

Additive Actions on Projective Hypersurfaces

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Abstract By an additive action on a hypersurface H in \mathbb{P}^{n+1} we mean an effective action of a commutative unipotent group on \mathbb{P}^{n+1} which leaves H invariant and acts on H with an open orbit. Brendan Hassett and Yuri Tschinkel have shown that actions of commutative unipotent groups on projective spaces can be described in terms of local algebras with some additional data. We prove that additive actions on projective hypersurfaces correspond to invariant multilinear symmetric forms on local algebras. It allows us to obtain explicit classification results for non-degenerate quadrics and quadrics of corank one.

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1 Introduction

Let \mathbb{K} be an algebraically closed field of characteristic zero and \mathbb{G}_a the additive group $(\mathbb{K}, +)$. Consider the commutative unipotent affine algebraic group \mathbb{G}_a^n . In other words, \mathbb{G}_a^n is the additive group of an n -dimensional vector space over \mathbb{K} . The first aim of this paper is to survey recent results on actions of \mathbb{G}_a^n with an open orbit on projective algebraic varieties. To this end we include a detailed proof of the Hassett–Tschinkel correspondence, discuss its corollaries, interpretations, and related examples. Also we develop the method of Hassett and Tschinkel to show that the generically transitive actions of the group \mathbb{G}_a^n on projective hypersurfaces correspond to invariant multilinear symmetric forms on finite-dimensional local algebras. This leads to explicit classification results for non-degenerate quadrics and quadrics of corank one.

By an additive action on a variety X we mean a faithful regular action of the group \mathbb{G}_a^n on X such that one of the orbits is open in X . The study of such actions was initiated by Brendan Hassett and Yuri Tschinkel [11]. They showed that additive actions on the projective space \mathbb{P}^n up to equivalence are in bijection with isomorphism classes of local algebras of \mathbb{K} -dimension $n + 1$. In particular, the number of additive actions on \mathbb{P}^n is finite if and only if $n \leq 5$.

Additive actions on projective subvarieties $X \subseteq \mathbb{P}^m$ induced by an action $\mathbb{G}_a^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m$ can be described in terms of local $(m + 1)$ -dimensional algebras equipped with some additional data. This approach was used in [2, 15] to classify additive actions on projective quadrics. Elena Sharoiko proved in [15, Theorem 4] that an additive action on a non-degenerate quadric $Q \subseteq \mathbb{P}^{n+1}$ is unique up to equivalence. Recently Baohua Fu and Jun-Muk Hwang [9] used a different technique to show the uniqueness of additive action on a class of Fano varieties with Picard number 1. This result shows that an abundance of additive actions on the projective space should be considered as an exception.

A variety with a given additive action looks like an “additive analogue” of a toric variety. Unfortunately, it turns out that two theories have almost no parallels, see [2, 11].

Generalized flag varieties G/P of a semisimple algebraic group G admitting an additive action are classified in [1]. Roughly speaking, an additive action on G/P exists if and only if the unipotent radical P^u of the parabolic subgroup P is commutative. The uniqueness result in this case follows from [9]. In particular, it covers the case of Grassmannians and thus answers a question posed in [2]. Another proof of the uniqueness of additive actions on generalized flag varieties is obtained by Rostislav Devyatov [7]. It uses nilpotent multiplications on certain finite-dimensional modules over semisimple Lie algebras.

Evgeny Feigin proposed a construction based on the PBW-filtration to degenerate an arbitrary generalized flag variety G/P to a variety with an additive action, see [8] and further publications.

In [5], Ulrich Derenthal and Daniel Loughran classified singular del Pezzo surfaces with additive actions; see also [6]. By the results of [4], Manin’s Conjecture is true for such surfaces.

In this paper we prove that additive actions on projective hypersurfaces of degree d in \mathbb{P}^{n+1} are in bijection with invariant d -linear symmetric forms on $(n + 2)$ -dimensional local algebras. The corresponding form is the polarization of the equation defining the hypersurface. As an application, we give a short proof of uniqueness of additive action on non-degenerate quadrics and classify additive actions on quadrics of corank one. The case of cubic projective hypersurfaces is studied in the recent preprint of Ivan Bazhov [3].

The paper is organized as follows. In Sect. 2 we define additive actions and consider the problem of extension of an action $\mathbb{G}_a^n \times X \rightarrow X$ on a projective hypersurface X to the ambient space \mathbb{P}^{n+1} . The Hassett–Tschinkel correspondence is discussed in Sect. 3. Section 4 is devoted to invariant multilinear symmetric forms on local algebras. Our main result (Theorem 2) describes additive actions on projective hypersurfaces in these terms. Also we give an explicit formula for an invariant multilinear symmetric form (Lemma 1) and prove that if a hypersurface X in \mathbb{P}^{n+1} admits an additive action and the group $\text{Aut}(X)^0$ is reductive, then X is either a hyperplane or a non-degenerate quadric (Proposition 5). Additive actions on non-degenerate quadrics and on quadrics of corank one are classified in Sect. 5 and Sect. 6, respectively.

2 Additive Actions on Projective Varieties

Let X be an irreducible algebraic variety of dimension n and \mathbb{G}_a^n be the commutative unipotent group.

Definition 1. An *inner additive action* on X is an effective action $\mathbb{G}_a^n \times X \rightarrow X$ with an open orbit.

It is well known that for an action of a unipotent group on an affine variety all orbits are closed, see, e.g., [14, Sect. 1.3]. It implies that if an affine variety X admits an additive action, then X is isomorphic to the group \mathbb{G}_a^n with the \mathbb{G}_a^n -action by left translations.

