Brascamp–Lieb-Type Inequalities on Weighted Riemannian Manifolds with Boundary

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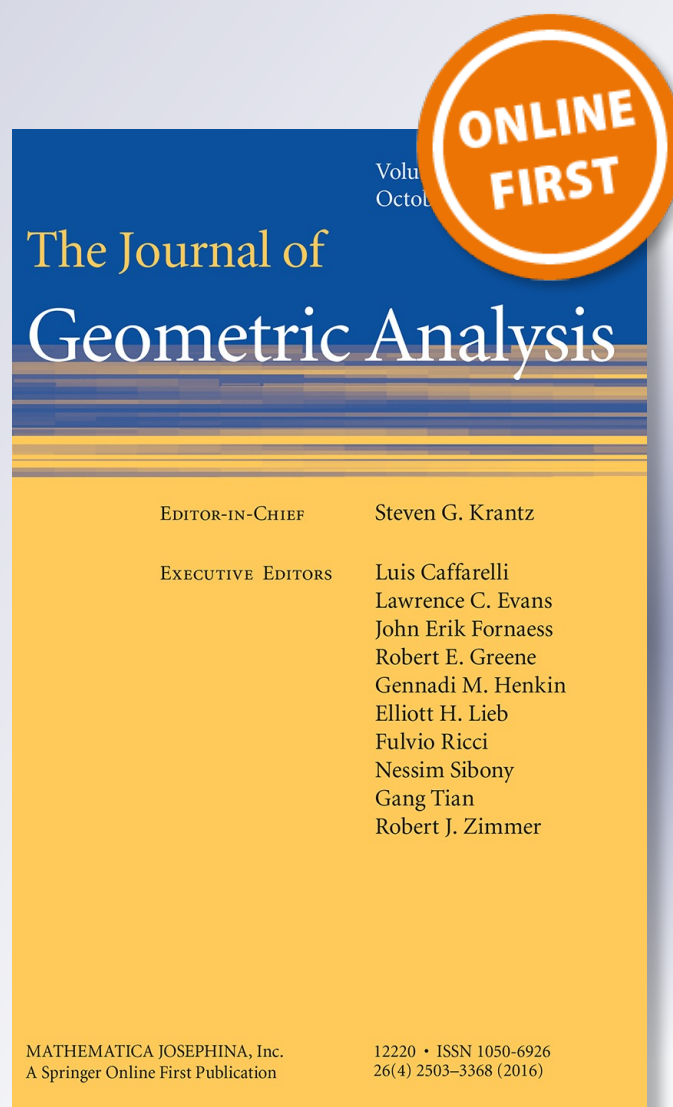
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Abstract It is known that by dualizing the Bochner–Lichnerowicz–Weitzenböck formula, one obtains Poincaré-type inequalities on Riemannian manifolds equipped with a density, which satisfy the Bakry–Émery Curvature-Dimension condition (combining a lower bound on its generalized Ricci curvature and an upper bound on its generalized dimension). When the manifold has a boundary, an appropriate generalization of the Reilly formula may be used instead. By systematically dualizing this formula for various combinations of boundary conditions of the domain (convex, mean-convex) and the function (Neumann, Dirichlet), we obtain new Brascamp–Lieb-type inequalities on the manifold. All previously known inequalities of Lichnerowicz, Brascamp–Lieb, Bobkov–Ledoux, and Veysseire are recovered, extended to the Riemannian setting and generalized into a single unified formulation, and their appropriate versions in the presence of a boundary are obtained. Our framework allows to encompass the entire class of Borell's convex measures, including heavy-tailed measures, and extends the latter class to weighted-manifolds having negative generalized dimension.

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1 Introduction

Throughout the paper, we consider a compact *weighted-manifold* (M, g, μ), namely a compact smooth connected and oriented n-dimensional Riemannian manifold (M, g) with boundary ∂M, equipped with a measure:

$$\mu = \exp(-V)d\mathrm{Vol}_M,$$

where Vol_M is the Riemannian volume form on M and $V \in C^2(M)$ is twice continuously differentiable. The boundary ∂M is assumed to be a C^2 manifold with outer unit-normal $\nu = \nu_{\partial M}$. The corresponding symmetric diffusion operator with invariant measure μ, which is called the weighted-Laplacian, is given by

$$L = L_{(M,g,\mu)} := \exp(V)\mathrm{div}(\exp(-V)\nabla) = \Delta - \langle \nabla V, \nabla \rangle ,$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric g, $\nabla = \nabla_g$ denotes the Levi–Civita connection, $\mathrm{div} = \mathrm{div}_g = tr(\nabla \cdot)$ denotes the Riemannian divergence operator, and $\Delta = \mathrm{div}\nabla$ is the Laplace-Beltrami operator. Indeed, note that with these generalized notions, the usual integration by parts formula is satisfied for $f, g \in C^2(M)$:

$$\int_M L(f)g\,d\mu = \int_{\partial M} f_\nu g\,d\mu_{\partial M} - \int_M \langle \nabla f, \nabla g \rangle \,d\mu$$
$$= \int_{\partial M} (f_\nu g - g_\nu f)\,d\mu_{\partial M} + \int_M L(g)f\,d\mu,$$

where $u_\nu = \nu \cdot u$ and $\mu_{\partial M} := \exp(-V)d\mathrm{Vol}_{\partial M}$.

The second fundamental form $\mathrm{II} = \mathrm{II}_{\partial M}$ of $\partial M \subset M$ at $x \in \partial M$ is as usual (up to sign) defined by $\mathrm{II}_x(X, Y) = \langle \nabla_X \nu, Y \rangle$, $X, Y \in T\partial M$. The quantities

$$H_g(x) := tr(\mathrm{II}_x), \quad H_\mu(x) := H_g(x) - \langle \nabla V(x), \nu(x) \rangle$$

are called the Riemannian mean-curvature and *generalized* mean-curvature of ∂M at $x \in \partial M$, respectively. It is well known that H_g governs the first variation of $\mathrm{Vol}_{\partial M}$ under the normal-map $t \mapsto \exp(t\nu)$, and similarly H_μ governs the first variation of $\exp(-V)d\mathrm{Vol}_{\partial M}$ in the weighted-manifold setting, see e.g. [30].

In the purely Riemannian setting, it is classical that positive lower bounds on the Ricci curvature tensor Ric_g and upper bounds on the topological dimension n play a fundamental role in governing various Sobolev-type inequalities on (M, g), see e.g. [7,9,10,22,52] and the references therein. In the weighted-manifold setting, the pertinent information on *generalized* curvature and *generalized* dimension may be incorporated into a single tensor, which was put forth by Bakry and Émery [1,2] following Lichnerowicz [24,25]. The N-dimensional Bakry–Émery Curvature tensor ($N \in (-\infty, \infty]$) is defined as (setting $\Psi = \exp(-V)$):

$$\mathrm{Ric}_{\mu,N} := \mathrm{Ric}_g + \nabla^2 V - \frac{1}{N-n}dV \otimes dV = \mathrm{Ric}_g - (N-n)\frac{\nabla^2 \Psi^{\frac{1}{N-n}}}{\Psi^{\frac{1}{N-n}}}, \qquad (1.1)$$

and the Bakry–Émery Curvature-Dimension condition $CD(\rho, N)$, $\rho \in \mathbb{R}$, is the requirement that as 2 tensors on M:

$$\text{Ric}_{\mu,N} \geq \rho g.$$

Here $\nabla^2 V$ denotes the Riemannian Hessian of V. Note that the case $N = n$ is only defined when V is constant, i.e. in the classical non-weighted Riemannian setting where μ is proportional to Vol_M, in which case $\text{Ric}_{\mu,n}$ boils down to Ric_g. When $N = \infty$ we set

