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Towards a cluster structure on trigonometric zastava

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Abstract We study a moduli problem on a nodal curve of arithmetic genus 1, whose solution is an open subscheme in the zastava space for projective line. This moduli space is equipped with a natural Poisson structure, and we compute it in a natural coordinate system. We compare this Poisson structure with the trigonometric Poisson structure on the transversal slices in an affine flag variety. We conjecture that certain generalized minors give rise to a cluster structure on the trigonometric zastava.

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To Sasha Beilinson with love and gratitude.

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1 Introduction

1.1 Zastava and Euclidean monopoles

Let G be an almost simple simply connected algebraic group over $\mathbb{C}$. We denote by $\mathcal{B}$ the flag variety of G. Let us also fix a pair of opposite Borel subgroups B, B_- whose intersection is a maximal torus T. Let Λ denote the cocharacter lattice of T; since G is assumed to be simply connected, this is also the coroot lattice of G. We denote by $\Lambda_+ \subset \Lambda$ the sub-semigroup spanned by positive coroots.

It is well-known that $H_2(\mathcal{B}, \mathbb{Z}) = \Lambda$ and that an element $\alpha \in H_2(\mathcal{B}, \mathbb{Z})$ is representable by an effective algebraic curve if and only if $\alpha \in \Lambda_+$. Let $\overset{\circ}{Z}{}^\alpha$ denote the space of maps $C = \mathbb{P}^1 \to \mathcal{B}$ of degree α sending $\infty \in \mathbb{P}^1$ to $B_- \in \mathcal{B}$. It is known [10] that this is a smooth symplectic affine algebraic variety, which can be identified with the hyperkähler moduli space of framed G-monopoles on $\mathbb{R}^3$ with maximal symmetry breaking at infinity of charge α [15,16].

The monopole space $\overset{\circ}{Z}{}^\alpha$ has a natural partial compactification Z^α (*zastava* scheme). It can be realized as the moduli space of based *quasi-maps* of degree α; set-theoretically it can be described in the following way:

$$Z^\alpha = \bigsqcup_{0 \le \beta \le \alpha} \overset{\circ}{Z}{}^\beta \times \mathbb{A}^{\alpha-\beta},$$

where for $\gamma \in \Lambda_+$ we denote by $\mathbb{A}^\gamma$ the space of all colored divisors $\sum \gamma_i x_i$ with $x_i \in \mathbb{A}^1$, $\gamma_i \in \Lambda_+$ such that $\sum \gamma_i = \gamma$.

The zastava space is equipped with a *factorization* morphism $\pi_\alpha \colon Z^\alpha \to \mathbb{A}^\alpha$ whose restriction to $\overset{\circ}{Z}{}^\alpha \subset Z^\alpha$ has a simple geometric meaning: for a based map $\varphi \in \overset{\circ}{Z}{}^\alpha$ the colored divisor $\pi_\alpha(\phi)$ is just the pullback of the colored Schubert divisor $D \subset \mathcal{B}$ equal to the complement of the open B-orbit in $\mathcal{B}$. The morphism $\pi_\alpha \colon \overset{\circ}{Z}{}^\alpha \to \mathbb{A}^\alpha$ is the *Atiyah–Hitchin* integrable system (with respect to the above symplectic structure): all the fibers of π_α are Lagrangian.

A system of étale birational coordinates on $\overset{\circ}{Z}{}^\alpha$ was introduced in [10]. Let us recall the definition for $G = SL(2)$. In this case $\overset{\circ}{Z}{}^\alpha$ consists of all maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree α which send ∞ to 0. We can represent such a map by a rational function $\frac{R}{Q}$ where Q is a monic polynomial of degree α and R is a polynomial of degree $< \alpha$. Let $w_1, \dots, w_\alpha$ be the zeros of Q. Set $y_r = R(w_r)$. Then the functions $(y_1, \dots, y_\alpha, w_1, \dots, w_\alpha)$ form a system of étale birational coordinates on $\overset{\circ}{Z}{}^\alpha$, and the above mentioned symplectic form in these coordinates reads $\Omega_{\mathrm{rat}} = \sum_{r=1}^{\alpha} \frac{\mathrm{d}y_r \wedge \mathrm{d}w_r}{y_r}$.

For general G the definition of the above coordinates is quite similar. In this case given a point in $\overset{\circ}{Z}{}^\alpha$ we can define polynomials R_i, Q_i where i runs through the set I of vertices of the Dynkin diagram of G, $\alpha = \sum a_i \alpha_i$, and

(1) Q_i is a monic polynomial of degree a_i,
(2) R_i is a polynomial of degree $< a_i$.

Hence, we can define (étale, birational) coordinates $(y_{i,r}, w_{i,r})$ where $i \in I$ and $r = 1, \dots, a_i$. Namely, $w_{i,r}$ are the roots of Q_i, and $y_{i,r} = R_i(w_{i,r})$. The Poisson brackets of these coordinates with respect to the above symplectic form are as follows: $\{w_{i,r}, w_{j,s}\}_{\mathrm{rat}} = 0$, $\{w_{i,r}, y_{j,s}\}_{\mathrm{rat}} = \check{d}_i \delta_{ij} \delta_{rs} y_{j,s}$, $\{y_{i,r}, y_{j,s}\}_{\mathrm{rat}} = (\check{\alpha}_i, \check{\alpha}_j) \frac{y_{i,r} y_{j,s}}{w_{i,r} - w_{j,s}}$ for $i \neq j$, and finally $\{y_{i,r}, y_{i,s}\}_{\mathrm{rat}} = 0$. Here $\check{\alpha}_i$ is a simple root, $(,)$ is the invariant scalar product on $(\mathrm{Lie}\, T)^*$ such that the square length of a short root is 2, and $\check{d}_i = (\check{\alpha}_i, \check{\alpha}_i)/2$.

Now recall that the standard *rational* r-matrix for $\mathfrak{g} = \mathrm{Lie}\, G$ gives rise to a Lie bialgebra structure on $\mathfrak{g}[z^{\pm 1}]$ corresponding to the Manin triple $\mathfrak{g}[z], z^{-1}\mathfrak{g}[z^{-1}], \mathfrak{g}[z^{\pm 1}]$. This in turn gives rise to a Poisson structure on the affine Grassmannian $\mathrm{Gr}_G = G[z^{\pm 1}]/G[z]$. The transversal slices $\mathcal{W}^\lambda_\mu$ from a $G[z]$-orbit Gr^μ_G to another orbit Gr^λ_G (here $\mu \leq \lambda$ are dominant coweights of G) are examples of symplectic leaves of the above Poisson structure. According to [2], the zastava spaces are "stable limits" of the above slices. More precisely, for $\alpha = \lambda - \mu$ there is a birational Poisson map $s^{\lambda^*}_{\mu^*} \colon \mathcal{W}^{\lambda^*}_{\mu^*} \to Z^\alpha$ (here $\lambda^* := -w_0\lambda$, $\mu^* = -w_0\mu$, and w_0 is the longest element of the Weyl group $W = W(G, T)$).

1.2 Trigonometric zastava and periodic monopoles

We have an open subset $\mathbb{G}^\alpha_m \subset \mathbb{A}^\alpha$ (colored divisors not meeting $0 \in \mathbb{A}^1$), and we introduce the open subscheme of *trigonometric zastava* ${}^\dagger Z^\alpha := \pi_\alpha^{-1} \mathbb{G}^\alpha_m \subset Z^\alpha$, and its smooth open affine subvariety of *periodic monopoles* ${}^\dagger\overset{\circ}{Z}{}^\alpha := {}^\dagger Z^\alpha \cap \overset{\circ}{Z}{}^\alpha$. These schemes are solutions of the following modular problems.

Let $C^\dagger$ be an irreducible nodal curve of arithmetic genus 1 obtained by gluing the points $0, \infty \in C = \mathbb{P}^1$, so that $\pi \colon C \to C^\dagger$ is the normalization. Let $\mathfrak{c} \in C^\dagger$ be the singular point. The moduli space $\mathrm{Bun}^0_T(C^\dagger)$ of T-bundles on $C^\dagger$ of degree 0 is canonically identified with the Cartan torus T itself. We fix a T-bundle $\mathcal{F}_T$ which corresponds to a *regular* point $t \in T^{\mathrm{reg}}$. Then ${}^\dagger\overset{\circ}{Z}{}^\alpha$ is the moduli space of the following data:

(a) a trivialization $\tau_\mathfrak{c}$ of the fiber of $\mathcal{F}_T$ at the singular point $\mathfrak{c} \in C^\dagger$;

(b) a B-structure ϕ in the induced G-bundle $\mathcal{F}_G = \mathcal{F}_T \overset{T}{\times} G$ of degree α which is transversal to $\mathcal{F}_B = \mathcal{F}_T \overset{T}{\times} B$ at $\mathfrak{c}$.

The scheme ${}^\dagger Z^\alpha$ is the moduli space of the similar data with the only difference: we allow a B-structure in (b) to be *generalized* (i.e. to acquire defects at certain points of $C^\dagger$), but require it to have no defect at $\mathfrak{c}$.

As a regular Cartan element t varies, the above moduli spaces become fibers of a single family. More precisely, we consider the following moduli problem:
(t) a T-bundle $\mathcal{F}_T$ of degree 0 on $C^\dagger$ corresponding to a regular element of T;
(a,b) as above;
(c) a trivialization $f_\mathfrak{c}$ at $\mathfrak{c}$ of the T-bundle ϕ_T induced from the B-bundle ϕ in (b).

This moduli problem is represented by a scheme $\overset{\circ}{Y}{}^\alpha \subset Y^\alpha$ (depending on whether the B-structure in (b) is genuine or generalized). Note that Y^α is equipped with an action of

$T \times T$ changing the trivializations in (a,c). We prove that $\overset{\circ}{Y}{}^\alpha$ is a smooth affine variety equipped with a natural projection $\varpi\colon \overset{\circ}{Y}{}^\alpha \to {}^\dagger\overset{\circ}{Z}{}^\alpha$, and we construct a nondegenerate bivector field on $\overset{\circ}{Y}{}^\alpha$ arising from a differential in a spectral sequence involving the tangent and cotangent bundles of $\overset{\circ}{Y}{}^\alpha$ (this construction is a trigonometric degeneration of the construction [9] for elliptic curves; its rational analogue was worked out in [10]). This bivector field descends to ${}^\dagger\overset{\circ}{Z}{}^\alpha$ under the projection $\varpi\colon \overset{\circ}{Y}{}^\alpha \to {}^\dagger\overset{\circ}{Z}{}^\alpha$ and gives rise to a nondegenerate Poisson structure, i.e. a symplectic form on ${}^\dagger\overset{\circ}{Z}{}^\alpha$. It would be interesting to obtain this symplectic form by the method of [24, 4.2]. The Poisson brackets of the coordinates on ${}^\dagger\overset{\circ}{Z}{}^\alpha$ are as follows: $\{w_{i,r}, w_{j,s}\}_{\mathrm{trig}} = 0$, $\{w_{i,r}, y_{j,s}\}_{\mathrm{trig}} = \check{d}_i \delta_{ij}\delta_{rs} w_{j,s} y_{j,s}$, $\{y_{i,r}, y_{j,s}\}_{\mathrm{trig}} = (\check{\alpha}_i, \check{\alpha}_j)\frac{(w_{i,r}+w_{j,s})y_{i,r}y_{j,s}}{2(w_{i,r}-w_{j,s})}$ for $i \neq j$, and finally $\{y_{i,r}, y_{i,s}\}_{\mathrm{trig}} = 0$. In particular, the projection $\pi_\alpha\colon {}^\dagger\overset{\circ}{Z}{}^\alpha \to \mathbb{G}_m^\alpha$ is the *trigonometric Atiyah–Hitchin* integrable system (for $G = SL(2)$ this system goes back at least to [8]).

Now recall that the standard *trigonometric* r-matrix for $\mathfrak{g}$ gives rise to a Lie bialgebra structure on $\mathfrak{g}((z^{-1})) \oplus \mathfrak{t}$ which in turn gives rise to a Poisson structure on the affine flag variety $\mathcal{F}\ell_G$ (the quotient of $G[z^{\pm 1}]$ with respect to an Iwahori subgroup). The intersections $\mathcal{F}\ell_y^w$ of the opposite Iwahori orbits (aka *open Richardson varieties*) are Poisson subvarieties of $\mathcal{F}\ell_G$. Here w, y are elements of the affine Weyl group $W_a = W \ltimes \Lambda$. For dominant coweights $\mu \leq \lambda \in \Lambda$ such that $\lambda - \mu = \alpha$, and the longest element w_0 of the finite Weyl group W, the Richardson variety $\mathcal{F}\ell_{w_0\times\mu^*}^{w_0\times\lambda^*}$ is a symplectic leaf of $\mathcal{F}\ell_G$, the projection $\mathrm{pr}\colon \mathcal{F}\ell_{w_0\times\mu^*}^{w_0\times\lambda^*} \to \mathcal{W}_{\mu^*}^{\lambda^*}$ is an open embedding, and the composition $s_{\mu^*}^{\lambda^*} \circ \mathrm{pr}\colon \mathcal{F}\ell_{w_0\times\mu^*}^{w_0\times\lambda^*} \to Z^\alpha$ is a *symplectomorphism* onto its image ${}^\dagger\overset{\circ}{Z}{}^\alpha \subset Z^\alpha$ equipped with the trigonometric symplectic structure.

1.3 Cluster aspirations

It seems likely that the construction due to Leclerc [20] extends from the open Richardson varieties in the type ADE finite flag varieties to the case of the affine flag varieties, and provides $\mathcal{F}\ell_y^w$ with a cluster structure (even in the nonsimply laced case, cf. [22]). This structure can be transferred from $\mathcal{F}\ell_{w_0\times\mu^*}^{w_0\times\lambda^*}$ to ${}^\dagger\overset{\circ}{Z}{}^\alpha$ via the above symplectomorphism. If α is a dominant coweight of G, a reduced decomposition of $w_0 \times \mu^*$ is the beginning of a reduced decomposition of $w_0 \times \lambda^*$, and the existence of cluster structure on $\mathcal{F}\ell_{w_0\times\mu^*}^{w_0\times\lambda^*}$ is known for arbitrary symmetric affine Kac-Moody algebra. In case of $G = SL(2)$ the resulting cluster structure on the moduli space of periodic monopoles was discovered in [12], which served as the starting point of the present note. It seems likely that for general G the Gaiotto–Witten superpotential on Z^α (see e.g. [4]) restricted to ${}^\dagger\overset{\circ}{Z}{}^\alpha$ is totally positive in the above cluster structure (see Sect. 5 for details).

Remark 1.4 Implicit in the above discussion when α is *dominant* (as a coweight of G) is an affine open embedding ${}^\dagger\overset{\circ}{Z}{}^\alpha \subset \overset{\circ}{Z}{}^\alpha \hookrightarrow \mathbb{A}^{2|\alpha|}$ into an affine space. Indeed, in this case the lengths of the affine Weyl group elements satisfy $\ell(w_0\times\mu^*)+\ell(\alpha^*) = \ell(w_0\times\lambda^*)$, and we are in the situation of [20, Sect. 5]; hence according to *loc. cit.* and [20, Theorem 2.12], $\mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*}$ is an open subvariety of an affine space. Here is a modular interpretation of the above open embedding: $\mathbb{A}^{2|\alpha|}$ is the moduli space of B-bundles ϕ_B on $\mathbb{P}^1$ equipped with a trivialization $(\phi_B)_\infty \xrightarrow{\sim} B$ of the fiber at $\infty \in \mathbb{P}^1$, such that the induced T-bundle (under projection $B \twoheadrightarrow T$) has degree α.

1.5 Relation to Coulomb branches of 4d $\mathcal{N}=2$ quiver gauge theories

Let $G = SL(2)$, and $\alpha = a \in \mathbb{N}$. Then the methods of [1] establish an isomorphism $\mathbb{C}[{}^\dagger\overset{\circ}{Z}{}^a] \simeq K^{GL(a)_\mathcal{O}}(\mathrm{Gr}_{GL(a)})$ (where $\mathcal{O} = \mathbb{C}[[z]]$). More generally, for G of type ADE let us orient its Dynkin diagram, and for $\alpha = \sum_{i\in I} a_i\alpha_i$ let us consider a representation V of the Dynkin quiver such that $\dim V_i = a_i$. The group $GL(V) := \prod_{i\in I} GL(V_i)$ acts in $\mathbf{N} = \bigoplus_{i\to j} \mathrm{Hom}(V_i, V_j)$. Following [5] we consider the moduli space $\mathcal{R}$ of triples $(\mathcal{F}, \sigma, s)$ where $\mathcal{F}$ is a $GL(V)$-bundle on the formal disc $D = \mathrm{Spec}\,\mathbb{C}[[z]]$, σ is its trivialization over the punctured disc $D^* = \mathrm{Spec}\,\mathbb{C}((z))$, and s is a section of the associated vector bundle $\mathcal{F}\times^G \mathbf{N}$ such that it is sent to a regular section of the trivial bundle under σ. The group $GL(V)_\mathcal{O}$ acts naturally on $\mathcal{R}$, and as in [5] one can define the equivariant K-theory $K^{GL(V)_\mathcal{O}}(\mathcal{R})$ and equip it with the convolution algebra structure. Moreover, as in [6] one can establish an isomorphism $\mathbb{C}[{}^\dagger\overset{\circ}{Z}{}^\alpha] \simeq K^{GL(V)_\mathcal{O}}(\mathcal{R})$ such that the factorization morphism $\pi_\alpha\colon {}^\dagger\overset{\circ}{Z}{}^\alpha \to \mathbb{G}_m^\alpha$ corresponds to the embedding of the equivariant K-theory of the point: $K^{GL(V)_\mathcal{O}}(pt) \hookrightarrow K^{GL(V)_\mathcal{O}}(\mathcal{R})$.

