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BRIEF COMMUNICATIONS

FINITE-DIMENSIONAL SUBALGEBRAS IN POLYNOMIAL LIE ALGEBRAS OF RANK ONE

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Let $W_n(\mathbb{K})$ be the Lie algebra of derivations of the polynomial algebra $\mathbb{K}[X] := \mathbb{K}[x_1, \ldots, x_n]$ over an algebraically closed field $\mathbb{K}$ of characteristic zero. A subalgebra $L \subseteq W_n(\mathbb{K})$ is called polynomial if it is a submodule of the $\mathbb{K}[X]$-module $W_n(\mathbb{K})$. We prove that the centralizer of every nonzero element in L is abelian, provided that L is of rank one. This fact allows one to classify finite-dimensional subalgebras in polynomial Lie algebras of rank one.

Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $\mathbb{K}[X] := \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial algebra over $\mathbb{K}$. Recall that a *derivation* of $\mathbb{K}[X]$ is a linear operator $D\colon \mathbb{K}[X] \to \mathbb{K}[X]$ such that

$$D(fg) = D(f)g + fD(g) \quad \text{for all} \quad f, g \in \mathbb{K}[X].$$

Every derivation of the algebra $\mathbb{K}[X]$ has the form

$$P_1 \frac{\partial}{\partial x_1} + \ldots + P_n \frac{\partial}{\partial x_n} \quad \text{for some} \quad P_1, \ldots, P_n \in \mathbb{K}[X].$$

A derivation D can be extended to the derivation $\overline{D}$ of the field of rational functions $\mathbb{K}(X) := \mathbb{K}(x_1, \ldots, x_n)$ as follows:

$$\overline{D}\left(\frac{f}{g}\right) := \frac{D(f)g - fD(g)}{g^2}.$$

The kernel S of $\overline{D}$ is an algebraically closed subfield of $\mathbb{K}(X)$ (see Lemma 2.1 in [6]).

Denote the Lie algebra of all derivations of $\mathbb{K}[X]$ with respect to the standard commutator by $W_n(\mathbb{K})$. The study of the structure of the Lie algebra $W_n(\mathbb{K})$ and of its subalgebras is an important problem arising in various contexts (note that, in the case where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, we have the Lie algebra $W_n(\mathbb{K})$ of all vector fields with polynomial coefficients on $\mathbb{R}^n$ or $\mathbb{C}^n$). Since $W_n(\mathbb{K})$ is a free $\mathbb{K}[X]$-module, with basis $\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}$, it

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is natural to consider the subalgebras $L \subseteq W_n(\mathbb{K})$ that are $\mathbb{K}[X]$-submodules. Following the work of Buchstaber and Leykin [1], we call such subalgebras *polynomial Lie algebras*. In [1], the polynomial Lie algebras of maximal rank were considered. Earlier, Jordan studied the subalgebras of the Lie algebra $\mathrm{Der}(R)$ for a commutative ring R that are R-submodules in the R-module $\mathrm{Der}(R)$ (see [4]).

In this note, we study polynomial Lie algebras L of rank one. In Sec. 2, we prove that the centralizer of every nonzero element in L is abelian. Clearly, this property is inherited by any subalgebra in L. It is not difficult to describe all finite-dimensional Lie algebras with this property (see Proposition 2). In Theorem 1, we give a classification of finite-dimensional subalgebras in polynomial Lie algebras of rank one: every such subalgebra is either abelian, or solvable with an abelian ideal of codimension one and trivial center, or isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. Moreover, for all these three types, we construct an explicit realization in some L. Applying the obtained results to the Lie algebra $W_1(\mathbb{K})$, we give a description of all finite-dimensional subalgebras of $W_1(\mathbb{K})$ (Proposition 3). In the case $\mathbb{K} = \mathbb{C}$, this description can easily be deduced from the classic results of Lie (see [5]) about realizations (up to local diffeomorphisms) of finite-dimensional Lie algebras by vector fields on the complex line. In [5], Lie also classified analogous realizations on the complex plane and on the real line. On the real plane, such a classification was given in [2].

1. Lie Algebras with Abelian Centralizers

We begin with an elementary lemma on submodules of a free module. Let A be a unique factorization domain and let $N = Ae_1 \oplus \ldots \oplus Ae_n$ be a free A-module. An element $x \in N$ is said to be *reduced* if the condition $x = ax'$ with $a \in A$ and $x' \in N$ implies that the element a is invertible in A.

