Random Walks in Nonhomogeneous Poisson Environment

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Abstract In the first part of the paper, we consider a "random flight" process in R^d and obtain the weak limits under different transformations of the Poissonian switching times. In the second part, we construct diffusion approximations for this process and investigate their accuracy. To prove the weak convergence result, we use the approach of [15]. We consider more general model which may be called "random walk over ellipsoids in R^d". For this model, we establish the Edgeworth-type expansion. The main tool in this part is the parametrix method [5, 7].

Keywords Random walks · Random flights
Random nonhomogeneous environment · Diffusion approximation
Parametrix method

1 Introduction

We consider the moving particle process in R^d which is defined in the following way. There are two independent sequences (T_k) and (ε_k) of random variables.

The variables T_k are nonnegative and $\forall k \quad T_k \leq T_{k+1}$, while variables ε_k form an i.i.d sequence with common distribution concentrated on the unit sphere S^{d-1}.

The values ε_k are interpreted as the directions, and T_k as the moments of change of directions.

A particle starts from zero and moves in the direction ε_1 up to the moment T_1. It then changes direction to ε_2 and moves on within the time interval of length $T_2 - T_1$,

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etc. The speed is constant at all sites. The position of the particle at time t is denoted by $X(t)$.

The study of the processes of this type has a long history. The first work dates back probably to [13] and continued by [6, 14]. In [8] the case was considered where the increments $T_n - T_{n-1}$ form i.i.d. sequence with the common law having a heavy tail. The term "Levy flights" later changed to "Random flights".

To date, a large number of works were accumulated, devoted to the study of such processes, we mention here only articles by [4, 9, 11, 12] which contain an extensive bibliography and where for different assumptions on (T_k) and (ε_k) the exact formulas for the distribution of $X(t)$ were derived.

Our goals are different.

First, we are interested in the global behavior of the process $X = \{X(t),\ t \in R_+\}$, namely, we are looking for conditions under which the processes $\{Y_T,\ T > 0\}$,

$$Y_T(t) = \frac{1}{B(T)} X(tT), \quad t \in [0, 1],$$

weakly converges in $C[0, 1]:\ Y_T \Longrightarrow Y,\ B_T \longrightarrow \infty,\ T \longrightarrow \infty.$

From now on, we suppose that the points $(T_k),\ T_k \le T_{k+1}$, form a Poisson point process in R_+ denoted by $\mathbf{T}$.

It is clear that in the homogeneous case the process $X(t)$ is a conventional random walk because the spacings $T_{k+1} - T_k$ are independent, and then the limit process is Brownian motion.

In the nonhomogeneous case, the situation is more complicated as these spacings are not independent. Nevertheless, it was possible to distinguish three modes that determine different types of limiting processes.

For a more precise description of the results, it is convenient to assume that $T_k = f(\Gamma_k)$, where $\mathbf{\Pi} = (\Gamma_k)$ is a standard homogeneous Poisson point process on R_+ with intensity 1. In this case,

$$(\Gamma_k) \stackrel{\mathcal{L}}{=} (\gamma_1 + \gamma_2 + \cdots + \gamma_k),$$

where (γ_k) are i.i.d standard exponential random variables.

If the function f has power growth,

$$f(t) = t^\alpha,\ \alpha > 1/2,$$

the behavior of the process is analogous to the uniform case and then in the limit we obtain a Gaussian process which is a linearly transformed Brownian motion

$$Y(t) = \int_0^t K_\alpha(s) dW(s),$$

where W is a process of Brownian motion, for which the covariance matrix of $W(1)$ coincides with the covariance matrix of ε_1 and $K_\alpha(s)$ is a nonrandom kernel, and its exact expression is given below.

In the case of exponential growth,

$$f(t) = e^{t\beta}, \;\; \beta > 0,$$

the limiting process is piecewise linear with an infinite number of units, but $\forall \epsilon > 0$ the number of units in the interval $[\varepsilon, 1]$ will be a.s. finite.

Finally, with the super exponential growth of f, the process degenerates: its trajectories are linear functions:

$$Y(t) = \varepsilon t, \quad t \in [0, 1], \quad \varepsilon \stackrel{Law}{=} \varepsilon_1.$$

In the second part of the paper, the process $X(t)$ is assumed to be a Markov chain. We construct diffusion approximations for this process and investigate their accuracy. To prove the weak convergence, we use the approach of [15]. Under our assumptions the diffusion coefficients a and b have the property that for each $x \in R^d$ the martingale problem for a and b has exactly one solution P_x starting from x (that is well posed). It remains to check the conditions from [15] which imply the weak convergence of our sequence of Markov chains to this unique solution P_x. We consider also the more general model which may be called as "random walk over ellipsoids in R^d". For this model, we establish the convergence of the transition densities and obtain the Edgeworth-type expansion up to the order $n^{-3/2}$, where n is a number of switching. The main tool in this part is the parametrix method [5, 7].

2 Random Flights in Poissonian Environment

The reader is reminded that we suppose $T_k = f(\Gamma_k)$, where (Γ_k) is a standard homogeneous Poisson point process on R_+. Assume also that $E\varepsilon_1 = 0$.

It is more convenient to consider at first the behavior of the processes

$$Z_n(t) = Y_{T_n}(t),$$

as for $T = T_n$ the paths of Z_n have an integer number of full segments on the interval [0,1]. The typical path of $\{Z_n(t), t \in [0, 1]\}$ is a continuous broken line with vertices $\{(t_{n,k}, \frac{S_k}{B_n}), \;\; k = 0, 1, \ldots, n\}$, where $t_{n,k} = \frac{T_k}{T_n}$, $T_0 = 0$, $B_n = B(T_n)$, $S_k = \sum_1^k \varepsilon_i (T_i - T_{i-1})$.

Theorem 1 *Under the previous assumptions*

(1) If the function f has power growth: $f(t) = t^\alpha$, $\alpha > 1/2$, we take $B(T) = T^{\frac{2\alpha-1}{2\alpha}}$.

Then $Z_n \Longrightarrow Y$*, where* Y *is a Gaussian process*

$$Y(t) = \sqrt{2\alpha}\int_0^t s^{\frac{\alpha-1}{2\alpha}}\, dW(s),$$

and W *is a process of Brownian motion, for which the covariance matrix of* $W(1)$ *coincides with the covariance matrix of* ε_1*.*

(2) *If the function* f *has exponential growth:* $f(t) = e^{t\beta}$, $\beta > 0$, *we take* $B(T) = T$.
Then $Z_n \Longrightarrow Y$*, where* Y *is a continuous piecewise linear process with the vertices at the points* $(t_k, Y(t_k))$,

$$t_k = e^{-\beta\Gamma_{k-1}}, \quad \Gamma_0 = 0,$$

$$Y(t_k) = \sum_{i=k}^{\infty} \varepsilon_k (e^{-\beta\Gamma_{i-1}} - e^{-\beta\Gamma_i}), \quad Y(0) = 0.$$

(3) *In the super exponential case, suppose that* f *is increasing absolutely continuous and such that*

$$\lim_{t\to\infty} \frac{f'(t)}{f(t)} = +\infty.$$

We take $B(T) = T$.
Then $\frac{T_n}{T_{n+1}} \to 0$ *in probability, and* $Z_n \Longrightarrow Y$*, where the limiting process* Y *degenerates:*

$$Y(t) = \varepsilon_1 t, \quad t \in [0, 1].$$

Remark 1 In the case of power growth, the limiting process admits the following representation:

$$Y(t) \stackrel{\mathcal{L}}{=} \alpha\sqrt{\frac{2}{2\alpha-1}}\, W(t^{\frac{2\alpha-1}{\alpha}}),$$

where, as before, W is a Brownian motion, for which the covariance matrix of $W(1)$ coincides with the covariance matrix of ε_1.