$$\text{Ric}_\mu := \text{Ric}_{\mu,\infty} = \text{Ric}_g + \nabla^2 V.$$

It is customary to only treat the case when $N \in [n, \infty]$, with the interpretation that N is an upper bound on the "generalized dimension" of the weighted-manifold (M, g, μ); however, our method also applies with no extra effort to the case when $N \in (-\infty, 0]$, and so our results are treated in this greater generality, which in the Euclidean setting encompasses the entire class of Borell's convex (or "$1/N$-concave") measures [5] (cf. [4,6]). It will be apparent that the more natural parameter is actually $1/N$, with $N = \infty, 0$ interpreted as $1/N = 0, -\infty$, respectively, and so our results hold in the range $1/N \in [-\infty, 1/n]$. As $dV \otimes dV$ appearing in (1.1) is a positive semi-definite tensor, the $CD(\rho, N)$ condition is clearly monotone in $\frac{1}{N-n}$ and hence in $\frac{1}{N}$ in the latter range, so for all $N_+ \in [n, \infty]$, $N_- \in (-\infty, 0]$:

$$CD(\rho, n) \Rightarrow CD(\rho, N_+) \Rightarrow CD(\rho, \infty) \Rightarrow CD(\rho, N_-) \Rightarrow CD(\rho, 0);$$

note that $CD(\rho, 0)$ is the weakest condition in this hierarchy. It seems that outside the Euclidean setting, this extension of the Curvature–Dimension condition to the range $N \leq 0$ has not attracted much attention in the weighted-Riemannian and more general metric-measure space setting (cf. [27,44]); an exception is the work of Ohta and Takatsu [40,41]. We expect this gap in the literature to be quickly filled (in fact, concurrently to posting our work on the arXiv, Ohta [39] has posted a first attempt of a systematic treatise of the range $N \leq 0$, and subsequently other authors have also begun treating this extended range [14,17,31,32,50]).

A convenient equivalent form of the $CD(\rho, N)$ condition may be formulated as follows. Let Γ_2 denote the iterated carré-du-champ operator of Bakry–Émery:

$$\Gamma_2(u) := \left\| \nabla^2 u \right\|^2 + \langle \text{Ric}_\mu \nabla u, \nabla u \rangle,$$

where $\left\| \nabla^2 u \right\|$ denotes the Hilbert-Schmidt norm of $\nabla^2 u$. Then the $CD(\rho, N)$ condition is equivalent when $1/N \in (-\infty, 1/n]$ (see [1, Sect. 6] for the case $N \in [n, \infty]$ or Lemma 2.3 in the general case) to the requirement that

$$\Gamma_2(u) \geq \rho \, |\nabla u|^2 + \frac{1}{N}(Lu)^2 \quad \forall u \in C^2(M). \tag{1.2}$$

Denote by $\mathcal{S}_0(M)$ the class of functions u on M which are C^2 smooth in the interior of M and C^1 smooth on the entire compact M. Denote by $\mathcal{S}_N(M)$ the subclass of functions which in addition satisfy that u_ν is C^1 smooth on ∂M. The main tool we employ in this work is the following:

Theorem 1.1 [Generalized Reilly Formula] *For any function $u \in \mathcal{S}_N(M)$:*

$$\int_M (Lu)^2 \mathrm{d}\mu = \int_M \left\| \nabla^2 u \right\|^2 \mathrm{d}\mu + \int_M \langle Ric_\mu \nabla u, \nabla u \rangle \mathrm{d}\mu + \int_{\partial M} H_\mu (u_\nu)^2 \mathrm{d}\mu_{\partial M}$$
$$+ \int_{\partial M} \langle II_{\partial M} \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \, d\mu_{\partial M} - 2 \int_{\partial M} \langle \nabla_{\partial M} u_\nu, \nabla_{\partial M} u \rangle \, \mathrm{d}\mu_{\partial M}. \tag{1.3}$$

Here $\nabla_{\partial M}$ denotes the Levi-Civita connection on ∂M with its induced Riemannian metric.

This natural generalization of the (integrated) Bochner–Lichnerowicz–Weitzenböck formula for manifolds with boundary was first obtained by Reilly [43] in the classical Riemannian setting ($\mu = \text{Vol}_M$). The version above is a modification (obtained by integrating by parts on ∂M) of a previous version due to Ma and Du [28]. For completeness, we sketch in Sect. 2 the proof of the version (1.3) which we require for deriving our results.

It is known that by dualizing the Bochner–Lichnerowicz–Weitzenböck formula, various Poincaré-type inequalities such as the Lichnerowicz [23], Brascamp–Lieb [6,20] and Veysseire [47] inequalities may be obtained under appropriate bounds on curvature and dimension. Recently, heavy-tailed versions of the Brascamp–Lieb inequalities have been obtained in the Euclidean setting by Bobkov–Ledoux [4] and sharpened by Nguyen [37]. By employing the generalized Reilly formula, we unify, extend and generalize many of these previously known results to various new combinations of boundary conditions on the domain (locally convex, mean-convex) and the function (Neumann, Dirichlet) in the weighted-Riemannian setting. We mention in passing another celebrated application of the latter duality argument in the Complex setting, namely Hörmander's L^2 estimate [13], but we refrain from attempting to generalize it here; further more recent applications may be found in [3,12,15,16,20].

Given a finite measure ν on a measurable space Ω, and a ν-integrable function f on Ω, we denote

$$\fint_\Omega f d\nu := \frac{1}{\nu(\Omega)} \int_\Omega f \mathrm{d}\nu, \quad \text{Var}_\nu(f) := \int_\Omega \left(f - \fint_\Omega f \mathrm{d}\nu \right)^2 \mathrm{d}\nu.$$

The following theorem, obtained in Sect. 3, is the main result of this work:

Theorem 1.2 (Generalized dimensional Brascamp–Lieb with boundary) *Assume that $Ric_{\mu,N} > 0$ on M with $1/N \in (-\infty, 1/n]$. Then for any $f \in C^1(M)$:*

(1) *(Neumann Dimensional Brascamp–Lieb inequality on locally convex domain)*
Assume that $II_{\partial M} \geq 0$ (M is locally convex). Then,

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle Ric_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu.$$

(2) *(Dirichlet Dimensional Brascamp–Lieb inequality on generalized mean-convex domain)*
Assume that $H_\mu \geq 0$ (M is generalized mean-convex), $f \equiv 0$ on $\partial M \neq \emptyset$. Then,

$$\frac{N}{N-1} \int_M f^2 d\mu \leq \int_M \left\langle Ric_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu.$$

(3) *(Neumann Dimensional Brascamp–Lieb inequality on strictly generalized mean-convex domain)*
Assume that $H_\mu > 0$ (M is strictly generalized mean-convex). Then for any $C \in \mathbb{R}$:

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle Ric_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu + \int_{\partial M} \frac{1}{H_\mu} \left(f - C\right)^2 d\mu_{\partial M}.$$

In other words,

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle Ric_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu + Var_{\mu_{\partial M}/H_\mu}(f|_{\partial M}).$$

Restricting to Euclidean space ($\mathbb{R}^n$, $|\cdot|$) and setting $N = \infty$ in Case (1), the tensor $Ric_{\mu,\infty}$ boils down to the (Euclidean) Hessian $\nabla^2 V$, and we recover the celebrated Poincaré-type inequality obtained by H. J. Brascamp and E. H. Lieb [6] as an infinitesimal version of the Prekopá–Leindler inequality. When $Ric_{\mu,N} \geq \rho g$ with $\rho > 0$ (i.e. (M, g, μ) satisfies the CD(ρ, N) condition), by replacing the $\int_M \langle Ric_{\mu,N}^{-1} \nabla f, \nabla f \rangle d\mu$ term with the *looser* $\frac{1}{\rho} \int_M |\nabla f|^2 d\mu$ in all occurrences above, we obtain various generalizations of the classical Lichnerowicz estimate [23] on the spectral-gap of the weighted-Laplacian $-L$ under different boundary conditions; in particular, in the non-weighted classical case $N = n$, this recovers the spectral-gap estimate of Escobar [8] and Xia [51] under Neumann boundary conditions, and the one by Reilly [43] under Dirichlet conditions. When $N \leq -1$, Case (1) was obtained in the *Euclidean setting* (and under the stronger assumption that $Ric_{\mu,\infty} = \nabla^2 V > 0$) with a constant better than $\frac{N}{N-1}$ on the left-hand-side above by V. H. Nguyen [37], improving a previous estimate of S. Bobkov and M. Ledoux [4] valid when $N \leq 0$. However, on a general *weighted Riemannian manifold*, our constant $\frac{N}{N-1}$ is *best possible* in Case (1) for the entire range $N \in (-\infty, -1] \cup [n, \infty]$, see Sect. 3.2.

