Homogeneous Toric Varieties

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Abstract. A description of transitive actions of a semisimple algebraic group $G$ on toric varieties is obtained. Every toric variety admitting such an action lies between a product of punctured affine spaces and a product of projective spaces. The result is based on the Cox realization of a toric variety as a quotient space of an open subset of a vector space $V$ by a quasitorus action and on investigation of the $G$-module structure of $V$.

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1. Introduction

We study toric varieties $X$ equipped with a transitive action of a connected semisimple algebraic group $G$. In this case $X$ is called a homogeneous toric variety. The ground field $\mathbb{K}$ is algebraically closed and of characteristic zero.

Consider a quasiaffine variety

$$\mathcal{X} = \mathcal{X}(n_1, \ldots, n_m) := (\mathbb{K}^{n_1} \setminus \{0\}) \times \cdots \times (\mathbb{K}^{n_m} \setminus \{0\})$$

with $n_i \geq 2$. The group $G = G_1 \times \cdots \times G_m$, where every component $G_i$ is either $\text{SL}(n_i)$ or $\text{Sp}(n_i)$, and $n_i$ is even in the second case, acts on $\mathcal{X}$ transitively and effectively. Let $S = (\mathbb{K}^*)^m$ be an algebraic torus acting on $\mathcal{X}$ by component-wise scalar multiplication, and

$$p : \mathcal{X} \to \mathcal{Y} := \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_m-1}$$

be the quotient morphism. Fix a closed subgroup $S \subseteq S$. The action of the group $S$ on $\mathcal{X}$ admits a geometric quotient $p_X : \mathcal{X} \to \mathcal{X} := \mathcal{X}/S$. The variety $X$ is toric, it carries the induced action of the quotient group $\overline{S}/S$, and there is a quotient morphism $p_X : X \to \mathcal{Y}$ for this action closing the commutative diagram

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The induced action of the group $G$ on $X$ is transitive and locally effective. We say that the $G$-variety $X$ is obtained from $\mathcal{X}$ by central factorization. The following theorem gives a classification of transitive actions of semisimple groups on toric varieties up to a twist by a diagram automorphism of the acting group.

**Theorem 1.1.** Let $X$ be a toric variety with a transitive locally effective action of a connected simply connected semisimple algebraic group $G$. Then $G = G_1 \times \ldots \times G_m$, where every simple component $G_i$ is either $\text{SL}(n_i)$ or $\text{Sp}(n_i)$, and the variety $X$ is obtained from $\mathcal{X} = \mathcal{X}(n_1, \ldots, n_m)$ by central factorization. Conversely, any variety obtained from $\mathcal{X}$ by central factorization is a homogeneous toric variety.

Theorem 1.1 also describes homogeneous spaces of a semisimple group that have a toric structure. It is natural to apply the Cox realization of a variety in order to search for toric varieties in a given class of varieties. This idea is already used in [8], where toric affine $\text{SL}(2)$-embeddings are characterized.

In Section 2 we recall basic facts on the Cox realization and its generalization. Criteria of existence of an open $G$-orbit on $X$ in terms of $G$- and $(G \times S)$-actions on the total coordinate space $Z$ are also given there. In Section 3 we prove Theorem 1.1. The next section is devoted to special classes of toric homogeneous varieties and to a characterization of their fans. In the last section we consider transitive actions of reductive groups on toric varieties.

Our results are closely connected with the results of E.B. Vinberg [17], where algebraic transformation groups of maximal rank were classified. Recall that an algebraic transformation group of maximal rank is an effective generically transitive (i.e., with an open orbit) action of an algebraic group $\mathcal{G}$ on an algebraic variety $X$ such that $\dim X = \text{rk} \mathcal{G}$, where $\text{rk} \mathcal{G}$ is the rank of a maximal torus $T$ of the group $\mathcal{G}$. In this situation the induced action of the torus $T$ on $X$ is effective and generically transitive, see [6]. If the group $\mathcal{G}$ is semisimple, then an open $\mathcal{G}$-orbit on $X$ is a homogeneous toric variety. It turns out that in this case $X$ is a product of projective spaces and $\mathcal{G}$ acts on $X$ transitively. Theorem 1.1 implies that every homogeneous toric variety determines a reductive transformation group of maximal rank; here $\mathcal{G}$ is the quotient group $(\text{GL}(n_1) \times \ldots \times \text{GL}(n_m))/S$.