Yet more generally, given a framing W_i, $\dim W_i = l_i$, we set $\lambda = \sum l_i\omega_i$ and consider a representation $\mathbf{N}' = \mathbf{N} \oplus \bigoplus_i \mathrm{Hom}(W_i, V_i)$ of $GL(V)$. It gives rise to the space $\mathcal{R}'$ of triples as above, and one can prove as in [6] that the convolution algebra $K^{GL(V)_\mathcal{O}}(\mathcal{R}')$ is isomorphic to the coordinate ring of the moduli space ${}^\dagger\overline{\mathcal{W}}{}^{\lambda^*}_{\mu^*}$ of the triples $(\mathcal{P}, \sigma, \phi)$ where $\mathcal{P}$ is a G-bundle on C; σ is a trivialization of $\mathcal{P}$ off $1 \in C$ having a pole of degree $\leq \lambda^*$ at $1 \in C$, and ϕ is a B-structure on $\mathcal{P}$ of degree $-\mu$ having the fiber B_- at $\infty \in C$ and transversal to B at $0 \in C$ (a trigonometric slice). Note that the relation of ${}^\dagger\overline{\mathcal{W}}{}^{\lambda^*}_{\mu^*}$ to the Richardson varieties $\mathcal{F}\ell^w_y$ of Sect. 1.2 is unclear since the former "knows" about 3 points $0, 1, \infty \in \mathbb{P}^1$ while the latter only "knows" about 2 points.

1.6 Contents

In Sect. 2 the moduli spaces ${}^\dagger\overset{\circ}{Z}{}^\alpha \subset {}^\dagger Z^\alpha$ and $\overset{\circ}{Y}{}^\alpha \subset Y^\alpha$ are introduced. In Sect. 3 a bivector field on $\overset{\circ}{Y}{}^\alpha$ is introduced and the resulting Poisson bracket $\{,\}_{\mathrm{trig}}$ on ${}^\dagger\overset{\circ}{Z}{}^\alpha$ is computed in coordinates $(w_{i,r}, y_{i,r})$. The technicalities of this computation occupy the bulk of the present note. In Sect. 4 we compare the rational Poisson bracket on the slices $\mathcal{W}^{\lambda^*}_{\mu^*} \subset \mathrm{Gr}_G$ with the Poisson bracket $\{,\}_{\mathrm{rat}}$ on the monopole moduli space $\overset{\circ}{Z}{}^\alpha$.

We also compare the trigonometric Poisson bracket on the slices $\mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*} \subset \mathcal{F}\ell_G$ with the Poisson bracket $\{,\}_{\mathrm{trig}}$ on the periodic monopole moduli space ${}^\dagger\mathring{Z}^\alpha$. Section 5 contains a few comments on the cluster structures. Finally, the "Appendix" by Galyna Dobrovolska identifies our cluster structure on ${}^\dagger\mathring{Z}^\alpha$ for $G = SL_2$ with the one of [12].

2 Trigonometric zastava

2.1 Line bundles on a nodal curve $C^\dagger$

Let $C^\dagger$ be an irreducible nodal curve of arithmetic genus 1, and let $\pi : C \to C^\dagger$ be its normalization. Then C is a projective line. We fix a coordinate function z on C such that the preimage of the node $\mathfrak{c} \in C^\dagger$ consists of the points $0, \infty \in C$. We have $\mathrm{Pic}^0(C^\dagger) = \mathbb{G}_m$: any line bundle on $C^\dagger$ is obtained by descent from the one on C gluing its fibers at 0 and ∞; a degree 0 line bundle on C is trivial, so its fibers at 0 and ∞ are canonically identified. Moreover, the above choice of a coordinate z on C gives rise to an identification $\mathrm{Pic}^n(C^\dagger) \cong \mathbb{G}_m$ for any $n \in \mathbb{Z}$: if $n > 0$, and s is a section of $\mathcal{L} \in \mathrm{Pic}^n(C^\dagger)$ not vanishing at $\mathfrak{c}$, then $\mathrm{div}(s) \in \mathrm{Sym}^n(C^\dagger \setminus \mathfrak{c}) = \mathrm{Sym}^n(\mathbb{G}_m)$ (this identification makes use of the coordinate z); we have a multiplication morphism $m\colon\ \mathrm{Sym}^n(\mathbb{G}_m) \to \mathbb{G}_m$, and finally $\mathcal{L} \mapsto m(\mathrm{div}(s)) \in \mathbb{G}_m$ (the result is independent of the choice of a section s). If $\mathcal{L} \in \mathrm{Pic}^n(C^\dagger)$ for $n < 0$, then $\mathcal{L}$ goes to the inverse of the class of $\mathcal{L}^{-1}$.

For the canonical line bundle $\omega_{C^\dagger}$, we have the following exact sequence:

$$0 \to \omega_{C^\dagger} \to \pi_*\omega_C(\{0\} + \{\infty\}) \xrightarrow{\rho_+} \mathbb{C}_\mathfrak{c} \to 0 \tag{2.1}$$

where $\rho_+(\xi) = \mathrm{Res}_0(\xi) + \mathrm{Res}_\infty(\xi)$. The line bundle $\omega_{C^\dagger}$ is trivial, with trivializing section $z^{-1}\mathrm{d}z$. In what follows we will freely use the above identification $\omega_{C^\dagger} \cong \mathcal{O}_{C^\dagger}$.

We define the *theta-characteristic* $\theta \in \mathrm{Pic}^0(C^\dagger)$ as a unique *nontrivial* line bundle such that $\theta^2 = \omega_{C^\dagger}$. It enters the following exact sequence:

$$0 \to \theta \to \pi_*\omega_C(\{0\} + \{\infty\}) \xrightarrow{\rho_-} \mathbb{C}_\mathfrak{c} \to 0 \tag{2.2}$$

where $\rho_-(\xi) = \mathrm{Res}_0(\xi) - \mathrm{Res}_\infty(\xi)$.

From the above two sequences we have the natural embeddings $\pi_*\omega_C \hookrightarrow \omega_{C^\dagger} \hookrightarrow \pi_*\omega_C(\{0\}+\{\infty\})$ and $\pi_*\omega_C \hookrightarrow \theta \hookrightarrow \pi_*\omega_C(\{0\}+\{\infty\})$. Noting that $\omega_C(\{0\}+\{\infty\}) = \mathcal{O}_C$ we combine the above embeddings into the following exact sequence:

$$0 \to \pi_*\omega_C \to \theta \oplus \omega_{C^\dagger} \to \pi_*\mathcal{O}_C \to 0 \tag{2.3}$$

We also have natural embeddings $\pi_*\omega_C = \pi_*\mathcal{O}_C(-\{0\}-\{\infty\}) \hookrightarrow \pi_*\pi_*\mathcal{O}_C(-\{0\}) \hookrightarrow \pi_*\mathcal{O}_C$ and $\pi_*\omega_C = \pi_*\mathcal{O}_C(-\{0\} - \{\infty\}) \hookrightarrow \pi_*\pi_*\mathcal{O}_C(-\{\infty\}) \hookrightarrow \pi_*\mathcal{O}_C$. They combine into the following exact sequence:

$$0 \to \pi_*\omega_C \to \pi_*\Xi \to \pi_*\mathcal{O}_C \to 0 \tag{2.4}$$

where Ξ stands for $\mathcal{O}_C(-\{0\}) \oplus \mathcal{O}_C(-\{\infty\})$.

2.2 A group G

Let G be an almost simple simply connected algebraic group over $\mathbb{C}$. We denote by $\mathcal{B}$ the flag variety of G. Let us also fix a pair of opposite Borel subgroups B, B_- whose intersection is a maximal torus T (thus we have $\mathcal{B} = G/B = G/B_-$). We denote by $T^{\mathrm{reg}} \subset T$ the open subset formed by the regular elements.

Let Λ (resp. $\Lambda^\vee$) denote the cocharacter (resp. character) lattice of T; since G is assumed to be simply connected, this is also the coroot lattice of G. We denote by $\Lambda_+ \subset \Lambda$ the sub-semigroup spanned by positive coroots. We say that $\alpha \geq \beta$ (for $\alpha, \beta \in \Lambda$) if $\alpha - \beta \in \Lambda_+$. The simple coroots are $\{\alpha_i\}_{i \in I}$; the simple roots are $\{\check{\alpha}_i\}_{i\in I}$; the fundamental weights are $\{\check{\omega}_i\}_{i\in I}$. We consider the invariant bilinear form $(,)$ on the weight lattice $\Lambda^\vee$ such that the square length of a *short* simple root $(\check{\alpha}_i, \check{\alpha}_i) = 2$. We set $\check{d}_i := \frac{(\check{\alpha}_i, \check{\alpha}_i)}{2}$. We fix the Chevalley generators $(E_i, F_i, H_i)_{i\in I}$ of $\mathfrak{g}$. An irreducible G-module with a dominant highest weight $\check{\lambda} \in \Lambda^\vee_+$ is denoted $V_{\check{\lambda}}$; we fix its highest vector $v_{\check{\lambda}}$. For a weight $\check{\mu} \in \Lambda^\vee$ the $\check{\mu}$-weight subspace of a G-module V is denoted $V(\check{\mu})$. Finally, W is the Weyl group of G, T; the simple reflections are denoted $s_i,\ i \in I$, and $w_0 \in W$ is the longest element.

The identification $\mathrm{Pic}^0(C^\dagger) = \mathbb{G}_m$ (resp. $\mathrm{Pic}^n(C^\dagger) \cong \mathbb{G}_m$, depending on the choice of coordinate z) of Sect. 2.1 gives rise to the identification $\mathrm{Bun}_T^0(C^\dagger) = T$ (resp. $\mathrm{Bun}_T^\alpha(C^\dagger) \cong T$). We denote by $\mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger) \subset \mathrm{Bun}_T^0(C^\dagger)$ the open subset corresponding to $T^{\mathrm{reg}} \subset T$ under the above identification.

2.3 The moduli space Y^α

Given a T-bundle $\mathcal{F}_T \in \mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger)$ we denote by $\mathcal{F}_B$ (resp. $\mathcal{F}_G$) the corresponding induced B-bundle (resp. G-bundle).

Definition 2.4 Given $\alpha \in \Lambda_+$, we define $\overset{\circ}{Y}{}^\alpha$ as the moduli space of the following data:

(a) a regular T-bundle $\mathcal{F}_T \in \mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger)$;
(b) a trivialization $\tau_{\mathfrak{c}}$ of the fiber of $\mathcal{F}_T$ at $\mathfrak{c} \in C^\dagger$;
(c) a B-structure ϕ in $\mathcal{F}_G$ of degree α (that is, the induced T-bundle ϕ_T has degree α), such that ϕ is transversal to $\mathcal{F}_B$ at $\mathfrak{c}$;
(d) a trivialization $f_{\mathfrak{c}}$ of the induced T-bundle ϕ_T at $\mathfrak{c}$.

We also define Y^α as the moduli space of the data (a–d) above where we allow a B-structure in (c) to be generalized (see e.g. [2]) but require that it does not have a defect at $\mathfrak{c} \in C^\dagger$.

We have a natural action of $T \times T$ on $\overset{\circ}{Y}{}^\alpha \subset Y^\alpha$: the first (resp. second) copy of T acts via the change of trivialization $\tau_{\mathfrak{c}}$ (resp. $f_{\mathfrak{c}}$).

We also have a morphism $(p, q)\colon Y^\alpha \to \mathrm{Bun}_T^\alpha(C^\dagger) \times \mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger) \cong T \times T^{\mathrm{reg}}$ sending $(\mathcal{F}_T, \tau_{\mathfrak{c}}, \phi, f_{\mathfrak{c}})$ to $(\phi_T, \mathcal{F}_T)$.

Finally, we have a morphism $\varpi\colon Y^\alpha \to Z^\alpha$ to the zastava space (see e.g. [4]) defined as follows. Recall that Z^α is the moduli space of triples $(\mathcal{F}_G^C, \phi_\pm^C)$ where $\mathcal{F}_G^C$

is a G-bundle on C, while ϕ_-^C (resp. ϕ_+^C) is a U-structure in $\mathcal{F}_G^C$ (resp. a generalized B-structure in $\mathcal{F}_G^C$ of degree α) such that ϕ_+^C has no defect at $\infty \in C$, and is transversal to ϕ_-^C at $\infty \in C$. Now ϖ sends $(\mathcal{F}_T, \tau_\mathfrak{c}, \phi, f_\mathfrak{c})$ to a triple $\mathcal{F}_G^C := \pi^*\mathcal{F}_G$, $\phi_+^C := \pi^*\phi$, and ϕ_-^C defined as follows: $\mathcal{F}_G^C$ is induced from $\mathcal{F}_T^C := \pi^*\mathcal{F}_T$, and the latter T-bundle is trivial and trivialized at $\infty \in \pi^{-1}(\mathfrak{c})$. This trivialization extends uniquely to the whole of C, and induces a trivialization of $\mathcal{F}_G^C$. At last, ϕ_-^C is a trivial U-structure in the trivial G-bundle $\mathcal{F}_G^C$ corresponding to the point $1 \in G/U$. Note that $\mathring{Y}^\alpha = \varpi^{-1}(\mathring{Z}^\alpha)$ (recall that the open subset $\mathring{Z}^\alpha \subset Z^\alpha$ is formed by the triples $(\mathcal{F}_G^C, \phi_\pm^C)$ such that ϕ_+^C has no defect, i.e. is a usual as opposed to generalized B-structure. The moduli space $\mathring{Z}^\alpha$ is isomorphic to the moduli space of degree α based maps from (C, ∞) to $(\mathcal{B}, B_-)$).

Proposition 2.5 *Y^α is represented by a scheme.*

Proof Recall the scheme $\widehat{\mathcal{QM}}_\mathfrak{g}^\alpha$ introduced in [3, 2.3]. It is the moduli space of degree α generalized B-structures ϕ^C in the trivial G-bundle on C, equipped with a trivialization f_∞ at $\infty \in C$ of the induced T-bundle ϕ_T^C. We claim that Y^α is a locally closed subscheme in $T^{\mathrm{reg}} \times \widehat{\mathcal{QM}}_\mathfrak{g}^\alpha$. In effect, given a regular T-bundle $\mathcal{F}_T \in \mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger) = T^{\mathrm{reg}}$, its trivialization $\tau_\mathfrak{c}$ at $\mathfrak{c} \in C^\dagger$ gives rise to a trivialization τ_∞ of $\pi^*\mathcal{F}_T$ at $\infty \in C$ which extends uniquely to a trivialization of $\pi^*\mathcal{F}_T$ on C, and hence to a trivialization of $\pi^*\mathcal{F}_G$ on C. Now $\phi^C := \pi^*\phi$ is a generalized B-structure in $\pi^*\mathcal{F}_G$, and the trivialization $f_\mathfrak{c}$ gives rise to a trivialization f_∞ of ϕ_T^C at $\infty \in C$. Note that ϕ^C has no defect neither at $0 \in C$ nor at $\infty \in C$, and its values $\phi^C(0), \phi^C(\infty) \in \mathcal{B}$ are both transversal to $B \in \mathcal{B}$. Conversely, given $(t, \phi^C, f_\infty) \in T^{\mathrm{reg}} \times \widehat{\mathcal{QM}}^\alpha$ such that ϕ^C has no defect neither at $0 \in C$ nor at $\infty \in C$, and $\phi^C(0) = t\phi^C(\infty) \in \mathcal{B}$ is transversal to B, we construct $(\mathcal{F}_T, \tau_\mathfrak{c}, \phi, f_\mathfrak{c}) \in Y^\alpha$ by descent from C to $C^\dagger$. □

2.6 A reduction of Y^α

Recall the *factorization* morphism $\pi_\alpha\colon Z^\alpha \to \mathbb{A}^\alpha = (C \setminus \{\infty\})^\alpha$ (see e.g. [4]). We have an open embedding $\mathbb{G}_m^\alpha = (C \setminus \{0, \infty\})^\alpha \subset \mathbb{A}^\alpha$.

Definition 2.7 We define the trigonometric zastava space as ${}^\dagger Z^\alpha := \pi_\alpha^{-1}(\mathbb{G}_m^\alpha) \subset Z^\alpha$. We define the periodic monopole moduli space as ${}^\dagger\mathring{Z}^\alpha := {}^\dagger Z^\alpha \cap \mathring{Z}^\alpha$: a dense open smooth subscheme of the trigonometric zastava ${}^\dagger Z^\alpha$.

Recall the action of $T \times T$ on Y^α, and the morphism $(p, q)\colon Y^\alpha \to T \times T^{\mathrm{reg}}$ introduced in Sect. 2.3. The action of $1 \times T$ on Y^α is clearly free. The morphism $\varpi\colon Y^\alpha \to Z^\alpha$ of Sect. 2.3 is clearly $(1 \times T)$-equivariant and gives rise to the same named morphism $\varpi\colon Y^\alpha/(1 \times T) \to Z^\alpha$. We fix $t_0 \in T^{\mathrm{reg}}$.