In general, the existence of an inner additive action implies that the variety X is rational. For X normal, the divisor class group $\text{Cl}(X)$ is freely generated by prime divisors in the complement of the open \mathbb{G}_a^n -orbit. In particular, $\text{Cl}(X)$ is a free finitely generated abelian group.

The most interesting case is the study of inner additive actions on complete varieties X . In this case an inner additive action determines a maximal commutative unipotent subgroup of the linear algebraic group $\text{Aut}(X)^0$. Two inner additive actions are said to be *equivalent*, if the corresponding subgroups are conjugate in $\text{Aut}(X)$.

Proposition 1. *Let X be a complete variety with an inner additive action. Assume that the group $\text{Aut}(X)^0$ is reductive. Then X is a generalized flag variety G/P , where G is a linear semisimple group and P is a parabolic subgroup.*

Proof. Let X' be the normalization of X . Then the action of $\text{Aut}(X)^0$ lifts to X' . By the assumption, some unipotent subgroup of $\text{Aut}(X)^0$ acts on X' with an open orbit. Then a maximal unipotent subgroup of the reductive group $\text{Aut}(X)^0$ acts on X' with an open orbit. It means that X' is a spherical variety of rank zero, see [16, Sect. 1.5.1] for details. It yields that X' is a generalized flag variety G/P , see [16, Proposition 10.1], and $\text{Aut}(X)^0$ acts on X' transitively. The last condition implies that $X = X'$. \square

A classification of generalized flag varieties admitting an inner additive action is obtained in [1]. In particular, the parabolic subgroup P is maximal in this case.

Definition 2. Let X be a closed subvariety of dimension n in the projective space \mathbb{P}^m . Then an *additive action* on X is an effective action $\mathbb{G}_a^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m$ such that X is \mathbb{G}_a^n -invariant and the induced action $\mathbb{G}_a^n \times X \rightarrow X$ has an open orbit. Two additive actions on X are said to be *equivalent* if one is obtained from another via automorphism of \mathbb{P}^m preserving X .

Clearly, any additive action on a projective subvariety X induces an inner additive action on X . The converse is not true, i.e., not any action $\mathbb{G}_a^n \times X \rightarrow X$ with an open orbit on a projective subvariety X can be extended to the ambient space \mathbb{P}^m .

Example 1. Consider a subvariety

$$X = V(x^2z - y^3) \subseteq \mathbb{P}^2$$

and a rational \mathbb{G}_a -action on X given by

$$\left(\frac{y}{x}, a \right) \mapsto \frac{y}{x} + a.$$

Using affine charts one can check that this action is regular. On the other hand, it cannot be extended to \mathbb{P}^2 , because the closure of a \mathbb{G}_a -orbit on \mathbb{P}^2 cannot be a cubic, see Example 2.

At the same time, if the subvariety X is linearly normal in \mathbb{P}^m and X is normal, then an extension of a \mathbb{G}_a^n -action to \mathbb{P}^m exists. Indeed, the restriction $L =: \mathcal{O}(1)|_X$ of the line bundle $\mathcal{O}(1)$ on \mathbb{P}^m can be linearized with respect to the action $\mathbb{G}_a^n \times X \rightarrow X$, see, e.g., [12]. The linearization defines a structure of a rational \mathbb{G}_a^n -module on the space of sections $H^0(X, L)$. Since X is linearly normal, the restriction $H^0(\mathbb{P}^m, \mathcal{O}(1)) \rightarrow H^0(X, L)$ is surjective. Consider a vector space decomposition $H^0(\mathbb{P}^m, \mathcal{O}(1)) = V_1 \oplus V_2$, where V_1 is the kernel of the restriction. The complementary subspace V_2 projects to $H^0(X, L)$ isomorphically. This isomorphism induces a structure of a rational \mathbb{G}_a^n -module on V_2 . Further, we regard V_1 as the trivial \mathbb{G}_a^n -module. This gives a structure of a rational \mathbb{G}_a^n -module on

$H^0(\mathbb{P}^m, \mathcal{O}(1))$. Since \mathbb{P}^m is the projectivization of the dual space to $H^0(\mathbb{P}^m, \mathcal{O}(1))$, we obtain a required extended action $\mathbb{G}_a^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m$.

From now on we consider additive actions on projective subvarieties $X \subseteq \mathbb{P}^m$.

3 The Hassett–Tschinkel Correspondence

In [11] Brendan Hassett and Yuri Tschinkel established a remarkable correspondence between additive actions on the projective space \mathbb{P}^n and local algebras of \mathbb{K} -dimension $n + 1$. Moreover, they described rational cyclic \mathbb{G}_a^n -modules in terms of local algebras. In this section we recall these results. The proofs given here are taken from [2]. By a local algebra we always mean a commutative associative local algebra with unit.

Let $\rho : \mathbb{G}_a^n \rightarrow \mathrm{GL}_{m+1}(\mathbb{K})$ be a faithful rational representation. The differential defines a representation $d\rho : \mathfrak{g} \rightarrow \mathfrak{gl}_{m+1}(\mathbb{K})$ of the tangent algebra $\mathfrak{g} = \mathrm{Lie}(\mathbb{G}_a^n)$ and the induced representation $\tau : U(\mathfrak{g}) \rightarrow \mathrm{Mat}_{m+1}(\mathbb{K})$ of the universal enveloping algebra $U(\mathfrak{g})$. Since the group \mathbb{G}_a^n is commutative, the algebra $U(\mathfrak{g})$ is isomorphic to the polynomial algebra $\mathbb{K}[x_1, \dots, x_n]$, where \mathfrak{g} is identified with the subspace $\langle x_1, \dots, x_n \rangle$. The algebra $R := \tau(U(\mathfrak{g}))$ is isomorphic to the factor algebra $U(\mathfrak{g}) / \mathrm{Ker} \tau$. As $\tau(x_1), \dots, \tau(x_n)$ are commuting nilpotent operators, the algebra R is finite-dimensional and local. Let us denote by X_1, \dots, X_n the images of the elements x_1, \dots, x_n in R . Then the maximal ideal of R is $\mathfrak{m} := (X_1, \dots, X_n)$. The subspace $W := \tau(\mathfrak{g}) = \langle X_1, \dots, X_n \rangle$ generates R as an algebra with unit.