Finally, let us mention a related result from toric topology. A torus manifold is a smooth real even-dimensional manifold $M^{2m}$ with an effective action of a compact torus $(S^1)^n$ such that the set of $(S^1)^n$-fixed points is nonempty. In [15], homogeneous torus manifolds are studied. The latter are torus manifolds $M^{2m}$ with a transitive action of a compact Lie group $K$ such that the induced action of a maximal torus of $K$ coincides with the given $(S^1)^n$-action. It is proved that every homogeneous torus manifold may be realized as

$$M = \mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_m} \times (S^{2m_1} \times \ldots \times S^{2m_l})/F,$$
where $S^{2m}$ is a sphere of dimension $2m$, $F$ is a subgroup of $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ ($l$ copies), and each copy of $\mathbb{Z}_2$ acts on the corresponding sphere by central symmetry. A compact Lie group

$$K = \text{PSU}(n_1 + 1) \times \ldots \times \text{PSU}(n_k + 1) \times \text{SO}(2m_1 + 1) \times \ldots \times \text{SO}(2m_l + 1)$$

acts on $M$ transitively. Moreover, the manifold $M$ is orientable if and only if $F \subset \text{SO}(2m_1 + 2m_2 + \ldots + 2m_l + l)$.

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### 2. The Cox construction

A toric variety is a normal algebraic variety with an effective generically transitive action of an algebraic torus $T$. A toric variety $X$ is non-degenerate if any invertible regular function on $X$ is constant.

Let $\text{Cl}(X)$ be the divisor class group of the variety $X$. It is well-known that the group $\text{Cl}(X)$ of a toric variety $X$ is finitely generated, see [7, Section 3.4]. Recall that a quasitorus is an affine algebraic group $S$ isomorphic to a direct product of an algebraic torus $S^0$ and a finite abelian group $\Gamma$. Every closed subgroup of a torus is a quasitorus. The group of characters of a quasitorus $S$ is a finitely generated abelian group. The Neron-Severi quasitorus of a toric variety $X$ is a quasitorus $S$ whose group of characters is identified with $\text{Cl}(X)$.

We come to a canonical quotient realization of a non-degenerate toric variety $X$ obtained in [5]. Let $d$ be the number of prime $T$-invariant Weil divisors on $X$. Consider the vector space $\mathbb{K}^d$ and the torus $T = (\mathbb{K}^*)^d$ of all invertible diagonal matrices acting on $\mathbb{K}^d$. Then there are a closed embedding of the Neron-Severi quasitorus $S$ into $T$ and an open subset $U \subseteq \mathbb{K}^d$ such that

- the complement $\mathbb{K}^d \setminus U$ is a union of some coordinate subspaces of dimension $\leq d - 2$;
- there exist a categorical quotient $p_X: U \to U//S$ and an isomorphism $\varphi: X \to U//S$;
- via isomorphism $\varphi$, the $T$-action on $X$ corresponds to the action of the quotient group $T//S$ on $U//S$.

Later this realization was generalized to a wider class of normal algebraic varieties, see [11], [4], [10]. One of the conditions that determines this class is finite generation of the divisor class group $\text{Cl}(X)$. This allows to define the Neron-Severi quasitorus $S$ of the variety $X$. The space $\mathbb{K}^d$ is replaced by an affine factorial (or, more generally, factorially graded, see [1]) $S$-variety $Z$. It is called the total coordinate space of the variety $X$. Further, $X$ appears as the quotient space of the categorical quotient $p_X: U \to U//S$, where $U$ is an open $S$-invariant subset of $Z$ such that the complement $Z \setminus U$ is of codimension at least two in $Z$. The morphism $p_X: U \to X \cong U//S$ is called the universal torsor over $X$.

