *in the derived category.*

Proof We will show that Getzler's formula comes from a section of π on the level of complexes.

∇' gives rise to a section φ of $\pi : \bigoplus CP_i(p_{1*}p_2^*A_\bullet) \to \bigoplus CP_i(A_\bullet)$ because ∇' yields a connection on any $A^{i_1} \otimes \cdots \otimes A^{i_k}$ by the Leibniz rule. Note that

$$\begin{aligned}[][\varphi, b](a_0 \otimes \cdots \otimes a_n) &= (\sum 1 \otimes \cdots \otimes \overset{i}{\nabla'} \otimes \cdots \otimes 1)(\sum a_0 \otimes \cdots \otimes da_i \otimes \cdots \otimes a_n + \\ &+ \sum (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) - b(\sum a_0 \otimes \cdots \otimes \nabla' a_i \otimes \cdots \otimes a_n) \\ &= \sum a_0 \otimes \cdots \otimes (\nabla' d - d\nabla')a_i \otimes \cdots \otimes a_n\end{aligned} \tag{3.2}$$

This computation shows that $[\varphi, b + uB] = \mathcal{L}_{\tilde{\kappa}}$ (because, clearly $[\varphi, B] = 0$) where $\mathcal{L}_{\tilde{\kappa}} : CP_\bullet(A_\bullet) \to CP_\bullet(I^1(p_{1*}p_2^*A_\bullet)^\#/I^2(p_{1*}p_2^*A_\bullet)^\#)$. By [25], §4.1.11 we have $[u^{-1}\iota_{\tilde{\kappa}}, b + uB] = \mathcal{L}_{\tilde{\kappa}}$. Hence, $\varphi - u^{-1}\iota_{\tilde{\kappa}}$ is a morphism of complexes and a section of π so, in the derived category, $\pi^{-1} = \varphi - u^{-1}\iota_{\tilde{\kappa}}$. Applying m we get precisely the map (3.1) considered as a map $CP_\bullet(A_\bullet) \to p_{1*}p_2^*CP_\bullet(A_\bullet)$. □

3.3 Proof of Theorem 5

As explained in [17, §3.3 and §5.1] we have a canonical morphism

$$\mathcal{B}(A_\bullet)^\natural \to \pi^\flat_{(-2(p-1),0]} i_p^* A_\bullet{}^\natural \tag{3.3}$$

in $D(\Lambda, R)$. This induces a morphism of cyclic complexes

$$\begin{aligned} CC_\bullet(\mathcal{B}(A_\bullet)) = CC_\bullet(\mathcal{B}(A_\bullet)^\natural) &\to CC_\bullet(\pi^\flat_{(-2(p-1),0]} i_p^* A_\bullet{}^\natural), \\ V_{-1}CC_\bullet(\mathcal{B}(A_\bullet)) \to CC_\bullet(\pi^\flat_{(-2(p-1),-1]} i_p^* A_\bullet{}^\natural) &\xrightarrow{\sim} V_{[-p+2,-1]}\overline{C}P_\bullet(A_\bullet) \end{aligned} \tag{3.4}$$

We have to check that (3.4) factors through $V_{[-p+2,-1]}CC_\bullet(\mathcal{B}(A_\bullet))$ and that the resulted morphism is a quasi-isomorphism.

Recall that any complete resolution (in the sense of Sect. 2.1) has the structure of an algebra over an E_∞ operad (see [22], §2, for an explicit construction of this operad, or [28], Appendix S, for a more abstract approach). This makes $\mathcal{B}(R)^\natural$ and $\pi^\flat_{(-2(p-1),0]} i_p^* R^\natural$ into E_∞ algebras in the category of complexes over $Fun(\Lambda, R)$ and

$k[\epsilon]^\#$. Thus, the Lie derivative $\mathcal{L}_\theta$ with respect to the derivation $\theta = \epsilon \frac{d}{d\epsilon}$ acts on the cyclic complex of Q as the identity morphism. On the other hand, by the Cartan formula, for any derivation θ, the action of $u\mathcal{L}_\theta$ on the cyclic complex is homotopic to 0.