Assume that \mathbb{K}^{m+1} is a cyclic \mathbb{G}_a^n -module with a cyclic vector v , i.e., $\langle \rho(\mathbb{G}_a^n)v \rangle = \mathbb{K}^{m+1}$. The subspace $\tau(U(\mathfrak{g}))v$ is \mathfrak{g} - and \mathbb{G}_a^n -invariant; it contains the vector v and therefore coincides with the space \mathbb{K}^{m+1} . Let $I = \{y \in U(\mathfrak{g}) : \tau(y)v = 0\}$. Since the vector v is cyclic, the ideal I coincides with $\mathrm{Ker} \tau$, and we obtain identifications

$$R \cong U(\mathfrak{g})/I \cong \tau(U(\mathfrak{g}))v = \mathbb{K}^{m+1}.$$

Under these identifications the action of an element $\tau(y)$ on \mathbb{K}^{m+1} corresponds to the operator of multiplication by $\tau(y)$ on the factor algebra R , and the vector $v \in \mathbb{K}^{m+1}$ goes to the residue class of unit. Since $\mathbb{G}_a^n = \exp(\mathfrak{g})$, the \mathbb{G}_a^n -action on \mathbb{K}^{m+1} corresponds to the multiplication by elements of $\exp(W)$ on R .

Conversely, let R be a local $(m + 1)$ -dimensional algebra with a maximal ideal \mathfrak{m} , and $W \subseteq \mathfrak{m}$ be a subspace that generates R as an algebra with unit. Fix a basis X_1, \dots, X_n in W . Then R admits a presentation $\mathbb{K}[x_1, \dots, x_n]/I$, where I is the kernel of the homomorphism

$$\mathbb{K}[x_1, \dots, x_n] \rightarrow R, x_i \mapsto X_i.$$

These data define a faithful representation ρ of the group $\mathbb{G}_a^n := \exp(W)$ on the space R : the operator $\rho((a_1, \dots, a_n))$ acts as multiplication by the element $\exp(a_1 X_1 + \dots + a_n X_n)$. Since W generates R as an algebra with unit, one checks that the representation is cyclic with unit in R as a cyclic vector.

Summarizing, we obtain the following result.

Theorem 1 ([11, Theorem 2.14]). *The correspondence described above establishes a bijection between*

(1) *equivalence classes of faithful cyclic rational representations*

$$\rho : \mathbb{G}_a^n \rightarrow \mathrm{GL}_{m+1}(\mathbb{K});$$

(2) *isomorphism classes of pairs (R, W) , where R is a local $(m + 1)$ -dimensional algebra with the maximal ideal \mathfrak{m} and W is an n -dimensional subspace of \mathfrak{m} that generates R as an algebra with unit.*

Remark 1. Let $\rho : \mathbb{G}_a^n \rightarrow \mathrm{GL}_{m+1}(\mathbb{K})$ be a faithful cyclic rational representation. The set of cyclic vectors in \mathbb{K}^{m+1} is an open orbit of a commutative algebraic group C with $\rho(\mathbb{G}_a^n) \subseteq C \subseteq \mathrm{GL}_{m+1}(\mathbb{K})$, and the complement of this set is a hyperplane. In our notation, the group C is the extension of the commutative unipotent group $\exp(\mathfrak{m}) \cong \mathbb{G}_a^m$ by scalar matrices.

A faithful linear representation $\rho : \mathbb{G}_a^n \rightarrow \mathrm{GL}_{m+1}(\mathbb{K})$ determines an effective action of the group \mathbb{G}_a^n on the projectivization \mathbb{P}^m of the space \mathbb{K}^{m+1} . Conversely, let G be a connected affine algebraic group with the trivial Picard group, and X be a normal G -variety. By [12, Sect. 2.4], every line bundle on X admits a G -linearization. Moreover, if G has no non-trivial characters, then a G -linearization is unique. This shows that every effective \mathbb{G}_a^n -action on \mathbb{P}^m comes from a (unique) faithful rational $(m + 1)$ -dimensional \mathbb{G}_a^n -module.

An effective \mathbb{G}_a^n -action on \mathbb{P}^m has an open orbit if and only if $n = m$. In this case the corresponding \mathbb{G}_a^n -module is cyclic. In terms of Theorem 1 the condition $n = m$ means $W = \mathfrak{m}$, and we obtain the following theorem.

Proposition 2 ([11, Proposition 2.15]). *There is a one-to-one correspondence between*

(1) *equivalence classes of additive actions on \mathbb{P}^n ;*

(2) *isomorphism classes of local $(n + 1)$ -dimensional algebras.*

Remark 2. It follows from Remark 1 that if the group \mathbb{G}_a^n acts on \mathbb{P}^m and some orbit is not contained in a hyperplane, then the action can be extended to an additive action $\mathbb{G}_a^m \times \mathbb{P}^m \rightarrow \mathbb{P}^m$. It seems that such an extension exists without any extra assumption.