We refer to Sect. 3.1 for a longer exposition on the previously known generalizations in these directions; with few exceptions, Cases (2) and (3) and also Case (1) when $N \neq \infty$ seem new. We note that while the heat semi-group approach of Bakry–Émery is a very powerful tool in Case (1), namely under Neumann convex boundary

conditions, we are not aware of an analogous semi-group approach under the Case (2) Dirichlet mean-convex boundary conditions, let alone the mixed boundary conditions of Case (3), and thus confine our analysis to the L^2-duality approach.

To conclude this work, we extend in Sect. 4 a result of L. Veysseire [47], who obtained a spectral-gap estimate of $1/\int_M (1/\rho) d\mu$ assuming that $\mathrm{Ric}_\mu \geq \rho g$ for a function $\rho : M \to \mathbb{R}_+$ which is not necessarily bounded away from zero, to the case of Neumann boundary conditions when M is locally convex.

Remark 1.3 Although all of our results are formulated for compact weighted-manifolds with boundary, the results easily extend to the non-compact case, if the manifold M can be exhausted by compact submanifolds $\{M_k\}$ so that each $(M_k, g|_{M_k}, \mu|_{M_k})$ has an appropriate boundary (locally convex or generalized mean-convex, in accordance with the desired result). In the Dirichlet case, the asserted inequalities then extend to all functions in $C_0^1(M)$ having compact support and vanishing on the boundary ∂M. In the Neumann cases, the asserted inequalities extend to all functions $f \in C_{loc}^1(M) \cap L^2(M, \mu)$ when μ is a finite measure. When such an exhaustion is not available but the manifold is complete, one may alternatively apply a functional-analytic argument to obtain analogous results on non-compact manifolds—more details may be found in [18, Appendix].

2 Generalized Reilly Formula and Other Preliminaries

2.1 Notation

We denote by $int(M)$ the interior of M. Given a compact differentiable manifold Σ (which is at least C^k smooth), we denote by $C^k(\Sigma)$ the space of real-valued functions on Σ with continuous (and bounded) derivatives $\left(\frac{\partial}{\partial x}\right)^a f$, for every multi-index a of order $|a| \leq k$ in a given coordinate system. Similarly, the space $C^{k,\alpha}(\Sigma)$ denotes the subspace of functions whose k-th order derivatives are uniformly Hölder continuous of order α on the $C^{k,\alpha}$ smooth manifold Σ. When Σ is non-compact, we may use $C_{loc}^{k,\alpha}(\Sigma)$ to denote the class of functions u on M so that $u|_{\Sigma_0} \in C^{k,\alpha}(\Sigma_0)$ for all compact subsets $\Sigma_0 \subset \Sigma$. These spaces are equipped with their usual corresponding topologies.

Throughout this work, we employ Einstein summation convention. By abuse of notation, we denote different covariant and contravariant versions of a tensor in the same manner. So for instance, Ric_μ may denote the 2-covariant tensor $(\mathrm{Ric}_\mu)_{\alpha,\beta}$, but also may denote its 1-covariant 1-contravariant version $(\mathrm{Ric}_\mu)_\beta^\alpha$, as in

$$\langle \mathrm{Ric}_\mu \nabla f, \nabla f \rangle = g_{i,j}(\mathrm{Ric}_\mu)_k^i \nabla^k f \nabla^j f = (\mathrm{Ric}_\mu)_{i,j} \nabla^i f \nabla^j f = \mathrm{Ric}_\mu(\nabla f, \nabla f).$$

Similarly, inverse tensors are interpreted according to the appropriate context. For instance, the 2-contravariant tensor $(\mathrm{II}^{-1})^{\alpha,\beta}$ is defined by

$$(\mathrm{II}^{-1})^{i,j} \mathrm{II}_{j,k} = \delta_k^i.$$

We freely raise and lower indices by contracting with the metric. Since we only deal with 2-tensors, the only possible contraction is often denoted by using the trace notation tr.

Finally, when studying consequences of the $CD(\rho, N)$ condition, the various expressions in which N appears are interpreted in the limiting sense when $1/N = 0$. For instance, $N/(N - 1)$ is interpreted as 1, and $N f^{1/N}$ is interpreted as $\log f$ (since $\lim_{1/N \to 0} N(x^{1/N} - 1) = \log(x)$; the constant -1 in the latter limit does not influence our application of this convention).

2.2 Proof of the Generalized Reilly Formula

For completeness, we sketch the proof of our main tool, Theorem 1.1 from the Introduction, following the proof given in [28].

Proof of Theorem 1.1 The generalized Bochner–Lichnerowicz–Weitzenböck formula [2,24] states that for any $u \in C^3_{loc}(int(M))$, we have

$$\frac{1}{2} L |\nabla u|^2 = \left\| \nabla^2 u \right\|^2 + \langle \nabla Lu, \nabla u \rangle + \langle \mathrm{Ric}_\mu \nabla u, \nabla u \rangle. \tag{2.1}$$

We introduce an orthonormal frame of vector fields $e_1, \ldots, e_n$ so that $e_n = \nu$ on ∂M, and denote $u_i = du(e_i)$, $u_{i,j} = \nabla^2 u(e_i, e_j)$. Assuming in addition that $u \in C^2(M)$, we may integrate by parts:

$$\int_M \frac{1}{2} L |\nabla u|^2 \, d\mu = \int_{\partial M} \sum_{i=1}^{n} u_i u_{i,n} d\mu_{\partial M},$$

$$\int_M \langle \nabla Lu, \nabla u \rangle \, d\mu = \int_{\partial M} u_n (Lu) d\mu_{\partial M} - \int_M (Lu)^2 d\mu.$$

Consequently, integrating (2.1) over M, we obtain

$$\int_M \left((Lu)^2 - \left\| \nabla^2 u \right\|^2 - \langle \mathrm{Ric}_\mu \nabla u, \nabla u \rangle \right) d\mu = \int_{\partial M} \left(u_n(Lu) - \sum_{i=1}^{n} u_i u_{i,n} \right) d\mu_{\partial M}.$$

Now,

$$u_n(Lu) - \sum_{i=1}^{n} u_i u_{i,n} = \sum_{i=1}^{n-1} \left(u_n u_{i,i} - u_i u_{i,n} \right) - u_n \langle \nabla u, \nabla V \rangle.$$

Computing the different terms,

$$\sum_{i=1}^{n-1} u_{i,i} = \sum_{i=1}^{n-1} \left(e_i(e_i u) - (\nabla_{e_i} e_i)u \right)$$

$$= \sum_{i=1}^{n-1} \left(e_i(e_i u) - ((\nabla_{\partial M})_{e_i} e_i) u \right) + \left(\sum_{i=1}^{n-1} (\nabla_{\partial M})_{e_i} e_i - \nabla_{e_i} e_i \right) u$$

$$= \Delta_{\partial M} u + \left(\sum_{i=1}^{n-1} \mathrm{II}_{i,i} \right) e_n u = \Delta_{\partial M} u + tr(\mathrm{II}) u_n;$$

$$\sum_{i=1}^{n-1} u_i u_{i,n} = \sum_{i=1}^{n-1} u_i \left(e_i(e_n u) - (\nabla_{e_i} e_n) u \right) = \langle \nabla_{\partial M} u, \nabla_{\partial M} u_n \rangle - \langle \mathrm{II} \, \nabla_{\partial M} u, \nabla_{\partial M} u \rangle .$$