Let a connected affine algebraic group $G$ act on a normal variety $X$. Passing to a finite covering we may assume that $\text{Cl}(G) = 0$ [12, Proposition 4.6]. Then the
action of $G$ on $X$ can be lifted to an action of $G$ on the total coordinate space $Z$ that commutes with the $S$-action, see [3, Section 4]. It turns out that the set $U$ is $(G \times S)$-invariant and $p_X : U \to X$ is a $G$-equivariant morphism.

**Lemma 2.1.** The following conditions are equivalent.

(i) The action of the group $G$ on $X$ is generically transitive.

(ii) The action of the group $G \times S$ on $Z$ is generically transitive.

**Proof.** Let $X_0 \subseteq X$ be an open $G$-orbit. Each point $x \in X_0$ is smooth on $X$, and thus the fiber $p_X^{-1}(x)$ is isomorphic to the quasitorus $S$ [10, Proposition 2.2, (iii)]. It shows that the group $G \times S$ acts on $p_X^{-1}(X_0)$ transitively.

Conversely, if $Z_0 \subseteq Z$ is an open $(G \times S)$-orbit, then $Z_0 \subseteq U$ and the action of $G$ on the quotient space $U/S$ is generically transitive.

Assume that the group $G$ has trivial group of characters. Then the lifting of the action of the group $G$ to $Z$ is unique, compare [3, Remark 4.1] and [9, Proposition 1.8]. Let $H$ be a closed subgroup of $G$. Every invertible regular function on the homogeneous space $G/H$ is constant, see [13, Proposition 1.2].

**Proposition 2.2.** The following conditions are equivalent.

(i) The action of the group $G$ on $X$ is generically transitive and the complement of an open $G$-orbit has codimension at least two in $X$.

(ii) The action of the group $G$ on the total coordinate space $Z$ is generically transitive.

(iii) The action of the group $G$ on the total coordinate space $Z$ is generically transitive and the complement of an open $G$-orbit has codimension at least two in $Z$.

**Proof.** We check "(i) $\Rightarrow$ (iii)". Let $X_0 \subseteq X$ be an open $G$-orbit. The condition $\text{codim}_X(X \setminus X_0) \geq 2$ implies that $p_X : p_X^{-1}(X_0) \to X_0$ is the universal torsor over $X_0$ and that the complement to $p_X^{-1}(X_0)$ in $Z$ does not contain divisors, see [2, Section 2]. By [2, Lemma 3.14] (see also [1, Theorem 4.1]), the universal torsor over a homogeneous space $G/H$ is the projection $G/H_1 \to G/H$, where $H_1$ is the intersection of kernels of all characters of the subgroup $H$. This shows that the group $G$ acts on $p_X^{-1}(X_0)$ transitively.

In order to obtain "(iii) $\Rightarrow$ (i)" note that $p_X(Z_0)$, where $Z_0$ is the open $G$-orbit in $Z$, is an open $G$-orbit in $X$ whose complement does not contain divisors. The implication "(iii) $\Rightarrow$ (ii)" is obvious.

To verify "(ii) $\Rightarrow$ (iii)" let $Z_0 \subseteq Z$ be an open $G$-orbit. Since the subset $Z_0$ is $S$-invariant, for every prime divisor $D \subset Z$ in the complement to $Z_0$ the set $S \cdot D$ is an $S$-invariant Weil divisor. Each $S$-invariant Weil divisor on $Z$ is a principal divisor $\text{div}(f)$ of a regular function $f \in \mathbb{K}[Z]$, see [10, Proposition 2.2, (iv)]. Then the non-constant function $f$ is invertible on $Z_0$, a contradiction.
The same arguments lead to the following result.

**Proposition 2.3.** The action of the group $G$ on $X$ is transitive if and only if the open subset $U \subseteq Z$ is a $G$-orbit.