Given the projectivization \mathbb{P}^m of a faithful rational \mathbb{G}_a^n -module and a point $x \in \mathbb{P}^m$ with the trivial stabilizer, the closure X of the orbit $\mathbb{G}_a^n \cdot x$ is a projective variety equipped with an additive action. Closures of generic orbits are hypersurfaces if and only if $n = m - 1$. If such a hypersurface is not a hyperplane, then \mathbb{P}^m comes from the projectivization of a cyclic \mathbb{G}_a^n -module, it is given by a pair (R, W) , and the condition $n = m - 1$ means that W is a hyperplane in \mathfrak{m} . We obtain the following result.

Proposition 3. *There is a one-to-one correspondence between*

- (1) *equivalence classes of additive actions on hypersurfaces in \mathbb{P}^{n+1} of degree at least two;*
- (2) *isomorphism classes of pairs (R, W) , where R is a local $(n + 2)$ -dimensional algebra with the maximal ideal \mathfrak{m} and W is a hyperplane in \mathfrak{m} that generates R as an algebra with unit.*

It is shown in [2, Theorem 5.1] that the degree of the hypersurface corresponding to a pair (R, W) is the maximal exponent d such that the subspace W does not contain the ideal \mathfrak{m}^d .

Example 2. There exist two 3-dimensional local algebras,

$$\mathbb{K}[x]/(x^3) \quad \text{and} \quad \mathbb{K}[x, y]/(x^2, xy, y^2).$$

In the first case $\mathfrak{m}^3 = 0$, and in the second one we have $\mathfrak{m}^2 = 0$. This shows that for every \mathbb{G}_a -action on \mathbb{P}^2 the orbit closures are either lines or quadrics.

4 Invariant Multilinear Forms on Local Algebras

Consider a pair (R, W) as in Proposition 3 and let $H \subseteq \mathbb{P}^{n+1}$ be the corresponding hypersurface. Let us fix a coordinate system on $R = \langle 1 \rangle \oplus \mathfrak{m}$ such that x_0 is the coordinate along $\langle 1 \rangle$ and x_1, \dots, x_{n+1} are coordinates on \mathfrak{m} .

Assume that H is defined by a homogeneous equation

$$f(x_0, x_1, \dots, x_{n+1}) = 0$$

of degree d . Since H is invariant under the action of \mathbb{G}_a^n , the polynomial f is \mathbb{G}_a^n -semi-invariant [14, Theorem 3.1]. But the group \mathbb{G}_a^n has no non-trivial characters, and the polynomial f is \mathbb{G}_a^n -invariant. Equivalently, f is annihilated by the Lie algebra \mathfrak{g} .

It is well known that for a given homogeneous polynomial f of degree d on a vector space R there exists a unique d -linear symmetric map

$$F: R \times R \times \dots \times R \rightarrow \mathbb{K}$$

such that $f(v) = F(v, v, \dots, v)$ for all $v \in R$, see, e.g., [14, Sect. 9.1]. The map F is called the *polarization* of the polynomial f .

Since the representation $d\rho$ of the Lie algebra \mathfrak{g} on R is given by multiplication by elements of W , a homogeneous polynomial f on R is annihilated by \mathfrak{g} if and only if

$$F(ab_1, b_2, \dots, b_d) + F(b_1, ab_2, \dots, b_d) + \dots + F(b_1, b_2, \dots, ab_d) = 0 \quad \forall a \in W, \\ b_1, \dots, b_d \in R. \tag{1}$$

Definition 3. Let R be a local algebra with the maximal ideal \mathfrak{m} . An *invariant d -linear form* on R is a d -linear symmetric map

$$F: R \times R \times \cdots \times R \rightarrow \mathbb{K}$$

such that $F(1, 1, \dots, 1) = 0$, the restriction of F to \mathfrak{m} is nonzero, and there exists a hyperplane W in \mathfrak{m} which generates R as an algebra with unit and such that condition (1) holds.

If F_1 (resp. F_2) are invariant d_1 -linear (resp. d_2 -linear) forms on R with respect to the same hyperplane W , then the product $F_1 F_2$ defines an invariant $(d_1 + d_2)$ -linear form. An invariant multilinear form is said to be *irreducible*, if it cannot be represented as such a product.

One can show that there is no invariant linear form. It implies that any invariant bilinear or 3-linear form is irreducible.

We are ready to formulate our main result.

Theorem 2. *Additive actions on hypersurfaces of degree $d \geq 2$ in \mathbb{P}^{n+1} are in natural one-to-one correspondence with pairs (R, F) , where R is a local algebra of dimension $n + 2$ and F is an irreducible invariant d -linear form on R up to a scalar.*

Proof. An additive action on a hypersurface H in \mathbb{P}^{n+1} is given by a faithful rational representation $\rho: \mathbb{G}_a^n \rightarrow \mathrm{GL}_{n+2}(\mathbb{K})$ making \mathbb{K}^{n+2} a cyclic \mathbb{G}_a^n -module. In our correspondence we identify \mathbb{K}^{n+2} with the local algebra R . We choose coordinates x_0, x_1, \dots, x_{n+1} compatible with the decomposition $R = \langle 1 \rangle \oplus \mathfrak{m}$. Let $f(x_0, x_1, \dots, x_{n+1}) = 0$ be the equation of the hypersurface H , where f is irreducible. Then the algebra of \mathbb{G}_a^n -invariants on \mathbb{K}^{n+2} is freely generated by x_0 and f .

Every \mathbb{G}_a^n -invariant hypersurface in \mathbb{P}^{n+1} is given by

$$\alpha f(x_0, x_1, \dots, x_{n+1}) + \beta x_0^d = 0, \quad (\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\}.$$

So we may assume that f does not contain the term x_0^d . Let F be the polarization of f . Then condition (1) holds and $F(1, \dots, 1) = 0$. If the restriction of F to \mathfrak{m} is zero, then x_0 divides f , and f is not irreducible, a contradiction.