Putting everything together,

$$\int_M \left((Lu)^2 - \left\| \nabla^2 u \right\|^2 - \langle \mathrm{Ric}_\mu \nabla u, \nabla u \rangle \right) d\mu$$

$$= \int_{\partial M} \left(u_n (\Delta_{\partial M} u - \langle \nabla u, \nabla V \rangle) + tr(\mathrm{II})(u_n)^2 \right) d\mu_{\partial M}$$

$$- \int_{\partial M} \langle \nabla_{\partial M} u, \nabla_{\partial M} u_n \rangle \, d\mu_{\partial M} + \int_{\partial M} \langle \mathrm{II} \, \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \, d\mu_{\partial M}.$$

This is the formula obtained in [28] for smooth functions. To conclude the proof, simply note that

$$\langle \nabla u, \nabla V \rangle = \langle \nabla_{\partial M} u, \nabla_{\partial M} V \rangle + u_n V_n,$$
$$L_{\partial M} = \Delta_{\partial M} - \langle \nabla_{\partial M} V, \nabla_{\partial M} \rangle ,$$
$$H_\mu = tr(\mathrm{II}) - V_n,$$

and thus,

$$\int_{\partial M} \left(u_n (\Delta_{\partial M} u - \langle \nabla u, \nabla V \rangle) + tr(\mathrm{II})(u_n)^2 \right) d\mu_{\partial M}$$

$$= \int_{\partial M} \left(u_n L_{\partial M} u + H_\mu u_n^2 \right) d\mu_{\partial M}.$$

Integrating by parts one last time, this time on ∂M, we obtain

$$\int_{\partial M} u_n L_{\partial M} u \, d\mu_{\partial M} = - \int_{\partial M} \langle \nabla_{\partial M} u_n, \nabla_{\partial M} u \rangle \, d\mu_{\partial M}.$$

Finally, plugging everything back, we obtain the asserted formula for u:

$$\int_M \left((Lu)^2 - \left\| \nabla^2 u \right\|^2 - \langle \mathrm{Ric}_\mu \nabla u, \nabla u \rangle \right) d\mu$$

$$= \int_{\partial M} H_\mu u_n^2 d\mu_{\partial M} - 2 \int_{\partial M} \langle \nabla_{\partial M} u_n, \nabla_{\partial M} u \rangle \, d\mu_{\partial M}$$

$$+ \int_{\partial M} \langle \mathrm{II} \, \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \, d\mu_{\partial M}.$$

To conclude that the assertion in fact holds for $u \in \mathcal{S}_N(M)$, we employ a standard approximation argument using a partition of unity and mollification. Since the metric is assumed at least C^3 and ∂M is C^2, we may approximate any $u \in \mathcal{S}_N(M)$ by functions $u_k \in C^3_{loc}(int(M)) \cap C^2(M)$, so that $u_k \to u$ in $C^2_{loc}(int(M))$ and $C^1(M)$, and $(u_k)_\nu \to u_\nu$ in $C^1(\partial M)$. The assertion then follows by passing to the limit. □

Remark 2.1 For minor technical reasons, it will be useful to record the following variants of the generalized Reilly formula, which are obtained by analogous approximation arguments to the one given above:

- If u_ν or u are constant on ∂M and $u \in \mathcal{S}_0(M)$ (recall $\mathcal{S}_0(M) := C^2_{loc}(int(M)) \cap C^1(M)$), then

$$
\int_M (Lu)^2 d\mu = \int_M \left\| \nabla^2 u \right\|^2 d\mu + \int_M \langle \text{Ric}_\mu \, \nabla u, \nabla u \rangle d\mu \\
+ \int_{\partial M} H_\mu(u_\nu)^2 d\mu_{\partial M} + \int_{\partial M} \langle \text{II}_{\partial M} \, \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \, d\mu_{\partial M}.
$$

(2.2)

- If $u \in \mathcal{S}_D(M) := \mathcal{S}_0(M) \cap C^2(\partial M)$, then integration by parts yields

$$
\int_M (Lu)^2 d\mu = \int_M \left\| \nabla^2 u \right\|^2 d\mu + \int_M \langle \text{Ric}_\mu \, \nabla u, \nabla u \rangle d\mu \\
+ \int_{\partial M} H_\mu(u_\nu)^2 d\mu_{\partial M} + \int_{\partial M} \langle \text{II}_{\partial M} \, \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \, d\mu_{\partial M} \\
+ 2 \int_{\partial M} u_\nu L_{\partial M} u \, d\mu_{\partial M}.
$$

(2.3)

Remark 2.2 Throughout this work, when integrating by parts, we employ a slightly more general version of the textbook Stokes Theorem $\int_M d\omega = \int_{\partial M} \omega$, in which one only assumes that ω is a continuous differential $(n-1)$-form on M which is differentiable on $int(M)$ (and so that $d\omega$ is integrable there); a justification may be found in [29]. This permits us to work with the classes $C^k_{loc}(int(M))$ occurring throughout this work.

2.3 The CD(ρ, N) Condition for $1/N \in [-\infty, 1/n]$

The results in this subsection for $1/N \in [0, 1/n]$ are due to Bakry (e.g. [1, Sect. 6]).

Lemma 2.3 *For any $u \in C^2_{loc}(M)$ and $1/N \in [-\infty, 1/n]$,*

$$
\Gamma_2(u) = \langle Ric_\mu \, \nabla u, \nabla u \rangle + \left\| \nabla^2 u \right\|^2 \geq \langle Ric_{\mu, N} \, \nabla u, \nabla u \rangle + \frac{1}{N}(Lu)^2. \tag{2.4}
$$

Our convention throughout this work is that $-\infty \cdot 0 = 0$, and so if $Lu = 0$ at a point $p \in M$, the assertion when $\frac{1}{N} = -\infty$ is that

$$\Gamma_2(u) \geq \langle Ric_{\mu,0}\, \nabla u, \nabla u \rangle,$$

at that point.

Proof Recalling the definitions, this is equivalent to showing that

$$\left\| \nabla^2 u \right\|^2 + \frac{1}{N-n} \langle \nabla u, \nabla V \rangle^2 \geq \frac{1}{N}(Lu)^2.$$

Clearly the case that $1/N = 0$ $(N = \infty)$ follows. But by Cauchy–Schwarz,

$$\left\| \nabla^2 u \right\|^2 \geq \frac{1}{n}(\Delta u)^2,$$

and so the case $N = n$, which corresponds to a constant function V so that $Ric_\mu = Ric_{\mu,n} = Ric_g$ and $L = \Delta$, also follows. It remains to show that

$$\frac{1}{n}(\Delta u)^2 + \frac{1}{N-n} \langle \nabla u, \nabla V \rangle^2 \geq \frac{1}{N}(Lu)^2.$$

The case $1/N = -\infty$ $(N = 0)$ follows since when $0 = Lu = \Delta u - \langle \nabla u, \nabla V \rangle$ then

$$\frac{1}{n}(\Delta u)^2 - \frac{1}{n} \langle \nabla u, \nabla V \rangle^2 = \frac{1}{n}(\Delta u + \langle \nabla u, \nabla V \rangle)(\Delta u - \langle \nabla u, \nabla V \rangle) = 0.$$

In all other cases, the assertion follows from another application of Cauchy–Schwarz:

$$\frac{1}{\alpha}A^2 + \frac{1}{\beta}B^2 \geq \frac{1}{\alpha + \beta}(A + B)^2 \quad \forall A, B \in \mathbb{R},$$

valid as soon as (α, β) lay in either the set $\{\alpha, \beta > 0\}$ or the set $\{\alpha + \beta < 0 \text{ and } \alpha\beta < 0\}$.