3. Classification of homogeneous toric varieties

In this section we prove Theorem 1.1. Since the variety $X$ is toric, its total coordinate space $Z$ is an affine space.

**Lemma 3.1.** Let a semisimple group $G$ act on a toric variety $X$ with an open orbit. Then $X$ is non-degenerate and the action of the group $G \times S$ on the affine space $Z$ is equivalent to a linear one.

**Proof.** Since any invertible function of the open $G$-orbit is constant, the variety $X$ is non-degenerate. By Lemma 2.1, the action of the group $G \times S$ on the space $Z$ is generically transitive, and the second statement follows from [14, Proposition 5.1].

Later on we assume that $G = G_1 \times \ldots \times G_m$ acts on $X$ transitively. Denote by $V$ the total coordinate space $Z$ of the variety $X$ regarded as the $(G \times S)$-module. We proceed with a description of the $G$-module structure on $V$.

**Proposition 3.2.** Let $V = V_1 \oplus \ldots \oplus V_s$ be a decomposition into irreducible summands. Then every simple component $G_i$ acts not identically only on one summand $V_i$ (up to renumbering), and thus $m = s$. Moreover, every $G_i$ acts on the set of nonzero vectors in $V_i$ transitively.

**Proof.** By Proposition 2.3, the complement of the open $G$-orbit $U$ in $V$ is a union of coordinate subspaces (in some, possibly nonlinear, coordinate system). Thus each irreducible component of the complement is a smooth variety. The linear action of the group $G$ on $V$ commutes with the group $\mathbb{K}^\times$ of scalar operators, and the open orbit $U$ as well as any component of the complement $V \setminus U$ is $(G \times \mathbb{K}^\times)$-invariant. But a cone is a smooth variety if and only if it is a subspace. This shows that each component of $V \setminus U$ is a maximal proper submodule of $V$. In particular, the number of maximal proper submodules is finite and thus the $G$-modules $V_1, \ldots, V_s$ are pairwise non-isomorphic. The orbit $U$ is the set of vectors $v \in V$ whose projection on each $V_i$ is nonzero. This implies that the group $G$ acts on the set of nonzero vectors of each submodule $V_i$ transitively.

If several components of $G$ act on some $V_i$ not identically, then $V_i$ is isomorphic to the tensor product of simple modules of these components. Then the cone of decomposable tensors in $V_i$ is $G$-invariant, a contradiction.

Suppose that a simple component $G_i$ acts on both $V_i$ and $V_j$ not identically. Then $G_i$ acts transitively on the set of pairs $(v_i, v_j)$ with nonzero $v_i$ and $v_j$. In particular, any such pair is an eigenvector of a Borel subgroup of $G_i$. Fix a Borel subgroup $B \subset G_i$ and a highest vector for $B$ in $V_i$ as $v_i$ and a lowest vector for $B$ in $V_j$ as $v_j$. Since the intersection of two opposite parabolic subgroups of $G_i$
does not contain a Borel subgroup, we get a contradiction.

The following lemma is well known. We give a short self-contained proof suggested by the referee.

**Lemma 3.3.** Finite-dimensional rational modules of a simple group $G$ such that $G$ acts on the set of nonzero vectors transitively are

1. the tautological $\text{SL}(n)$-module $\mathbb{K}^n$ and $\text{Sp}(2n)$-module $\mathbb{K}^{2n}$;
2. the dual $\text{SL}(n)$-module $(\mathbb{K}^n)^*$.

**Proof.** Since $G$ acts on $V \setminus \{0\}$ transitively, $V$ is a simple $G$-module of highest weight $\lambda$ and $V = g_\lambda$, where $g$ is the tangent algebra of the group $G$ and $v_\lambda$ is a highest weight vector. In particular, a lowest weight vector is $v_{-\lambda} = e^{-\alpha} v_\lambda$, where $\alpha$ is a positive root, whence $\alpha = \lambda + \lambda^*$ is the highest root. This occurs only for $G = \text{SL}(n)$ with fundamental weights $\lambda = \omega_1, \omega_{n-1}$, and $G = \text{Sp}(2n)$ with $\lambda = \omega_1$.