Conversely, let (R, F) be such a pair and W be a subspace from the definition of F . Then (R, W) gives rise to a structure of a rational \mathbb{G}_a^n -module on R . Consider the hypersurface $f = 0$ in $\mathbb{P}(R) \cong \mathbb{P}^{n+1}$, where $f(v) = F(v, v, \dots, v)$. It is invariant under the action of \mathbb{G}_a^n . By the assumptions, f is irreducible and thus the hypersurface $f = 0$ coincides with the closure of a generic \mathbb{G}_a^n -orbit. \square

Given a hyperplane W in the maximal ideal \mathfrak{m} of a local algebra R that generates R as an algebra with unit, let d be the maximal exponent such that the subspace W does not contain the ideal \mathfrak{m}^d . By Theorem 2 and [2, Theorem 5.1], there exists a unique up to a scalar irreducible invariant (with respect to W) d -linear form F_W on R . Let us write down this form explicitly.

By linearity, we may assume that each argument of F_W is either the unit 1 or an element of \mathfrak{m} . Fix an isomorphism $\mathfrak{m}/W \cong \mathbb{K}$ and consider the projection $\pi: \mathfrak{m} \rightarrow \mathfrak{m}/W \cong \mathbb{K}$. We define the form

$$F_W(b_1, \dots, b_d) := (-1)^k k!(d-k-1)! \pi(b_1 \dots b_d),$$

where k is the number of units among b_1, \dots, b_d , and for $k = d$ we let $F_W(1, \dots, 1) = 0$.

Lemma 1. *The form F_W is an irreducible invariant d -linear form on R .*

Proof. We begin with condition (1). Since $ab_i \in \mathfrak{m}$ for all $a \in W$ and $b_i \in R$, with $0 < k < d$ this condition can be rewritten as

$$(k(-1)^{k-1}(k-1)!(d-k)! + (d-k)(-1)^k k!(d-k-1)!) \pi(ab_1 \dots b_d) = 0,$$

and it is obvious. For $k = 0$ we have $ab_1 \dots b_d \in \mathfrak{m}^{d+1} \subseteq W$, and thus $\pi(ab_1 \dots b_d) = 0$. For $k = d$ we have $-(d-2)! \pi(a) = 0$, because $a \in W$.

The restriction of F to \mathfrak{m} is nonzero since \mathfrak{m}^d is not contained in W . It follows from [2, Theorem 5.1] that the form F is irreducible. Finally, we have $F_W(1, \dots, 1) = 0$ by definition. \square

The next proposition follows immediately from [13, Theorems 1 and 4]. Let us obtain this result using our technique.

Proposition 4. *Let H be a smooth hypersurface in \mathbb{P}^{n+1} admitting an additive action. Then H is either a hyperplane or a non-degenerate quadric.*

Proof. Assume that an additive action on H is given by a triple (R, W, F) . Let e_0, e_1, \dots, e_{n+1} be a basis of R compatible with the decomposition

$$R = \langle 1 \rangle \oplus W \oplus \langle e_{n+1} \rangle.$$

Moreover, we may assume that $e_{n+1} \in \mathfrak{m}^d$, where \mathfrak{m}^{d+1} is contained in W . Then in the notation of Lemma 1 we have $\pi(be_{n+1}) = 0$ for all $b \in \mathfrak{m}$. It means that the variable x_{n+1} can appear in the equation $f(x_0, \dots, x_{n+1}) = 0$ of the hyperplane H only in the term $x_0^{d-1} x_{n+1}$. Thus the point $[0 : \dots : 0 : 1]$ lies on H and it is singular provided $d \geq 3$. It remains to note that the only smooth quadric is a non-degenerate one. \square

Proposition 5. *Let H be a hypersurface in \mathbb{P}^{n+1} which admits an additive action and such that the group $\text{Aut}(H)^0$ is reductive. Then H is either a hyperplane or a non-degenerate quadric.*

Proof. By Proposition 1, the variety H is smooth, and the assertion follows from Proposition 4. \square

Remark 3. Take a triple (R, W, F) as in Definition 3 and consider the sum I of all ideals of the algebra R contained in W . It is the biggest ideal of R contained in W . Taking a compatible basis of R , we see that the equation of the corresponding hypersurface does not depend on the coordinates in I . Moreover, for the factor algebra $R' = R/I$ we have

$$R' = \langle 1' \rangle \oplus W' \oplus (\mathfrak{m}')^d$$

with $\mathfrak{m}' = \mathfrak{m}/I$, $W' = W/I$, and $\dim(\mathfrak{m}')^d = 1$. The invariant form F descends to R' , the subspace W' contains no ideal of R' , and the algebra R' is Gorenstein. Such a reduction is useful in classification problems.

5 Non-degenerate Quadrics

In this section we classify non-degenerate invariant bilinear symmetric forms on local algebras. These results are obtained in [15], but we give a short elementary proof.

Let R be a local algebra of dimension $n + 2$ with the maximal ideal \mathfrak{m} and F a non-degenerate bilinear symmetric form on R such that $F(1, 1) = 0$. Assume that for some hyperplane W in \mathfrak{m} generating R we have

$$F(ab_1, b_2) + F(b_1, ab_2) = 0 \quad \text{for all } b_1, b_2 \in R \text{ and } a \in W. \quad (2)$$

We choose a basis $e_0 = 1, e_1, \dots, e_n, e_{n+1}$ of R such that $W = \langle e_1, \dots, e_n \rangle$ and $\mathfrak{m} = \langle e_1, \dots, e_{n+1} \rangle$. For any $b \in R$ let $b = b^{(0)} + b^{(1)} + \dots + b^{(n+1)}$ be the decomposition corresponding to this basis.