Lemma 1. *For every submodule $M \subseteq N$ of rank one, there exist an ideal $I \subseteq A$ and a reduced element $m_0 \in N$ such that $M = Im_0$. The submodule M defines the element m_0 uniquely up to multiplication by an invertible element of A.*

Proof. Take a nonzero element $m \in M$, $m = a_1e_1 + \ldots + a_ne_n$. Let a be the greatest common divisor of $a_1, \ldots, a_n$, and $m_0 = a_1^0e_1 + \ldots + a_n^0e_n$, where $a_i^0 = a_i/a$. Since M has rank one, for every nonzero $m' \in M$ there are nonzero $c, d \in A$ such that $cm + dm' = 0$. Then $acm_0 + dm' = 0$. If $m' = a_1'e_1 + \ldots + a_n'e_n$, then $aca_i^0 + da_i' = 0$ for all $i = 1, \ldots, n$. If d does not divide ac, then some prime $p \in A$ divides all elements $a_1^0, \ldots, a_n^0$. However, the elements $a_1^0, \ldots, a_n^0$ are coprime, a contradiction. Thus, m' is equal to bm_0 with $b = ac/d$. This proves that all elements of M have the form bm_0 for some $b \in A$. Clearly, all elements $b \in A$ such that $bm_0 \in M$ form an ideal I of A. The second assertion follows from the fact that a free A-module has no torsion.

Lemma 1 is proved.

We say that a derivation $P_1 \dfrac{\partial}{\partial x_1} + \ldots + P_n \dfrac{\partial}{\partial x_n}$ is *reduced* if the polynomials $P_1, \ldots, P_n$ are coprime. Setting $A = \mathbb{K}[X]$ and $N = W_n(\mathbb{K})$, we get the following version of Lemma 1:

Lemma 2. *For every submodule $M \subseteq W_n(\mathbb{K})$ of rank one, there exist an ideal $I \subseteq \mathbb{K}[X]$ and a reduced derivation $D_0 \in W_n(\mathbb{K})$ such that $M = ID_0$. The submodule M defines the derivation D_0 uniquely up to a nonzero scalar.*

We now study the centralizers of elements in a polynomial Lie algebra of rank one.

Proposition 1. *Let L be a subalgebra of the Lie algebra $W_n(\mathbb{K})$. Assume that L is a submodule of rank one in the $\mathbb{K}[X]$-module $W_n(\mathbb{K})$. Then the centralizer of any nonzero element in L is abelian.*

Proof. According to Lemma 2, the subalgebra L has the form ID_0 for some reduced derivation $D_0 \in W_n(\mathbb{K})$. Denote the extension of D_0 to the field $\mathbb{K}(X)$ by $\overline{D_0}$ and let S be the kernel of $\overline{D_0}$. Take any nonzero element $fD_0 \in L$, $f \in I$, and consider its centralizer $C = C_L(fD_0)$. For every nonzero element $gD_0 \in C$, one has

$$[fD_0, gD_0] = (fD_0(g) - gD_0(f))D_0 = 0.$$

This yields $D_0(f)g - fD_0(g) = 0$, whence $\overline{D_0}(f/g) = 0$ and $f/g \in S$. Take another nonzero element $hD_0 \in C$. By the same arguments, we get $f/h \in S$. This shows that $g/h \in S$. The latter condition is equivalent to $[gD_0, hD_0] = 0$, and so the subalgebra C is abelian.

Proposition 1 is proved.

The next proposition seems to be known, but, having no precise reference, we supply it with a complete proof. By $Z(F)$ we denote the center of a Lie algebra F.

Proposition 2. *Let F be a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{K}$ of characteristic zero. Assume that the centralizers of all nonzero elements in F are abelian. Then either F is abelian, or $F \cong A \rightthreetimes \langle b \rangle$, where $b \in F$, $A \subset F$ is an abelian ideal, and $Z(F) = 0$, or $F \cong \mathfrak{sl}_2(\mathbb{K})$.*

Proof. If the centralizers of all nonzero elements of a Lie algebra F are abelian, then the same property holds for every subalgebra of F. Assume that F is not abelian and the centralizers of all elements of F are abelian. Then the center $Z(F)$ is trivial.

Case 1. F is solvable. Then F contains a noncentral one-dimensional ideal $\langle a \rangle$ (see [3] (II.4.1, Corollary B)). Let A be the centralizer of a in F. Clearly, A is an abelian ideal of codimension one in F. Then $F \cong A \rightthreetimes \langle b \rangle$ for any $b \in F \setminus A$.

Case 2. F is semisimple. Then $F = F_1 \oplus \ldots \oplus F_k$ is the sum of simple ideals. Since the centralizer of every element $x \in F_1$ contains $F_2 \oplus \ldots \oplus F_k$, we conclude that F is simple. Let H be a Cartan subalgebra in F and let $F = N_- \oplus H \oplus N_+$ be the Cartan decomposition with opposite maximal nilpotent subalgebras N_- and N_+ in F (see [3] (II.8.1)). Since the centralizer of every element in N_+ is abelian, either the subalgebra N_+ is abelian or $Z(N_+) = 0$. The second possibility is excluded because N_+ is nilpotent. Thus, N_+ is abelian. This is the case if and only if the root system of the Lie algebra F has rank one, or, equivalently, $F \cong \mathfrak{sl}_2(\mathbb{K})$.