$\square$

Remark 2.4 It is immediate to deduce from Lemma 2.3 that for $1/N \in (-\infty, 1/n]$, $Ric_{\mu,N} \geq \rho g$ on M, $\rho \in \mathbb{R}$, if and only if:

$$\Gamma_2(u) \geq \rho\, |\nabla u|^2 + \frac{1}{N}(Lu)^2, \quad \forall u \in C^2_{loc}(M).$$

Indeed, the necessity follows from Lemma 2.3. The sufficiency follows by locally constructing given $p \in M$ and $X \in T_p M$ a function u so that $\nabla u = X$ at p and equality holds in both applications of the Cauchy–Schwarz inequality in the proof above, as this implies that $Ric_{\mu,N}(X, X) \geq \rho |X|^2$. Indeed, equality in the first application implies that $\nabla^2 u$ is a multiple of g at p, whereas the equality in the second implies when $1/N \notin \{0, 1/n\}$ that $\langle \nabla u, \nabla V \rangle$ and Δu are appropriately proportional at p; clearly all three requirements can be simultaneously met. The cases $1/N \in \{0, 1/n\}$ follow by approximation.

2.4 Solution to Poisson Equation on Weighted Riemannian Manifolds

As our manifold is smooth, connected, compact, with C^2 smooth boundary and strictly positive C^2-density all the way up to the boundary, all of the classical elliptic existence, uniqueness and regularity results (e.g. [11, Chap. 8], [26, Chap. 5], [19, Chap. 3]) immediately extend from the Euclidean setting to our weighted-manifold one (see e.g. [36,45]); for more general situations (weaker regularity of metric, Lipschitz domains, etc.) see e.g. [35] and the references therein. We summarize the results we require in the following:

Theorem 2.5 *Given a weighted-manifold (M, g, μ), $\mu = \exp(-V)dVol_M$, we assume that ∂M is C^2 smooth. Let $\alpha \in (0, 1)$, and assume that g is $C^{2,\alpha}$ smooth and $V \in C^{1,\alpha}(M)$. Let $f \in C^{0,\alpha}(M)$, $\varphi_D \in C^2(\partial M)$ and $\varphi_N \in C^1(\partial M)$. Then there exists a function $u \in C^{2,\alpha}_{loc}(int(M)) \cap C^{1,\beta}(M)$ for all $\beta \in (0, 1)$, which solves*

$$Lu = f \quad on \quad M,$$

with either of the following boundary conditions on ∂M:

(1) *Dirichlet: $u|_{\partial M} = \varphi_D$, assuming $\partial M \neq \emptyset$.*
(2) *Neumann: $u_\nu|_{\partial M} = \varphi_N$, assuming the following compatibility condition is satisfied:*

$$\int_M f \, d\mu = \int_{\partial M} \varphi_N \, d\mu_{\partial M}.$$

In particular, $u \in \mathcal{S}_0(M)$ in either case. Moreover, $u \in \mathcal{S}_N(M)$ in the Neumann case and $u \in \mathcal{S}_D(M)$ in the Dirichlet case.

Remark 2.6 For future reference, we remark that it is enough to only assume in the proof of the generalized Reilly formula (including the final approximation argument) that the metric g is C^3 smooth, so in particular the above regularity results apply.

We will not require the uniqueness of u above, but for completeness we mention that this is indeed the case for Dirichlet boundary conditions, and up to an additive constant in the Neumann case.

2.5 Spectral-gap on Weighted Riemannian Manifolds

Let λ_1^N denote the best constant in the Neumann Poincaré inequality:

$$\lambda_1^N \text{Var}_\mu(f) \leq \int_M |\nabla f|^2 \, d\mu, \quad \forall f \in H^1(\mu),$$

and let λ_1^D denote the best constant in the Dirichlet Poincaré inequality:

$$\lambda_1^D \int_M f^2 d\mu \leq \int_M |\nabla f|^2 \, d\mu, \quad \forall f \in H_0^1(\mu).$$

Here $H^1(\mu)$ and $H^1_0(\mu)$ denote the Sobolev spaces obtained by completing $C^\infty(M)$ and $C^\infty_0(M)$ in the $H^1(\mu)$-norm $\sqrt{\int_M f^2 d\mu + \int_M |\nabla f|^2 \, d\mu}$. It is well known (e.g. [46]) that the symmetric operator $-L$ on $L^2(\mu)$ with domain $C^\infty(M)$ or $C^\infty_0(M)$ admits a (unique) self-adjoint positive semi-definite extension, called the Neumann and Dirichlet (negative) Laplacian, respectively. Both instances have discrete non-negative spectra with corresponding complete orthonormal bases of eigenfunctions. In the first case, λ^N_1 is the first positive eigenvalue of the (negative) Neumann Laplacian:

$$-Lu = \lambda^N_1 u \text{ on } M, \quad u_\nu \equiv 0 \text{ on } \partial M;$$

the zero eigenvalue corresponds to the eigenspace of constant functions, and so only functions u orthogonal to constants are considered. In the second case, λ^D_1 is the first (positive) eigenvalue of the (negative) Dirichlet Laplacian:

$$-Lu = \lambda^D_1 u \text{ on } M, \quad u \equiv 0 \text{ on } \partial M.$$

Our assumptions on the smoothness of M, its boundary, and the density $\exp(-V)$, guarantee by elliptic regularity theory that in either case, all eigenfunctions are in $\mathcal{S}_0(M)$ (in fact, in $\mathcal{S}_N(M)$ in the Neumann case and in $\mathcal{S}_D(M)$ in the Dirichlet case).

3 Generalized Brascamp–Lieb-Type Inequalities on M

In this section, we provide a proof of Theorem 1.2 from the Introduction, which we repeat here for convenience:

Theorem 3.1 (Generalized dimensional Brascamp–Lieb with boundary) *Assume that* $Ric_{\mu,N} > 0$ *on* M *with* $1/N \in (-\infty, 1/n]$. *Then for any* $f \in C^1(M)$:

(1) *(Neumann Dimensional Brascamp–Lieb inequality on locally convex domain)*
   Assume that $II_{\partial M} \geq 0$ *(M is locally convex). Then*

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle Ric^{-1}_{\mu,N} \nabla f, \nabla f \right\rangle d\mu.$$

(2) *(Dirichlet Dimensional Brascamp–Lieb inequality on generalized mean-convex domain)*
   Assume that $H_\mu \geq 0$ *(M is generalized mean-convex), $f \equiv 0$ on $\partial M \neq \emptyset$. Then*

$$\frac{N}{N-1} \int_M f^2 d\mu \leq \int_M \left\langle Ric^{-1}_{\mu,N} \nabla f, \nabla f \right\rangle d\mu.$$

(3) *(Neumann Dimensional Brascamp–Lieb inequality on strictly generalized mean-convex domain)*

Assume that $H_\mu > 0$ (M is strictly generalized mean-convex). Then for any $C \in \mathbb{R}$:

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle Ric_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu + \int_{\partial M} \frac{1}{H_\mu} \left(f - C \right)^2 d\mu_{\partial M}.$$

In other words,

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle Ric_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu + Var_{\mu_{\partial M}/H_\mu}(f|_{\partial M}).$$

3.1 Previously Known Particular Cases

3.1.1 $1/N = 0$: Generalized Brascamp–Lieb Inequalities

Recall that when $1/N = 0$, $\mathrm{Ric}_{\mu,N} = \mathrm{Ric}_\mu$, and $\frac{N}{N-1} = 1$. When (M, g) is Euclidean space $\mathbb{R}^n$ and $\mu = \exp(-V) dx$ is a finite measure, the Brascamp–Lieb inequality [6] asserts that

$$\mathrm{Var}_\mu(f) \leq \int_{\mathbb{R}^n} \left\langle (\nabla^2 V)^{-1} \nabla f, \nabla f \right\rangle d\mu, \quad \forall f \in C^1(\mathbb{R}^n).$$

Observe that in this case, $\mathrm{Ric}_\mu = \nabla^2 V$, and so taking into account Remark 1.3, we see that the Brascamp–Lieb inequality follows from Case (1). The latter is easily seen to be sharp, as witnessed by testing the Gaussian measure in Euclidean space.