- Lemma 2.** (1) $F(1, a) = 0$ for all $a \in W$;
(2) $F(1, b) = F(1, b^{(n+1)})$ for all $b \in R$;
(3) If $a, a' \in W$ and $aa' \in W$, then $aa' = 0$;
(4) The restriction of the form F to W is non-degenerate.

Proof. Assertion (1) follows from (2) with $b_1 = b_2 = 1$. For (2), note that

$$F(1, b) = F(1, b^{(0)}) + F(1, b^{(1)}) + \dots + F(1, b^{(n)}) + F(1, b^{(n+1)}).$$

The first term is 0 because of $F(1, 1) = 0$, and the second one is 0 by (1). If $a, a', aa' \in W$, then for any $b \in R$ we have

$$F(b, aa') = -F(ab, a') = F(aa'b, 1) = -F(b, aa'),$$

and $F(b, aa') = 0$. Since F is non-degenerate, we obtain (3).

For (4), assume that for some $0 \neq a \in W$ we have $F(a, a') = 0$ for all $a' \in W$. Since F is non-degenerate, it yields $F(a, e_{n+1}) = \lambda$ for some nonzero $\lambda \in \mathbb{K}$. If $F(1, e_{n+1}) = \mu$, then $F(\mu a - \lambda 1, e_{n+1}) = 0$, and the vector $\mu a - \lambda 1$ is in the kernel of the form F . \square

Let us denote by $M(F)$ the matrix of a bilinear form F in a given basis.

Proposition 6. *In the notation as above, the triple (R, W, F) can be transformed into the form*

$$R = \mathbb{K}[e_1, \dots, e_n]/(e_i^2 - e_j^2, e_i e_j; 1 \leq i < j \leq n), \quad W = \langle e_1, \dots, e_n \rangle,$$

$$M(F) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Proof. Since F is non-degenerate, we may assume that $F(1, e_{n+1}) = 1$. Using Lemma 2, (4), we may suppose that $F(e_i, e_j) = -\delta_{ij}$ for all $1 \leq i, j \leq n$. Now the matrix of the form F looks like

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & -1 & \dots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & * \\ 1 & * & \dots & * & * \end{pmatrix}$$

For all $1 \leq i < j \leq n$ we have $F(1, e_i e_j) = -F(e_i, e_j) = 0$. It follows from Lemma 2, (2) that $(e_i e_j)^{(n+1)} = 0$. We conclude that $e_i e_j \in W$ and thus $e_i e_j = 0$ by Lemma 2, (3).

Since $F(1, e_i^2) = -F(e_i, e_i) = 1$, we have $e_i^2 = e_{n+1} + f_i$, where $f_i \in W$. Then

$$(e_1 + e_i)(e_1 - e_i) = e_1^2 - e_i^2 = f_1 - f_i \in W.$$

By Lemma 2, (3) we obtain $e_1^2 = e_i^2$.

Without loss of generality it can be assumed that $e_{n+1} = e_1^2$. Let $n \geq 2$. Then $e_{n+1} e_i = e_1^2 e_i = 0$, where $1 \leq i \neq j \leq n$. If $n = 1$ then $e_{n+1} e_1 = e_1^3 \in \mathfrak{m}^3 = 0$.

Hence $e_{n+1} b = 0$ for any element $b \in \mathfrak{m}$, and R is isomorphic to $\mathbb{K}[e_1, \dots, e_n]/(e_i^2 - e_j^2, e_i e_j)$.

It remains to prove that $e_{n+1} \perp_F \mathfrak{m}$. Indeed, we have

$$F(e_{n+1}, b) = F(e_1^2, b) = -F(e_1, e_1 b) = F(1, e_1^2 b) = F(1, 0) = 0 \quad \forall b \in \mathfrak{m}.$$

This completes the proof of the proposition. \square

As a corollary we obtain the result of [15, Theorem 4].

Corollary 1. *A non-degenerate quadric $Q_n \subseteq \mathbb{P}^{n+1}$ admits a unique additive action up to equivalence.*

6 Quadrics of Corank One

Let us classify invariant bilinear symmetric forms of rank $n + 1$ on local $(n + 2)$ -dimensional algebras. Geometrically these results can be interpreted as a classification of additive actions on quadrics of corank one in \mathbb{P}^{n+1} .

Let R be a local algebra of dimension $n + 2$, $n \geq 2$, with the maximal ideal \mathfrak{m} and F a bilinear symmetric form of rank $n + 1$ on R such that $F(1, 1) = 0$. Assume that for some hyperplane W in \mathfrak{m} condition (2) holds. We choose a basis $e_0 = 1, e_1, \dots, e_n, e_{n+1}$ of R such that $W = \langle e_1, \dots, e_n \rangle$ and $\mathfrak{m} = \langle e_1, \dots, e_{n+1} \rangle$.

Lemma 3. *The kernel $\text{Ker } F$ is contained in W .*

Proof. Let $\text{Ker } F = \langle l \rangle$. Assume that l is not in W . Then we should consider four alternatives.