Applying an outer automorphism of $G$, we may assume that

$$G = G_1 \times \ldots \times G_m \quad \text{and} \quad V = V_1 \oplus \ldots \oplus V_m,$$

where every component $G_i$ is either $\text{SL}(n_i)$ or $\text{Sp}(n_i)$, and $V_i$ is the tautological $G_i$-module with identical action of other components. The open $G$-orbit $U$ in $V$ coincides with the subvariety $\mathcal{X} = \mathcal{X}(n_1, \ldots, n_m)$. Therefore the variety $X$ is obtained from $\mathcal{X}$ by central factorization.

Let $S = (\mathbb{K}^*)^m$ be an algebraic torus acting on $V = V_1 \oplus \ldots \oplus V_m$ by component-wise scalar multiplication. It remains to explain why for any subgroup $S \subseteq S$ there exists a geometric quotient $\mathcal{X} \to \mathcal{X}/S$. This follows from the fact that $X$ is a homogeneous space of the group $\tilde{G} := \text{GL}(n_1) \times \ldots \times \text{GL}(n_m)$, and $S$ is a central subgroup of $\tilde{G}$. The proof of Theorem 1.1 is completed.

**Remark 3.4.** The collection $(n_1, \ldots, n_m)$ is determined by a homogeneous toric variety $X$ uniquely. Indeed, if $\mathbb{K}^d \supset U \to X$ is the Cox realization of $X$ and $C_1, \ldots, C_m$ are irreducible components of the complement $\mathbb{K}^d \setminus U$, then $n_i = d - \dim C_i$.

### 4. Properties of homogeneous toric varieties

In this section we use standard notation of toric geometry, see [7]. Let $\mathcal{N}$ be the lattice of one-parameter subgroups of a $d$-dimensional torus $T$ and $\mathcal{M}$ be the lattice of characters of $T$. The torus $T$ acts diagonally on the space $\mathbb{K}^d = V = V_1 \oplus \ldots \oplus V_m$, and $S \subseteq T$ is the $m$-dimensional subtorus acting on every $V_i$ by scalar multiplication. Identification of $T$ with $(\mathbb{K}^*)^d$ defines standard bases in $\mathcal{N}$ and $\mathcal{M}$. Moreover, the decomposition $V = V_1 \oplus \ldots \oplus V_m$ divides the standard basis of $\mathcal{N}$ into $m$ groups $I_1, \ldots, I_m$, where each group $I_j$ contains $n_j$ basis vectors and $n_j := \dim V_j$. The open subvariety $\mathcal{X}(n_1, \ldots, n_m) = U \subset V$ is a toric $T$-variety. Its fan $\mathcal{C} = \mathcal{C}(n_1, \ldots, n_m)$ in the lattice $\mathcal{N}$ consists of the cones generated by all collections of standard basis vectors that do not contain any subset $I_j$. 

Let $S \leq S$ be a closed subgroup. There is a sequence of lattices of one-parameter subgroups $\mathcal{N}_S \subseteq \mathcal{N}_S \subset \mathcal{N}$, where the lattice $\mathcal{N}_S$ is determined by the connected component $S^0$ of the quasitorus $S$. The fan $\mathcal{C}_S$ of the quotient space $\mathcal{X}/S^0$ is the image of the fan $\mathcal{C}$ under the projection

$\mathcal{N}_S \rightarrow (\mathcal{N}/\mathcal{N}_S)_Q$.

The fan $\mathcal{C}_S$ of the variety $\mathcal{X}/S$ coincides with the fan $\mathcal{C}_S$ considered with regard to an over-lattice of $\mathcal{N}/\mathcal{N}_S$ of finite index, see [7, Section 2.2]. In particular, the fan $\mathcal{C}_S$ coincides with the fan $\mathcal{P}$ of the product of projective spaces $\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_m-1}$, and $\mathcal{C}_S$ may be considered as an intermediate step of the projection:

$\mathcal{C} \rightarrow \mathcal{C}_S \rightarrow \mathcal{P}$.