Proposition 2.8 *For any $t_0 \in T^{\mathrm{reg}}$ we have an isomorphism $\varpi\colon q^{-1}(t_0)/(1 \times T) \xrightarrow{\sim} {}^\dagger Z^\alpha$, and $\varpi\colon (q^{-1}(t_0) \cap \mathring{Y}^\alpha)/(1 \times T) \xrightarrow{\sim} {}^\dagger\mathring{Z}^\alpha$.*

Proof The locally closed embedding $Y^\alpha \hookrightarrow T^{\mathrm{reg}} \times \widehat{\mathcal{QM}}^\alpha_{\mathfrak{g}}$ constructed in the proof of Proposition 2.5 gives rise to the locally closed embedding $q^{-1}(t_0)/(1 \times T) \hookrightarrow \{t_0\} \times \mathcal{QM}^\alpha_{\mathfrak{g}}$ where $\mathcal{QM}^\alpha_{\mathfrak{g}}$ is the moduli space of degree α quasimaps from C to $\mathcal{B}$. More precisely, the image of $q^{-1}(t_0)/(1 \times T)$ is the locally closed subscheme ${}^{t_0}\mathcal{QM}^\alpha_{\mathfrak{g}} \subset \mathcal{QM}^\alpha_{\mathfrak{g}}$ formed by the quasimaps that have no defects at $0, \infty \in C$, their values $\phi^C(0)$ and $\phi^C(\infty)$ are both transversal to B, and $\phi^C(0) = t_0\phi^C(\infty)$.

An open subset of $\mathcal{B}$ formed by the Borels transversal to B (the big Schubert cell) is a free orbit $U \cdot \{B_-\}$, and we will identify it with U (the unipotent radical of B). So for $\phi^C \in {}^{t_0}\mathcal{QM}^\alpha_{\mathfrak{g}}$ we have $\phi^C(\infty) = n \cdot \{B_-\}$ for $n \in U$, and we will simply write $\phi^C(\infty) = n \in U$. Note that G acts on $\mathcal{QM}^\alpha_{\mathfrak{g}}$, and U acts on ${}^{t_0}\mathcal{QM}^\alpha_{\mathfrak{g}}$, and $n^{-1}\phi^C(\infty) = B_-$, i.e. $n^{-1}\phi^C \in Z^\alpha$ is a based quasimap.

A moment of reflection shows that $\varpi(t_0, \tau_{\mathfrak{c}}, \phi, f_{\mathfrak{c}}) = (\phi^C(\infty))^{-1}\phi^C$, and the condition of transversality of B and $\phi^C(0)$ guarantees that the value at $0 \in C$ of $(\phi^C(\infty))^{-1}\phi^C$ is also transversal to B, i.e. $(\phi^C(\infty))^{-1}\phi^C \in {}^\dagger Z^\alpha$. Thus we have a well defined morphism $q^{-1}(t_0)/(1\times T) \cong {}^{t_0}\mathcal{QM}^\alpha_{\mathfrak{g}} \to {}^\dagger Z^\alpha$, and we have to prove that it is an isomorphism, i.e. that for a based quasimap $\varphi\colon (C, \infty) \to (\mathcal{B}, B_-)$ without defect at $0 \in C$ with $\varphi(0) \in U \cdot \{B_-\} \subset \mathcal{B}$ there exists a unique $\phi^C \in {}^{t_0}\mathcal{QM}^\alpha_{\mathfrak{g}}$ such that $\varphi = (\phi^C(\infty))^{-1}\phi^C$.

Let $\varphi(0) = n' \cdot \{B_-\}$ for some $n' \in U$. We are looking for the desired ϕ^C in the form $n^{-1}\varphi,\ n \in U$. So we must have $\mathrm{Ad}_{t_0}(n) = n^{-1}n'$, that is $t_0 n^{-1} t_0^{-1} = n^{-1}n' \Leftrightarrow [n, t_0] = n'$. It remains to recall the following well known

Lemma 2.9 *Let $t_0 \in T^{\mathrm{reg}}$. Then the commutator map $U \to U,\ n \mapsto [n, t_0]$ is an isomorphism (of algebraic varieties).*

Proof Filtering U by its lower central series, one can introduce a system of coordinates $(x_{i,j})^{1\le j\le b_i}_{1\le i\le h-1}$ on the affine space U such that for the inversion morphism $U \to U,\ n \mapsto n^{-1}$ we have $(x_{i,j})^{-1} = (y_{i,j}(\underline{x}))$, and $y_{i,j}(\underline{x}) = -x_{i,j} + P_{i,j}(x_{i',j'})^{1\le j'\le b_{i'}}_{1\le i'<i}$ for a certain polynomial $P_{i,j}$. Moreover, for the multiplication morphism $m\colon U \times U \to U$ we have $m((x'_{i,j}), (x''_{i,j})) = (x_{i,j}(\underline{x'}, \underline{x''}))$, and $x_{i,j}(\underline{x'}, \underline{x''}) = x'_{i,j} + x''_{i,j} + Q_{i,j}(x'_{i',j'}, x''_{i'',j''})^{1\le j'\le b_{i'}, 1\le j''\le b_{i''}}_{1\le i', i''<i}$ for a certain polynomial $Q_{i,j}$. Finally, for the adjoint action $\mathrm{Ad}_{t_0}\colon U \to U$ we have $\mathrm{Ad}_{t_0}(x_{i,j}) = (w_{i,j}(\underline{x}))$, and $w_{i,j}(\underline{x}) = a_{i,j}x_{i,j}$ for a certain number $a_{i,j} \ne 1$ (due to the regularity assumption on t_0).

Now given $n' = (x'_{i,j}) \in U$ we can construct a unique $n = (x_{i,j}) \in U$ such that $[n, t_0] = n'$ recursively starting from $i = 1$, and going to $i = h - 1$. □

The proposition is proved. □

Corollary 2.10 *Y^α is an irreducible scheme with an open dense smooth subscheme $\overset{\circ}{Y}{}^\alpha$.*

Proof We have seen that $q\colon Y^\alpha \to T^{\mathrm{reg}}$ is a fibration with a smooth irreducible base and a fiber F that is a T-torsor over ${}^\dagger Z^\alpha$. Now ${}^\dagger Z^\alpha$ is open in the irreducible zastava scheme Z^α possessing an open dense smooth subscheme $\overset{\circ}{Z}{}^\alpha$. Finally, $q\colon \overset{\circ}{Y}{}^\alpha \to T^{\mathrm{reg}}$ is a fibration with a smooth irreducible base and a fiber $\overset{\circ}{F}$ that is a T-torsor over ${}^\dagger\overset{\circ}{Z}{}^\alpha = \overset{\circ}{Z}{}^\alpha \cap {}^\dagger Z^\alpha$. □

3 A trigonometric symplectic structure

3.1 Coordinates on Y^α

Recall the locally closed embedding $Y^\alpha \hookrightarrow \widehat{\mathcal{QM}}{}^\alpha_{\mathfrak{g}}$ introduced in the proof of Proposition 2.5. Via the Plücker embedding, $\widehat{\mathcal{QM}}{}^\alpha_{\mathfrak{g}}$ is a locally closed subscheme in $\prod_{i\in I} V_{\check\omega_i} \otimes \Gamma(C, \mathcal{O}_C(a_i))$ (notations of Sect. 2.2) where $\alpha = \sum_{i\in I} a_i\alpha_i$. In particular, we have the coefficients $Q_i, R_i, S_{ij} \in \Gamma(C, \mathcal{O}_C(a_i))$ of the highest, prehighest and next highest vectors $v_{\check\omega_i}, F_i v_{\check\omega_i}, F_j F_i v_{\check\omega_i}$. Thus Q_i, R_i, S_{ij} are the regular functions on Y^α with coefficients in the space of degree $\leq a_i$ polynomials in z. The conditions in Sect. 2.3(c) ensure that $\deg Q_i = a_i$, and $Q_i(0) \neq 0$.

It follows from [10, Remark 2] and Proposition 2.8 that (the coefficients of) $(Q_i, R_i)_{i\in I}$ form a rational coordinate system on Y^α. Let us denote by B_i (resp. b_i) the leading coefficient (resp. constant term) of Q_i, so that $Q_i = B_i z^{a_i} + \cdots + b_i$. Similarly, we have $R_i = C_i z^{a_i} + \cdots + c_i$. Note that $B_i \neq 0 \neq b_i$. Following [10, 3.3], we introduce a rational étale coordinate system on Y^α. Namely, $(w_{i,r})_{i\in I}^{1\leq r\leq a_i}$ are the ordered roots of Q_i, and $y_{i,r} := B_i^{-1} R_i(w_{i,r})$. The desired coordinate system is formed by $(B_i, C_i, w_{i,r}, y_{i,r})_{i\in I}^{1\leq r\leq a_i}$. It follows from [10, Remark 2] and Proposition 2.8 that these functions do form a coordinate system on an unramified covering of the open subset of $\overset{\circ}{Y}{}^\alpha$ where all the roots of all the polynomials Q_i, $i \in I$, are distinct.

We describe the $T \times T$-action on Y^α, and the morphisms $(p, q)\colon Y^\alpha \to T \times T$ (see Sect. 2.3) in the above coordinates. Note that the collection of fundamental weights $\check\omega_i\colon T \to \mathbb{G}_m$ identifies T with $\mathbb{G}_m^I$. We have $(t_1, t_2) \cdot (Q_i, R_i) = (\check\omega_i(t_1 t_2) Q_i, \check\omega_i(t_1 t_2)\check\alpha_i(t_1)^{-1} R_i)$, and $\check\omega_j(p(Q_i, R_i)_{i\in I}) = B_j^{-1} b_j$, and $\check\alpha_j(q(Q_i, R_i)_{i\in I}) = B_j^{-1} c_j^{-1} b_j C_j$.

3.2 The tangent bundle

Our goal in this section is to describe the tangent space $T_y \overset{\circ}{Y}{}^\alpha$ at $y = (\mathcal{F}_T, \tau_{\mathfrak{c}}, \phi, f_{\mathfrak{c}}) \in \overset{\circ}{Y}{}^\alpha$. We denote by $\mathfrak{g}, \mathfrak{b}, \mathfrak{u}, \mathfrak{t}$ the Lie algebras of G, B, U, T. Given a T-bundle $\mathcal{F}_T$ we denote the vector bundle associated to the adjoint action of T on $\mathfrak{g}$ by $\mathfrak{g}^{\mathcal{F}}$. It is a direct sum of two subbundles corresponding to the trivial (resp. nontrivial) eigenvalues of T on $\mathfrak{g}$: $\mathfrak{g}^{\mathcal{F}} = \mathfrak{t}^{\mathcal{F}} \oplus \mathfrak{r}^{\mathcal{F}}$. Note that $\mathfrak{t}^{\mathcal{F}} = \mathfrak{t} \otimes \mathcal{O}_{C^\dagger}$. A B-structure ϕ on $\mathcal{F}_G$ gives rise to a vector subbundle $\mathfrak{b}^\phi \subset \mathfrak{g}^{\mathcal{F}}$. The adjoint action of B on $\mathfrak{u}$ gives rise to a subbundle $\mathfrak{u}^\phi \subset \mathfrak{b}^\phi$. We denote the quotient bundle by $\mathfrak{h}^\phi = \mathfrak{h} \otimes \mathcal{O}_{C^\dagger}$: a trivial bundle where $\mathfrak{h} = \mathfrak{b}/\mathfrak{u}$ is the abstract Cartan. The Killing form identifies the dual vector bundle $\mathfrak{b}^{\phi *}$ with the quotient bundle $\mathfrak{g}^{\mathcal{F}}/\mathfrak{u}^\phi =: (\mathfrak{g}/\mathfrak{u})^\phi$. For a vector bundle $\mathcal{V}$ on $C^\dagger$ we denote by $\mathcal{V}_{\mathfrak{c}}$ the skyscraper quotient of $\mathcal{V}$ by the ideal sheaf of the point $\mathfrak{c} \in C^\dagger$.

We consider the following complex $K_y^\bullet$ of coherent sheaves on $C^\dagger$: it lives in degrees $-1, 0$, and $K_y^{-1} = (\mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\omega_C$, while $K_y^0 = (\mathfrak{g}/\mathfrak{u})^\phi$. The differential $d\colon K_y^{-1} \to K_y^0$ is a direct sum of $d'\colon \mathfrak{t} \otimes \pi_*\omega_C \to (\mathfrak{g}/\mathfrak{u})^\phi$ and $d''\colon \mathfrak{h} \otimes \pi_*\omega_C \to (\mathfrak{g}/\mathfrak{u})^\phi$. Here d' is the composition of $\mathfrak{t} \otimes \pi_*\omega_C \hookrightarrow \mathfrak{t} \otimes \mathcal{O}_{C^\dagger} = \mathfrak{t}^{\mathcal{F}}$ (see Sect. 2.1) and $\mathfrak{t}^{\mathcal{F}} \hookrightarrow \mathfrak{g}^{\mathcal{F}} \twoheadrightarrow (\mathfrak{g}/\mathfrak{u})^\phi$, while d'' is the composition $\mathfrak{h} \otimes \pi_*\omega_C \hookrightarrow \mathfrak{h} \otimes \mathcal{O}_{C^\dagger} = \mathfrak{h}^\phi \hookrightarrow (\mathfrak{g}/\mathfrak{u})^\phi$.

Proposition 3.3 *There is a canonical isomorphism* $T_y \overset{\circ}{Y}{}^\alpha \cong H^0(C^\dagger, K_y^\bullet)$.

Proof We consider the following complex ${}'K_y^\bullet$ of coherent sheaves on $C^\dagger$: it lives in degrees $-1, 0$, and ${}'K_y^{-1} = \mathfrak{b}^\phi$, while ${}'K_y^0 = \mathfrak{t}_{\mathfrak{c}}^{\mathcal{F}} \oplus \mathfrak{h}_{\mathfrak{c}}^\phi$. The differential from ${}'K_y^{-1}$ to ${}'K_y^0$ is a direct sum of $d'\colon \mathfrak{b}^\phi \to \mathfrak{t}_{\mathfrak{c}}^{\mathcal{F}}$ and $d''\colon \mathfrak{b}^\phi \to \mathfrak{h}_{\mathfrak{c}}^\phi$ where d' is the composition $\mathfrak{b}^\phi \hookrightarrow \mathfrak{g}^{\mathcal{F}} \twoheadrightarrow \mathfrak{t}^{\mathcal{F}} \twoheadrightarrow \mathfrak{t}_{\mathfrak{c}}^{\mathcal{F}}$, and d'' is the composition $\mathfrak{b}^\phi \twoheadrightarrow \mathfrak{h}^\phi \twoheadrightarrow \mathfrak{h}_{\mathfrak{c}}^\phi$.

Then $T_y \overset{\circ}{Y}{}^\alpha = H^0(C^\dagger, {}'K_y^\bullet)$. Now consider yet another complex ${}''K_y^\bullet$ living in degrees $-1, 0$ such that ${}''K_y^{-1} = {}'K_y^{-1}$ and ${}''K_y^0 = {}'K_y^0 \oplus \mathfrak{r}^{\mathcal{F}}$, and the differential equals $d' + d'' + d'''$ where $d'''\colon \mathfrak{b}^\phi \to \mathfrak{r}^{\mathcal{F}}$ is the composition $\mathfrak{b}^\phi \hookrightarrow \mathfrak{g}^{\mathcal{F}} \twoheadrightarrow \mathfrak{r}^{\mathcal{F}}$. We have a canonical morphism ${}''K_y^\bullet \to {}'K_y^\bullet$ inducing an isomorphism on cohomology $H^0(C^\dagger, {}''K_y^\bullet) \xrightarrow{\sim} H^0(C^\dagger, {}'K_y^\bullet)$ since $H^\bullet(C^\dagger, \mathfrak{r}^{\mathcal{F}}) = 0$ due to the regularity assumption on $\mathcal{F}_T$.

Also we have a canonical quasiisomorphism ${}'''K_y^\bullet \to {}''K_y^\bullet$ where ${}'''K_y^{-1} = {}''K_y^{-1} \oplus (\mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\omega_C$, and ${}'''K_y^0 = (\mathfrak{t} \oplus \mathfrak{h}) \otimes \mathcal{O}_{C^\dagger} \oplus \mathfrak{r}^{\mathcal{F}}$. The new components of the differential are as follows: $(\mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\omega_C \hookrightarrow (\mathfrak{t} \oplus \mathfrak{h}) \otimes \mathcal{O}_{C^\dagger}$ (see Sect. 2.1), and $\mathfrak{b}^\phi \twoheadrightarrow \mathfrak{h}^\phi$, and the composition $\mathfrak{b}^\phi \hookrightarrow \mathfrak{g}^{\mathcal{F}} \twoheadrightarrow \mathfrak{t}^{\mathcal{F}}$.

Finally, note that ${}'''K_y^0 = \mathfrak{h}^\phi \oplus \mathfrak{g}^{\mathcal{F}}$, and $(\mathfrak{h}^\phi \oplus \mathfrak{g}^{\mathcal{F}})/d(\mathfrak{b}^\phi) \cong (\mathfrak{g}/\mathfrak{u})^\phi$, so we have a canonical quasiisomorphism ${}'''K_y^\bullet \to K_y^\bullet$.