It is clear that we can also express Y in another way:

$$Y(t) \stackrel{\mathcal{L}}{=} \alpha\sqrt{\frac{2}{2\alpha-1}}\, K^{\frac{1}{2}} w(t^{\frac{2\alpha-1}{\alpha}}),$$

where w is a standard Brownian motion and K is the covariance matrix of ε_1.

Remark 2 In the case of exponential growth, it is possible to describe the limiting process Y in the following way:

We take a Poisson point process $\mathbf{T} = (t_k)$, $t_k = e^{-\beta\Gamma_{k-1}}$, defined on $(0, 1]$, and define a step process $\{Z(t),\ t \in (0, 1]\}$,

$$Z(t) = \varepsilon_k \quad \text{for} \ \ t \in (t_{k+1}, t_k].$$

Then

$$Y(t) = \int_0^t Z(s)\,\mathrm{d}s.$$

3 Diffusion Approximation

In this section, first we consider a model of random flight which is equivalent to the study of random broken lines $\{X_n(t),\ t \in [0, 1]\}$ with the vertices $(\frac{k}{n},\ X_n(\frac{k}{n}))$, and such that $(h = \frac{1}{n})$

$$X_n\left((k+1)h\right) = X_n(kh) + hb(X_n(kh)) + \sqrt{h}\xi_k(X(kh)),$$

$$X_n(0) = x_0, \quad \xi_k(X_n(kh)) = \rho_k \sigma(X_n(kh))\varepsilon_k, \tag{1}$$

where $\{\varepsilon_k\}$ and $\{\rho_k\}$ are two independent sequences and

$\{\varepsilon_k\}$ are i.i.d. r. v. uniformly distributed on the unit sphere S^{d-1};

$\{\rho_k\}$ are i.i.d. r. v. having an absolutely continuous distribution, $\rho_k \geq 0$, $E\rho_k^2 = d$;

$b : R^d \longrightarrow R^d$ is a bounded measurable function and $\sigma : R^d \longrightarrow R^d \times R^d$ is a bounded measurable matrix function.

Theorem 2 *Let* $X = \{X(t),\ t \in [0, 1]\}$ *be a solution of stochastic equation*

$$X(t) = x_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dw(s).$$

Suppose that b and σ *are continuous functions satisfying the Lipschitz condition*

$$|b(t) - b(s)| + |\sigma(t) - \sigma(s)| \leq K|t - s|.$$

Moreover, it is supposed that $b(x)$ *and* $\frac{1}{\det(\sigma(x))}$ *are bounded.*

Then,

$$X_n \Longrightarrow X \quad in \quad \mathbb{C}[0, 1].$$

Our next result is about the approximation of the transition density. We consider now more general models given by a triplet $(b(x), \sigma(x), f(r;\theta))$, $x \in R^d$, $r \geq 0$, $\theta \in R^+$, where $b(x)$ is a vector field, $\sigma(x)$ is a $d \times d$ matrix, $a(x) := \sigma\sigma^T(x) > \delta I$, $\delta > 0$, and $f(r;\theta)$ is a radial density depending on a parameter θ controlling the frequency of changes of directions, namely, the frequency increases when θ decreases. Suppose $X(0) = x_0$. The vector $b(x_0)$ acts by shifting a particle from x_0 to $x_0 + \Delta(\theta)b(x_0)$, where $\Delta(\theta) = c_d\theta^2$, $c_d > 0$. Several examples of such functions $\Delta(\theta)$ for different models will be given below. Define

$$\mathcal{E}_{x_0}(r) := \{x : \left|a^{-1/2}(x_0)(x - x_0 - \Delta(\theta)b(x_0))\right|^2 = r^2\},$$

$$\mathcal{S}^d_{x_0}(r) := \{y : |y - x_0 - \Delta(\theta)b(x_0)|^2 = r^2\}.$$

The initial direction is defined by a random variable ξ_0, and the law of ξ_0 is a pushforward of the spherical measure on $\mathcal{S}^d_{x_0}(1)$ under affine change of variables

$$x - x_0 - \Delta(\theta)b(x_0) = a^{1/2}(x_0)(y - x_0 - \Delta(\theta)b(x_0)).$$

Then particle moves along the ray l_{x_0} corresponding to the directional unit vector

$$\varepsilon_0 := \frac{\xi_0 - x_0 - \Delta(\theta)b(x_0)}{|\xi_0 - x_0 - \Delta(\theta)b(x_0)|},$$

and changes the direction in $(r, r + dr)$ with probability

$$\det\left(a^{-1/2}(x_0)\right) \cdot f(r\left|a^{-1/2}(x_0)e_0\right|)dr. \tag{2}$$

Let ρ_0 be a random variable independent of ξ_0 and distributed on l_{x_0} with the radial density (2). We consider the point $x_1 = x_0 + \Delta(\theta)b(x_0) + \rho_0\varepsilon_0$. Let (ε_k, ρ_k) be independent copies of (ε_0, ρ_0). Starting from x_1, we repeat the previous construction to obtain $x_2 = x_1 + \Delta(\theta)b(x_1) + \rho_1\varepsilon_1$. After n switches, we arrive at the point x_n,

$$x_n = x_{n-1} + \Delta(\theta)b(x_{n-1}) + \rho_{n-1}\varepsilon_{n-1}.$$

To obtain the one-step characteristic function $\Psi_1(t)$, we make use of formula (6) from [17] (see also the proof of Theorem 2.1 in [10]):

$$\Psi_1(t) = Ee^{i\langle t, \rho_0\varepsilon_0\rangle} = \int_0^\infty \int_{\mathcal{E}_{x_0}(r)} e^{i\left\langle t, a^{1/2}(x_0)a^{-1/2}(x_0)\xi\right\rangle} \mu_{\mathcal{E}_{x_0}(r)}(d\xi)d\Phi_{\mathcal{E}}(r) =$$

$$= \int_0^\infty \int_{\mathcal{S}^d_{x_0}(r)} e^{i\left\langle a^{1/2}(x_0)t, y\right\rangle} \lambda_r^d(dy) f(r; \theta)dr =$$

$$= 2^{\frac{d-2}{2}}\Gamma\left(\frac{d}{2}\right)\int_0^\infty \frac{J_{\frac{d-2}{2}}(r\left|a^{1/2}(x_0)t\right|)}{\left(r\left|a^{1/2}(x_0)t\right|\right)^{\frac{d-2}{2}}} f(r; \theta)dr, \tag{3}$$

where $J_\nu(z)$ is the Bessel function, $d\Phi_{\mathcal{E}}(r)$ is the F-measure of the layer between $\mathcal{E}_{x_0}(r)$ and $\mathcal{E}_{x_0}(r+dr)$, *and* F is the law of $\rho_0\varepsilon_0$. Now we make our main assumption about the radial density:

(A1) The function $f(r; \theta)$ is homogeneous of degree -1, that is

$$f(\lambda r; \lambda\theta) = \lambda^{-1} f(r; \theta), \ \forall\lambda \neq 0.$$