1. Let $\langle l \rangle = \langle 1 \rangle$. Then $F(a, b) = -F(1, ab) = 0$ for all $a \in W, b \in R$, and $\dim \text{Ker } F \geq 2$, a contradiction.
2. Let $\langle l \rangle \subseteq \mathfrak{m} \setminus W$. Without loss of generality it can be assumed that $l = e_{n+1}$. As we have seen, $F(1, b) = F(1, b^{(n+1)}) = 0$ for all $b \in R$. Thus we have $1 \in \text{Ker } F$, which leads to a contradiction.
3. Let $\langle l \rangle \subseteq R \setminus (\mathfrak{m} \cup \langle 1, W \rangle)$. Without loss of generality it can be assumed that $l = 1 + e_{n+1}$. We have $0 = F(1, l) = F(1, 1) + F(1, e_{n+1}) = F(1, e_{n+1})$. It again follows that $1 \in \text{Ker } F$.
4. Let $\langle l \rangle \subseteq \langle 1, W \rangle \setminus W$. We can assume that $l = 1 + f$, where $W \ni f \neq 0$. Then

$$F(1, b) = -F(f, b) = F(1, fb) = \dots = F(1, f^{n+3}b) = 0 \quad \forall b \in R.$$

Thus we again have $1 \in \text{Ker } F$.

This completes the proof of the lemma. \square

Proposition 7. *In the notation as above, the triple (R, W, F) can be transformed into the form*

$$M(F) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \quad W = \langle e_1, \dots, e_n \rangle,$$

and R is isomorphic to one of the following algebras:

1. $\mathbb{K}[e_1, \dots, e_n]/(e_i e_j - \lambda_{ij} e_n, e_i^2 - e_j^2 - (\lambda_{ii} - \lambda_{jj}) e_n, e_s e_n, 1 \leq i < j \leq n-1, 1 \leq s \leq n, n \geq 3)$ where λ_{ij} are elements of a symmetric block diagonal $(n-1) \times (n-1)$ -matrix Λ such that each block Λ_k is

$$\lambda_k \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & \ddots \\ \dots & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & \ddots & \ddots \\ \dots & \ddots & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} 0 & 1 & 0 \\ \dots & \ddots & -1 \\ 1 & \ddots & \dots \\ 0 & -1 & 0 \end{pmatrix}$$

with some $\lambda_k \in \mathbb{K}$;

2. $\mathbb{K}[e_1, e_2]/(e_1^3, e_1 e_2, e_2^2)$ or $\mathbb{K}[e_1]/(e_1^4)$ with $e_2 = e_1^3, e_3 = e_1^2$.

Remark 4. Blocks Λ_k of size 1 are (λ_k) . Blocks Λ_k of size 2 are

$$\begin{pmatrix} \lambda_k + \frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & \lambda_k - \frac{i}{2} \end{pmatrix}$$

Proof of Proposition 7. By Lemma 3 we may assume that $\text{Ker } F = \langle e_n \rangle$ and $F(1, e_{n+1}) = 1$, because of $F(1, a) = 0$ for all $a \in W$. Let V be the linear span $\langle e_1, \dots, e_{n-1} \rangle$. As in Lemma 2, (4) one can show that the restriction of F to V is non-degenerate. Thus we can assume that the matrix of F has the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & -1 & \dots & 0 & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 & * \\ 0 & 0 & \dots & 0 & 0 & * \\ 1 & * & \dots & * & * & * \end{pmatrix}.$$

We have $1 = -F(e_i, e_i) = F(1, e_i^2) = F(1, (e_i^2)^{(n+1)}) \Rightarrow e_i^2 = e_{n+1} + f_i$ for some $f_i \in W$ and every $i = 1, \dots, n-1$. We may assume that $e_{n+1} = e_1^2$.

Then $f_i = e_i^2 - e_1^2 = (e_i + e_1)(e_i - e_1)$ and, as in Lemma 2, (3), we obtain $f_i = \lambda_{ii}e_n$.

Again as in Lemma 2, (3), we have $e_i e_j = \lambda_{ij}e_n$ for all $1 \leq i < j \leq n-1$.

Thus multiplication on the subspace V is given by the matrix $I_n e_{n+1} + \Lambda e_n$, where I_n is the identity matrix and a symmetric matrix Λ is defined up to adding a scalar matrix.

It is easy to check that the symmetric matrix $\Lambda = (\lambda_{ij})$ under orthogonal transformations on V transforms as the matrix of a bilinear symmetric form. It follows from [10, Chap. 11, Sect. 3] that Λ can be transformed into the canonical block diagonal form by orthogonal transformation. Here each block Λ_k has the form

$$\lambda_k \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & \ddots \\ \cdot & \ddots & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & \ddots & \ddots \\ \cdot & \ddots & \ddots & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} 0 & 1 & 0 \\ \cdot & \ddots & \ddots & -1 \\ 1 & \ddots & \ddots \\ 0 & -1 & 0 \end{pmatrix}, \quad \lambda_k \in \mathbb{K}.$$

We claim that $e_n \mathfrak{m} = 0$. Indeed, $F(ae_n, b) = -F(e_n, ab) = 0$ for all $a \in W$ and $b \in R$, and thus $ae_n = \alpha e_n$ for some $\alpha \in \mathbb{K}$. But a is nilpotent, and $\alpha = 0$. Finally, we have $e_n e_{n+1} = e_n e_1^2 = 0$.

Further,

$$\begin{aligned} F(e_{n+1}a, 1) &= -F(e_{n+1}, a) = -F(e_1^2, a) = -F(1, e_1^2 a) \\ &= -F(1, e_{n+1}a) \Rightarrow F(e_{n+1}, a) = 0 \end{aligned} \quad (3)$$

for all $a \in W$.