The composition of the morphisms induced on $H^0(C^\dagger, ?)$ by the above quasiisomorphisms is the desired isomorphism $T_y \overset{\circ}{Y}{}^\alpha = H^0(C^\dagger, {}'K_y^\bullet) \cong H^0(C^\dagger, K_y^\bullet)$. □

3.4 The cotangent bundle

Let us describe the Serre dual complex $L_y^\bullet := \mathcal{D}K_y^\bullet$. It lives in degrees $0, 1$, and $L_y^0 = \mathfrak{b}^\phi$, while $L_y^1 = (\mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\mathcal{O}_C$. The differential $d\colon L_y^0 \to L_y^1$ is a direct sum of $d'\colon \mathfrak{b}^\phi \to \mathfrak{t} \otimes \pi_*\mathcal{O}_C$ and $d''\colon \mathfrak{b}^\phi \to \mathfrak{h} \otimes \pi_*\mathcal{O}_C$. Here d'' is the composition $\mathfrak{b}^\phi \twoheadrightarrow \mathfrak{h}^\phi = \mathfrak{h} \otimes \mathcal{O}_{C^\dagger} \hookrightarrow \mathfrak{h} \otimes \pi_*\mathcal{O}_C$ (see Sect. 2.1), while d' is the composition $\mathfrak{b}^\phi \hookrightarrow \mathfrak{g}^{\mathcal{F}} \twoheadrightarrow \mathfrak{t}^{\mathcal{F}} = \mathfrak{t} \otimes \mathcal{O}_{C^\dagger} \hookrightarrow \mathfrak{t} \otimes \pi_*\mathcal{O}_C$. Now Proposition 3.3 has the following immediate

Corollary 3.5 *There is a canonical isomorphism* $T_y^* \overset{\circ}{Y}{}^\alpha \cong H^0(C^\dagger, L_y^\bullet)$.

3.6 Some differentials

Here we describe the differentials of the morphism $(p, q)\colon \overset{\circ}{Y}{}^\alpha \to T \times T^{\mathrm{reg}}$ and of the action of $T \times T$ on $\overset{\circ}{Y}{}^\alpha$ introduced in Sect. 2.3.

Note that for a regular T-bundle $\mathcal{F}_T$ on $C^\dagger$ the tangent space $T_{\mathcal{F}_T}\mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger)$ is canonically isomorphic to $H^1(C^\dagger, \mathfrak{t}^{\mathcal{F}}) = H^1(C^\dagger, \mathfrak{t}\otimes\mathcal{O}_{C^\dagger}) \cong \mathfrak{t}$ (here the second isomorhism is $\mathrm{Id}_\mathfrak{t}\otimes\mathrm{Tr}$ for the trace isomorphism $\mathrm{Tr}\colon H^1(C^\dagger, \mathcal{O}_{C^\dagger}) = H^1(C^\dagger, \omega_{C^\dagger}) \xrightarrow{\sim} \mathbb{C}$). The scalar product $(,)$ on $\mathfrak{t}^*$ (Sect. 2.2) identifies $\mathfrak{t}$ with $\mathfrak{t}^*$. Together with the Serre duality $H^1(C^\dagger, \mathcal{O}_{C^\dagger})^* = H^0(C^\dagger, \mathcal{O}_{C^\dagger})$ this gives rise to a canonical isomorphism $T^*_{\mathcal{F}_T}\mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger) \cong H^0(C^\dagger, \mathfrak{t}\otimes\mathcal{O}_{C^\dagger}) \cong \mathfrak{t}$. Similarly, for a degree α T-bundle ϕ_T we have canonical isomorphisms $T_{\phi_T}\mathrm{Bun}_T^\alpha(C^\dagger) \cong H^1(C^\dagger, \mathfrak{h}^\phi) = H^1(\mathfrak{h}\otimes\mathcal{O}_{C^\dagger}) = \mathfrak{h}$ and $T^*_{\phi_T}\mathrm{Bun}_T^\alpha(C^\dagger) \cong H^0(C^\dagger, \mathfrak{h}\otimes\mathcal{O}_{C^\dagger}) = \mathfrak{h}$.

We have distinguished triangles $(\mathfrak{g}/\mathfrak{u})^\phi \to K_y^\bullet \to (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\omega_C[1]$ and $(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C[-1] \to L_y^\bullet \to \mathfrak{b}^\phi$. They give rise to a morphism

$$\begin{aligned}\mathfrak{d}_p\colon T_y\overset{\circ}{Y}{}^\alpha &= H^0(C^\dagger, K_y^\bullet) \to H^1(C^\dagger, (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\omega_C)\\ &\to H^1(C^\dagger, (\mathfrak{t}\oplus\mathfrak{h})\otimes\omega_{C^\dagger}) \to H^1(C^\dagger, \mathfrak{h}\otimes\omega_{C^\dagger}) = T_{\phi_T}\mathrm{Bun}_T^\alpha(C^\dagger)\end{aligned}$$

where the middle arrow arises from the natural morphism $\pi_*\omega_C \to \omega_{C^\dagger}$ (see Sect. 2.1), and the next arrow arises from the projection $\mathfrak{t}\oplus\mathfrak{h} \to \mathfrak{h}$. Similarly, we have

$$\begin{aligned}\mathfrak{d}_q\colon T_y\overset{\circ}{Y}{}^\alpha &= H^0(C^\dagger, K_y^\bullet) \to H^1(C^\dagger, (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\omega_C)\\ &\to H^1(C^\dagger, (\mathfrak{t}\oplus\mathfrak{h})\otimes\omega_{C^\dagger}) \to H^1(C^\dagger, \mathfrak{t}\otimes\omega_{C^\dagger}) = T_{\mathcal{F}_T}\mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger).\end{aligned}$$

Dually, we have

$$\begin{aligned}\mathfrak{d}_p^*\colon T^*_{\phi_T}\mathrm{Bun}_T^\alpha(C^\dagger) &= H^0(C^\dagger, \mathfrak{h}\otimes\mathcal{O}_{C^\dagger}) \to H^0(C^\dagger, (\mathfrak{t}\oplus\mathfrak{h})\otimes\mathcal{O}_{C^\dagger})\\ &\to H^0(C^\dagger, (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C) \to H^0(C^\dagger, L_y^\bullet) = T_y^*\overset{\circ}{Y}{}^\alpha\end{aligned}$$

and

$$\begin{aligned}\mathfrak{d}_q^*\colon T^*_{\mathcal{F}_T}\mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger) &= H^0(C^\dagger, \mathfrak{t}\otimes\mathcal{O}_{C^\dagger}) \to H^0(C^\dagger, (\mathfrak{t}\oplus\mathfrak{h})\otimes\mathcal{O}_{C^\dagger})\\ &\to H^0(C^\dagger, (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C) \to H^0(C^\dagger, L_y^\bullet) = T_y^*\overset{\circ}{Y}{}^\alpha.\end{aligned}$$

We also have a morphism

$$\begin{aligned}\mathfrak{a}_1\colon T_eT &= \mathfrak{t} = H^0(C^\dagger, \mathfrak{t}\otimes\mathcal{O}_{C^\dagger}) = H^0(C^\dagger, \mathfrak{t}^{\mathcal{F}}) \to H^0(C^\dagger, (\mathfrak{g}/\mathfrak{u})^\phi)\\ &\to H^0(C^\dagger, K_y^\bullet) = T_y\overset{\circ}{Y}{}^\alpha,\end{aligned}$$

and

$$\begin{aligned}\mathfrak{a}_2\colon T_eT &\cong \mathfrak{h} = H^0(C^\dagger, \mathfrak{h}\otimes\mathcal{O}_{C^\dagger}) = H^0(C^\dagger, \mathfrak{h}^\phi) \to H^0(C^\dagger, (\mathfrak{g}/\mathfrak{u})^\phi)\\ &\to H^0(C^\dagger, K_y^\bullet) = T_y\overset{\circ}{Y}{}^\alpha.\end{aligned}$$

Lemma 3.7 *a)* $(\mathfrak{a}_1, \mathfrak{a}_2)\colon \mathfrak{t}\oplus\mathfrak{h} \to T_y\mathring{Y}^\alpha$ *is the differential of the action* $T\times T\times \mathring{Y}^\alpha \to \mathring{Y}^\alpha$ *(see Sect. 2.3);*

b) $(\mathfrak{d}_p, \mathfrak{d}_q)\colon T_y\mathring{Y}^\alpha \to T_{\phi_T}\mathrm{Bun}_T^\alpha(C^\dagger) \oplus T_{\mathcal{F}_T}\mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger)$ *is the differential of* $(p,q)\colon \mathring{Y}^\alpha \to \mathrm{Bun}_T^\alpha(C^\dagger)\times \mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger)$.

c) $(\mathfrak{d}_p^*, \mathfrak{d}_q^*)\colon T^*_{\phi_T}\mathrm{Bun}_T^\alpha(C^\dagger) \oplus T^*_{\mathcal{F}_T}\mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger) \to T_y^*\mathring{Y}^\alpha$ *is the codifferential of* $(p,q)\colon \mathring{Y}^\alpha \to \mathrm{Bun}_T^\alpha(C^\dagger)\times \mathrm{Bun}_T^{\mathrm{reg}}(C^\dagger)$.

Proof Clear from the construction. □

3.8 A bivector field

We consider the following bicomplex $M_y^{\bullet,\bullet}$:

$$\begin{array}{ccccc}
 & & \mathfrak{b}^\phi & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C \\
 & & \downarrow & & \downarrow \\
(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\omega_C & \longrightarrow & \mathfrak{r}^{\mathcal{F}}\oplus(\mathfrak{t}\oplus\mathfrak{h})\otimes(\theta\oplus\mathcal{O}_{C^\dagger}) & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C \\
\downarrow & & \downarrow & & \\
(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\omega_C & \longrightarrow & (\mathfrak{g}/\mathfrak{u})^\phi & &
\end{array} \tag{3.1}$$

Here the middle term lives in bidegree $(0,0)$, the first line is nothing but $L_y^\bullet$ of Sect. 3.4, while the last line is nothing but $K_y^\bullet$ of Sect. 3.2. The left vertical arrow is the identity morphism, as well as the right vertical arrow. The middle line is a direct sum of the complex consisting of $\mathfrak{r}^{\mathcal{F}}$ in degree 0, and of the exact complex (2.3) tensored with $\mathfrak{t}\oplus\mathfrak{h}$. Note that the middle term can be rearranged as $\mathfrak{h}^\phi\oplus\mathfrak{g}^{\mathcal{F}}\oplus(\mathfrak{t}\oplus\mathfrak{h})\otimes\theta$. Now the middle column is a direct sum of the complex consisting of $(\mathfrak{t}\oplus\mathfrak{h})\otimes\theta$ in degree 0, and of the exact complex $\mathfrak{b}^\phi\to\mathfrak{h}^\phi\oplus\mathfrak{g}^{\mathcal{F}}\to(\mathfrak{g}/\mathfrak{u})^\phi$.

It follows (looking at the columns of (3.1)) that the total complex $\mathrm{Tot}\, M_y^{\bullet,\bullet}$ has only one cohomology in degree 0, and $H^0(\mathrm{Tot}\, M_y^{\bullet,\bullet}) = (\mathfrak{t}\oplus\mathfrak{h})\otimes\theta$. Since $H^\bullet(C^\dagger, (\mathfrak{t}\oplus\mathfrak{h})\otimes\theta)=0$, we deduce that $H^\bullet(C^\dagger, \mathrm{Tot}\, M_y^{\bullet,\bullet})=0$. Now let us look at the rows of $M_y^{\bullet,\bullet}$. The hypercohomology of $C^\dagger$ with coefficients in the middle row vanishes since $H^\bullet(C^\dagger,\mathfrak{r}^{\mathcal{F}})=0$ (due to the regularity assumption on $\mathcal{F}_T$). Hence the second differential in the spectral sequence converging to $H^\bullet(C^\dagger, \mathrm{Tot}\, M_y^{\bullet,\bullet})=0$ from the hypercohomology of $C^\dagger$ with coefficients in the rows is $d_2\colon H^0(C^\dagger, L_y^\bullet)\xrightarrow{\sim} H^0(C^\dagger, K_y^\bullet)$.

Finally, due to Proposition 3.3 and Corollary 3.5 we can view the above differential as $d_2\colon T_y^*\mathring{Y}^\alpha \xrightarrow{\sim} T_y\mathring{Y}^\alpha$.

3.9 Calculation of the bivector field: preparation

We follow the strategy of [10], and eventually reduce our calculation to the one of *loc. cit.* The result of the somewhat lengthy calculation of d_2 is contained in Lemma 3.10, Remark 3.11 and Proposition 3.17.

Given a character $\check{\lambda} \in \Lambda^\vee$, we consider the composed homomorphism $B \to T \to \mathbb{G}_m$, and denote the associated (to the B-torsor ϕ) line bundle on $C^\dagger$ by $\mathcal{L}^\phi_{\check{\lambda}}$. For an irreducible G-module $V_{\check{\lambda}}$, the associated (to the G-torsor $\mathcal{F}_G$) vector bundle on $C^\dagger$ is denoted $\mathcal{V}^{\mathcal{F}}_{\check{\lambda}}$. If $\check{\lambda}$ is a fundamental weight $\check{\omega}_i$, then we have an isomorphism $V^*_{\check{\omega}_i} \cong V_{\check{\omega}_{i^*}}$ for an involution $I \xrightarrow{\sim} I,\ i \mapsto i^*$. If we extend the involution $\check{\omega}_i \mapsto \check{\omega}_{i^*}$ by linearity to the weight lattice $\Lambda^\vee$, $\check{\lambda} \mapsto \check{\lambda}^*$, then this involution preserves the scalar product $(,)$ of Sect. 2.2.

We have a natural embedding of vector bundles on $C^\dagger$: $(\mathfrak{g}/\mathfrak{u})^\phi \hookrightarrow \bigoplus_{i\in I} \mathcal{V}^{\mathcal{F}}_{\check{\omega}_{i^*}} \otimes \mathcal{L}^\phi_{\check{\omega}_i}$, and the dual surjection $\bigoplus_{i\in I} \mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i^*}} \otimes \mathcal{L}^\phi_{-\check{\omega}_i} \twoheadrightarrow \mathfrak{b}^\phi$. They give rise to the following morphisms of two-term complexes of coherent sheaves on $C^\dagger$:

$$
\begin{array}{ccc}
(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\omega_C & \longrightarrow & (\mathfrak{g}/\mathfrak{u})^\phi \\
\downarrow & & \downarrow \\
(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi & \longrightarrow & (\mathfrak{g}/\mathfrak{u})^\phi\otimes\pi_*\mathcal{O}_C \\
\| & & \downarrow \\
(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi & \longrightarrow & (\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check{\omega}_{i^*}}\otimes\mathcal{L}^\phi_{\check{\omega}_i})\otimes\pi_*\mathcal{O}_C
\end{array}
\tag{3.2}
$$

(the upper vertical arrows arise from the morphisms $\pi_*\omega_C \to \pi_*\Xi = \pi_*(\mathcal{O}_C(-\{0\})\oplus \mathcal{O}_C(-\{\infty\}))$ and $\mathcal{O}_{C^\dagger} \to \pi_*\mathcal{O}_C$ of Sect. 2.1), and dually

$$
\begin{array}{ccc}
(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i^*}}\otimes\mathcal{L}^\phi_{-\check{\omega}_i})\otimes\pi_*\omega_C & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi \\
\downarrow & & \| \\
\mathfrak{b}^\phi\otimes\pi_*\omega_C & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi \\
\downarrow & & \downarrow \\
\mathfrak{b}^\phi & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C
\end{array}
\tag{3.3}
$$

Note that the top row of (3.2) coincides with the bottom row $K^\bullet_y$ of the bicomplex (3.1), while the bottom row of (3.3) coincides with the top row $L^\bullet_y$ of the bicomplex (3.1). So composing the vertical arrows of (3.3), (3.2) with the vertical arrows of (3.1) we obtain a bicomplex ${}'M^{\bullet,\bullet}_y$:

$$
\begin{array}{ccccc}
 & & (\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{-\check{\omega}_i})\otimes\pi_*\omega_C & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi \\
 & & \downarrow & & \downarrow \\
(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\omega_C & \longrightarrow & \mathfrak{r}^{\mathcal{F}}\oplus(\mathfrak{t}\oplus\mathfrak{h})\otimes(\theta\oplus\mathcal{O}_{C^\dagger}) & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C \\
\downarrow & & \downarrow & & \\
(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi & \longrightarrow & (\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{\check{\omega}_i})\otimes\pi_*\mathcal{O}_C & &
\end{array}
\tag{3.4}
$$

Note that $H^\bullet(C^\dagger, (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi)=0$, so just as in Sect. 3.8, the second differential in the spectral sequence converging to $H^\bullet(C^\dagger, \mathrm{Tot}\ {}'M_y^{\bullet,\bullet})$ from the hypercohomology of $C^\dagger$ with coefficients in the rows is

$$
d_2'\colon H^1(C^\dagger, (\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{-\check{\omega}_i})\otimes\pi_*\omega_C)\to H^0(C^\dagger, (\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{\check{\omega}_i})\otimes\pi_*\mathcal{O}_C). \tag{3.5}
$$

Lemma 3.10 *The following diagram commutes:*

$$
\begin{array}{ccc}
H^0(C^\dagger, L_y^\bullet) & \longleftarrow & H^1\Big(C^\dagger, \Big(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{-\check{\omega}_i}\Big)\otimes\pi_*\omega_C\Big) \\
{\scriptstyle d_2}\downarrow & & {\scriptstyle d_2'}\downarrow \\
H^0(C^\dagger, K_y^\bullet) & \longrightarrow & H^0\Big(C^\dagger, \Big(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{\check{\omega}_i}\Big)\otimes\pi_*\mathcal{O}_C\Big)
\end{array}
$$

Proof Clear. □

Remark 3.11 In what follows we will be occupied with the calculation of

$$
d_2'\colon H^1\left(C^\dagger, (\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{-\check{\omega}_i})\otimes\pi_*\omega_C\right)\to H^0\left(C^\dagger, (\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{\check{\omega}_i})\otimes\pi_*\mathcal{O}_C\right).
$$

Let us presently comment in which sense does it calculate the desired $d_2\colon H^0(C^\dagger, L_y^\bullet)\to H^0(C^\dagger, K_y^\bullet)$. It is easy to see that for $y\in\mathring{Y}^\alpha$ lying in the open subset $U^\alpha\subset\mathring{Y}^\alpha$ formed by all the quadruples $(\mathcal{F}_T, \tau_{\mathfrak{c}}, \phi, f_{\mathfrak{c}})$ such that $\mathfrak{t}_{\mathfrak{c}}^{\mathcal{F}}\cap\mathfrak{b}_{\mathfrak{c}}^{\phi}=0\subset\mathfrak{g}_{\mathfrak{c}}^{\mathcal{F}}$ the morphism $H^1(C^\dagger, (\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{-\check{\omega}_i})\otimes\pi_*\omega_C)\to H^0(C^\dagger, L_y^\bullet)$ is surjective, and the morphism $H^0(C^\dagger, K_y^\bullet)\to H^0(C^\dagger, (\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{\check{\omega}_i})\otimes\pi_*\mathcal{O}_C)$ is injective. Since we are only going to calculate our d_2 generically, the only trouble is that for some α the open subset U^α may happen to be empty. Indeed, for $\alpha=\sum_{i\in I}a_i\alpha_i$, we have $U^\alpha=\emptyset$ iff $a_i=0$ for some $i\in I$. So in what follows we assume $a_i>0\ \forall i\in I$ (otherwise the moduli space $\mathring{Y}^\alpha$ essentially reduces to the one for a semisimple Lie algebra $\mathfrak{g}'$ of smaller rank).