Denote by $p_{\mathcal{E}}(n,x,y)$ the transition density after n switches in the RF model described above. To obtain the one-step transition density $p_{\mathcal{E}}(1,x,y)$ (we write (x,y) instead of (x_0,x_1)), we use the inverse Fourier transform, (3) and **(A1).** We have

$$p_{\mathcal{E}}(1,x,y)=\Delta^{-d/2}(\theta)q_x\left(\frac{y-x-\Delta(\theta)b(x)}{\sqrt{\Delta(\theta)}}\right), \tag{4}$$

where

$$q_x(z)=\frac{2^{\frac{d-2}{2}}\Gamma\left(\frac{d}{2}\right)}{(2\pi)^d}\int_{R^d}\cos\langle\tau,z\rangle\left[\int_0^\infty \frac{J_{\frac{d-2}{2}}(\rho\left|a^{1/2}(x)\tau\right|)}{\left(\rho\left|a^{1/2}(x)\tau\right|\right)^{\frac{d-2}{2}}}f(\rho;c_d)d\rho\right]d\tau. \tag{5}$$

Consider two examples.

Example 1 We put $\Delta(\theta)=(d+1)^2\theta^2$ and

$$f(r;\theta)=\frac{1}{\Gamma(d)}r^{-1}\left(\frac{r}{\theta}\right)^d\exp\left(-\frac{r}{\theta}\right).$$

Using (3), formula 6.623 (2) on p. 694 from [3], and the doubling formula for the Gamma function, we obtain

$$p_{\mathcal{E}}(1,x,y)=\Delta^{-d/2}(\theta)q_x\left(\frac{y-x-\Delta(\theta)b(x)}{\sqrt{\Delta(\theta)}}\right),$$

where

$$q_x(z)=\frac{(d+1)^{d/2}}{2^d\pi^{(d-1)/2}\Gamma\left(\frac{d+1}{2}\right)\left|\det a^{1/2}(x)\right|}e^{-\sqrt{d+1}\left|a^{-1/2}(x)z\right|}.$$

It is easy to check that

$$\int z_i q_x(z)=0,\quad \int z_i z_j q_x(z)dz=a_{ij}(x).$$

Example 2 We put $\Delta(\theta)=\theta^2/2$ and

$$f(r;\theta)=C_d r^{-1}\left(\frac{r}{\theta}\right)^d\exp\left(-\frac{r^2}{\theta^2}\right),$$

where $C_d=\frac{2^{(d+1)/2}}{(d-2)!!\sqrt{\pi}}$ if d is odd, and $C_d=\frac{2}{[(d-2)/2]!}$ if d is even. From (3) and formula 6.631 (4) on p. 698 of [3], we obtain

$$p_{\mathcal{E}}(1,x,y)=\Delta^{-d/2}(\theta)\phi_x(\frac{y-x-\Delta(\theta)b(x)}{\sqrt{\Delta(\theta)}}),$$

where

$$\phi_x(z) = \frac{1}{(2\pi)^{d/2}\sqrt{\det a(x)}} \exp\left(-\frac{1}{2}\left\langle a^{-1}(x)z, z\right\rangle\right).$$

It is easy to see that the transition density (4) corresponds to the one-step transition density in the following Markov chain model:

$$X_{(k+1)\Delta(\theta)} = X_{k\Delta(\theta)} + \Delta(\theta)\, b(X_{k\Delta(\theta)}) + \sqrt{\Delta(\theta)}\xi_{(k+1)\Delta(\theta)},$$

where the conditional density (under $X_{k\Delta(\theta)} = x$) of the innovations $\xi_{(k+1)\Delta(\theta)}$ is equal to $q_x(\cdot)$. If we put $\theta = \theta_n = \sqrt{\frac{2}{n}}$, then $\Delta(\theta_n) = \frac{1}{n}$ and we obtain a sequence of Markov chains defined on an equidistant grid

$$X_{\frac{k+1}{n}} = X_{\frac{k}{n}} + \frac{1}{n}b(X_{\frac{k}{n}}) + \frac{1}{\sqrt{n}}\xi_{\frac{k+1}{n}}, \quad X_0 = x_0. \tag{6}$$

Note that the triplet $(b(x), \sigma(x), f(r;\theta)),\ x \in R^d,\ r \geq 0,\ \theta \in R^+$, of Example 2 corresponds to the classical Euler scheme for the d-dimensional SDE

$$dX(t) = b(X_t)dt + \sigma(X_t)dW(t), \quad X(0) = x_0. \tag{7}$$

Let $p(1, x, y)$ be transition density from 0 to 1 in the model (7). We make the following assumptions.

(A2) The function $a(x) = \sigma\sigma^T(x)$ is uniformly elliptic.

(A3) The functions $b(x)$ and $\sigma(x)$ and their derivatives up to the sixth order are continuous and bounded uniformly in x. The sixth derivative is globally Lipschitz.

Theorem 3 *Under the assumptions (A2) and (A3,) we have the following expansion: for any positive integer S as $n \to \infty$*

$$\sup_{x,y \in R^d}\left(1 + |y-x|^S\right) \cdot \left| p_{\mathcal{E}}(n, x, y) - p(1, x, y) - \frac{1}{2n} p \otimes \left(L_*^2 - L^2\right) p(1, x, y)\right| = O(n^{-3/2}), \tag{8}$$

where

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial^2_{x_i x_j} + \sum_{i=1}^{d} b_i(x)\partial_{x_i}. \tag{9}$$

The operator L_* in (8) is the same operator as in (9) but with coefficients "frozen" at x. It means that when calculating degrees of the operator we do not differentiate coefficients and we consider them as constants, taking them out of the derivative.

Clearly, $L = L_*$ but, in general, $L^2 \neq L_*^2$. The convolution-type binary operation $\otimes$ is defined for functions f and g in the following way:

$$(f \otimes g)(t, x, y) = \int_0^t ds \int_{R^d} f(s, x, z) g(t-s, z, y) dz.$$

Proof It follows immediately from Theorem 1 of [7].

4 Proof of Theorem 1

4.1 Asymptotic Behavior in Case (3)

We have, by taking $B_n = B(T_n) = T_n$:

$$\sup_{t\in\left[0,\frac{T_{n-1}}{T_n}\right]} \|X_n(t)\|_\infty \le \sum_{k=1}^{n-1} \frac{T_k - T_{k-1}}{T_n} = \frac{T_{n-1}}{T_n} \xrightarrow[n\to\infty]{} 0 \text{ a.s.}$$

At the same time,

$$X_n(1) = \frac{S_{n-1} + \varepsilon_n (T_n - T_{n-1})}{T_n} = \varepsilon_n + o(1) \Rightarrow \mathcal{P}_{\varepsilon_1}$$

Therefore, the process X_n converges weakly to the process $\{Y(t)\}$, $Y(t) = \varepsilon_1 t$, $t \in [0, 1]$.

This process is in some sense degenerate. Hence, this case is not very interesting.