$\mathcal{B}(A_\bullet)^\natural$, $\pi^\flat_{(-2(p-1),0]} i_p^* A_\bullet{}^\natural$ are modules over these algebras respectively. The morphism 3.3 can be promoted to

$$\mathcal{B}(A_\bullet)^\natural \overset{L}{\otimes}_{\mathcal{B}(R)^\natural} \pi^\flat_{(-2(p-1),0]} i_p^* R^\natural \to \pi^\flat_{(-2(p-1),0]} i_p^* A_\bullet{}^\natural \tag{3.5}$$

Moreover, if we endow the left-hand side of (3.5) with the filtration induced by the canonical filtration on $\pi^\flat_{(-2(p-1),0]} i_p^* R^\natural$ and the right-hand side with τ^{dec}, then (3.5) is a filtered quasi-isomorphism. Pass to mixed complexes[21]:

$$C(\mathcal{B}(A_\bullet)) \overset{L}{\otimes}_{C(\mathcal{B}(R))} C(\pi^\flat_{(-2(p-1),0]} i_p^* R^\natural) \to C(\pi^\flat_{(-2(p-1),0]} i_p^* A_\bullet{}^\natural) \tag{3.6}$$

Now Theorem 5 follows from an easy Lemma below.

Lemma 3.3 *The homomorphism of E_∞ algebras*

$$C(\mathcal{B}(R)) \to C(\pi^\flat_{(-2(p-1),0]} i_p^* R^\natural)$$

induces a quasi-isomorphism

$$\tau_{(-2(p-1),0]} C(\mathcal{B}(R)) \xrightarrow{\sim} C(\pi^\flat_{(-2(p-1),0]} i_p^* R^\natural).$$

4 The local monodromy theorem

In this section we prove Theorem 3 in a stronger and more general form. We start by recalling some results of Katz from [19].

4.1 Katz's Theorem

Let S be a smooth geometrically connected complete curve over a field K of characteristic 0, $K(S)$ the field of rational functions on S, and let E be a finite-dimensional vector space over $K(S)$ with a K-linear connection

$$\nabla : E \to E \otimes \Omega^1_{K(S)/K}.$$

Recall that ∇ is said to have regular singularities if E can be extended to a vector bundle $\mathcal{E}$ over S such that ∇ extends to a connection on $\mathcal{E}$, which has at worst simple poles at some finite closed subset $D \subset S$:

$$\overline{\nabla} : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_S(\log D).$$

[21] The Hochschild chains functor C is viewed here as the symmetric ∞-functor from the derived category of cyclic modules to the derived category of chain complexes with S^1-action [28, Appendix S].

One says that the local monodromy of (E, ∇) is quasi-unipotent if the pair $(\mathcal{E}, \overline{\nabla})$ as above can be chosen so that the residue of $\overline{\nabla}$

$$\operatorname{Res} \overline{\nabla} : \mathcal{E}_{|D} \to \mathcal{E}_{|D}$$

has rational eigenvalues.[22] Let $\operatorname{Res} \overline{\nabla} = D + N$, with $[D, N] = 0$, be the Jordan decomposition of $\operatorname{Res} \overline{\nabla}$ as a sum of a semi-simple operator D and a nilpotent operator N. If the local monodromy of (E, ∇) is quasi-unipotent we say its exponent of nilpotence is $\leq \nu$ if $N^{\nu} = 0$.

If $K = \mathbb{C}$ then the category of finite-dimensional $K(S)$-vector spaces with K-linear connections with regular singularities and quasi-unipotent local monodromy is equivalent to the category of local systems (in the topological sense) over S take off finitely many points whose local monodromy around every puncture is quasi-unipotent (i.e., all its eigenvalues are roots of unity). The exponent of nilpotence of local monodromy is the size of its largest Jordan block.

In [19, Th. 13.0.1], Katz proved the following result.