3.12 Reduction to a calculation on *C*

The goal of this section is a description of d_2' (3.5) in terms of C, see Corollary 3.15.

Note that $\pi^*(\mathfrak{h}^\phi \oplus \mathfrak{g}^\mathcal{F}) \cong (\mathfrak{h} \oplus \mathfrak{g}) \otimes \mathcal{O}_C$, and $\pi^*((\mathfrak{t} \oplus \mathfrak{h}) \otimes \theta) \cong (\mathfrak{t} \oplus \mathfrak{h}) \otimes \mathcal{O}_C$. Hence $\pi_*\pi^*(\mathfrak{h}^\phi \oplus \mathfrak{g}^\mathcal{F} \oplus (\mathfrak{t} \oplus \mathfrak{h}) \otimes \theta) \cong (\mathfrak{h} \oplus \mathfrak{g} \oplus \mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\mathcal{O}_C$. The morphisms

$$\begin{aligned}
&\mathfrak{r}^\mathcal{F} \oplus (\mathfrak{t} \oplus \mathfrak{h}) \otimes (\theta \oplus \mathcal{O}_{C^\dagger}) = \mathfrak{h}^\phi \oplus \mathfrak{g}^\mathcal{F} \oplus (\mathfrak{t} \oplus \mathfrak{h}) \otimes \theta \to (\mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\mathcal{O}_C,\\
&\text{resp. } \mathfrak{r}^\mathcal{F} \oplus (\mathfrak{t} \oplus \mathfrak{h}) \otimes (\theta \oplus \mathcal{O}_{C^\dagger}) = \mathfrak{h}^\phi \oplus \mathfrak{g}^\mathcal{F} \oplus (\mathfrak{t} \oplus \mathfrak{h}) \otimes \theta\\
&\quad \to \left(\bigoplus_{i\in I} \mathcal{V}^\mathcal{F}_{\check{\omega}_{i*}} \otimes \mathcal{L}^\phi_{\check{\omega}_i}\right) \otimes \pi_*\mathcal{O}_C
\end{aligned}$$

of (3.4) factor through the canonical morphism

$$\begin{aligned}
\mathfrak{h}^\phi \oplus \mathfrak{g}^\mathcal{F} \oplus (\mathfrak{t} \oplus \mathfrak{h}) \otimes \theta &\to \pi_*\pi^*(\mathfrak{h}^\phi \oplus \mathfrak{g}^\mathcal{F} \oplus (\mathfrak{t} \oplus \mathfrak{h}) \otimes \theta)\\
&\cong (\mathfrak{h} \oplus \mathfrak{g} \oplus \mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\mathcal{O}_C
\end{aligned}$$

and

$$\begin{aligned}
&(\mathfrak{h} \oplus \mathfrak{g} \oplus \mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\mathcal{O}_C \to (\mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\mathcal{O}_C,\\
&\text{resp. } (\mathfrak{h} \oplus \mathfrak{g} \oplus \mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\mathcal{O}_C \to \left(\bigoplus_{i\in I} \mathcal{V}^\mathcal{F}_{\check{\omega}_{i*}} \otimes \mathcal{L}^\phi_{\check{\omega}_i}\right) \otimes \pi_*\mathcal{O}_C.
\end{aligned}$$

Hence we obtain a morphism from the bicomplex ${}'M_y^{\bullet,\bullet}$ (3.4) to the following bicomplex ${}''M_y^{\bullet,\bullet}$:

$$\begin{array}{ccccc}
 & & \left(\bigoplus_{i\in I} \mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}} \otimes \mathcal{L}^\phi_{-\check{\omega}_i}\right) \otimes \pi_*\omega_C & \longrightarrow & (\mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\Xi\\
 & & \downarrow & & \downarrow\\
(\mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\omega_C & \longrightarrow & (\mathfrak{h} \oplus \mathfrak{g} \oplus \mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\mathcal{O}_C & \longrightarrow & (\mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\mathcal{O}_C\\
\downarrow & & \downarrow & & \\
(\mathfrak{t} \oplus \mathfrak{h}) \otimes \pi_*\Xi & \longrightarrow & \left(\bigoplus_{i\in I} \mathcal{V}^\mathcal{F}_{\check{\omega}_{i*}} \otimes \mathcal{L}^\phi_{\check{\omega}_i}\right) \otimes \pi_*\mathcal{O}_C & &
\end{array} \tag{3.6}$$

(the morphisms from all the terms of ${}'M_y^{\bullet,\bullet}$ to the corresponding terms of ${}''M_y^{\bullet,\bullet}$ except for the middle ones are identities). Just as in Sect. 3.9 we obtain the second differential

$$d_2'' \colon H^1\left(C^\dagger, \left(\bigoplus_{i\in I} \mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}} \otimes \mathcal{L}^\phi_{-\check{\omega}_i}\right) \otimes \pi_*\omega_C\right) \tag{3.7}$$

$$\to H^0\left(C^\dagger, \left(\bigoplus_{i\in I} \mathcal{V}^\mathcal{F}_{\check{\omega}_{i*}} \otimes \mathcal{L}^\phi_{\check{\omega}_i}\right) \otimes \pi_*\mathcal{O}_C\right) \tag{3.8}$$

in the spectral sequence converging to $H^\bullet(C^\dagger, \mathrm{Tot}\ {}''M_y^{\bullet,\bullet})$ from the hypercohomology of $C^\dagger$ with coefficients in the rows. It follows that $d_2'' = d_2'$ of (3.5).

Now the morphisms

$$\left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{-\check{\omega}_i}\right)\otimes\pi_*\omega_C\to(\mathfrak{h}\oplus\mathfrak{g}\oplus\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C,$$
$$\text{resp. } (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\omega_C\to(\mathfrak{h}\oplus\mathfrak{g}\oplus\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C$$

of (3.6) factor through the natural morphism

$$(\mathfrak{h}\oplus\mathfrak{g}\oplus\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi\to(\mathfrak{h}\oplus\mathfrak{g}\oplus\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C$$

(see Sect. 2.1) and

$$\left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{-\check{\omega}_i}\right)\otimes\pi_*\omega_C\to(\mathfrak{h}\oplus\mathfrak{g}\oplus\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi,$$
$$\text{resp. } (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\omega_C\to(\mathfrak{h}\oplus\mathfrak{g}\oplus\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi.$$

Hence we obtain a morphism to the bicomplex ${}''M_y^{\bullet,\bullet}$ (3.4) to the following bicomplex ${}'''M_y^{\bullet,\bullet}$:

$$\begin{array}{ccccc}
 & & \left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{-\check{\omega}_i}\right)\otimes\pi_*\omega_C & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi \\
 & & \downarrow & & \downarrow \\
(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\omega_C & \longrightarrow & (\mathfrak{h}\oplus\mathfrak{g}\oplus\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C \\
\downarrow & & \downarrow & & \\
(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi & \longrightarrow & \left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{\check{\omega}_i}\right)\otimes\pi_*\mathcal{O}_C & &
\end{array}\tag{3.9}$$

(the morphisms from all the terms of ${}'''M_y^{\bullet,\bullet}$ to the corresponding terms of ${}''M_y^{\bullet,\bullet}$ except for the middle ones are identities). Just as in Sect. 3.9 we obtain the second differential

$$d_2''':\ H^1\left(C^\dagger,\left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{-\check{\omega}_i}\right)\otimes\pi_*\omega_C\right)\to H^0\left(C^\dagger,\left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check{\omega}_{i*}}\otimes\mathcal{L}^{\phi}_{\check{\omega}_i}\right)\otimes\pi_*\mathcal{O}_C\right)\tag{3.10}$$

in the spectral sequence converging to $H^\bullet(C^\dagger,\mathrm{Tot}\ {}'''M_y^{\bullet,\bullet})$ from the hypercohomology of $C^\dagger$ with coefficients in the rows. It follows that $d_2'''=d_2''$ of (3.7).

Note that the bicomplex $'''M_y^{\bullet,\bullet}$ is obtained from the following bicomplex $^\circ M_y^{\bullet,\bullet}$:

$$\begin{array}{ccccc} & & \left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\pi_*\Xi & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi \\ & & \downarrow & & \downarrow \\ (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\omega_C & \longrightarrow & (\mathfrak{h}\oplus\mathfrak{g}\oplus\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\mathcal{O}_C \\ \downarrow & & \downarrow & & \\ (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi & \longrightarrow & \left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\pi_*\Xi & & \end{array} \tag{3.11}$$

by composing the vertical arrows of (3.11) with the vertical arrows of the following commutative diagrams:

$$\begin{array}{ccc} \left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\pi_*\omega_C & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi \\ \downarrow & & \| \\ \left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\pi_*\Xi & \longrightarrow & (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi \end{array} \tag{3.12}$$

$$\begin{array}{ccc} (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi & \longrightarrow & \left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\pi_*\Xi \\ \| & & \downarrow \\ (\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi & \longrightarrow & \left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\pi_*\mathcal{O}_C \end{array} \tag{3.13}$$

Similarly to Lemma 3.10 we deduce the following

Lemma 3.13 *The following diagram commutes:*

$$\begin{array}{ccc} H^1(C^\dagger,\left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\pi_*\Xi) & \longleftarrow & H^1(C^\dagger,\left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\pi_*\omega_C) \\ {\scriptstyle d_2^\circ}\downarrow & & {\scriptstyle d_2'''}\downarrow \\ H^0(C^\dagger,\left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\pi_*\Xi) & \longrightarrow & H^0(C^\dagger,\left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\pi_*\mathcal{O}_C) \end{array}$$

where $d_2^\circ\colon H^1(C^\dagger,\left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\pi_*\Xi)\to H^0(C^\dagger,(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{\check\omega_i})\otimes\pi_*\Xi)$ *is the second differential in the spectral sequence converging to* $H^\bullet(C^\dagger,\mathrm{Tot}\ ^\circ M_y^{\bullet,\bullet})$ *from the hypercohomology of* $C^\dagger$ *with coefficients in the rows.*

By the projection formula,

$$\left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\pi_*\mathcal{O}_C\cong\pi_*\pi^*\left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_i*}\otimes\mathcal{L}^{\phi}_{\check\omega_i}\right)=\pi_*\bigoplus_{i\in I}V_{\check\omega_i*}\otimes\pi^*\mathcal{L}^{\phi}_{\check\omega_i},$$

and similarly

$$\left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_{i*}}\otimes\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\pi_*\Xi\cong\pi_*\left(\left(\bigoplus_{i\in I}V_{\check\omega_{i*}}\otimes\pi^*\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\Xi\right),$$
$$\left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check\omega_{i*}}\otimes\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\pi_*\omega_C\cong\pi_*\left(\left(\bigoplus_{i\in I}V^*_{\check\omega_{i*}}\otimes\pi^*\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\omega_C\right),$$
$$\left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check\omega_{i*}}\otimes\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\pi_*\Xi\cong\pi_*\left(\left(\bigoplus_{i\in I}V^*_{\check\omega_{i*}}\otimes\pi^*\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\Xi\right).$$

Now since the upper-right and lower-left terms $(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi$ of the bicomplex (3.11) are acyclic sheaves on $C^\dagger$, the differential d_2° coincides with the second differential from the spectral sequence arising from the following *complex* on $C^\dagger$:

$$\left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check\omega_{i*}}\otimes\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\pi_*\Xi\to(\mathfrak{h}\oplus\mathfrak{g}\oplus\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi\to\left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_{i*}}\otimes\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\pi_*\Xi \tag{3.14}$$

which is by construction a direct sum of the one term complex $(\mathfrak{t}\oplus\mathfrak{h})\otimes\pi_*\Xi$ in degree zero and the following complex:

$$\left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check\omega_{i*}}\otimes\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\pi_*\Xi\to(\mathfrak{h}\oplus\mathfrak{g})\otimes\pi_*\Xi\to\left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_{i*}}\otimes\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\pi_*\Xi \tag{3.15}$$

which in turn is nothing but the direct image $\pi_*N_y^\bullet$ of the following complex $N_y^\bullet$ of vector bundles on C:

$$\left(\bigoplus_{i\in I}V^*_{\check\omega_{i*}}\otimes\pi^*\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\Xi\to(\mathfrak{h}\oplus\mathfrak{g})\otimes\Xi\to\left(\bigoplus_{i\in I}V_{\check\omega_{i*}}\otimes\pi^*\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\Xi \tag{3.16}$$

Lemma 3.14 *The following diagram commutes:*

$$\begin{array}{ccc} H^1\left(C^\dagger,\left(\bigoplus_{i\in I}\mathcal{V}^{*\mathcal{F}}_{\check\omega_{i*}}\otimes\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\pi_*\Xi\right) & = & H^1\left(C,\left(\bigoplus_{i\in I}V^*_{\check\omega_{i*}}\otimes\pi^*\mathcal{L}^{\phi}_{-\check\omega_i}\right)\otimes\Xi\right)\\ \downarrow d_2^\circ & & \downarrow d_2^C\\ H^0\left(C^\dagger,\left(\bigoplus_{i\in I}\mathcal{V}^{\mathcal{F}}_{\check\omega_{i*}}\otimes\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\pi_*\Xi\right) & = & H^0(C,(\bigoplus_{i\in I}V_{\check\omega_{i*}}\otimes\pi^*\mathcal{L}^{\phi}_{\check\omega_i})\otimes\Xi)\end{array}$$

where $d_2^C: H^1\left(C,\left(\bigoplus_{i\in I}V_{\check\omega_{i*}}\otimes\pi^*\mathcal{L}^{\phi}_{\check\omega_i}\right)\otimes\Xi\right)\to H^0(C,(\bigoplus_{i\in I}V_{\check\omega_{i*}}\otimes\pi^*\mathcal{L}^{\phi}_{\check\omega_i})\otimes\Xi)$ *is the second differential in the spectral sequence converging to* $H^\bullet(C,N_y^\bullet)$ *from the cohomology of* C *with coefficients in its terms.*

Recall that our goal is to compute the differential (3.5):

$$H^1\left(C, \left(\bigoplus_{i\in I} V^*_{\check{\omega}_{i*}} \otimes \pi^*\mathcal{L}^{\phi}_{-\check{\omega}_i}\right)\otimes \omega_C\right) = H^1\left(C^\dagger, \left(\bigoplus_{i\in I} \mathcal{V}^{*\mathcal{F}}_{\check{\omega}_{i*}} \otimes \mathcal{L}^{\phi}_{-\check{\omega}_i}\right)\otimes \pi_*\omega_C\right)$$
$$\xrightarrow{d_2'} H^0\left(C^\dagger, \left(\bigoplus_{i\in I} \mathcal{V}^{\mathcal{F}}_{\check{\omega}_{i*}} \otimes \mathcal{L}^{\phi}_{\check{\omega}_i}\right)\otimes \pi_*\mathcal{O}_C\right) = H^0\left(C, \bigoplus_{i\in I} V_{\check{\omega}_{i*}} \otimes \pi^*\mathcal{L}^{\phi}_{\check{\omega}_i}\right).$$

The bottom line of the present section is the following

Corollary 3.15 *The following diagram commutes:*

$$\begin{array}{ccc} H^1\left(C, \left(\bigoplus_{i\in I} V^*_{\check{\omega}_{i*}} \otimes \pi^*\mathcal{L}^{\phi}_{-\check{\omega}_i}\right)\otimes \omega_C\right) & \longrightarrow & H^1\left(C, \left(\bigoplus_{i\in I} V^*_{\check{\omega}_{i*}} \otimes \pi^*\mathcal{L}^{\phi}_{-\check{\omega}_i}\right)\otimes \Xi\right) \\ {\scriptstyle d_2'}\downarrow & & {\scriptstyle d_2^C}\downarrow \\ H^0\left(C, \bigoplus_{i\in I} V_{\check{\omega}_{i*}} \otimes \pi^*\mathcal{L}^{\phi}_{\check{\omega}_i}\right) & \longleftarrow & H^0\left(C, \left(\bigoplus_{i\in I} V_{\check{\omega}_{i*}} \otimes \pi^*\mathcal{L}^{\phi}_{\check{\omega}_i}\right)\otimes \Xi\right) \end{array}$$