The extension to the weighted-Riemannian setting for $1/N = 0$, at least when (M, g) has no boundary, is well known to experts, although we do not know who to accredit this to (see e.g. the Witten Laplacian method of Helffer–Sjöstrand [12] as exposed by Ledoux [20]). The case of a locally convex boundary with Neumann boundary conditions (Case 1 above) can easily be justified in Euclidean space by a standard approximation argument, but this is less clear in the Riemannian setting; probably this can be achieved by employing the Bakry–Émery semi-group formalism (see Qian [42] and Wang [48,49]). To the best of our knowledge, the other Cases (2) and (3) are new even for $1/N = 0$.

3.1.2 $Ric_{\mu,N} \geq \rho g$ with $\rho > 0$: Generalized Lichnerowicz Inequalities

Assume that $\mathrm{Ric}_{\mu,N} \geq \rho g$ with $\rho > 0$, so that (M, g, μ) satisfies the $\mathrm{CD}(\rho, N)$ condition. It follows that

$$\int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu \leq \frac{1}{\rho} \int_M |\nabla f|^2 d\mu, \tag{3.1}$$

and so we may replace in all three cases of Theorem 3.1 every occurrence of the left-hand term in (3.1) by the right-hand one. So for instance, Case (1) implies that

$$\frac{N}{N-1}\mathrm{Var}_\mu(f) \leq \frac{1}{\rho}\int_M |\nabla f|^2 \, d\mu, \tag{3.2}$$

and similarly for the other two cases; we refer to the resulting inequalities as Cases $(1')$, $(2')$, and $(3')$. Clearly, Cases $(1')$ and $(2')$ are spectral-gap estimates for $-L$ with Neumann and Dirichlet boundary conditions, respectively.

Recall that in the non-weighted Riemannian setting ($\mu = \mathrm{Vol}_M$ and $N = n$), $\mathrm{Ric}_{\mu,N} = \mathrm{Ric}_g$. In this classical setting, the above spectral-gap estimates are due to the following authors: when $\partial M = \emptyset$, Cases $(1')$ and $(3')$ degenerate to a single statement, due to Lichnerowicz [23], and by Obata's theorem [38] equality is attained if and only if M is the n-sphere. When $\partial M \neq \emptyset$, Case $(1')$ is due to Escobar [8] and independently Xia [51] ; Case $(2')$ is due to Reilly [43] ; in both cases, one has equality if and only if M is the n-hemisphere ; Case $(3')$ seems new even in the classical case.

On weighted-manifolds with $N \in [n, \infty]$, Case $(1')$ is certainly known, see e.g. [21] (in fact, a stronger log-Sobolev inequality goes back to Bakry and Émery [2]); Case $(2')$ was recently obtained under a slightly stronger assumption by Ma and Du [28, Theorem 2]; for an adaptation to the CD(ρ, N) condition see Li and Wei [21, Theorem 3], who also showed that in both cases one has equality if and only if $N = n$ and M is the n-sphere or n-hemisphere endowed with its Riemannian volume form, corresponding to whether ∂M is empty or non-empty, respectively. As already mentioned, Case $(3')$ seems new.

To the best of our knowledge, the case of $N < 0$ has not been previously treated in the Riemannian setting. Concurrently to posting our work on the arXiv, Ohta [39] has also obtained Case $(1')$ for $N < 0$ when $\partial M = \emptyset$.

3.1.3 Generalized Bobkov–Ledoux–Nguyen Inequalities

In the *Euclidean setting* with $N \leq -1$ (and under the stronger assumption that $\mathrm{Ric}_\mu = \nabla^2 V > 0$), Case (1) with a better constant of $\frac{n-N-1}{n-N}$ instead of our $\frac{N}{N-1} = \frac{-N}{-N+1}$ is due to Nguyen [37, Proposition 10], who generalized and sharpened a previous version valid for $N \leq 0$ by Bobkov–Ledoux [4]. However, on a general *weighted Riemannian manifold*, our constant $\frac{N}{N-1}$ is best possible in the range $N \in (-\infty, -1] \cup [n, \infty]$, see Subsection 3.2 below.

Note that in the Euclidean case, the CD(0, N) condition with $N \in \mathbb{R}$ corresponds to Borell's class of convex measures [5], also known as "$1/N$-concave measures" (cf. [34]). When $N < 0$, these measures are heavy-tailed, having tails decaying to zero only polynomially fast, and consequently the corresponding generator $-L$ may not have a strictly positive spectral-gap. This is compensated by the weight $\mathrm{Ric}_{\mu,N}^{-1}$ in the resulting Poincaré-type inequality. A prime example is given by the Cauchy measure in $\mathbb{R}^n$, which satisfies CD(0, 0) (it is $-\infty$-concave). See [4,37] for more information.

Still in the Euclidean setting with $N \geq n$ (in fact $N > n - 1$), a dimensional version of the Brascamp–Lieb inequality which is reminiscent of Case (1) was obtained

by Nguyen [37, Theorem 9]. The Bobkov–Ledoux results were obtained as an infinitesimal version of the Borell–Brascamp–Lieb inequality [5,6]—a generalization of the Brunn-Minkowski inequality, which is strictly confined to the Euclidean setting. Nguyen's approach is already more similar to our own, dualizing an ad hoc Bochner formula obtained for a non-stationary diffusion operator.

In any case, our unified formulation (and treatment) of both regimes $N \leq 0$ and $N \in [n, \infty]$, the weaker assumption that $\mathrm{Ric}_{\mu,N} > 0$, the extension to the Riemannian setting with sharp constant $\frac{N}{N-1}$ and the treatment of the different boundary conditions in Cases (1), (2), and (3) seem new.

3.2 Sharpness of the $\frac{N}{N-1}$ Constant in the Riemannian Setting

We briefly comment on the sharpness of the constant $\frac{N}{N-1}$ for the range $N \in (-\infty, -1] \cup [n, \infty]$ in the more traditional setting of Case (1); the sharpness of Case (2) is also shown for $N \geq n$. This constant is no longer sharp in Case (1) for $N < 0$ with $|N| \ll 1$, since under the $\mathrm{CD}(\rho, N)$ condition with $\rho > 0$, the spectral-gap remains bounded below as $N < 0$ increases to 0, see [31].

As described in Sect. 3.1.2, it is classical that equality in the Lichnerowicz estimate (3.2) is attained by the n-sphere and n-hemisphere in Cases (1) (and (3)) and by the n-hemisphere in Case (2), both endowed with the usual Riemannian volume. This demonstrates the sharpness of the constant $\frac{N}{N-1}$ when $N = n$.

For general $N \in (-\infty, -1] \cup (n, \infty]$, the sharpness may be shown as follows. Given $\rho > 0$, set $\delta = \frac{\rho}{N-1}$ and

$$\beta := \begin{cases} \frac{\pi}{2\sqrt{\delta}} & \delta > 0 \\ \infty & \delta < 0 \end{cases}, \quad \alpha := \begin{cases} -\beta & \text{Case (1)} \\ 0 & \text{Case (2)} \end{cases}.$$

Define the following functions of $t \in [\alpha, \beta]$:

$$R(t) := \begin{cases} \cos(\sqrt{\delta}t) & \delta > 0 \\ \cosh(\sqrt{-\delta}t) & \delta < 0 \end{cases}, \quad \Psi_{N-1}(t) := R^{N-1}(t).$$

If we extend our setup to include the case of one-dimensional ($n = 1$) weighted manifolds, namely the case of the real line endowed with a density, then it is immediate to check that $([\alpha, \beta], |\cdot|, \mu = \Psi_{N-1}(t)dt)$ satisfies the $\mathrm{CD}(\rho, N)$ condition, since

$$\mathrm{Ric}_{\mu,N} = -(N-1)\frac{\left(\Psi_{N-1}^{\frac{1}{N-1}}\right)''}{\Psi_{N-1}^{\frac{1}{N-1}}} = -(N-1)\frac{R''}{R} = (N-1)\delta = \rho.$$