Case 3. F is neither solvable nor semisimple. Consider the Levi decomposition $F = R \rightthreetimes G$, where G is a maximal semisimple subalgebra and R is the radical of F. According to Case 2, the algebra G is isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. Let A denote the ideal of R that coincides with R if R is abelian, and is given by $A = [R, R]$ otherwise. According to Case 1, the ideal A is abelian. Consider the decomposition $A = A_1 \oplus \ldots \oplus A_s$ into simple G-modules with respect to the adjoint representation. If $\dim A_1 = 1$, then the centralizer of a nonzero element in A_1 contains G, a contradiction. Suppose that $\dim A_1 \geq 2$. Fix an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ in G and take the highest vector $x \in A_1$ with respect to the Borel subalgebra $\langle e, h \rangle$. Then $[e, x] = 0$ and the centralizer $C_F(x)$ contains the subalgebra $A \rightthreetimes \langle e \rangle$. The latter is not abelian because the adjoint action of the element e on A_1 is not trivial. This contradiction concludes the proof.

2. Main Results

In this section, we obtain a classification of finite-dimensional subalgebras in polynomial Lie algebras of rank one.

Theorem 1. *Let L be a polynomial Lie algebra of rank one in $W_n(\mathbb{K})$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero, and let $F \subset L$ be a finite-dimensional subalgebra. Then one of the following conditions is satisfied:*

(1) F is abelian;

(2) $F \cong A \rightthreetimes \langle b \rangle$, where $A \subset F$ is an abelian ideal and $[b, a] = a$ for every $a \in A$;

(3) F is a three-dimensional simple Lie algebra, i.e., $F \cong \mathfrak{sl}_2(\mathbb{K})$.

Proof. According to Propositions 1 and 2, every finite-dimensional subalgebra $F \subset L$ either is abelian, or has the form $A \rightthreetimes \langle b \rangle$, or is isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. It remains to prove that, in the second case, we can find $b \in F$ with $[b, a] = a$ for every $a \in A$. Take any element b with $F = A \rightthreetimes \langle b \rangle$.

Let us prove that the operator $\mathrm{ad}(b)$ is diagonalizable. Assuming the contrary, let $a_0, a_1 \in A$ be nonzero elements such that $[b, a_1] = \lambda a_1 + a_0$ and $[b, a_0] = \lambda a_0$ for some $\lambda \in \mathbb{K}$. According to Lemma 2, the subalgebra L has the form ID_0 for some ideal $I \subseteq \mathbb{K}[X]$ and some reduced derivation $D_0 \in W_n(\mathbb{K})$. We set $a_0 = fD_0$, $a_1 = gD_0$, and $b = hD_0$, $f, g, h \in I$. The relations $[b, a_1] = \lambda a_1 + a_0$, $[b, a_0] = \lambda a_0$, and $[a_0, a_1] = 0$ are equivalent to

$$hD_0(g) - gD_0(h) = \lambda g + f, \qquad hD_0(f) - fD_0(h) = \lambda f, \qquad fD_0(g) - gD_0(f) = 0.$$

Multiplying the second relation by g, we get

$$hgD_0(f) - fgD_0(h) = \lambda fg.$$

This and the third relation imply that

$$hfD_0(g) - fgD_0(h) = \lambda fg \Rightarrow hD_0(g) - gD_0(h) = \lambda g.$$

Together with the first relation, this gives $f = 0$, a contradiction.

Now assume that $[b, a_1] = \lambda_1 a_1$ and $[b, a_2] = \lambda_2 a_2$ for some $\lambda_1, \lambda_2 \in \mathbb{K}$. If $a_1 = fD_0$, $a_2 = gD_0$, and $b = hD_0$, then

$$hD_0(f) - fD_0(h) = \lambda_1 f, \qquad hD_0(g) - gD_0(h) = \lambda_2 g, \qquad fD_0(g) - gD_0(f) = 0.$$

Consequently,

$$ghD_0(f) = gf(\lambda_1 + D_0(h)) = fhD_0(g) = fg(\lambda_2 + D_0(h)).$$

This proves that $\lambda_1 = \lambda_2$, and, hence, $\mathrm{ad}(b)$ is a scalar operator. Since F is not abelian, we conclude that $\mathrm{ad}(b)$ is nonzero, and, multiplying b by a suitable scalar, we may assume that $\mathrm{ad}(b)$ is the identical operator.

Theorem 1 is proved.