3.16 Calculation on *C*

The differential d_2^C of Corollary 3.15 was computed in [10]. To formulate the result, we introduce homogeneous coordinates z_1, z_2 on C such that $z = z_1/z_2$, so that $z_1 = 0$ (resp. $z_2 = 0$) is an equation of $0 \in C$ (resp. $\infty \in C$). We also introduce another copy of the curve C with homogeneous coordinates u_1, u_2, and $u := u_1/u_2$. The differential d_2^C has "matrix elements"

$$\widetilde{D}_{\check{\omega}_i,\check{\omega}_j} : H^1\left(C, V^*_{\check{\omega}_{i*}} \otimes \pi^*\mathcal{L}^{\phi}_{-\check{\omega}_i} \otimes \Xi\right) \to H^0\left(C, V_{\check{\omega}_{j*}} \otimes \pi^*\mathcal{L}^{\phi}_{\check{\omega}_j} \otimes \Xi\right).$$

Note that $\pi^*\mathcal{L}_{-\check{\omega}_i} \cong \mathcal{O}_C(-a_i)$, and $\pi^*\mathcal{L}_{\check{\omega}_j} \cong \mathcal{O}_C(a_j)$, while $\Xi \cong \mathcal{O}_C(-1)\oplus\mathcal{O}_C(-1)$ is Serre selfdual, so that

$$\widetilde{D}_{\check{\omega}_i,\check{\omega}_j} \in V_{\check{\omega}_{i*}} \otimes V_{\check{\omega}_{j*}} \otimes H^0(C, \mathcal{O}_C(a_i-1) \oplus \mathcal{O}_C(a_i-1))$$
$$\otimes H^0(C, \mathcal{O}_C(a_j-1) \oplus \mathcal{O}_C(a_j-1)).$$

Decomposing $V_{\check{\omega}_{i*}} \otimes V_{\check{\omega}_{j*}}$ according to the weights of T, for $\check{\lambda} \in \Lambda^\vee$ we obtain a matrix element $\widetilde{D}^{\check{\lambda}}_{\check{\omega}_i,\check{\omega}_j}$ which is defined as the weight $\check{\lambda}^*$-component of $\widetilde{D}_{\check{\omega}_i,\check{\omega}_j}$. Then according to [10, 3.8], $\widetilde{D}^{\check{\omega}_i+\check{\omega}_j}_{\check{\omega}_i,\check{\omega}_j} = 0 = \widetilde{D}^{2\check{\omega}_i-2\check{\alpha}_i}_{\check{\omega}_i,\check{\omega}_i}$, and if $i \neq j$, then $\widetilde{D}^{\check{\omega}_i+\check{\omega}_j-\check{\alpha}_i}_{\check{\omega}_i,\check{\omega}_j} = 0 = \widetilde{D}^{\check{\omega}_i+\check{\omega}_j-\check{\alpha}_j}_{\check{\omega}_i,\check{\omega}_j}$, while

$$\widetilde{D}^{2\check{\omega}_i-\check{\alpha}_i}_{\check{\omega}_i,\check{\omega}_i} = \check{d}_i(F_{i^*}v_{\check{\omega}_{i^*}} \otimes v_{\check{\omega}_{i^*}} - v_{\check{\omega}_{i^*}} \otimes F_{i^*}v_{\check{\omega}_{i^*}}) \otimes \frac{Q_i(z_1,z_2)R_i(u_1,u_2) - R_i(z_1,z_2)Q_i(u_1,u_2)}{z_1u_2 - z_2u_1} \tag{3.17}$$

$$\widetilde{D}^{\check{\omega}_i+\check{\omega}_j-\check{\alpha}_i-\check{\alpha}_j}_{\check{\omega}_i,\check{\omega}_j} = (\check{\alpha}_i,\check{\alpha}_j)(\langle\alpha_i,\check{\alpha}_j\rangle^{-1}v_{\check{\omega}_{i^*}} \otimes F_{i^*}F_{j^*}v_{\check{\omega}_{j^*}} + F_{i^*}v_{\check{\omega}_{i^*}} \otimes F_{j^*}v_{\check{\omega}_{j^*}} + \langle\alpha_j,\check{\alpha}_i\rangle^{-1}F_{j^*}F_{i^*}v_{\check{\omega}_{i^*}} \otimes v_{\check{\omega}_{j^*}}) \otimes \frac{R_i(z_1,z_2)R_j(u_1,u_2) - Q_i(z_1,z_2)S_{ji}(u_1,u_2) - S_{ij}(z_1,z_2)Q_j(u_1,u_2)}{z_1u_2 - z_2u_1} \tag{3.18}$$

Here the homogeneous polynomials Q_i, R_i, S_{ij} are but the homogenizations of the same named polynomials of one variable introduced in Sect. 3.1, and the above matrix coefficients are "scalar" 2×2-matrices with respect to the decomposition $\Xi \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$.

Now to compute the desired d_2' it remains to describe the horizontal arrows of the commutative diagram of Corollary 3.15. The lower one

$$H^0\left(C, \left(\bigoplus_{i\in I} V_{\check{\omega}_{i^*}} \otimes \pi^*\mathcal{L}^{\phi}_{\check{\omega}_i}\right) \otimes \Xi\right) \to H^0\left(C, \bigoplus_{i\in I} V_{\check{\omega}_{i^*}} \otimes \pi^*\mathcal{L}^{\phi}_{\check{\omega}_i}\right)$$

arises from the surjection

$$H^0(C, \mathcal{O}_C(a_i-1) \oplus \mathcal{O}_C(a_i-1)) \cong H^0\left(C, \pi^*\mathcal{L}^{\phi}_{\check{\omega}_i} \otimes \Xi\right) \to H^0(C, \pi^*\mathcal{L}^{\phi}_{\check{\omega}_i}) \cong H^0(C, \mathcal{O}_C(a_i))$$

which takes a pair $(P_1(z_1,z_2), P_2(z_1,z_2))$ of homogeneous degree a_i-1 polynomials to $z_1P_1 + z_2P_2$. The upper arrow

$$H^1\left(C, \left(\bigoplus_{i\in I} V^*_{\check{\omega}_{i^*}} \otimes \pi^*\mathcal{L}^{\phi}_{-\check{\omega}_i}\right) \otimes \omega_C\right) \to H^1\left(C, \left(\bigoplus_{i\in I} V^*_{\check{\omega}_{i^*}} \otimes \pi^*\mathcal{L}^{\phi}_{-\check{\omega}_i}\right) \otimes \Xi\right)$$

arises from the dual embedding

$$H^0(C, \mathcal{O}_C(a_i))^* \cong H^1\left(C, \pi^*\mathcal{L}^{\phi}_{-\check{\omega}_i} \otimes \omega_C\right) \to H^1(C, \pi^*\mathcal{L}^{\phi}_{-\check{\omega}_i} \otimes \Xi) \cong H^0(C, \mathcal{O}_C(a_i-1) \oplus \mathcal{O}_C(a_i-1))^*.$$

Namely, if we think of $H^0(C, \mathcal{O}_C(a_i))^*$ as of the homogeneous degree a_i differential operators in (u_1,u_2), then an operator O goes to (O_1,O_2) such that $O_1(P) := O(u_2 \cdot P)/2$, while $O_2(P) := O(u_1 \cdot P)/2$.

Composing with the matrix elements of (3.17), (3.18) we obtain the corresponding matrix elements ${}'\widetilde{D}_{\check{\omega}_i,\check{\omega}_j} \in V_{\check{\omega}_{i^*}} \otimes V_{\check{\omega}_{j^*}} \otimes H^0(C, \mathcal{O}_C(a_i)) \otimes H^0(C, \mathcal{O}_C(a_j))$ of d_2':

$$
\begin{aligned}
{}'\widetilde{D}^{2\check{\omega}_i-\check{\alpha}_i}_{\check{\omega}_i,\check{\omega}_i} &= \check{d}_i(F_{i^*}v_{\check{\omega}_{i^*}} \otimes v_{\check{\omega}_{i^*}} - v_{\check{\omega}_{i^*}} \otimes F_{i^*}v_{\check{\omega}_{i^*}}) \\
&\quad \otimes \frac{(z_1u_2+z_2u_1)(Q_i(z_1,z_2)R_i(u_1,u_2) - R_i(z_1,z_2)Q_i(u_1,u_2))}{2(z_1u_2-z_2u_1)}
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
&{}'\widetilde{D}^{\check{\omega}_i+\check{\omega}_j-\check{\alpha}_i-\check{\alpha}_j}_{\check{\omega}_i,\check{\omega}_j} = (\check{\alpha}_i,\check{\alpha}_j)(\langle\alpha_i,\check{\alpha}_j\rangle^{-1}v_{\check{\omega}_{i^*}} \\
&\quad \otimes F_{i^*}F_{j^*}v_{\check{\omega}_{j^*}} + F_{i^*}v_{\check{\omega}_{i^*}} \otimes F_{j^*}v_{\check{\omega}_{j^*}} + \langle\alpha_j,\check{\alpha}_i\rangle^{-1}F_{j^*}F_{i^*}v_{\check{\omega}_{i^*}} \otimes v_{\check{\omega}_{j^*}}) \\
&\quad \otimes \frac{(z_1u_2+z_2u_1)(R_i(z_1,z_2)R_j(u_1,u_2) - Q_i(z_1,z_2)S_{ji}(u_1,u_2) - S_{ij}(z_1,z_2)Q_j(u_1,u_2))}{2(z_1u_2-z_2u_1)}
\end{aligned} \tag{3.20}
$$

Going back from the homogeneous polynomials in (z_1, z_2) (resp. (u_1, u_2)) to the polynomials in $z = z_1/z_2$ (resp. $u = u_1/u_2$) we arrive at the following

Proposition 3.17 *The matrix elements of the differential d_2' (3.5) are*

$$
\begin{aligned}
{}'\widetilde{D}^{2\check{\omega}_i-\check{\alpha}_i}_{\check{\omega}_i,\check{\omega}_i} &= \check{d}_i(F_{i^*}v_{\check{\omega}_{i^*}} \otimes v_{\check{\omega}_{i^*}} - v_{\check{\omega}_{i^*}} \otimes F_{i^*}v_{\check{\omega}_{i^*}}) \\
&\quad \otimes \frac{(z+u)(Q_i(z)R_i(u) - R_i(z)Q_i(u))}{2(z-u)}
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
{}'\widetilde{D}^{\check{\omega}_i+\check{\omega}_j-\check{\alpha}_i-\check{\alpha}_j}_{\check{\omega}_i,\check{\omega}_j} &= (\check{\alpha}_i,\check{\alpha}_j)(\langle\alpha_i,\check{\alpha}_j\rangle^{-1}v_{\check{\omega}_{i^*}} \otimes F_{i^*}F_{j^*}v_{\check{\omega}_{j^*}} \\
&\quad + F_{i^*}v_{\check{\omega}_{i^*}} \otimes F_{j^*}v_{\check{\omega}_{j^*}} + \langle\alpha_j,\check{\alpha}_i\rangle^{-1}F_{j^*}F_{i^*}v_{\check{\omega}_{i^*}} \otimes v_{\check{\omega}_{j^*}}) \\
&\quad \otimes \frac{(z+u)(R_i(z)R_j(u) - Q_i(z)S_{ji}(u) - S_{ij}(z)Q_j(u))}{2(z-u)}
\end{aligned} \tag{3.22}
$$

while ${}'\widetilde{D}^{\check{\omega}_i+\check{\omega}_j}_{\check{\omega}_i,\check{\omega}_j} = 0 = {}'\widetilde{D}^{2\check{\omega}_i-2\check{\alpha}_i}_{\check{\omega}_i,\check{\omega}_i}$, *and if $i \neq j$, then* ${}'\widetilde{D}^{\check{\omega}_i+\check{\omega}_j-\check{\alpha}_i}_{\check{\omega}_i,\check{\omega}_j} = 0 = {}'\widetilde{D}^{\check{\omega}_i+\check{\omega}_j-\check{\alpha}_j}_{\check{\omega}_i,\check{\omega}_j}$.

3.18 Calculation of the Poisson bracket

The differential d_2 (Sect. 3.8) defines a bivector field on $\overset{\circ}{Y}{}^\alpha$ (i.e. a bidifferential operation on the coordinate ring of $\overset{\circ}{Y}{}^\alpha$). We denote the bivector by $\mathfrak{B}$ and the corresponding bidifferential operation on the coordinate ring of $\overset{\circ}{Y}{}^\alpha$ simply by $\{\cdot,\cdot\}$ (though it is *not* a Poisson bracket on $\overset{\circ}{Y}{}^\alpha$).

Proposition 3.19 *We have*

$$
\begin{aligned}
&\{w_{i,r}, w_{j,s}\} = 0, \\
&\{w_{i,r}, y_{j,s}\} = \check{d}_i\delta_{ij}\delta_{rs}w_{i,r}y_{j,s}, \\
&\{y_{i,r}, y_{j,s}\} = (1-\delta_{ij})(\check{\alpha}_i,\check{\alpha}_j)\frac{w_{i,r}+w_{j,s}}{2(w_{i,r}-w_{j,s})}y_{i,r}y_{j,s}.
\end{aligned}
$$

Proof By Proposition 3.17 on $\overset{\circ}{Y}{}^\alpha$ we have

$$\{Q_i(z), Q_j(u)\} = 0, \tag{3.23}$$

$$\{Q_i(z), R_j(u)\} = -\check{d}_i\delta_{ij}\frac{z+u}{2(z-u)}(Q_i(z)R_j(u) - R_i(z)Q_j(u)), \tag{3.24}$$

$$\{R_i(z), R_j(u)\} = (1-\delta_{ij})(\check{\alpha}_i, \check{\alpha}_j)\frac{z+u}{2(z-u)}(R_i(z)R_j(u) - Q_i(z)S_{ji}(u) - S_{ij}(z)Q_j(u)). \tag{3.25}$$

The relation $\{w_{i,r}, w_{j,s}\} = 0$ is obvious from (3.23). We have $\{B_i, B_j\} = 0$ and $\{w_{i,r}, B_j\} = 0$ from (3.23) as well. Substituting $u = w_{j,s}$ to (3.24), we get $\{B_i, y_{j,s}\} = -\frac{\check{d}_i\delta_{ij}}{2}B_i y_{j,s}$ and $\{w_{i,r}, y_{j,s}\} = \check{d}_i\delta_{ij}\delta_{rs}w_{i,r}y_{j,s}$. Finally, substituting $z = w_{i,r}$, $u = w_{j,s}$ to (3.25), we get $\{y_{i,r}, y_{j,s}\} = (1-\delta_{ij})(\check{\alpha}_i, \check{\alpha}_j)\frac{w_{i,r}+w_{j,s}}{2(w_{i,r}-w_{j,s})}y_{i,r}y_{j,s}$. □

The $1 \times T$ action on $\overset{\circ}{Y}{}^\alpha$ preserves this bivector field, hence it gives a well-defined bivector field on $\overset{\circ}{Y}{}^\alpha/(1 \times T)$. Moreover, the following is true:

Corollary 3.20 *The map $\varpi : \overset{\circ}{Y}{}^\alpha \to {}^\dagger\overset{\circ}{Z}{}^\alpha$ agrees with the bivector field $\mathfrak{B}$ on $\overset{\circ}{Y}{}^\alpha$ (in the sense that for $f_1, f_2 \in \mathbb{C}[{}^\dagger\overset{\circ}{Z}{}^\alpha]$ we have $\{\varpi^*(f_1), \varpi^*(f_2)\} = \varpi^*(f)$ for some $f \in \mathbb{C}[{}^\dagger\overset{\circ}{Z}{}^\alpha]$). So we get a bivector field on ${}^\dagger\overset{\circ}{Z}{}^\alpha = \varpi(\overset{\circ}{Y}{}^\alpha)$.*

Proof Note that the functions $w_{i,r}, y_{j,s}$ form a (rational étale) coordinate system on ${}^\dagger\overset{\circ}{Z}{}^\alpha$. So the only thing to be checked is that the bracket of any pair of pullbacks of these functions is a pullback of some function on ${}^\dagger\overset{\circ}{Z}{}^\alpha$. But this immediately follows from Proposition 3.19. □

Slightly abusing notation we denote the image of $\mathfrak{B}$ on ${}^\dagger\overset{\circ}{Z}{}^\alpha$ also by $\mathfrak{B}$.