4.2 Asymptotic Behavior in Case (2)

Take $B_n = T_n$ and show that the limit process Y is not trivial. For simplicity fix $\beta = 1$. We have now $t_{n,k} := \frac{T_k}{T_n} = e^{-(\Gamma_n - \Gamma_k)} = e^{-(\gamma_{k+1}+\cdots+\gamma_n)}$, and

$$X_n\left(t_{n,k}\right) = \sum_{i=1}^{k} \varepsilon_i (e^{-(\gamma_{i+1}+\cdots+\gamma_n)} - e^{-(\gamma_i+\cdots+\gamma_n)}), \quad k = 1, \dots, n.$$

The process X_n is completely defined by two independent vectors $(\varepsilon_1, \dots, \varepsilon_n)$ and $(\gamma_1, \dots, \gamma_n)$. Hence, its distribution will be the same if we replace these vectors by $(\varepsilon_n, \dots, \varepsilon_1)$ and $(\gamma_n, \dots, \gamma_1)$. In another words, the process $(X_n(\cdot)) \stackrel{\mathcal{L}}{=} (Y_n(\cdot))$, where $Y_n(\cdot)$ is a broken line with vertices $(\tau_{n,k}, Y_n(\tau_{n,k}))$, $(\tau_{n,k}) \downarrow$, $\tau_{n,1} = 1$, $\tau_{n,k} = e^{-(\gamma_1+\cdots+\gamma_{k-1})}$, $k = 2, \dots, n$, and

$$Y_n(\tau_{n,k}) = \sum_{i=k}^{n-1} \varepsilon_i \left(e^{-(\gamma_1+\cdots+\gamma_{i-1})} - e^{-(\gamma_1+\cdots+\gamma_i)}\right) + \varepsilon_n e^{-(\gamma_1+\cdots+\gamma_{n-1})};$$

$Y_n(0) = 0$, and $\gamma_0 := 0$.

Using the notation $\Gamma_k = \gamma_1 + \cdots + \gamma_k$, we get the more compact formula:

$$Y_n(\tau_{n,k}) = \sum_{i=k}^{n-1} \varepsilon_i \left(e^{-\Gamma_{i-1}} - e^{-\Gamma_i}\right) + \varepsilon_n e^{-\Gamma_{n-1}}.$$

Consider now the process $\{Y(t), t \in [0, 1]\}$ defined as follows:

$$Y(0) = 0, \quad Y(t_k) = \sum_{i=k}^{\infty} \varepsilon_i \left(e^{-\Gamma_{i-1}} - e^{-\Gamma_i}\right), \tag{10}$$

where $t_k = e^{-\Gamma_{k-1}}$, $k = 2, 3, \ldots, t_1 = 1$; for $t \in [t_{k+1}, t_k]$, $Y(t)$ is defined by linear interpolation. The paths of Y are continuous broken lines, starting at 0 and having an infinite number of segments in the neighborhood of zero.

The evident estimation

$$\sup_{t\in[0,1]} |Y(t) - Y_n(t)| \le \left|\sum_{i=n}^{\infty} \varepsilon_i \left(e^{-\Gamma_{i-1}} - e^{-\Gamma_i}\right)\right| + e^{-\Gamma_{n-1}} \le$$

$$\le \sum_{i=n}^{\infty} \left(e^{-\Gamma_{i-1}} - e^{-\Gamma_i}\right) + e^{-\Gamma_{n-1}} = 2e^{-\Gamma_{n-1}} \longrightarrow 0 \quad \text{a.s.}$$

shows that a.s. $Y_n(\cdot) \xrightarrow{\mathbb{C}[0,1]} Y(\cdot)$.

Conclusion: In case (2), the process X_n converges weakly to $Y(\cdot)$.

Remark 3 In the case where $\beta \neq 1$, it is simply necessary to replace $e^{-\Gamma_k}$ by $e^{-\frac{\Gamma_k}{\beta}}$.

Remark 4 It seems that the last result could be expanded by considering more general sequences (ε_k).

Interpretation: $\frac{\varepsilon_k}{|\varepsilon_k|}$ defines the direction and $|\varepsilon_k|$ defines the velocity of displacement in this direction on the step S_k.

4.3 Asymptotic Behavior in Case of Power Growth

In this case, $T_k = \Gamma_k^\alpha, \ \alpha > 1/2, \ t_{n,k} = \frac{T_k}{T_n} = \left(\frac{\Gamma_k}{\Gamma_n}\right)^\alpha$, and

$$X_n(t_{n,k}) = \frac{1}{B_n} \sum_{i=1}^{k} \varepsilon_i (\Gamma_i^\alpha - \Gamma_{i-1}^\alpha); \ \Gamma_0 = 0, \ k = 0, 1, \ldots, n. \tag{11}$$

Let $x \in \mathbb{R}^d$ be such that $|x| = 1$. We will show below that

$$\operatorname{Var}\left(\sum_{i=1}^{n} \langle \varepsilon_i, x\rangle (\Gamma_i^\alpha - \Gamma_{i-1}^\alpha)\right) = E\langle \varepsilon_i, x\rangle^2 \sum_{i=1}^{n} \mathrm{E}(\Gamma_i^\alpha - \Gamma_{i-1}^\alpha)^2 \sim C(x) n^{2\alpha-1}, \ n \to \infty,$$

where $C(x) = \frac{2\alpha^2}{2\alpha-1} E\langle \varepsilon_1, x\rangle^2$. Therefore it is natural to take $B_n^2 = n^{2\alpha-1}$.

We proceed in five steps:

Step 1: Lemmas

Step 2: We compare $X_n(\cdot)$ with $Z_n(\cdot)$ where $Z_n(t_{n,k}) = \frac{\alpha}{B_n} \sum_{i=1}^{k} \varepsilon_i \gamma_i \Gamma_{i-1}^{\alpha-1}$ and show that $\|X_n - Z_n\|_\infty \xrightarrow{\mathbb{P}} 0$.

Step 3: We compare $Z_n(\cdot)$ with $W_n(\cdot)$ where $W_n(t_{n,k}) = \frac{\alpha}{B_n} \sum_{i=1}^{k} \varepsilon_i \gamma_i (i-1)^{\alpha-1}$ and state that $\|Z_n - W_n\|_\infty \xrightarrow{\mathbb{P}} 0$.

Step 4: We show that process $U_n(\cdot)$,

$$U_n\left(\left(\frac{k}{n}\right)^\alpha\right) = \frac{\alpha}{B_n} \sum_{i=1}^{k} \varepsilon_i \gamma_i (i-1)^{\alpha-1},$$

converges weakly to the limiting process

$$Y(t) = \sqrt{2\alpha} \int_0^t s^{\frac{\alpha-1}{2\alpha}}\, \mathrm{d}W(s);$$

here $W(\cdot)$ is a process of Brownian motion, for which the covariance matrix of $W(1)$ coincides with the covariance matrix of ε_1.

Step 5: We show that the convergence $W_n \Rightarrow Y$ follows from the convergence $U_n \Rightarrow Y$.

Finally: We get the convergence $X_n \Rightarrow Y$.

4.3.1 Step 1

This section contains several technical lemmas necessary for realization of subsequent steps.