Note that when $n = 1$, our constant $\frac{N}{N-1}$ and Nguyen's one $\frac{n-N-1}{n-N}$ coincide. As we have learned from Nguyen, his constant is sharp in the Euclidean setting for any $n \geq 1$. One consequently verifies the sharpness for $n = 1$ by using the same test function

used by Nguyen in [37], namely $f(t) = \frac{d}{dt} R(t)$. Indeed, when $N < -1$ or $N > 1$ (to ensure convergence of the integrals below) we have

$$\int f(t) d\mu = \int_{-\beta}^{\beta} R'(t) R^{N-1}(t) dt = \frac{1}{N} \int_{-\beta}^{\beta} (R^N(t))' dt = 0,$$

since $\lim_{t \to \beta} R^N(t) = 0$, and since also $f(0) = R'(0) = 0$ (so that the Dirichlet boundary condition at $t = 0$ is satisfied in Case (2)), we may integrate by parts:

$$\int f^2(t) d\mu = \frac{1}{N} \int_{\alpha}^{\beta} R'(t)(R^N(t))' dt = -\frac{1}{N} \int_{\alpha}^{\beta} R''(t) R^N(t) dt$$

$$= \frac{\rho}{N(N-1)} \int_{\alpha}^{\beta} R^{N+1}(t) dt.$$

On the other hand,

$$\int \mathrm{Ric}_{\mu,N}^{-1} f'(t)^2 d\mu = \frac{1}{\rho} \int_{\alpha}^{\beta} (R''(t))^2 R^{N-1}(t) dt = \frac{\rho}{(N-1)^2} \int_{\alpha}^{\beta} R^{N+1}(t) dt.$$

Comparing the last two expressions, we conclude the sharpness of the constant $\frac{N}{N-1}$ for $n = 1$ in Case (1) when $|N| > 1$ and in Case (2) when $N > 1$ (the function $f(t)$ does not vanish at infinity when $N < 0$ so this range is excluded in Case (2)). When $N = -1$, one uses an appropriately truncated version of the above test function. In any case, to assert sharpness for a *compact* weighted manifold with strictly positive density, we truncate the above construction at a finite $\beta_\epsilon \in (0, \beta)$, and let β_ϵ tend to β.

To see the sharpness for $n \geq 2$, we proceed by repeating the construction from [33], which emulates the above 1-dimensional model space on a thin weighted n-dimensional manifold of revolution. For $n \geq 3$, define

$$\Psi_{N-n}(t) := R^{N-n}(t),$$

and given $\epsilon > 0$, consider the n-dimensional manifold $M := [\alpha, \beta] \times S^{n-1}$ endowed with the metric g_ϵ and measure μ_ϵ given by

$$g_\epsilon := dt^2 + \epsilon^2 R(t)^2 g_{S^{n-1}};$$
$$\mu_\epsilon := \Psi(t, \theta) dvol_{g_\epsilon}(t, \theta), \quad \Psi(t, \theta) = \Psi_{N-n}(t), \quad (t, \theta) \in [\alpha, \beta] \times S^{n-1}.$$

The intuition behind this construction is that when $\epsilon > 0$ is small enough, the geometry of (M, g_ϵ) will contribute (at least) $(n-1)\delta$ to the generalized Ricci curvature tensor $\mathrm{Ric}_{g,\mu,N}$, and a factor of $R^{n-1}(t)$ to the density $d\mu_\epsilon \left((-\infty, t] \times S^{n-1}\right)/dt$, whereas the measure μ_ϵ will contribute $(N-n)\delta g_\epsilon$ to the former and a factor of $\Psi_{N-n}(t) = R^{N-n}(t)$ to the latter, totaling $(N-1)\delta = \rho$ and $R^{N-1}(t) = \Psi_{N-1}(t)$, respectively. Consequently $(M, g_\epsilon, \mu_\epsilon)$ satisfies the CD(ρ, N) condition for small enough $\epsilon > 0$, and its measure projection onto the axis of revolution is $c_\epsilon \Psi_{N-1}(t)$; the sharpness of

the constant then follows from our previous one-dimensional analysis. Note that in Case (2), the boundary component $\{0\} \times S^{n-1}$ is totally geodesic and hence satisfies our boundary curvature assumptions. In practice, when $N \geq n$ (and thus $\beta < \infty$), we need to ensure that the resulting compact weighted manifold is smooth at its vertices (at $t \in \{-\beta, \beta\}$ in Case (1) and $t = \beta$ in Case (2)), and this is achieved as in [33] by gluing appropriate caps. When $N \leq -1$ (and thus $\beta = \infty$), in order to obtain a compact manifold as in the formulation of Theorem 3.1, we also need to truncate the above construction at a finite $\beta_\epsilon > 0$; the resulting boundary $\{-\beta_\epsilon, \beta_\epsilon\} \times S^{n-1}$ turns out to indeed be locally convex since $R'(\beta_\epsilon) = -R'(-\beta_\epsilon) > 0$, according to the calculation in [33]. The construction is even more complicated for the case $n = 2$; we refer to [33] for further precise details and rigorous justifications.

3.3 Proof of Theorem 3.1

Proof of Theorem 3.1 Plugging (2.4) into the generalized Reilly formula, we obtain for any $u \in \mathcal{S}_N(M)$:

$$\frac{N-1}{N} \int_M (Lu)^2 \mathrm{d}\mu \geq \int_M \langle \mathrm{Ric}_{\mu,N} \, \nabla u, \nabla u \rangle \mathrm{d}\mu$$
$$+ \int_{\partial M} H_\mu (u_\nu)^2 \mathrm{d}\mu_{\partial M} + \int_{\partial M} \langle \mathrm{II}_{\partial M} \, \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \, \mathrm{d}\mu_{\partial M}$$
$$- 2 \int_{\partial M} \langle \nabla_{\partial M} u_\nu, \nabla_{\partial M} u \rangle \, \mathrm{d}\mu_{\partial M}. \tag{3.3}$$

Recall that this remains valid for $u \in \mathcal{S}_0(M)$ if u or u_ν are constant on ∂M. Lastly, note that if $Lu = f$ in M with $f \in C^1(M)$ and $u \in \mathcal{S}_0(M)$, then

$$\int_M f^2 \mathrm{d}\mu = \int_M (Lu)^2 \mathrm{d}\mu = \int_M f Lu \, \mathrm{d}\mu = - \int_M \langle \nabla f, \nabla u \rangle \mathrm{d}\mu + \int_{\partial M} f u_\nu \mathrm{d}\mu_{\partial M}. \tag{3.4}$$

Consequently, by Cauchy–Schwarz

$$\int_M f^2 \mathrm{d}\mu \leq \left(\int_M \langle \mathrm{Ric}_{\mu,N} \, \nabla u, \nabla u \rangle \mathrm{d}\mu \right)^{1/2} \left(\int_M \langle \mathrm{Ric}_{\mu,N}^{-1} \, \nabla f, \nabla f \rangle \mathrm{d}\mu \right)^{1/2}$$
$$+ \int_{\partial M} f u_\nu \mathrm{d}\mu_{\partial M}. \tag{3.5}$$

We now proceed to treat the individual three cases.

(1) Assume that $\int_M f d\mu = 0$ and solve the Neumann Poisson problem for $u \in \mathcal{S}_0(M)$:

$$Lu = f \text{ on } M, \quad u_\nu \equiv 0 \text{ on } \partial M;$$

note that the compatibility condition $\int_{\partial M} u_\nu d\mu_{\partial M} = \int_M f d\mu = 0$ is indeed satisfied, so a solution exists. Since $u_\nu|_{\partial M} \equiv 0$ and $\mathrm{II}_{\partial M} \geq 0$, we obtain from (3.3):

$$\frac{N}{N-1} \int_M \langle \mathrm{Ric}_{\mu,N} \nabla u, \nabla u \rangle d\mu \leq \int_M (Lu)^2 d\mu = \int_M f^2 d\mu. \qquad (3.6)$$

Plugging this back into (3.5) and using that $u_\nu \equiv 0$ yields the assertion of Case (1).