Corollary 3.21 *In the coordinates $w_{i,r}, y_{j,s}$ on ${}^\dagger\overset{\circ}{Z}{}^\alpha$ the bivector field reads*

$$\mathfrak{B} = \sum_{i,r}\check{d}_i w_{i,r}y_{i,r}\frac{\partial}{\partial w_{i,r}} \wedge \frac{\partial}{\partial y_{i,r}} + \sum_{i\neq j}\sum_{r,s}\frac{(\check{\alpha}_i, \check{\alpha}_j)}{2}\frac{w_{i,r}+w_{j,s}}{w_{i,r}-w_{j,s}}y_{i,r}y_{j,s}\frac{\partial}{\partial y_{i,r}} \wedge \frac{\partial}{\partial y_{j,s}}.$$

Corollary 3.22 *The bivector field $\mathfrak{B}$ on ${}^\dagger\overset{\circ}{Z}{}^\alpha$ is Poisson, i.e.* $[\mathfrak{B}, \mathfrak{B}] = 0$. *This Poisson structure extends uniquely to ${}^\dagger Z^\alpha$.*

Proof The first claim is immediate from the explicit formula of Corollary 3.21. We have a smooth open subvariety ${}^\dagger\overset{\circ}{Z}{}^\alpha \subset {}^\dagger\overset{\bullet}{Z}{}^\alpha \subset {}^\dagger Z^\alpha$ formed by the based quasimaps with defect at most a simple coroot, see e.g. [2, proof of Proposition 5.1]. Its complement has codimension 2 in ${}^{\dagger}Z^{\alpha}$. Now ${}^{\dagger}Z^{\alpha}$ is normal by [2, Corollary 2.10], so it suffices to check that the bivector field on ${}^{\dagger}\overset{\circ}{Z}{}^{\alpha}$ extends as a Poisson structure to ${}^{\dagger}\overset{\bullet}{Z}{}^{\alpha}$. Moreover, it suffices to check this at the generic points of the boundary components ${}^{\dagger}\overset{\bullet}{Z}{}^{\alpha} \setminus {}^{\dagger}\overset{\circ}{Z}{}^{\alpha}$ (given by equations $y_{i,r} = 0$) where the claim is evident from the explicit formula of Corollary 3.21. □

Corollary 3.23 *The Poisson structure* $\mathfrak{B}$ *on* ${}^{\dagger}\overset{\circ}{Z}{}^{\alpha}$ *is nondegenerate. The corresponding symplectic form reads*

$$\Omega_{\text{trig}} := \mathfrak{B}^{-1} = \sum_{i,r} \frac{dy_{i,r} \wedge \mathrm{d}w_{i,r}}{\check{\mathrm{d}}_i w_{i,r} y_{i,r}} + \sum_{i\neq j} \sum_{r,s} \frac{(\check{\alpha}_i, \check{\alpha}_j)}{2\check{\mathrm{d}}_i \check{\mathrm{d}}_j} \frac{w_{i,r} + w_{j,s}}{w_{i,r} - w_{j,s}} \frac{\mathrm{d}w_{i,r} \wedge \mathrm{d}w_{j,s}}{w_{i,r} w_{j,s}}.$$

4 Transversal slices in the affine flag variety

4.1 Schubert cells in the affine flag varieties

We have an embedding of the affine Grassmannian of G (thin one: an ind-scheme) into the Kashiwara affine Grassmannian (thick one: an infinite type scheme): $\mathrm{Gr} = G((z))/G[[z]] \cong G[z^{\pm 1}]/G[z] \hookrightarrow G((z^{-1}))/G[z] = \mathbf{Gr}$. The subgroup of currents $G[[z]]$ (resp. $G[[z^{-1}]]$) taking value in B (resp. B_-) at $z = 0$ (resp. $z = \infty$) is the Iwahori group $\mathbf{Iw}$ (resp. $\mathbf{Iw}_-$). The unipotent radical of $\mathbf{Iw}$ (resp. $\mathbf{Iw}_-$) is denoted $\mathbf{N}$ (resp. $\mathbf{N}_-$). We have an embedding of the affine flag variety of G (thin one: an ind-scheme) into the Kashiwara affine flag variety (thick one: an infinite type scheme): $\mathcal{F}\ell = G((z))/\mathbf{Iw} \cong G[z^{\pm 1}]/(G[z] \cap \mathbf{Iw}) \hookrightarrow G((z^{-1}))/(G[z] \cap \mathbf{Iw}) = \mathbf{Fl}$. The natural projection $\mathrm{pr}\colon \mathbf{Fl} \to \mathbf{Gr}$ (as well as its restriction $\mathrm{pr}\colon \mathcal{F}\ell \to \mathrm{Gr}$) is a fibration with fibers $\mathcal{B}$.

The set of T-fixed points in $\mathcal{F}\ell$ (resp. Gr) is in a natural bijection with the affine Weyl group $W_a = W \ltimes \Lambda$ (resp. the coweight lattice Λ). For $w \in W_a$ we will denote the corresponding T-fixed point by the same symbol w; its $\mathbf{N}$-orbit (resp. $\mathbf{N}_-$-orbit) will be denoted by $\mathcal{F}\ell^w \subset \mathcal{F}\ell$ (resp. $\mathbf{Fl}_w \subset \mathbf{Fl}$): a thin (resp. thick) Schubert cell. The intersection $\mathcal{F}\ell^w_y := \mathcal{F}\ell^w \cap \mathbf{Fl}_y$ (an open Richardson variety, aka transversal slice) is nonempty iff $w \geq y$ in the Bruhat order. Similarly, for $\lambda \in \Lambda \subset \mathrm{Gr}$ the $\mathbf{N}$-orbit $\mathbf{N} \cdot \lambda$ (resp. $\mathbf{N}_-$-orbit) will be denoted by $X^{\lambda} \subset \mathrm{Gr}$ (resp. $\mathbf{X}_{\lambda} \subset \mathbf{Gr}$): a thin (resp. thick) Schubert cell. For a dominant coweight $\lambda \in \Lambda^+ \subset \Lambda$ the $G((z))$-orbit $\mathrm{Gr}^{\lambda} := G((z)) \cdot \lambda$ is a union $\mathrm{Gr}^{\lambda} = \bigsqcup_{\nu \in W\cdot\lambda} X^{\nu}$.

Recall the notations of [2, 2.4]: $G_1 \subset G[[z^{-1}]]$ is the kernel of evaluation at $z = \infty$, and $\mathcal{W}_{\mu} := G_1 \cdot \mu \subset \mathbf{Gr}$ for $\mu \in \Lambda$. If μ is dominant, then $\mathcal{W}_{\mu} = \mathbf{X}_{\mu}$. If $\lambda \geq \mu$ is also dominant, then the transversal slice $\mathcal{W}^{\lambda}_{\mu} := \mathrm{Gr}^{\lambda} \cap \mathcal{W}_{\mu}$ of *loc. cit.* is a union $\mathcal{W}^{\lambda}_{\mu} = \bigsqcup_{\nu \in W\cdot\lambda} X^{\nu} \cap \mathbf{X}_{\mu}$.

Given a dominant $\eta \in \Lambda^+ \subset \Lambda \subset W_a$, we consider $-\eta$ as an element in the affine Weyl group; it is the *minimal* length representative of its *left* W-coset, and the *maximal* length representative of its *right* W-coset. Furthermore, η is the maximal length representative of its left W-coset, and the minimal length representative of its right W-coset. The projection $\mathrm{pr}\colon W_a = \mathcal{F}\ell^T \to \mathrm{Gr}^T = \Lambda$ realizes Λ as the set of *left* W-cosets in W_a. Hence for a dominant $\lambda \in W_a/W$, the affine Weyl group element $w_0(\lambda) \in \Lambda \subset W_a$ (resp. $w_0(\lambda) \times w_0 = w_0 \times \lambda \in W \ltimes \Lambda = W_a$) is the minimal length (resp. maximal length) representative of the left W-coset λ. In particular, $\mathrm{pr}\colon \mathcal{F}\ell^{w_0(\lambda)} \xrightarrow{\sim} X^\lambda$, and $\mathrm{pr}\colon \mathbf{Fl}_{w_0\times\lambda} \xrightarrow{\sim} \mathbf{X}_\lambda$.

Finally, for $\lambda \geq \mu \in \Lambda^+$, the intersection $X^\lambda \cap \mathbf{X}_\mu$ is open in the slice $\mathcal{W}^\lambda_\mu \subset \mathrm{Gr}$, and pr is an open embedding of the open Richardson variety $\mathcal{F}\ell^{w_0\times\lambda}_{w_0\times\mu} = \mathcal{F}\ell^{w_0\times\lambda} \cap \mathbf{Fl}_{w_0\times\mu}$ into $X^\lambda \cap \mathbf{X}_\mu$. All in all,

$$\mathrm{pr}\colon \mathcal{F}\ell^{w_0\times\lambda}_{w_0\times\mu} \hookrightarrow \mathcal{W}^\lambda_\mu \tag{4.1}$$

is an open embedding.

4.2 A modular interpretation

Recall the morphism $s^{\lambda^*}_{\mu^*}\colon \mathcal{W}^{\lambda^*}_{\mu^*} \to Z^\alpha$ of [2, Theorem 2.8] (here $\alpha = \lambda - \mu,\ \lambda^* = -w_0\lambda,\ \mu^* = -w_0\mu$). Recall the open subscheme of periodic monopoles ${}^\dagger\overset{\circ}{Z}{}^\alpha \subset Z^\alpha$ introduced in Definition 2.7.

Proposition 4.3 *The composition $s^{\lambda^*}_{\mu^*} \circ \mathrm{pr}\colon \mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*} \to Z^\alpha$ is an open embedding with the image ${}^\dagger\overset{\circ}{Z}{}^\alpha \subset Z^\alpha$.*

Proof Recall from the proof of [2, Theorem 2.8] that the slice closure $\overline{\mathcal{W}}{}^{\lambda^*}_{\mu^*} = \bigsqcup_{\mu^*\leq\nu^*\leq\lambda^*} \mathcal{W}^{\nu^*}_{\mu^*}$ is the moduli space of the following data: $(\mathcal{F}_{\mathrm{triv}} \xrightarrow{\sigma} \mathcal{F}_G)$ where $\mathcal{F}_G$ is a G-bundle of isomorphism class μ^*, and σ is an isomorphism from the trivial G-bundle away from $0 \in C$ with a pole of degree λ^* at 0, such that the value of the Harder-Narasimhan flag of $\mathcal{F}_G$ at $\infty \in C$ is compatible with the complete flag $\sigma(B_-)$. The bundle $\mathcal{F}_G$ has a unique complete flag (B-structure) ϕ of degree $w_0\mu^* = -\mu$ with value $B_- \in \mathcal{B}$ at $\infty \in C$ (with respect to the trivialization σ at ∞). This flag can be transformed via σ^{-1} to obtain a degree α generalized B-structure $\sigma^{-1}\phi$ in $\mathcal{F}_{\mathrm{triv}}$ without a pole but possibly with a defect at $0 \in C$. The morphism $s^{\lambda^*}_{\mu^*}\colon \overline{\mathcal{W}}{}^{\lambda^*}_{\mu^*} \to Z^\alpha$ takes $(\mathcal{F}_{\mathrm{triv}} \xrightarrow{\sigma} \mathcal{F}_G)$ to $\sigma^{-1}\phi$. The open subset $U \subset \overline{\mathcal{W}}{}^{\lambda^*}_{\mu^*}$ given by the condition that $\sigma^{-1}\phi$ has no defect at $0 \in C$, is mapped isomorphically onto $\overset{\circ}{Z}{}^\alpha$. We have a still smaller open subset $U' \subset U$ given by the condition that the fiber of $\sigma^{-1}\phi$ at $0 \in C$ is transversal to the flag $B \in \mathcal{B}$. The open subset U' is mapped isomorphically onto ${}^\dagger\overset{\circ}{Z}{}^\alpha$. Thus we have to check $\mathrm{pr}\colon \mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*} \xrightarrow{\sim} U'$.

Recall the semiinfinite orbit $\mathcal{S}^{\lambda^*}$ (whose intersection with $\mathcal{W}^{\lambda^*}_{\mu^*}$ is dense in $\mathcal{W}^{\lambda^*}_{\mu^*}$). It is formed by the data $(\mathcal{F}_{\mathrm{triv}} \xrightarrow{\sigma} \mathcal{F}_G)$ such that the transformation $\sigma\phi_{\mathrm{triv}}$ of the trivial complete flag with fibers $B \in \mathcal{B}$ in $\mathcal{F}_{\mathrm{triv}}$ via σ is a B-structure in $\mathcal{F}_G$ without defect at $0 \in C$. Note that U' lies inside $\mathcal{S}^{\lambda^*} \cap \overline{\mathcal{W}}{}^{\lambda^*}_{\mu^*}$ and is given there by the condition that the fibers of $\sigma\phi_{\mathrm{triv}}$ and of ϕ at $0 \in C$ are transversal. According to [23, Theorem 3.2, (3.6)], for $\nu^* < \lambda^*$ we have $\mathcal{S}^{\lambda^*} \cap \mathrm{Gr}^{\nu^*} = \emptyset$, and $\mathcal{S}^{\lambda^*} \cap \mathrm{Gr}^{\lambda^*} = X^{\lambda^*}$. It follows that $\mathcal{S}^{\lambda^*} \cap \overline{\mathcal{W}}{}^{\lambda^*}_{\mu^*} = X^{\lambda^*} \cap \mathbf{X}_{\mu^*}$. It remains to check that the open subset $\mathrm{pr}(\mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*}) \subset X^{\lambda^*} \cap \mathbf{X}_{\mu^*}$ is nothing but U'.

To this end recall the modular interpretation of our slices. First of all, **Gr** is the moduli space of G-bundles $\mathcal{F}_G$ on C equipped with a trivialization ς in the formal neighbourhood of $\infty \in C$. Second, **Fl** is the moduli space of triples $(\mathcal{F}_G, \varsigma, F)$ where $(\mathcal{F}_G, \varsigma) \in \mathbf{Gr}$, and F is a B-structure in the fiber of $\mathcal{F}_G$ at $0 \in C$. Third, $\mathbf{X}_{\mu^*} = \mathcal{W}_{\mu^*} \subset \mathbf{Gr}$ is formed by the pairs $(\mathcal{F}_G, \varsigma)$ such that the isomorphism type of $\mathcal{F}_G$ is μ^*, and the value of the Harder-Narasimhan flag of $\mathcal{F}_G$ at $\infty \in C$ is compatible with $B_- \in \mathcal{B}$ (with respect to the trivialization ς at $\infty \in C$). Now $\mathbf{Fl}_{w_0\times\mu^*} \subset \mathbf{Fl}$ is formed by the triples $(\mathcal{F}_G, \varsigma, F)$ such that $(\mathcal{F}_G, \varsigma) \in \mathbf{X}_{\mu^*}$, and F is the value $\phi|_0$ at $0 \in C$ of the unique degree $w_0\mu^* = -\mu$ complete flag ϕ in $\mathcal{F}_G$ such that $\phi|_\infty = B_- \in \mathcal{B}$ (so that ϕ is the refinement of the Harder-Narasimhan flag of $\mathcal{F}_G$). Furthermore, Gr is the moduli space of G-bundles $\mathcal{F}_G$ on C equipped with a trivialization σ over $C \setminus 0$, while $\mathcal{F}\ell$ is the moduli space of triples $(\mathcal{F}_G, \sigma, F)$ where $(\mathcal{F}_G, \sigma) \in \mathrm{Gr}$, and F is a B-structure in the fiber of $\mathcal{F}_G$ at $0 \in C$. The projection $\mathrm{pr}\colon \mathcal{F}\ell \to \mathrm{Gr}$ admits a section s over $X^{\lambda^*} = \mathcal{S}^{\lambda^*} \cap \mathrm{Gr}^{\lambda^*}$: we define F as the fiber at $0 \in C$ of the transformation $\sigma\phi_{\mathrm{triv}}$ of the trivial B-structure $B \in \mathcal{B}$ in the trivial G-bundle. Finally, $\mathcal{F}\ell^{w_0\times\lambda^*} \subset \mathcal{F}\ell$ is formed by the triples $(\mathcal{F}_G, \sigma, F)$ such that $(\mathcal{F}_G, \sigma) \in X^{\lambda^*}$, and F is transversal to $s(\mathcal{F}_G, \sigma)$.

Thus $\mathrm{pr}(\mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*}) = U' \subset X^{\lambda^*} \cap \mathbf{X}_{\mu^*}$. The proposition is proved. □

4.4 Stabilization

Let $\mu, \nu \in \Lambda^+$ be dominant coweights. According to [18, 2E], we have the inclusion of stabilizers $\mathrm{St}_\mu \subset \mathrm{St}_{\mu+\nu} \subset G_1 \subset G[[z^{-1}]]$, so the identity morphism $G_1 \to G_1$ induces a morphism $\varsigma_\mu^{\mu+\nu}\colon \mathbf{X}_\mu = G_1/\mathrm{St}_\mu \to G_1/\mathrm{St}_{\mu+\nu} = \mathbf{X}_{\mu+\nu}$. According to *loc. cit.*, $\varsigma_\mu^{\mu+\nu}$ restricts to the same named morphism $\mathcal{W}_\mu^\lambda \to \mathcal{W}_{\mu+\nu}^{\lambda+\nu}$ for any $\Lambda^+ \ni \lambda \geq \mu$. Similarly, we have $\mathrm{St}_{w_0\times\mu} \subset \mathrm{St}_{w_0\times(\mu+\nu)} \subset \mathbf{N}_-$, and the identity morphism $\mathbf{N}_- \to \mathbf{N}_-$ induces a morphism $\sigma_\mu^{\mu+\nu}\colon \mathbf{Fl}_{w_0\times\mu} = \mathbf{N}_-/\mathrm{St}_{w_0\times\mu} \to \mathbf{N}_-/\mathrm{St}_{w_0\times(\mu+\nu)} = \mathbf{Fl}_{w_0\times(\mu+\nu)}$ which restricts to the same named morphism $\mathcal{F}\ell^{w_0\times\lambda}_{w_0\times\mu} \to \mathcal{F}\ell^{w_0\times(\lambda+\nu)}_{w_0\times(\mu+\nu)}$ for any $\Lambda^+ \ni \lambda \geq \mu$. The following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}\ell^{w_0\times\lambda}_{w_0\times\mu} & \xrightarrow{\sigma_\mu^{\mu+\nu}} & \mathcal{F}\ell^{w_0\times(\lambda+\nu)}_{w_0\times(\mu+\nu)} \\ \mathrm{pr}\downarrow & & \mathrm{pr}\downarrow \\ \mathcal{W}_\mu^\lambda & \xrightarrow{\varsigma_\mu^{\mu+\nu}} & \mathcal{W}_{\mu+\nu}^{\lambda+\nu} \end{array} \tag{4.2}$$

Moreover, from the construction of $s_{\mu^*}^{\lambda^*}\colon \mathcal{W}_{\mu^*}^{\lambda^*} \to Z^\alpha$ in [2, Lemma 2.7, Theorem 2.8] (where $\alpha = \lambda - \mu$) it follows immediately that the following diagrams commute as well:

$$\begin{array}{ccc} \mathcal{W}_{\mu^*}^{\lambda^*} & \xrightarrow{\varsigma_{\mu^*}^{\mu^*+\nu^*}} & \mathcal{W}_{\mu^*+\nu^*}^{\lambda^*+\nu^*} \\ s_{\mu^*}^{\lambda^*}\downarrow & & s_{\mu^*+\nu^*}^{\lambda^*+\nu^*}\downarrow \\ Z^\alpha & = & Z^\alpha \end{array} \tag{4.3}$$

$$\begin{array}{ccc} \mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*} & \xrightarrow{\sigma^{\mu^*+\nu^*}_{\mu^*}} & \mathcal{F}\ell^{w_0\times(\lambda^*+\nu^*)}_{w_0\times(\mu^*+\nu^*)} \\ {\scriptstyle s^{\lambda^*}_{\mu^*}\circ\mathrm{pr}}\downarrow & & {\scriptstyle s^{\lambda^*+\nu^*}_{\mu^*+\nu^*}\circ\mathrm{pr}}\downarrow \\ {}^{\dagger}\overset{\circ}{Z}{}^{\alpha} & = & {}^{\dagger}\overset{\circ}{Z}{}^{\alpha} \end{array} \tag{4.4}$$

It follows in particular that $\sigma^{\mu^*+\nu^*}_{\mu^*}: \mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*} \xrightarrow{\sim} \mathcal{F}\ell^{w_0\times(\lambda^*+\nu^*)}_{w_0\times(\mu^*+\nu^*)}$ is an isomorphism.