Lemma 1 *Let* $\alpha > 0$ *and* $m \geq 1$. *Then* $\forall x > 0,\ h > 0$

$$(x+h)^\alpha - x^\alpha = \sum_{k=1}^{m} a_k h^k x^{\alpha-k} + R(x,h), \tag{12}$$

where

$$a_k = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!},$$

and

$$|R(x,h)| \leq |a_{m+1}| h^{m+1} \max\{x^{\alpha-(m+1)}, (x+h)^{\alpha-(m+1)}\}. \tag{13}$$

Proof By the formula of Taylor–Lagrange, we have (12) with

$$|R(x,y)| \le \frac{1}{(m+1)!} h^{m+1} \sup_{x \le t \le x+h} |f^{(m+1)}(t)|,$$

where $f(t) = t^\alpha$. As $f^{(m+1)}(t) = \alpha(\alpha-1)\ldots(\alpha-m)t^{\alpha-(m+1)}$, we get the claimed result. $\square$

Lemma 2 *For $\alpha \ge 0$ and $k \to \infty$*

$$\left(1+\frac{\alpha}{k}\right)^k = e^\alpha + O\left(\frac{1}{k}\right). \tag{14}$$

Proof It follows from the inequalities:

$$0 \le e^\alpha - \left(1+\frac{\alpha}{k}\right)^k \le \frac{e^\alpha \alpha^2}{k}.$$

$\square$

Lemma 3 *Let Γ be the Gamma function. Then as $k \to \infty$*

$$\frac{\Gamma(k+\alpha)}{\Gamma(k)} = k^\alpha + O(k^{\alpha-1}).$$

Proof It follows from Lemma 2 and well-known asymptotic (see a.e. [16], v. 2, 12.33)

$$\Gamma(t) = t^{t-\frac{1}{2}} e^{-t}\sqrt{2\pi}\left(1+\frac{1}{12t}+O\left(\frac{1}{t^2}\right)\right), \quad t \to \infty.$$

Lemma 4 *For any real β, we have as $k \to \infty$*

$$E(\Gamma_k^\beta) = k^\beta + O(k^{\beta-1}).$$

Proof The result follows from the well-known fact that

$$E(\Gamma_k^\beta) = \frac{\Gamma(k+\beta)}{\Gamma(k)}$$

and Lemma 3.

Lemma 5 *Let $\alpha \ge 0$. The following relations take place as $k \to \infty$:*

$$\Gamma_{k+1}^\alpha - \Gamma_k^\alpha = \alpha\gamma_{k+1}\Gamma_k^{\alpha-1} + \rho_k, \tag{15}$$

where $|\rho_k| = O(k^{\alpha-2})$ in probability;

$$E|\Gamma_{k+1}^\alpha - \Gamma_k^\alpha|^2 = 2\alpha^2 k^{2\alpha-2} + O(k^{2\alpha-3}); \tag{16}$$

$$E|\Gamma_{k+1}^\alpha - \Gamma_k^\alpha - \alpha\gamma_{k+1}\Gamma_k^{\alpha-1}|^2 = O(k^{2\alpha-4}). \tag{17}$$

Proof of Lemma 5 We find, by applying Lemma 1,

$$\Gamma_{k+1}^{\alpha} - \Gamma_k^{\alpha} = \alpha\gamma_{k+1}\Gamma_k^{\alpha-1} + R(\Gamma_k, \gamma_{k+1}), \tag{18}$$

where

$$R(\Gamma_k, \gamma_{k+1}) \leq \frac{1}{2}\gamma_{k+1}^2 \max_{\Gamma_k \leq s \leq \Gamma_{k+1}} |\alpha(\alpha-1)| s^{\alpha-2} \leq \frac{|\alpha(\alpha-1)|}{2}\gamma_{k+1}^2 \max\{\Gamma_{k+1}^{\alpha-2}, \Gamma_k^{\alpha-2}\}. \tag{19}$$

As $\Gamma_k \sim k$ a.s. when $k \to \infty$, we get (15).

The proofs of (16) and (17) follow directly from (18), (19) and Lemma 4. □

We deduce immediately from (16) the following relation.

Corollary 1 *We have*

$$\sum_1^{n-1} E|\Gamma_{k+1}^{\alpha} - \Gamma_k^{\alpha}|^2 = \frac{2\alpha^2}{2\alpha-1} n^{2\alpha-1} + O(n^{2\alpha-2}).$$

4.3.2 Step 2

We show that $\|X_n - Z_n\|_\infty \xrightarrow{\mathbb{P}} 0$, where

$$Z_n(t_{n,k}) = \frac{\alpha}{B_n}\sum_{i=1}^{k} \varepsilon_i\gamma_i\Gamma_{i-1}^{\alpha-1}.$$

It is clear that

$$\delta_n := \|X_n - Z_n\|_\infty = \sup_{t\in[0,1]} |X_n(t) - Z_n(t)| = \max_{k\leq n} |X(t_{n,k}) - Z_n(t_{n,k})| = \max_{k\leq n} |r_k|,$$

where

$$r_k = \frac{1}{B_n}\sum_{i=1}^{k} \varepsilon_i \left[\Gamma_i^{\alpha} - \Gamma_{i-1}^{\alpha} - \alpha\gamma_i\Gamma_{i-1}^{\alpha-1}\right] = \sum_{i=1}^{k} \varepsilon_i\xi_i,$$

and

$$\xi_i = \left(\Gamma_i^{\alpha} - \Gamma_{i-1}^{\alpha} - \alpha\gamma_i\Gamma_{i-1}^{\alpha-1}\right)\frac{1}{B_n}.$$

Let $\mathfrak{M} = \sigma(\xi_1, \xi_2, \ldots, \xi_n) = \sigma(\gamma_1, \gamma_2, \ldots, \gamma_n)$. Under condition $\mathfrak{M}$, the sequence (r_k) is the sequence of sums of independent random variables with mean zero. By Kolmogorov's inequality,

$$\mathbb{P}\{\max_{k\le n}|r_k|\ge t\} = \mathrm{E}\{\mathbb{P}\{\max_{k\le n}|r_k|\ge t \mid \mathfrak{M}\}\} \le \mathrm{E}\left(\frac{1}{t^2}\sum_{j=1}^{n}\xi_j^2\right) = \frac{1}{t^2}\sum_{j=1}^{n}\mathrm{E}\xi_j^2. \tag{20}$$

By Lemma 5, $\mathrm{E}\xi_j^2 = O(j^{-3})$. Therefore,

$$\sum_{j=1}^{n}\mathrm{E}\xi_j^2 = O(n^{-2}).$$

Finally, we get from (20): $\forall t > 0$

$$\mathbb{P}\{\delta_n \ge t\} \xrightarrow[n\to\infty]{} 0,$$

which gives the convergence $\|X_n - Z_n\|_\infty \xrightarrow{\mathbb{P}} 0$.

4.3.3 Step 3

We show now that $\|Z_n - W_n\|_\infty \xrightarrow[n\to\infty]{\mathbb{P}} 0$; where $W_n(t_{n,k}) = \frac{\alpha}{B_n}\sum_{i=1}^{k}\varepsilon_i\gamma_i(i-1)^{\alpha-1}$.

We have

$$\Delta_n = \sup_{t\in[0,1]}|Z_n(t) - W_n(t)| = \max_{k\le n}|Z_n(t_{n,k}) - W_n(t_{n,k})| = \max_{k\le n}\{|\beta_k|\},$$

where $\beta_k = \frac{\alpha}{B_n}\sum_{i=1}^{k}\varepsilon_i\gamma_i\left(\Gamma_{i-1}^{\alpha-1} - (i-1)^{\alpha-1}\right)$.