Let us show that all three possibilities indicated in Theorem 1 are realizable. Take a derivation $D_0 \in W_n(\mathbb{K})$ such that there exist nonconstant polynomials $p, q \in \mathbb{K}[X]$ with $D_0(p) = 0$ and $D_0(q) = 1$. For example, one may take $p = x_1$, $q = x_2$, and

$$D_0 = \frac{\partial}{\partial x_2} + P_3 \frac{\partial}{\partial x_3} + \ldots + P_n \frac{\partial}{\partial x_n}$$

with arbitrary $P_3, \ldots, P_n \in \mathbb{K}[X]$.

The subalgebra $\langle D_0, pD_0, \ldots, p^{m-1}D_0\rangle$ is an m-dimensional abelian subalgebra in $\mathbb{K}[X]D_0$ for every positive integer m.

The subalgebra $A \rtimes \langle b\rangle$ with $\dim A = m$ may be obtained by setting $A = \langle D_0, pD_0, \ldots, p^{m-1}D_0\rangle$ and $b = -qD_0$. Indeed,

$$[-qD_0, f(p)D_0] = (-D_0(f(p)) + f(p)D_0(q))D_0 = f(p)D_0 \quad \text{for every} \quad f(p) \in \mathbb{K}[p].$$

Finally, the derivations $e = q^2D_0$, $h = 2qD_0$, and $f = -D_0$ form an $\mathfrak{sl}_2$-triple in $\mathbb{K}[X]D_0$.

Remark 1. The structure of finite-dimensional subalgebras in a polynomial Lie algebra $L = ID_0$ depends on properties of the derivation D_0. In particular, if $\mathrm{Ker}(\overline{D_0}) = \mathbb{K}$, then all abelian subalgebras in $\mathbb{K}[X]D_0$ are one-dimensional.

Our last result concerns finite-dimensional subalgebras in the Lie algebra $W_1(\mathbb{K})$. According to Lemma 2, every polynomial Lie algebra in $W_1(\mathbb{K})$ has the form

$$L = q(x)\mathbb{K}[x]\frac{\partial}{\partial x}$$

with some polynomial $q(x) \in \mathbb{K}[x]$.

Proposition 3. *Let*

$$L = q(x)\mathbb{K}[x]\frac{\partial}{\partial x}$$

be a polynomial algebra. Then the following assertions are true:

1. *If* $\deg q(x) \geq 2$, *then every finite-dimensional Lie subalgebra in* L *is one-dimensional.*
2. *If* $\deg q(x) = 1$, *then every finite-dimensional Lie subalgebra in* L *either is one-dimensional or coincides with*

$$F_k = \left\langle q(x)\frac{\partial}{\partial x}, q(x)^k\frac{\partial}{\partial x}\right\rangle$$

   for some $k \geq 2$.

3. *If* $q(x) = \mathrm{const} \neq 0$ (*i.e.,* $L = W_1(\mathbb{K})$), *then every finite-dimensional Lie subalgebra in* L *either is one-dimensional, or coincides with*

$$F_{k,\beta} = \left\langle (x+\beta)\frac{\partial}{\partial x}, (x+\beta)^k\frac{\partial}{\partial x}\right\rangle$$

   for some $\beta \in \mathbb{K}$ *and* $k = 0, 2, 3, \ldots,$ *or is a three-dimensional subalgebra*

$$F(\beta) = \left\langle \frac{\partial}{\partial x},\ (x+\beta)\frac{\partial}{\partial x},\ (x+\beta)^2\frac{\partial}{\partial x}\right\rangle,$$

   where $\beta \in \mathbb{K}$.

Proof. Let us describe all two-dimensional subalgebras in $W_1(\mathbb{K})$. Every such subalgebra has the form

$$\left\langle f(x)\frac{\partial}{\partial x},\ g(x)\frac{\partial}{\partial x}\right\rangle \quad \text{with} \quad f(x),\ g(x) \in \mathbb{K}[x] \quad \text{and} \quad fg' - f'g = g. \tag{$*$}$$

If $\deg(f) \geq 2$, then, looking at the highest terms of fg' and $f'g$, we get $\deg(f) = \deg(g)$. However, the polynomials $(f + \lambda g, g)$ satisfy relation $(*)$ for every $\lambda \in \mathbb{K}$, and we may thus assume that f is linear. Each root of g is also a root of f, and so g is proportional to f^k for some $k = 0, 2, 3, \dots$. This observation, together with Theorem 1 and Remark 1, proves all the assertions.

Proposition 3 is proved.

If we consider the realizations obtained in Proposition 3 up to automorphisms of the polynomial ring $\mathbb{K}[x]$, then one can take $q(x) = x$ in the case $\deg q(x) = 1$ for the Lie algebra F_k, and $\beta = 0$ in the case $q(x) = \mathrm{const} \neq 0$.

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