Let us define a sub-lattice $\mathcal{M}_S \leq \mathcal{M}$ as the set of characters of the torus $T$ containing $S$ in the kernel. Elements of $\mathcal{M}_S$ are linear functions on the space $(\mathcal{N}/\mathcal{N}_S)_Q$.

**Proposition 4.1.** Let $X = \mathcal{X}/S$ be a homogeneous toric variety. Then

1. the variety $X$ is quasiprojective;
2. the variety $X$ is not affine;
3. the variety $X$ is projective if and only if it coincides with $\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_m-1}$;
4. the variety $X$ is quasiaffine if and only if the lattice $\mathcal{M}_S$ contains a vector with positive coordinates;
5. the variety $X$ has a nonconstant regular function if and only if the lattice $\mathcal{M}_S$ contains a nonzero vector with nonnegative coordinates.

**Proof.** (1) By Chevalley’s Theorem, any homogeneous space of an affine algebraic group is a quasiprojective variety.

(2) A toric variety obtained via Cox construction is affine if and only if $U = V$. In our situation this is not the case.

(3) Maximal dimension of a cone in the fan $\mathcal{C}$ equals $n_1 + \cdots + n_m - m$. Therefore the fan $\mathcal{C}_S$ is complete if and only if it is obtained from $\mathcal{C}$ by projection to $(\mathcal{N}/\mathcal{N}_S)_Q$, and thus $\mathcal{C}_S$ coincides with $\mathcal{P}$.

(4) A toric variety is quasiaffine if and only if its fan is a collection of faces of a strongly convex polyhedral cone. In our case, this condition implies that the projection $K$ of the support of the fan $\mathcal{C}$ to $(\mathcal{N}/\mathcal{N}_S)_Q$ is a strongly convex cone. The latter is equivalent to existence of a linear function on the space $(\mathcal{N}/\mathcal{N}_S)_Q$ that is positive on $K \setminus \{0\}$. This gives the desired element of the lattice $\mathcal{M}_S$.

Conversely, assume that the lattice $\mathcal{M}_S$ contains a vector $v$ with positive coordinates. We have to show that the projection of each cone of the fan $\mathcal{C}$ is a face of $K$. Fix proper subsets $J_1 \subset I_1, \ldots, J_m \subset I_m$ of the sets of standard basis vectors of the lattice $\mathcal{N}$. We claim that there is an element of the lattice $\mathcal{M}_S$, which vanishes on the vectors of $J_1 \cup \cdots \cup J_m$ and is positive on other standard basis.
vectors. Indeed, the sublattice $M_S$ is defined in terms of the sums of coordinates of a character over all $m$ groups of its coordinates. The desired vector should have the same sums of coordinates over the groups as the vector $v$.

(5) Since regular functions on $X$ form a rational $T$-module, one may consider only $T$-semiinvariant regular functions. Further, regular $T$-semiinvariants on $X$ correspond to characters from $M_S$ that are nonnegative on the rays of the fan $C$, see [7, Section 3.3].

**Remark 4.2.** Let $X$ be a homogeneous toric variety. Then $X$ is projective if and only if $X$ contains a $T$-fixed point. Indeed, the latter condition means that the fan $C_S$ contains a cone of full dimension, thus $N_S = N_S$ and $S = S$.

**Example 4.3.** Let $m = 2$ and $n_1 = n_2 = 2$. Then $X = (\mathbb{K}^2 \setminus \{0\}) \times (\mathbb{K}^2 \setminus \{0\})$. Set $S = \{(s, s, s, s) : s \in \mathbb{K}^\times\}$. Then

$$M_S = \{(x_1, x_2, x_3, x_4) : x_i \in \mathbb{Z}, x_1 + x_2 + x_3 + x_4 = 0\},$$

and the variety $X$ is $\mathbb{P}^3 \setminus (D_1 \cup D_2)$, where $D_i \cong \mathbb{P}^1$. If we set

$$S = \{(s, s, s^{-1}, s^{-1}) : s \in \mathbb{K}^\times\},$$

then

$$M_S = \{(x_1, x_2, x_3, x_4) : x_i \in \mathbb{Z}, x_1 + x_2 = x_3 + x_4\},$$

and $X$ is a three-dimensional quadratic cone with the apex removed.