Theorem (Katz) *Let C be a smooth scheme of relative dimension 1 over a domain R which is finitely generated (as a ring) over $\mathbb{Z}$, with fraction field K of characteristic zero. Assume that the generic fiber of C is geometrically connected. Let (M, ∇) be a locally free $\mathcal{O}_C$-module with a connection $\nabla : M \to M \otimes \Omega^1_{C/R}$. Assume that (M, ∇) is globally nilpotent of nilpotence ν, that is, for any prime number p, the $\mathcal{O}_{C \otimes \mathbb{F}_p}$-module $M \otimes \mathbb{F}_p$ with $R \otimes \mathbb{F}_p$-linear connection admits a filtration*

$$0 = V_0(M \otimes \mathbb{F}_p) \subset \cdots \subset V_\nu(M \otimes \mathbb{F}_p) = M \otimes \mathbb{F}_p$$

such that the p-curvature of each successive quotient V_i / V_{i-1} is 0. Then the pullback $M \otimes_{\mathcal{O}(C)} K(C)$ of M to the generic point of C has regular singularities and quasi-unipotent local monodromy of exponent $\leq \nu$.

4.2 Monodromy Theorem

Now we can prove the main result of this section.

Theorem 8 *Let $A_\bullet$ be a smooth and proper DG algebra over $K(S)$ and let d be a non-negative integer such that*

$$HH_m(A_\bullet, A_\bullet) = 0, \text{ for every } m \text{ with } |m| > d. \tag{4.1}$$

Then the Gauss–Manin connection on the relative periodic cyclic homology $HP_*(A_\bullet)$ has regular singularities and quasi-unipotent local monodromy of exponent $\leq d + 1$.

[22] One can show (see e.g., [19], §12) that if $\operatorname{Res} \overline{\nabla}$ has rational eigenvalues for one extension then it has rational eigenvalues for every extension $(\mathcal{E}, \overline{\nabla})$ of (E, ∇).

Proof Using Theorem 1 from [30], there exists a finitely generated $\mathbb{Z}$-algebra $R \subset K$, a smooth affine scheme C of relative dimension 1 over R with a geometrically connected generic fiber, and a smooth proper DG algebra $B_\bullet$ over $\mathcal{O}(C)$ together with an open embedding $C \otimes_R K \hookrightarrow S$ of curves over K and a quasi-isomorphism $A_\bullet = B_\bullet \otimes_{\mathcal{O}(C)} K(S)$ of DG algebras over $K(S)$. We can choose $B_\bullet$ to be term-wise flat over $\mathcal{O}(C)$. Since the Hochschild homology $\bigoplus_i HH_i(B_\bullet, B_\bullet)$ of a smooth proper DG algebra is finitely generated over $\mathcal{O}(C)$ replacing C by a dense open subscheme we may assume that $\bigoplus_i HH_i(B_\bullet, B_\bullet)$ and $HP_*(B_\bullet, B_\bullet)$ are free $\mathcal{O}(C)$-modules of finite rank. It follows that

$$HH_i(B_\bullet, B_\bullet) \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\sim} HH_i(B_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p, B_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p),$$
$$HH_i(B_\bullet, B_\bullet) \otimes_{\mathcal{O}(C)} K(S) \xrightarrow{\sim} HH_i(A_\bullet, A_\bullet).$$

Using the Hodge-to-de Rham spectral sequence it follows that the periodic cyclic homology also commutes with the base change. Then by Cor. 1.2 $(M, \nabla) = (HP_*(B_\bullet), \nabla_{GM})$ satisfies the assumptions of the theorem of Katz with $\nu = d + 1$ and we are done. □