Similar to the previous case, (β_k) under condition $\mathfrak{M}$ is the sequence of sums of independent random variables with mean zero. Therefore,

$$\mathbb{P}\{\max_{k\le n}\{|\beta_k|\}\ge t\} = \mathrm{E}\left(\mathbb{P}\{\max_{k\le n}\{|\beta_k|\}\ge t \mid \mathfrak{M}\}\right) \le \frac{1}{t^2}\sum_{j=1}^{n}\mathrm{E}\eta_j^2,$$

where $\eta_j = \frac{\alpha}{B_n}\gamma_j\left(\Gamma_{j-1}^{\alpha-1} - (j-1)^{\alpha-1}\right)$.

Estimation of $\mathrm{E}\eta_j^2$.

By independence of γ_j and Γ_{j-1}

$$\mathrm{E}\eta_j^2 = \frac{2\alpha^2}{B_n^2}\mathrm{E}\left(\Gamma_{j-1}^{\alpha-1} - (j-1)^{\alpha-1}\right)^2.$$

Let us change $j-1$ to k

$$\mathrm{E}\left(\Gamma_k^{\alpha-1}-k^{\alpha-1}\right)^2=\mathrm{E}\left(\Gamma_k^{2\alpha-2}\right)+k^{2\alpha-2}-2k^{\alpha-1}\mathrm{E}\left(\Gamma_k^{\alpha-1}\right)=$$
$$=\frac{\Gamma(k+2\alpha-2)}{\Gamma(k)}+k^{2\alpha-2}-2k^{\alpha-1}\frac{\Gamma(k+\alpha-1)}{\Gamma(k)}=(\text{by Lemma 3})=$$
$$=\left[k^{2\alpha-2}+O(k^{2\alpha-3})+k^{2\alpha-2}-2k^{2\alpha-2}\right]=O(k^{2\alpha-3}).$$

Hence,

$$\mathrm{E}\eta_j^2\le C\frac{j^{2\alpha-3}}{n^{2\alpha-1}}.$$

It follows from this estimation that

for $\alpha>1$

$$\sum_{j=1}^n \mathrm{E}\eta_j^2\le\frac{C}{n};$$

for $\alpha=1$

$$\sum_{j=1}^n \mathrm{E}\eta_j^2\le\frac{\log n}{n};$$

and for $1/2<\alpha<1$

$$\sum_{j=1}^n \mathrm{E}\eta_j^2\le\frac{C}{n^{2\alpha-1}}.$$

We have finally $\mathbb{P}\{\max_{k\le n}|\beta_k|\ge t\}\to 0,\ n\to\infty$, which gives the convergence $\|W_n-Z_n\|\xrightarrow{\mathbb{P}}0$.

4.3.4 Step 4

Let U_n be the process defined at the points $\frac{k}{n}$ by

$$U_n\left(\left(\frac{k}{n}\right)^{\alpha}\right)=\frac{\alpha}{B_n}\sum_{i=1}^k\varepsilon_i\gamma_i(i-1)^{\alpha-1},\ k=1,2,\ldots,n,$$

and by linear interpolation on the intervals $[\frac{k}{n},\frac{k+1}{n}],\ k=0,\ldots,n-1$. We now state the weak convergence of the processes U_n to the process Y,

$$Y(t)=\sqrt{2\alpha}\int_0^t s^{\frac{\alpha-1}{2\alpha}}\,\mathrm{d}W(s),$$

W is a Brownian motion, for which the covariance matrix of $W(1)$ coincides with the covariance matrix of ε_1.

The proof is standard because $U_n(\cdot)$ represents a (more or less) usual broken line constructed by the consecutive sums of independent (nonidentically distributed) random variables. One could apply Prokhorov's theorem (see [2], Chap. IX, Sect. 3, Theorem1).

Only one thing must be checked: the Lindeberg condition.

Let $\varepsilon > 0$. We have

$$\Lambda_n(\varepsilon) := \frac{1}{B_n^2} \sum_1^n E\left\{\|\varepsilon_i\gamma_i(i-1)^{\alpha-1}\|^2 \mathbf{1}_{\{\|\varepsilon_i\gamma_i(i-1)^{\alpha-1}\|\ge \varepsilon B_n\}}\right\} =$$

$$= \frac{1}{n^{2\alpha-1}} \sum_2^n (i-1)^{2\alpha-2} E\left\{\gamma_1^2 \mathbf{1}_{\{|\gamma_1|(i-1)^{\alpha-1}\|\ge \varepsilon n^{\alpha-1/2}\}}\right\}.$$

As

$$\{|\gamma_1|(i-1)^{\alpha-1}\| \ge \varepsilon n^{\alpha-1/2}\} \subset \{|\gamma_1| \ge \varepsilon\sqrt{n}\}$$

for $2 \le i \le n$, we get

$$\Lambda_n(\varepsilon) \le \frac{1}{2\alpha-1} E\gamma_1^2 \mathbf{1}_{\{|\gamma_1|\ge\varepsilon\sqrt{n}\}} \to 0,$$

as $n \to \infty$.

It means that the Lindeberg condition is fulfilled, and by the above-mentioned Prokhorov's theorem the process U_n is weakly converging. To identify the limiting process with Y, it is sufficient to state that for any $0 < s < t \le 1$, and for any $x \in \mathbb{R}^d$, $|x| = 1$, we have the convergence $\langle U_n(t) - U_n(s), x\rangle \Longrightarrow \langle Y(t) - Y(s), x\rangle$.

It is clear that

$$[U_n(t) - U_n(s)] - \left[U_n\left(\left(\frac{k}{n}\right)^\alpha\right) - U_n\left(\left(\frac{l}{n}\right)^\alpha\right)\right] \xrightarrow{\mathbb{P}} 0,$$

if $\left(\frac{k}{n}\right)^\alpha \to t, \quad \left(\frac{l}{n}\right)^\alpha \to s.$

Let $l < k$. As

$$\left\langle U_n\left(\left(\frac{k}{n}\right)^\alpha\right) - U_n\left(\left(\frac{l}{n}\right)^\alpha\right), x\right\rangle = \frac{\alpha}{B_n} \sum_{i=l+1}^k \langle\varepsilon_i, x\rangle\gamma_i(i-1)^{\alpha-1},$$

by the theorem of Lindeberg–Feller, it is sufficient to state the convergence of variances.

We have

$$\mathrm{Var}\left\langle U_n\left(\left(\frac{k}{n}\right)^\alpha\right) - U_n\left(\left(\frac{l}{n}\right)^\alpha\right), x\right\rangle =$$

$$= \frac{2\alpha^2}{n^{2\alpha-1}} E\langle\varepsilon_1, x\rangle^2 \sum_{i=l+1}^{k} (i-1)^{2\alpha-2} \xrightarrow[n\to\infty]{} \frac{2\alpha^2}{2\alpha-1} E\langle\varepsilon_1, x\rangle^2 [t^{\frac{2\alpha-1}{\alpha}} - s^{\frac{2\alpha-1}{\alpha}}],$$

and

$$\operatorname{Var}\langle Y(t) - Y(s), x\rangle = 2\alpha E\langle\varepsilon_1, x\rangle^2 \int_s^t u^{\frac{\alpha-1}{\alpha}}\, \mathrm{d}u = \frac{2\alpha^2}{2\alpha-1} E\langle\varepsilon_1, x\rangle^2 [t^{\frac{2\alpha-1}{\alpha}} - s^{\frac{2\alpha-1}{\alpha}}],$$

which are the same. Therefore, indeed $U_n \Rightarrow Y$.

4.3.5 Step 5: Convergence $X_n \Rightarrow Y$.

Due to the steps 2 and 3, it is sufficient to show that $W_n \Rightarrow Y$.