Let us characterize the fans of homogeneous toric varieties. Let $N$ be a lattice, $\Delta$ be a fan in $N_Q$ and $P$ be the set of primitive vectors on the rays of $\Delta$. Denote by $N_0$ a sublattice of $N$ generated by $P$. Fix a positive integer $m$.

**Definition 4.4.** A fan $\Delta$ is called $m$-partite if

- the set $P$ spans the vector space $N_Q$;
- the set $P$ can be decomposed into $m$ subsets $P = I_1 \cup \ldots \cup I_m$, where each $I_j$ contains at least two elements, and the cones of $\Delta$ are exactly the cones generated by subsets $J \subseteq P$ that do not contain any $I_j$.

Set $I_j = \{e_1^j, \ldots, e_{n_j}^j\}$ and $q_j = e_1^j + \ldots + e_{n_j}^j$. Let $Q$ be a sublattice of $N$ generated by $q_1, \ldots, q_m$, and $Q_Q = Q \otimes \mathbb{Q}$.

**Proposition 4.5.** A fan $\Delta$ is the fan of a homogeneous toric variety if and only if

1. $\Delta$ is $m$-partite for some $m \geq 1$;
2. every linear relation among elements of $P$ has the form $\lambda_1 q_1 + \ldots + \lambda_m q_m = 0$ for some rational $\lambda_i$;
3. $N \subseteq N_0 + Q_Q$. 

Proof. A fan is $m$-partite if and only if it is a projection of the fan $C(n_1, \ldots, n_m)$ with some $n_i \geq 2$. Condition 2 means that the kernel of the projection is of the form $(N_{S^0})_Q$, where $S \subseteq S$. Finally, condition 3 means that $N$ is generated by $P$ and some elements

$$\frac{r_{ji}}{R_i} q_i + \ldots + \frac{r_{mi}}{R_i} q_m, \quad \text{where} \quad r_{ji} \in \mathbb{Z}_{\geq 0}, \quad R_i \in \mathbb{Z}_{>0}, \quad r_{ji} < R_i, \quad \text{and} \quad i = 1, \ldots, l.$$

Equivalently, the corresponding toric variety is obtained as the quotient of the variety $\mathcal{X}(n_1, \ldots, n_m)/S^0$ by an action of the group $\Gamma = \Gamma_1 \times \ldots \times \Gamma_l$, where $\Gamma_i$ is the cyclic group of $R_i$-th roots of unity and an element $\varepsilon \in \Gamma_i$ multiplies the $j$-th factor of $\mathcal{X}(n_1, \ldots, n_m)$ by $\varepsilon^{n_i}$.

5. Some generalizations

Let a connected reductive group $G$ act on a toric variety $X$ transitively. One may assume that $G = G^s \times L$, where $G^s$ is a simply connected semisimple group, $L$ is a central torus, and the $G$-action on $X$ is locally effective. It is well known that any toric variety $X$ is isomorphic to a direct product $X_0 \times X_1$, where $X_0$ is a non-degenerate toric variety and $X_1$ is an algebraic torus.

Let us give a construction of a transitive $G$-action on a toric variety $X$. Take a $G^s$-homogeneous toric variety $X_0$ with a locally effective and $G^s$-equivariant action of a quasisitos $L'$. Fix an inclusion $L' \subseteq L$ into an algebraic torus $L$ as a closed subgroup. The group $G = G^s \times L$ acts on $X_0 \times L$, where $G^s$ acts on the first factor and $L$ acts on the second one by multiplication. Consider the $G$-equivariant action of $L'$ on $X_0 \times L$ given by $(x_0, l) \mapsto (sx_0, s^{-1}l)$ for every $s \in L'$. Then

$$X(X_0, G^s, L', L) := (X_0 \times L)/L'$$

is a $G$-homogeneous toric variety.