(2) Assume that $f|_{\partial M} \equiv 0$ and solve the Dirichlet Poisson problem for $u \in \mathcal{S}_0(M)$:

$$Lu = f \text{ on } M, \quad u \equiv 0 \text{ on } \partial M.$$

Observe that (3.6) still holds since $u|_{\partial M} \equiv 0$ and $H_\mu \geq 0$. Plugging (3.6) back into (3.5) and using that $f|_{\partial M} \equiv 0$ yields the assertion of Case (2).

(3) Assume that $\int_M f d\mu = 0$ and solve the Dirichlet Poisson problem:

$$Lu = f \text{ on } M, \quad u \equiv 0 \text{ on } \partial M.$$

The difference with the previous case is that the $\int f u_\nu d\mu_{\partial M}$ term in (3.4) does not vanish since we do not assume that $f|_{\partial M} \equiv 0$. Consequently, we cannot afford to omit the positive contribution of $\int_{\partial M} H_\mu (u_\nu)^2 d\mu_{\partial M}$ in (3.3):

$$\frac{N-1}{N} \int_M f^2 d\mu \geq \int_M \langle \mathrm{Ric}_{\mu,N} \nabla u, \nabla u \rangle d\mu + \int_{\partial M} H_\mu u_\nu^2 d\mu_{\partial M}.$$

Applying the duality argument, this time in additive form, we obtain for any $\lambda > 0$:

$$\begin{aligned}
\int_M f^2 d\mu &= -\int_M \langle \nabla f, \nabla u \rangle d\mu + \int_{\partial M} f u_\nu d\mu_{\partial M} \\
&\leq \frac{1}{2\lambda} \int_M \langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \rangle d\mu + \frac{\lambda}{2} \int_M \langle \mathrm{Ric}_{\mu,N} \nabla u, \nabla u \rangle d\mu \\
&\quad + \int_{\partial M} f u_\nu d\mu_{\partial M}.
\end{aligned}$$

Since $\int_{\partial M} u_\nu d\mu_{\partial M} = \int_M f d\mu = 0$, we may as well replace the last term by $\int_{\partial M}(f - C)u_\nu d\mu_{\partial M}$. Plugging in the previous estimate and applying the Cauchy–Schwarz inequality again to eliminate u_ν, we obtain

$$\left(1 - \frac{\lambda}{2}\frac{N-1}{N}\right)\int_M f^2 \mathrm{d}\mu$$

$$\leq \frac{1}{2\lambda}\int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1}\, \nabla f, \nabla f\right\rangle \mathrm{d}\mu + \int_{\partial M}(f-C)u_\nu \mathrm{d}\mu_{\partial M} - \frac{\lambda}{2}\int_{\partial M} H_\mu u_\nu^2 \mathrm{d}\mu_{\partial M}$$

$$\leq \frac{1}{2\lambda}\int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1}\, \nabla f, \nabla f\right\rangle \mathrm{d}\mu + \frac{1}{2\lambda}\int_{\partial M} \frac{1}{H_\mu}(f-C)^2 \mathrm{d}\mu_{\partial M}.$$

Multiplying by 2λ and using the optimal $\lambda = \frac{N}{N-1}$, we obtain the assertion of Case (3).

4 Generalized Veysseire Spectral-Gap Inequality on Convex M

The next result was recently obtained by L. Veysseire [47] for compact weighted-manifolds without boundary. It may be thought of as a spectral-gap version of the Generalized Brascamp–Lieb inequality. We provide an extension in the case that M is locally convex.

Theorem 4.1 (Veysseire spectral-gap inequality with locally convex boundary) *Assume that as 2-tensors on M:*

$$Ric_\mu \geq \rho g,$$

for some measurable function $\rho : M \to \mathbb{R}_+$. Then for any $f \in C^1(M)$:

(1) *(Neumann Veysseire inequality on locally convex domain)*
   Assume that $II_{\partial M} \geq 0$ (M is locally convex). Then

$$Var_\mu(f) \leq \fint_M \frac{1}{\rho}\mathrm{d}\mu \int_M |\nabla f|^2 \,\mathrm{d}\mu.$$

Remark 4.2 We do not know whether the analogous results for Dirichlet or Neumann boundary conditions (Cases (2) and (3) in the previous section) hold on a generalized mean-convex domain, as the proof given below breaks down in those cases.

Remark 4.3 As in Veysseire's work [47], further refinements are possible. For instance, if in addition the $CD(\rho_0, N)$ condition is satisfied for $\rho_0 > 0$ and $1/N \in [-\infty, 1/n]$, then one may obtain an estimate on the corresponding spectral-gap λ_1^N of the form:

$$\lambda_1^N \geq \frac{N}{N-1}\rho_0 + \frac{1}{\fint_M \frac{1}{\rho-\rho_0}\mathrm{d}\mu}.$$

As explained in [47], this may be obtained by using an appropriate convex combination of the Lichnerowicz estimate (Case (1) of Theorem 1.2 after replacing $\mathrm{Ric}_{\mu,N}^{-1}$ with $1/\rho_0$) and the estimates obtained in this section, with a final application of the Cauchy–Schwarz inequality. Similarly, it is possible to interpolate between the Lichnerowicz

estimates and the Dimensional Brascamp–Lieb ones of Theorem 1.2. We leave this to the interested reader.

Veysseire's proof in [47] is based on the Bochner formula and the following observation, valid for any $u \in C^2(M)$ at any point so that $\nabla u \neq 0$:

$$\left\| D^2 u \right\| \geq |\nabla |\nabla u||. \tag{4.1}$$

At a point where $\nabla u = 0$, we define $|\nabla |\nabla u|| := 0$.

Proof of Theorem 4.1 Plugging (4.1) into the generalized Reilly formula and integrating the $\int_M (Lu)^2 d\mu$ term by parts, we obtain for any $u \in S_N(M)$ so that $Lu \in C^1(M)$:

$$\int_{\partial M} u_\nu Lu d\mu - \int_M \langle \nabla u, \nabla Lu \rangle \, d\mu \geq \int_M |\nabla |\nabla u||^2 \, d\mu + \int_M \langle \text{Ric}_\mu \nabla u, \nabla u \rangle d\mu$$

$$+ \int_{\partial M} H_\mu (u_\nu)^2 d\mu_{\partial M} + \int_{\partial M} \langle \text{II}_{\partial M} \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \, d\mu_{\partial M}$$

$$- 2 \int_{\partial M} \langle \nabla_{\partial M} u_\nu, \nabla_{\partial M} u \rangle \, d\mu_{\partial M}. \tag{4.2}$$

Let $u \in S_N(M)$ denote an eigenfunction of $-L$ with zero Neumann boundary conditions corresponding to λ_1^N, so that in particular $Lu = -\lambda_1^N u \in C^1(M)$, and denote $h = |\nabla u| \in H^1(\mu)$. Applying (4.2) to u, using that $\text{II}_{\partial M} \geq 0$, and that $\int_{\{h=0\}} |\nabla h|^2 \, d\mu = 0$ for any $h \in H^1(\mu)$, we obtain

$$\lambda_1^N \int_M h^2 d\mu \geq \int_M |\nabla h|^2 \, d\mu + \int_M \rho h^2 d\mu.$$

Applying the Neumann Poincaré inequality to the function h, we obtain

$$\lambda_1^N \int_M h^2 d\mu \geq \lambda_1^N \left(\int_M h^2 d\mu - \frac{1}{\mu(M)} \left(\int_M h d\mu \right)^2 \right) + \int_M \rho h^2 d\mu.$$

It follows by Cauchy–Schwarz that

$$\lambda_1^N \geq \frac{\mu(M) \int_M \rho h^2 d\mu}{(\int_M h d\mu)^2} \geq \frac{\mu(M)}{\int_M \frac{1}{\rho} d\mu},$$

concluding the proof. □

Remark 4.4 The proof above actually yields a meaningful estimate on the spectral-gap λ_1^N even when $\text{II}_{\partial M}$ is negatively bounded from below. However, this estimate depends on upper bounds on $|\nabla u|$, where u is the first non-trivial Neumann eigenfunction, both in M and on its boundary.

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