Let $f_n : [0, 1] \to [0, 1]$, be a piecewise linear continuous function such that $f_n(t_{n,k}) = \left(\frac{k}{n}\right)^\alpha$; $t_{n,k} = \left(\frac{\Gamma_k}{\Gamma_n}\right)^\alpha$; $k = 0, 1, \ldots, n$.

By definition of W_n and U_n, we have

$$W_n(t) = U_n(f_n(t)), \ t \in [0, 1].$$

By the corollary to Lemma 6 (see below), the function f_n converges in probability uniformly to f, $f(t) = t$, and by previous step $U_n \Rightarrow Y$.

It means that we can apply Lemma 7 which gives the necessary convergence.

Lemma 6 *Let*

$$M_n = \max_{k\le n} \left\{ \left| \frac{\Gamma_k}{\Gamma_n} - \frac{k}{n} \right| \right\}.$$

Then $M_n \xrightarrow{\mathbb{P}} 0, \quad n \to \infty.$

Proof of Lemma 6 We have

$$\begin{aligned} \mathbb{P}\{M_n > \varepsilon\} &= \mathrm{E}\left\{ \mathbb{P}\left\{ \max_{k\le n} \left| \frac{\Gamma_k}{\Gamma_n} - \frac{k}{n} \right| > \varepsilon \,|\, \Gamma_n \right\} \right\} = \\ &= \int_0^\infty \mathbb{P}\left\{ \max_{k\le n} \left| \frac{\Gamma_k}{\Gamma_n} - \frac{k}{n} \right| > \varepsilon \,|\, \Gamma_n = t \right\} \mathcal{P}_{\Gamma_n}(\mathrm{d}t) = \\ &= \int_0^\infty \mathbb{P}\left\{ \max_{k\le n} \left| \xi_{n,k} - \frac{k}{n} \right| > \varepsilon \right\} \mathcal{P}_{\Gamma_n}(\mathrm{d}t) = \mathbb{P}\left\{ \max_{k\le n} \left| \xi_{n,k} - \frac{k}{n} \right| > \varepsilon \right\}, \end{aligned} \tag{21}$$

where $(\xi_{n,k})_{k=1,\ldots,n}$ are the order statistics from $[0, 1]$-uniform distribution.

Let $\delta_n := \max_{k\le n} \left|\xi_{n,k} - \frac{k}{n}\right|$. Evidently, $\delta_n \le \sup_{[0,1]} |F_n^*(x) - x|$, where F_n^* is the uniform empirical distribution function. By Glivenko–Cantelli theorem, $\sup_{[0,1]} |F_n^*(x) - x| \to 0$ a.s, which gives the convergence $M_n \to 0$ in probability. $\square$

Corollary 2 $M_n^{(1)} = \max_{k\le n}\left|\left(\frac{\Gamma_k}{\Gamma_n}\right)^\alpha - \left(\frac{k}{n}\right)^\alpha\right| \xrightarrow{\mathbb{P}} 0,\ n \to \infty.$

The proof follows directly from Lemma 6 due to the uniform continuity of the function $h(x) = x^\alpha$, $x \in [0, 1]$.

Lemma 7 *Let* $\{U_n\}$ *be a sequence of continuous processes on* $[0, 1]$ *weakly convergent to some limit process* U. *Let* $\{f_n\}$ *be a sequence of random continuous bijections* $[0, 1]$ *on* $[0, 1]$ *which in probability uniformly converges to the identity function* $f(t) \equiv t$. *Then the process* $W_n,\ W_n(t) = U_n(f_n(t)),\ t \in [0, 1]$, *will converge weakly to* U.

Proof of Lemma 7 By theorem 4.4 from [1], we have the weak convergence in $\mathbb{M} := \mathbb{C}[0, 1] \times \mathbb{C}[0, 1]$

$$(U_n, f_n) \Longrightarrow (U, f).$$

By Skorohod representation theorem, we can find random elements $(\tilde{U}_n, \tilde{f}_n)$ and $(\tilde{U}, \tilde{f})$ of $\mathbb{M}$ (defined probably on a new probability space) such that

$$(U_n, f_n) \stackrel{\mathcal{L}}{=} (\tilde{U}_n, \tilde{f}_n), \quad (U, f) \stackrel{\mathcal{L}}{=} (\tilde{U}, \tilde{f}),$$

and $(\tilde{U}_n, \tilde{f}_n) \to (\tilde{U}, \tilde{f})$ a.s. in $\mathbb{M}$.

As the last convergence implies evidently the a.s. uniform convergence of $\tilde{U}_n(\tilde{f}_n(t))$ to $\tilde{U}(\tilde{f}(t))$, we get the convergence in distribution of $U(f_n(\cdot))$ to $U(f(\cdot)) = U(\cdot)$. □

5 Proof of Theorem 2

Proof of Theorem 2. We need some facts from [15]. Consider $(\Omega, \mathcal{M})$, where $\Omega = \mathbb{C}([0, \infty); R^d)$ be the space of continuous trajectories from $[0, \infty)$ into R^d. Given $t \ge 0$ and $\omega \in \Omega$ let $x(t, \omega)$ denote the position of ω in R^d at time t. If we put

$$D(\omega, \omega') = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{0\le t\le n} \left|x(t, \omega) - x(t, \omega')\right|}{1 + \sup_{0\le t\le n} |x(t, \omega) - x(t, \omega')|}$$

then it is well known that D is a metric on Ω and (Ω, D) is a Polish space. The convergence induced by D is the uniform convergence on bounded t-intervals. For simplicity, we will omit ω in the future and we will be assuming that all our processes are homogeneous in time. Analogous results for time-inhomogeneous processes may be obtained by simply considering the time-space processes.

We will use $\mathcal{M}$ to denote the Borel σ-field of subsets of (Ω, D), $\mathcal{M} = \sigma[x(t) : t \ge 0]$. We also will consider an increasing family of σ-algebras $\mathcal{M}_t = \sigma[x(s) : 0 \le s \le t]$. The classical approach to the construction of diffusion processes corresponding to given coefficients a and b involves a transition probability function

$P(s, x; t, \cdot)$ which allows to construct for each $x \in R^d$, a probability measure P_x on $\Omega = \mathbb{C}([0, \infty); R^d)$ with the properties that

$$P_x(x(0) = x) = 1$$

and

$$P_x(x(t_2) \in \Gamma \,|\mathcal{M}_{t_1}) = P(t_1, x(t_1); t_2, \Gamma) \; a.s.P_x$$

for all $0 \leq t_1 < t_2$ and $\Gamma \in \mathcal{B}_{R^d}$ (the Borel σ-algebra in R^d). It appears that this measure is a martingale measure for a special martingale related with the second-order differential operator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(\cdot)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(\cdot)\frac{\partial}{\partial x_i},$$

namely, for all $f \in \mathbb{C}_0^\infty(R^d)$

$$P_x(x(0) = x) = 1,$$

$$(f(x(t)) - \int_0^t Lf(x(u))du, \mathcal{M}_t, P_x) \tag{22}$$

is a martingale. We will say that the martingale problem for a and b is *well posed* if, for each x, there is exactly one solution to that martingale problem starting from x. We will be working with the following setup. For each $h > 0$, let $\Pi_h(x, \cdot)$ be a transition function on R^d. Given $x \in R^d$, let P_x^h be the probability measure on Ω characterized by the properties that