Proposition 5.1. Let $X$ be a toric variety endowed with a transitive and locally effective action of a connected reductive group $G = G^s \times L$. Then the non-degenerate factor $X_0$ of $X$ is a $G^s$-homogeneous toric variety. Moreover, if $L'$ is the stabilizer of a $G^s$-orbit on $X$ in the torus $L$, then $X$ is $G$-equivariantly isomorphic to $X(X_0, G^s, L', L)$.

Proof. Since the $G^s$- and $L$-actions on $X$ commute, all $G^s$-orbits are of the same dimension. Let $Y$ be one of these orbits. Any invertible function on $Y$ is constant. Consider the above decomposition $X = X_0 \times X_1$. Since points on $X_1$ are separated by invertible functions, $Y$ is contained in a subvariety $X_0 \times \{x_1\}$, where $x_1 \in X_1$. Let $L'$ be the stabilizer of the subvariety $Y$ in the torus $L$. Then the stabilizer $H$ of a point $x \in Y$ is contained in the subgroup $G^s \times L'$ and the homogeneous space $G/H$ projects onto $G/(G^s \times L') \cong L/L'$. Points on $L/L'$ are separated by invertible functions, hence $X_0 \times \{x_1\}$ is contained in a fiber of the projection. But the fibers coincide with $G^s$-orbits on $X$. This implies $Y = X_0 \times \{x_1\}$.

Let us identify the variety $X_0$ with the subvariety $Y \subseteq X$. Consider the
morphism
\[ \varphi: X_0 \times L \to X, \quad (x_0, l) \mapsto lx_0. \]
Two pairs \((x_0, l)\) and \((\bar{x}_0, \bar{l})\) are in the same fiber of \(\varphi\) if and only if \((\bar{x}_0, \bar{l}) = (sx_0, s^{-1}l)\) with \(s = \bar{l}^{-1}l\). This shows that \(\varphi\) induces a bijective morphism \(X(X_0, G^a, L', L) \to X\). Clearly, this is an isomorphism of \(G\)-homogeneous spaces.

If the subgroup \(L'\) is connected, then \(L \cong L' \times L''\) with some complementary subtorus \(L''\), and \(X \cong X_0 \times L''\). But unlike the case of algebraic transformation groups of maximal rank [17, Theorem 2], this situation does not always occur. Indeed, one may consider a toric variety \((\mathbb{K}^2 \setminus \{0\}) \times \mathbb{K}^a\) with a transitive locally effective action of the group \(\text{SL}(2) \times \mathbb{K}^a\) given as \((g, t) \cdot (v, a) = (g(tv), t^a)\).

**Remark 5.2.** It would be interesting to generalize [17, Theorem 3] and to describe toric varieties with transitive actions of non-reductive affine algebraic groups.

Besides homogeneous toric varieties, our method allows to describe toric varieties with a generically transitive action of a semisimple group \(G\). By Lemma 2.1, they are quasitorus quotients of open subsets of generically transitive \((G \times S)\)-modules. Such modules are known as \((G \times S)\)-prehomogeneous vector spaces. For an explicit description, one needs a list of prehomogeneous vector spaces. The classification results here are known only under some restrictions on the group and on the module. For example, if \(G\) is simple and the number of irreducible summands of the module does not exceed three, the classification is given in a series of papers of M. Sato, T. Kimura, K. Ueda, T. Yoshigiaki and others.

If the complement of an open \(G\)-orbit on a toric variety \(X\) has codimension at least two in \(X\), then \(X\) comes from a \(G\)-prehomogeneous vector space (Proposition 2.2). When the group \(G\) is simple, the list of \(G\)-prehomogeneous vector spaces is obtained in [16, Theorems 7-8], and the corresponding toric varieties are described in [2, Proposition 4.7]. In contrast to the homogeneous case, here appear singular [2, Example 5.8] and non-quasiprojective [2, Example 5.9] varieties.

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