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References

1. Berthelot, P., Ogus, A.: Notes on Crystalline Cohomology. Princeton University Press, Princeton, NJ (1978)
2. Berthelot, P., Ogus, A.: Notes on Crystalline Cohomology. University of Tokyo Press, Tokyo (1978)
3. Bondal, A., van den Bergh, M.: Generators and representability of functors in commutative and non-commutative geometry. Mosc. Math. J. **3**(1), 1–36 (2003). 258
4. Braverman, A., Bezrukavnikov, R.: Geometric Langlands correspondence for D-modules in prime characteristic: the $GL(n)$ case. Pure Appl. Math. Q. **3**(1), 153–179 (2007). (Special Issue: In honor of R. MacPherson. Part 3)
5. Calaque, D., Van den Bergh, M.: Hochschild cohomology and Atiyah classes. Adv. Math. **224**(5), 1839–1889 (2010)
6. Cassels, J., Fröhlich, A. (eds.): Algebraic Number Theory. Academic Press, New York (1967)
7. Deligne, P.: Theorie de Hodge, II. Publ. Math. Inst. Hautes Études Sci. No. **40**, 5–57 (1971)
8. Deligne, P., Illusie, L.: Relèvements modulo p^2 et décomposition du complexe de de Rham. Invent. Math. **89**(2), 247–270 (1987)
9. Drinfeld, V.: DG quotients of DG categories. J. Algebra **272**(2), 643–691 (2004)
10. Faltings, G.: Crystalline cohomology and p-adic Galois-representations. In: Algebraic Analysis, Geometry, and Number Theory (Baltimore, MD, 1988), pp. 25–80. Johns Hopkins University Press, Baltimore, MD (1989)
11. Fontaine, J.-M., Laffaille, G.: Construction de représentations p-adiques. Ann. Sci. Ecole Norm. Sup. (4) **15**(4), 547–608 (1982)
12. Friedlander, E.M., Suslin, A.: Cohomology of finite group schemes over a field. Invent. Math. **127**(2), 209–270 (1997)

13. Getzler, E.: Cartan homotopy formulas and the Gauss–Manin connection in cyclic homology. Quantum Deformations of Algebras and Their Representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), Israel Mathematics Conference Proceedings, vol. 7, pp 65–78. Bar-Ilan University, Ramat Gan (1993)
14. Kaledin, D.: Cyclic Homology with Coefficients, Algebra, Arithmetic, and Geometry: in Honor of Yu. I. Manin, vol. II, pp. 23–47. Progress in Mathematics, vol 270. Birkhäuser Inc., Boston (2009)
15. Kaledin, D.: Bokstein homomorphism as a universal object. http://lanl.arxiv.org/pdf/1510.06258v1
16. Kaledin, D.: Co-periodic cyclic homology. arXiv:1509.08784
17. Kaledin, D.: Spectral sequences for cyclic homology. In: Auroux, D., Katzarkov, L., Pantev, T., Soibelman, Y., Tschinkel, Y. (eds.) Algebra, Geometry and Physics in the 21st Century (Kontsevich Festschrift). Progress in Mathematics, vol. 324, pp. 99–129. Birkhäuser (2017)
18. Kassel, C.: A Künneth formula for the cyclic cohomology of $\mathbb{Z}/2$-graded algebras. Math. Ann. **275**, 683–699 (1986)
19. Katz, N.: Nilpotent connections and the monodromy theorem: applications of a result of Turrittin. Inst. Hautes Études Sci. Publ. Math. No. **39**, 175–232 (1970)
20. Katz, N.: Algebraic solutions of differential equations (p-curvature and the Hodge filtration). Invent. Math. **18**, 1–118 (1972)
21. Keller, B.: On differential graded categories. International Congress of Mathematicians, vol. II, pp. 151–190. European Mathematical Society, Zürich (2006)
22. Langer, M.: Dyer–Lashof operations on Tate cohomology of finite groups. Algebr. Geom. Topol. **12**, 829–865 (2012)
23. Lazard, M.: Commutative Formal Groups. Lecture Notes in Mathematics, no. 443. Springer, Berlin (1975)
24. Lurie, J.: Rational and p-adic homotopy theory. http://www.math.harvard.edu/~lurie/
25. Loday, J.: Cyclic Homology, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301. Springer, Berlin (1992)
26. Lowen, W.: Obstruction theory for objects in abelian and derived categories. Commun. Algebra **33**(9), 3195–3223 (2005)
27. Mathew, A.: Kaledin's degeneration theorem and topological Hochschild homology. arXiv:1710.09045
28. Nikolaus, T., Scholze, P.: On topological cyclic homology. arXiv:1707.01799
29. Ogus, A., Vologodsky, V.: Nonabelian Hodge theory in characteristic p. Publ. Math. Inst. Hautes Études Sci. No. **106**, 1–138 (2007)
30. Toën, B.: Anneaux de définition des dg-algèbres propres et lisses. Bull. Lond. Math. Soc. **40**(4), 642–650 (2007)