# AN EASY PROOF OF POLYA'S THEOREM ON RANDOM WALKS

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ABSTRACT. We present an easy proof of Polya's theorem on random walks: with the probability one a random walk on the two-dimensional lattice returns to the starting point.

#### 1. Introduction

We consider two-dimensional random walks that start at the origin and performed on the lattice  $\mathbb{Z}^2$ . There are four directions for each step and the choice of a direction is random with probability 1/4 for each direction. We will consider only finite walks. A given walk of length n is performed with probability  $1/2^{2n}$ .

A *loop* is a walk that begins and ends at origin. It is convenient to consider the walk of length zero — the *trivial* loop. A loop is *simple*, if it is not a concatenation of two nontrivial loops. We will assume that a simple loop is nontrivial.

Obviously, a loop has an even length. Let  $P_n$  be the number of simple loops of length 2n. It is easy to see that  $P_1 = 4$  and  $P_2 = 20$ . The number

$$r = \frac{P_1}{2^4} + \frac{P_2}{2^8} + \dots = \sum_{n=1}^{\infty} \frac{P_n}{2^{4n}} \le 1$$

is the probability of returning to the origin.

# Polya's theorem on random walk. r = 1.

There are many proofs of this theorem. See, for example, [1], [2], [3], [4].

# 2. The one-dimensional case

In our approach the reasoning in the one-dimensional and the reasoning in the two-dimensional cases are the same, so we will study the one-dimensional case at first

Let  $B_n$  be the number of loops of the length 2n,  $P_n$  be the number of simple loops of the same length and

$$B(t) = 1 + B_1 t + B_2 t^2 + \dots$$
 and  $P(t) = P_1 t + P_2 t^2 + \dots$ 

be the corresponding generating functions. As each loop is the concatenation of the simple loop and a loop (maybe trivial), then  $B(t) = P(t) \cdot B(t) + 1$ . Thus, P(t) = 1 - 1/B(t) in the ring of formal series.

The probability of a given walk of the length n is  $1/2^n$ , hence, we will work with "weighted" generating functions

$$b(x) = B\left(\frac{x}{4}\right) = 1 + \frac{B_1}{2^2} x + \frac{B_2}{2^4} x^2 + \dots \quad \text{and} \quad p(x) = P\left(\frac{x}{4}\right) = \frac{P_1}{2^2} x + \frac{P_2}{2^4} x^2 + \dots$$

Then

$$r = p(1) = \frac{P_1}{2^2} + \frac{P_2}{2^4} + \dots \leq 1$$

is the probability of returning to the origin. So, the convergence radius of the series p(x) is  $\geqslant 1$ 

As  $B_n = \binom{2n}{n}$ , then the convergence radius of the series b(x) is one. Indeed,

$$\lim_{n \to \infty} \binom{2n+2}{n+1} \cdot 2^{-(2n+2)} : \binom{2n}{n} \cdot 2^{-2n} = 1.$$

It means that p(x) = 1 - 1/b(x) as functions, if 0 < x < 1. As  $\lim_{x\to 1} p(x) = r$ , then

$$r = 1 - \frac{1}{\lim_{x \to 1} b(x)}.$$

But

$$B_n/2^{2n} = \frac{(2n)!}{(n!)^2 2^{2n}} \sim \frac{\sqrt{4\pi n} \cdot 2^{2n} \cdot n^{2n}}{2\pi n \cdot n^{2n} \cdot 2^{2n}} = \frac{1}{\sqrt{\pi n}}.$$

Thus,  $\lim_{x\to 1} b(x) = \infty$  and r = 1.

# 3. The two-dimensional case

As above, let  $B_n$  be the number of loops of length 2n,  $P_n$  be the number of simple loops of the same length,

$$B(t) = 1 + B_1t + B_2t^2 + \dots$$
 and  $P(t) = P_1t + P_2t^2 + \dots$ 

be the corresponding generating functions. Then

$$r = \frac{P_1}{4^2} + \frac{P_2}{4^4} + \frac{P_3}{4^6} + \dots \leqslant 1$$

is the probability of the returning to the origin. As above we have that  $B(t) = P(t) \cdot B(t) + 1$  and P(t) = 1 - 1/B(t) in the ring of formal series. We have to compute the number  $B_n$ .

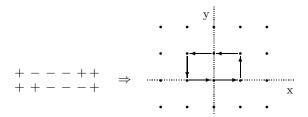
# Proposition 1. $B_n = \binom{2n}{n}^2$ .

*Proof.* Given two strings a and b of +1 and -1 of length 2n, such that there are equal number of plus and minus ones in each string, we must construct a loop of length 2n. It can be done in the following way: the pair (a[i],b[i]) corresponds to the i-th step

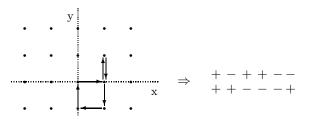
- in the direction (0,1), if (a[i],b[i])=(-1,+1);
- in the direction (0, -1), if (a[i], b[i]) = (+1, -1);
- in the direction (1,0), if (a[i],b[i]) = (+1,+1);
- and in the direction (-1,0), if (a[i],b[i]) = (-1,-1).

Thus constructed walk returns to the origin. Indeed, let the number of pairs (+1,+1) is k, the number of pairs (-1,-1) is l, the number of pairs (+1,-1) is m and the number of pairs (-1,+1) is n, then k+m=l+n and k+n=l+m. Hence, 2k=2l, i.e. k=l and m=n. So, the number of "right" steps is equal to the number of "left" steps and the number of "up" steps is equal to the number of "down" steps.

### Example 3.1.



In the opposite way,



Remark 3.1. A similar bijection does not exist in the three dimensional case, because there are eight triads of plus and minus ones, but only six directions in the three dimensional lattice.

As above, let

$$b(x) = B\left(\frac{x}{16}\right) = 1 + \frac{B_1}{4^2} x + \frac{B_2}{4^4} x^2 + \dots \quad \text{and} \quad p(x) = P\left(\frac{x}{16}\right) = \frac{P_1}{4^2} x + \frac{P_2}{4^4} x^2 + \dots$$

The proof of Polya's theorem. As coefficients of the series b(x) are squares of the coefficients of the same series in the one-dimensional case, then the radius of convergence of this series is one and its coefficients are equivalent to  $1/\pi n$ . Thus this series is divergent at one and

$$r = \lim_{x \to 1} p(x) = 1 - 1/\lim_{x \to 1} b(x) = 1.$$

#### References

[1] P.G. Doyle, J.L. Snell. Random walks and electrical networks. Mathematical Association of America, 1984.

- [2] M. Skopenkov, V. Smykalov, A. Ustinov. Random walks and electrical networks. Matematicheskoe Prosveshzhenie, 16, 2012, 25-47 [in Russian].
- [3] J. Novak. Polya's random walks theorem. Amer. Math. Monthly, 121(8), 2014.
- [4] K. Lange. Polya's random walks theorem revisited. Amer. Math. Monthly, 122(10), 2015, 1005-1007.

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