$$(i) \;\; P_x^h(x(0) = x) = 1, \tag{23}$$

$$(ii) \;\; P_x^h\left\{x(t) = \frac{(k+1)h - t}{h}x(kh) + \frac{t - kh}{h}x((k+1)h), \quad kh \leq t < (k+1)h\right\} = 1 \tag{24}$$

for all $k \geq 0$,

$$(iii) \;\; P_x^h(x((k+1)h) \in \Gamma \mid \mathcal{M}_{kh}) = \Pi_h(x(kh), \Gamma), \quad P_x^h - \text{ a.s.}$$

$$\text{for all } k \geq 0 \text{ and } \Gamma \in \mathcal{B}_{R^d}. \tag{25}$$

Define

$$a_h^{ij}(x) = \frac{1}{h}\int_{|y-x|\leq 1} (y_i - x_i)(y_j - x_j)\Pi_h(x, dy), \tag{26}$$

$$b_h^i(x) = \frac{1}{h}\int_{|y-x|\le 1}(y_i - x_i)\Pi_h(x, dy), \tag{27}$$

and

$$\Delta_h^\varepsilon(x) = \frac{1}{h}\Pi_h(x, R^d \backslash B(x,\varepsilon)), \tag{28}$$

where $B(x,\varepsilon)$ is the open ball with center x and radius ε. What we are going to assume is that for all $R > 0$

$$\lim_{h\searrow 0}\sup_{|x|\le R}\|a_h(x) - a(x)\| = 0, \tag{29}$$

$$\lim_{h\searrow 0}\sup_{|x|\le R}|b_h(x) - b(x)| = 0, \tag{30}$$

$$\sup_{h>0}\sup_{x\in R^d}(\|a_h(x)\| + |b_h(x)|) < \infty, \tag{31}$$

$$\lim_{h\searrow 0}\sup_{x\in R^d}\Delta_h^\varepsilon(x) = 0. \tag{32}$$

Theorem A. ([15], p. 272, Theorem 11.2.3). *Assume that in addition to* (29)–(32) *the coefficients a and b are continuous and have the property that for each $x \in R^d$ the martingale problem for a and b has exactly one solution P_x starting from x (that is well posed). Then P_x^h* converges weakly to P_x *uniformly in x on compact subsets of R^d.*

Sufficient conditions for the well posedness are given by the following theorem.

Let S_d be the set of symmetric nonnegative definite $d \times d$ real matrices.

Theorem B. ([15], p. 152, Theorem 6.3.4). *Let $a : R^d \longrightarrow S_d$ and $b : R^d \longrightarrow R^d$ be bounded measurable functions and suppose that $\sigma : R^d \longrightarrow R^d \times R^d$ is a bounded measurable function such that $a = \sigma\sigma^*$. Assume that there is an A such that*

$$\|\sigma(x) - \sigma(y)\| + |b(x) - b(y)| \le A\,|x - y| \tag{33}$$

for all $x, y \in R^d$. Then the martingale problem for a and b is well posed and the corresponding family of solutions $\{P_x : x \in R^d\}$ is Feller continuous (that is $P_{x_n} \to P_x$ weakly if $x_n \to x$).

Note that (33) and uniform ellipticity of $a(x)$ imply the existence of the transition density $p(s, x; t, y)$ ([15], Theorem 3.2.1, p. 71).

Consider the model

$$X((k+1)h) = X(kh) + hb(X(kh)) + \sqrt{h}\xi(X(kh)),$$

$$\xi(X(kh)) = \rho_k\sigma(X(kh))\varepsilon_k, \tag{34}$$

where $\{\varepsilon_k\}$ are i.i.d. random vectors uniformly distributed on the unit sphere S^{d-1}, and $\{\rho_k\}$ are i.i.d. random variables having a density, $\rho_k \geq 0$, $E\rho_k^2 = d$. Let us check the conditions (29)–(32). It is easy to see that

$$\Pi_h(x, dy) = p_h^x(y)dy, \text{ where } p_h^x(y) = h^{-d/2} f_\xi \left(\frac{y - x - hb(x)}{\sqrt{h}} \right). \tag{35}$$

Here, f_ξ denotes the density of the random vector ξ. Let us check (32). Note that $E\xi = 0$ and the covariance matrix of the vector ξ is equal to

$$Cov(\xi, \xi^T) = E(\rho_k^2 \sigma(x)\varepsilon_k \varepsilon_k^T \sigma^T(x)) = a(x). \tag{36}$$

We have

$$h\Delta_h^\varepsilon(x) = \Pi_h(x, R^d \backslash B(x, \varepsilon)) = \int_{R^d \backslash B(x,\varepsilon)} p_h^x(y)dy =$$

$$= \int_{v+\sqrt{h}b(x) \in R^d \backslash B(0, \frac{\varepsilon}{\sqrt{h}})} f_\xi(v)dv = P\left\{ \xi \in \overline{B\left(0, \frac{\varepsilon}{\sqrt{h}}\right)\right\} - \sqrt{h}b(x)} \right) \leq$$

$$\leq P\left\{ |\xi|^2 \geq \frac{\varepsilon^2}{4h} \right\} = o(h). \tag{37}$$

The last equality is a consequence of the Markov inequality. The equality (36), the uniform ellipticity of $a(x)$ and (37) imply (32). To prove (29), note that by (33)

$$a_h^{ij}(x) = \frac{1}{h} \int_{|y-x| \leq 1} (y_i - x_i)(y_j - x_j) p_h^x(y)dy =$$

$$= \int_{\left|v+\sqrt{h}b(x)\right| \leq \frac{1}{\sqrt{h}}} (v_i + \sqrt{h}b^i(x))(v_j + \sqrt{h}b^j(x)) f_\xi(v)dv =$$

$$= \int_{\left|v+\sqrt{h}b(x)\right| \leq \frac{1}{\sqrt{h}}} v_i v_j f_\xi(v)dv + o(\sqrt{h}) = a(x) + o(1). \tag{38}$$

To check (30), note that

$$b_h^i(x) = \frac{1}{h} \int_{|y-x| \leq 1} (y_i - x_i) p_h^x(y)dy =$$

$$= \frac{1}{\sqrt{h}} \int_{\left|v+\sqrt{h}b(x)\right| \leq \frac{1}{\sqrt{h}}} (v_i + \sqrt{h}b^i(x)) f_\xi(v)dv =$$

$$= b^i(x) \int_{\left|v+\sqrt{h}b(x)\right| \le \frac{1}{\sqrt{h}}} f_\xi(v)dv - \frac{1}{\sqrt{h}} \int_{\left|v+\sqrt{h}b(x)\right| > \frac{1}{\sqrt{h}}} v_i f_\xi(v)dv. \tag{39}$$

To estimate the second integral in (39), we apply the Cauchy–Schwarz inequality

$$\frac{1}{\sqrt{h}} \int_{\left|v+\sqrt{h}b(x)\right| > \frac{1}{\sqrt{h}}} |v| f_\xi(v)dv \le \frac{1}{\sqrt{h}} \left(\int |v|^2 f_\xi(v)dv \right)^{1/2} \left(P(|\xi|^2 \ge \frac{1}{4h} \right)^{1/2} = o(1), \tag{40}$$

and (39), (40) imply (30). Finally, (31) follows from our calculations and assumptions of Theorem B. Weak convergence P_x^h to P_x follows now from Theorems A and B cited above. □

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