1. Let $n \geq 3$. We claim that $e_{n+1} \mathfrak{m} = 0$. Indeed, for $1 \leq i \neq j \leq n-1$ we have

$$e_{n+1}e_i = (e_j^2 - \lambda_{jj}e_n)e_i = \lambda_{ij}e_j e_n - \lambda_{jj}e_n e_i = 0.$$

In this case the algebra R is isomorphic to

$$\begin{aligned} &\mathbb{K}[e_1, \dots, e_n]/(e_i e_j - \lambda_{ij}e_n, e_i^2 - e_j^2 - (\lambda_{ii} - \lambda_{jj})e_n, e_s e_n, \\ &1 \leq i < j \leq n-1, 1 \leq s \leq n). \end{aligned}$$

2. Let $n = 2$. We have $e_3^2 = e_1^4 \in \mathfrak{m}^4 = 0 \Rightarrow e_3^2 = 0$. Since $F(e_1 e_3, 1) = -F(e_3, e_1) = 0$, it follows that $e_1 e_3 \in W$. Thus $e_1 e_3 = \alpha e_1 + \beta e_2$ and we have

$$0 = e_1^4 = (e_1 e_3)e_1 = \alpha e_1^2 + \beta e_1 e_2 = \alpha e_3 \Rightarrow \alpha = 0.$$

If $\beta = 0$, then $R \cong \mathbb{K}[e_1, e_2]/(e_1^3, e_1 e_2, e_2^2)$. If $\beta \neq 0$, then we may assume that $\beta = 1$, and $R \simeq \mathbb{K}[e_1]/(e_1^4)$ with $e_2 = e_1^3, e_3 = e_1^2$.

In all cases $e_{n+1}^2 = e_1^2 e_{n+1} = 0$, and it follows that $F(e_{n+1}, e_{n+1}) = F(1, e_{n+1}^2) = 0$. Combining this with (3), we obtain

$$M(F) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 7 is proved. \square

Remark 5. The normal form of a symmetric matrix Λ is unique up to permutation of blocks. Indeed, we conjugate the matrix Λ by the symmetric block diagonal matrix T such that each block T_k is

$$\frac{1}{2} \begin{pmatrix} 1 & 0 & \dots & 0 & i \\ 0 & 1 & & i & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & i & & 1 & 0 \\ i & 0 & \dots & 0 & 1 \end{pmatrix},$$

and obtain the Jordan normal form of Λ with the same block sizes and the same eigenvalues.

We claim that the matrix Λ defining a triple (R, W, F) is unique up to permutation of blocks, scalar multiplication, and adding a scalar matrix. To see this, let two matrices Λ, Λ' define the same triple (R, W, F) . Notice that adding a scalar matrix to Λ we do not change the defining relations of R . Denote by ϕ an automorphism of R such that $W = \phi(W)$ and

$$F = \phi^{-1T} F \phi^{-1}.$$

It yields $\text{Ker } F = \phi(\text{Ker } F)$ and $\phi(e_n) = \alpha e_n$. Multiplying the matrix Λ' by α^{-1} we obtain $\phi(e_n) = e_n$. Moreover, ϕ induces on $W/\text{Ker } F$ an orthogonal transformation, and thus two canonical forms of the matrix Λ can differ only by the order of blocks.

Example 3. Two cases in Proposition 7, (2), correspond to two non-equivalent actions of \mathbb{G}_a^2 on the quadric $2x_0x_3 - x_1^2 = 0$ in \mathbb{P}^3 , namely,

$$(a_1, a_2) \cdot [x_0 : x_1 : x_2 : x_3] = \left[x_0 : x_1 + a_1x_0 : x_2 + a_2x_0 : x_3 + \frac{a_1^2}{2}x_0 + a_1x_1 \right]$$

and

$$(a_1, a_2) \cdot [x_0 : x_1 : x_2 : x_3] \\ = \left[x_0 : x_1 + a_1 x_0 : x_2 + \left(a_2 + \frac{a_1^3}{6} \right) x_0 + \frac{a_1^2}{2} x_1 + a_1 x_3 : x_3 + \frac{a_1^2}{2} x_0 + a_1 x_1 \right].$$

For the first action there is a line of fixed points, while the second one has three orbits.

Example 4. Let $n = 3$. If the matrix Λ is diagonal, then up to scalar addition and multiplication we have $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. With non-diagonal Λ we have

$\begin{pmatrix} i/2 & 1/2 \\ 1/2 & -i/2 \end{pmatrix}$. So there are three equivalence classes of additive actions in this case, and they can be easily written down explicitly.

Example 5. Consider the case $n = 4$. We have six types of the matrix Λ with one depending on a parameter. Namely, in the diagonal matrix $\Lambda = \text{diag}(0, 1, t)$, where $t \in \mathbb{K} \setminus \{0, 1\}$, the parameter t is defined up to transformations $\{t, \frac{1}{t}, 1-t, \frac{t-1}{t}, \frac{t}{t-1}, \frac{1}{1-t}\}$. Therefore, the parameters t and t' determine equivalent actions if and only if

$$\frac{(t^2 - t + 1)^3}{t^2(1-t)^2} = \frac{(t'^2 - t' + 1)^3}{t'^2(1-t')^2}.$$

The action of \mathbb{G}_a^4 on the quadric $2x_0x_5 - x_1^2 - x_2^2 - x_3^2 = 0$ in this case has the form

$$(a_1, a_2, a_3, a_4) \cdot [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \\ = \left[x_0 : x_1 + a_1 x_0 : x_2 + a_2 x_0 : x_3 + a_3 x_0 : x_4 + \frac{2a_4 + a_2^2 + ta_3^2}{2} x_0 \right. \\ \left. + a_2 x_2 + ta_3 x_3 : x_5 + \frac{a_1^2 + a_2^2 + a_3^2}{2} x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 \right].$$

This agrees with the results of [2, Sect. 4].

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