4.5 s^λ_μ in coordinates

We will use the generalized minors of [18, 2A] to construct regular functions on the open Richardson varieties. Namely, given an irreducible G-module $V_{\check\lambda}$ with highest weight $\check\lambda \in \Lambda^\vee_+$ and highest vector $v_{\check\lambda}$, its dual $V^*_{\check\lambda}$ is isomorphic to $V_{\check\lambda^*}$ with the lowest weight $-\check\lambda$, and the lowest vector $v_{-\check\lambda}$ such that $\langle v_{-\check\lambda}, v_{\check\lambda}\rangle = 1$. Given $w, y \in W$, we define the following regular function on G

$$\Delta_{w\check\lambda, y\check\lambda}(g) := \langle \overline{w} v_{-\check\lambda}, g\overline{y} v_{\check\lambda}\rangle \tag{4.5}$$

where $\overline{w}, \overline{y} \in G$ are the lifts of w, y defined in *loc. cit.*

Following *loc. cit.* we consider the regular functions $\Delta^{(s)}_{w\check\lambda, y\check\lambda}$, $s \in \mathbb{Z}$, on $G((z^{-1}))$ defined as follows:

$$\Delta_{w\check\lambda, y\check\lambda}(g(z)) = \sum_{s=-\infty}^{\infty} \Delta^{(s)}_{w\check\lambda, y\check\lambda}(g(z))z^{-s} \tag{4.6}$$

More generally, to any $v \in V_{\check\lambda}$ and $\beta \in V^*_{\check\lambda}$ we can assign the generalized minor $\Delta_{\beta,v}(z) := \langle \beta, g(z)v\rangle$. We also denote by $\Delta^{(s)}_{\beta,v}$ the coefficient at z^{-s} of the power series $\Delta_{\beta,v}(z)$.

Recall from [2, 2.6] that $\mathrm{St}_\mu \subset G_1 \subset G[[z^{-1}]]$ is the stabilizer of $\mu \in \Lambda^+ \subset \Lambda = \mathrm{Gr}^T$. Similarly, for $w \in W_a$ we denote by $\mathrm{St}_w \subset \mathbf{N}_-$ the stabilizer of $w \in W_a = \mathcal{F}\ell^T$. We have $\mathbf{X}_\mu = G_1/\mathrm{St}_\mu$, and $\mathbf{Fl}_w = \mathbf{N}_-/\mathrm{St}_w$. In case $w = w_0 \times \mu$ (see Sect. 4.1), we have $\mathrm{St}_\mu = G_1 \cap \mathrm{St}_{w_0\times\mu}$, and the natural morphism $\mathbf{Fl}_{w_0\times\mu} = \mathbf{N}_-/\mathrm{St}_{w_0\times\mu} \to G_1/\mathrm{St}_\mu = \mathbf{X}_\mu$ is an isomorphism. According to [18, Lemma 2.19], the functions $\Delta^{(s)}_{\check\omega_i,\check\omega_i}$, $\Delta^{(s)}_{s_i\check\omega_i,\check\omega_i}$, $s > 0$, $i \in I$ restricted to G_1 (resp. $\mathbf{N}_-$) are St_μ-invariant (resp. $\mathrm{St}_{w_0\times\mu}$-invariant); hence they may be viewed as the functions on $\mathbf{Fl}_{w_0\times\mu} \cong \mathbf{X}_\mu$.

Now let $\Lambda^+ \ni \lambda \geq \mu$, and $\Lambda_+ \ni \alpha = \sum_{i\in I} a_i\alpha_i := \lambda - \mu$. Recall the isomorphism $s^{\lambda^*}_{\mu^*} \circ \mathrm{pr}: \mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*} \xrightarrow{\sim} {}^{\dagger}\overset{\circ}{Z}{}^{\alpha}$ of Proposition 4.3. Recall also the regular polynomial-valued functions Q_i, R_i on Z^α (see e.g. [10, 3.3]): $Q_i = z^{a_i} + q_{i,a_i-1}z^{a_i-1} + \cdots$ (resp. $R_i = r_{i,a_i-1}z^{a_i-1} + \cdots$) is the highest (resp. prehighest) Plücker coordinate on the space of based quasimaps (in notations of *loc. cit.* $Q_i = \phi^{-\check\omega_i}_{\check\omega_i}$, $R_i = \phi^{\check\alpha_i-\check\omega_i}_{\check\omega_i}$). Following *loc. cit.* and Sect. 3.1 we also consider a rational étale coordinate system on Z^α. Namely, $(w_{i,r})^{1\leq r\leq a_i}_{i\in I}$ are the ordered roots of Q_i, and $y_{i,r} := R_i(w_{i,r})$.

Proposition 4.6 *Under the isomorphism $s^{\lambda^*}_{\mu^*} \circ \mathrm{pr}\colon \mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*} \xrightarrow{\sim} {}^{\dagger}\overset{\circ}{Z}{}^{\alpha}$ we have $Q_i = \sum_{s=0}^{a_i} \Delta^{(s)}_{\check{\omega}_i,\check{\omega}_i} z^{a_i-s}$, $R_i = \sum_{s=0}^{a_i} \Delta^{(s)}_{s_i\check{\omega}_i,\check{\omega}_i} z^{a_i-s}$.*

Proof Follows at once from the commutative diagram [2, (2.3)] (and the definition of $\pi_{\mu,n}$ in [2, Lemma 2.7]). ◻

4.7 Rational Poisson bracket revisited

We fix a basis $e_\alpha, e_{-\alpha}, h_i$ in $\mathfrak{g}$ where $i \in I$, and $\alpha \in R^+$ is a positive coroot (and the weight of e_α is the dual root $\check{\alpha}$; in particular, $e_{\alpha_i} = E_i$ of Sect. 2.2, and $e_{-\alpha_i} = \check{d}_i F_i$). We assume $(e_\alpha, e_{-\alpha}) = 1, (h_i, h_j) = \delta_{ij}$. Then the Lie bialgebra structure on $\mathfrak{g}((z^{-1}))$ is determined by the classical rational r-matrix

$$r_{\mathrm{rat}}(z,u) := \frac{1}{z-u}\left(\sum_{\alpha>0} e_\alpha \otimes e_{-\alpha} + e_{-\alpha}\otimes e_\alpha + \sum_{i\in I} h_i \otimes h_i\right), \qquad (4.7)$$

see e.g. [7, Sect. 6.4]. This determines a Poisson group structure on $G((z^{-1}))$ such that G_1 is a Poisson subgroup.

Proposition 4.8 [18, Proposition 2.13] *The rational Poisson bracket $\{,\}_{\mathrm{rat}}$ of the functions $\Delta^{(s)}_{\beta,v}$ on the subgroup G_1 is*

$$\begin{aligned}
&\{\Delta_{\beta_1,v_1}(z), \Delta_{\beta_2,v_2}(u)\}_{\mathrm{rat}}\\
&= \frac{1}{z-u}\Bigg(\sum_{\alpha>0} \Delta_{\beta_1,e_\alpha v_1}(z)\Delta_{\beta_2,e_{-\alpha}v_2}(u) + \sum_{\alpha>0} \Delta_{\beta_1,e_{-\alpha}v_1}(z)\Delta_{\beta_2,e_\alpha v_2}(u)\\
&\quad + \sum_{i\in I} \Delta_{\beta_1,h_i v_1}(z)\Delta_{\beta_2,h_i v_2}(u)\Bigg)\\
&\quad - \frac{1}{z-u}\Bigg(\sum_{\alpha>0} \Delta_{e_\alpha\beta_1,v_1}(z)\Delta_{e_{-\alpha}\beta_2,v_2}(u) + \sum_{\alpha>0} \Delta_{e_{-\alpha}\beta_1,v_1}(z)\Delta_{e_\alpha\beta_2,v_2}(u)\\
&\quad + \sum_{i\in I} \Delta_{h_i\beta_1,v_1}(z)\Delta_{h_i\beta_2,v_2}(u)\Bigg).
\end{aligned}$$

According to [18], this Poisson structure on G_1 induces a Poisson structure on transversal slices $\mathcal{W}^\lambda_\mu$ in the affine Grassmannian $\mathbf{Gr} = G((z^{-1}))/G[z]$. On the other hand, recall a symplectic structure on $\overset{\circ}{Z}{}^\alpha$ defined in [10]. It extends uniquely to a Poisson bracket $\{,\}^Z_{\mathrm{rat}}$ on Z^α by the same argument as in the proof of Corollary 3.22.

The following theorem confirms expectations of [18, Remark 2.11].

Theorem 4.9 *The map $s^{\lambda^*}_{\mu^*}\colon \mathcal{W}^{\lambda^*}_{\mu^*} \to Z^\alpha$ is Poisson.*

Proof The field of rational functions on Z^α coincides with the field of rational functions in the Fourier coefficients of the functions $Q_i(z)$, $R_i(z)$. Hence it is sufficient to show that the Poisson bracket of the coefficients $Q_i(z)$, $R_i(z)$ is the same on $\mathcal{W}^\lambda_\mu$ and Z^α. Let us introduce the following generalized minors: $S_{ij}(z) := \Delta_{E_jE_iv_{-\check\omega_i}, v_{\check\omega_i}} = \langle E_jE_iv_{-\check\omega_i}, g(z)v_{\check\omega_i}\rangle$. According to [10, (7) and (8)], the Poisson bracket of the (polynomial-valued) functions $Q_i(z)$, $R_i(z)$ is given by

$$\{Q_i(z), Q_j(u)\}^Z_{\text{rat}} = 0, \tag{4.8}$$

$$\{Q_i(z), R_j(u)\}^Z_{\text{rat}} = -\check d_i\delta_{ij}\left(\frac{1}{z-u}Q_i(z)R_j(u) - \frac{1}{z-u}R_i(z)Q_j(u)\right), \tag{4.9}$$

$$\begin{aligned}\{R_i(z), R_j(u)\}^Z_{\text{rat}} &= (1-\delta_{ij})((\check\alpha_i, \check\alpha_j)\frac{1}{z-u}R_i(z)R_j(u)\\ &\quad + \check d_i\check d_j\frac{1}{z-u}Q_i(z)S_{ji}(u)) + \check d_i\check d_j\frac{1}{z-u}S_{ij}(z)Q_j(u)).\end{aligned} \tag{4.10}$$

On the other hand, the Fourier coefficients of the pullbacks $(s^{\lambda^*}_{\mu^*})^*Q_i = z^{a_i} + \sum_{s=1}^{a_i}\Delta^{(s)}_{\check\omega_i,\check\omega_i}z^{a_i-s}$ and $(s^{\lambda^*}_{\mu^*})^*R_i = \sum_{s=1}^{a_i}\Delta^{(s)}_{s_i\check\omega_i,\check\omega_i}z^{a_i-s}$ obey the same relations by Proposition 4.8. ◻

4.10 Trigonometric Poisson bracket

The standard Lie bialgebra structure on $\mathfrak{g}((z^{-1}))\oplus\mathfrak{t}$ (see e.g. [7, 6.2.1, 6.5]) gives rise to a Poisson structure on **Fl** such that the open Richardson varieties $\mathcal{F}\ell^w_y$ are Poisson subvarieties of **Fl** (cf. [21, Corollary 2.9]).

This Lie bialgebra structure on $\mathfrak{g}((z^{-1}))\oplus\mathfrak{t}$ is determined by the classical r-matrix

$$\begin{aligned}r_{\text{trig}}(z,u) := \frac{1}{z-u}\Bigg(&z(\sum_{\alpha>0}e_\alpha\otimes e_{-\alpha} + \frac12\sum_{i\in I}h_i\otimes h_i)\\ &+ u\Bigg(\sum_{\alpha>0}e_{-\alpha}\otimes e_\alpha + \frac12\sum_{i\in I}h_i\otimes h_i\Bigg)\Bigg),\end{aligned} \tag{4.11}$$

see e.g. [7, (6.6)].

Proposition 4.11 *The Poisson bracket of the functions $\Delta^{(s)}_{\beta,v}$ on the Iwahori subgroup* **Iw**$_-$ *is*

$$\begin{aligned}&\{\Delta_{\beta_1,v_1}(z), \Delta_{\beta_2,v_2}(u)\}_{\text{trig}} =\\ &\frac{1}{z-u}\Bigg(z\sum_{\alpha>0}\Delta_{\beta_1,e_\alpha v_1}(z)\Delta_{\beta_2,e_{-\alpha}v_2}(u) + u\sum_{\alpha>0}\Delta_{\beta_1,e_{-\alpha}v_1}(z)\Delta_{\beta_2,e_\alpha v_2}(u)\\ &+\frac{z+u}{2}\sum_{i\in I}\Delta_{\beta_1,h_iv_1}(z)\Delta_{\beta_2,h_iv_2}(u)\Bigg)\end{aligned}$$

$$-\frac{1}{z-u}\left(z\sum_{\alpha>0}\Delta_{e_\alpha\beta_1,v_1}(z)\Delta_{e_{-\alpha}\beta_2,v_2}(u)+u\sum_{\alpha>0}\Delta_{e_{-\alpha}\beta_1,v_1}(z)\Delta_{e_\alpha\beta_2,v_2}(u)\right.$$
$$\left.+\frac{z+u}{2}\sum_{i\in I}\Delta_{h_i\beta_1,v_1}(z)\Delta_{h_i\beta_2,v_2}(u)\right)$$

Proof This follows from the Belavin–Drinfeld formula for trigonometric r-matrix (see e.g. [7, (6.6)]). Indeed, following [18, Proposition 2.13] we note that the cobracket on $\mathfrak{g}((z^{-1}))$ is coboundary, namely it is given by the map

$$a(t)\mapsto[a(z)\otimes 1+1\otimes a(u), r_{\mathrm{trig}}(z,u)],$$

where the r-matrix is given by (4.11). By the standard procedure this gives a structure of Poisson group on $G((z^{-1}))$. We note that the Iwahori subgroup $\mathbf{Iw}_-\subset G((z^{-1}))$ is a Poisson subgroup, hence the bracket of any two functions on it is the restriction of the bracket of any extensions of these functions to $G((z^{-1}))$. The rest of the proof is a word-to-word repetition of that of [18, Proposition 2.13]. □

By an abuse of notation, we will denote the rational étale functions $w_{i,r}\circ s^{\lambda^*}_{\mu^*}\circ$ pr, $y_{i,r}\circ s^{\lambda^*}_{\mu^*}\circ$ pr (notations of Sect. 4.5) on $\mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*}$ simply by $w_{i,r}$, $y_{i,r}$.

Proposition 4.12 *We have*

$$\begin{aligned}\{w_{i,r},w_{j,s}\}_{\mathrm{trig}}&=0,\\ \{w_{i,r},y_{j,s}\}_{\mathrm{trig}}&=\check{d}_i\delta_{ij}\delta_{rs}w_{i,r}y_{j,s},\\ \{y_{i,r},y_{j,s}\}_{\mathrm{trig}}&=(1-\delta_{ij})(\check{\alpha}_i,\check{\alpha}_j)\frac{w_{i,r}+w_{j,s}}{2(w_{i,r}-w_{j,s})}y_{i,r}y_{j,s}.\end{aligned}$$

Proof Consider the functions $Q_i(z)=\Delta^{(0)}_{\check{\omega}_i,\check{\omega}_i}\prod_{r=1}^{a_i}(z-w_{i,r})$, $R_i(z)=\Delta^{(0)}_{\check{\omega}_i,\check{\omega}_i}\sum_{r=1}^{a_i}y_{i,r}\frac{Q_i(z)}{(z-w_{i,r})Q_i'(w_{i,r})}$. According to Proposition 4.6 we have $Q_i=\sum_{s=0}^{a_i}\Delta^{(s)}_{\check{\omega}_i,\check{\omega}_i}z^{a_i-s}$, $R_i=\sum_{s=0}^{a_i}\Delta^{(s)}_{s_i\check{\omega}_i,\check{\omega}_i}z^{a_i-s}$.

Set $B_i:=\Delta^{(0)}_{\check{\omega}_i,\check{\omega}_i}$ and recall the generalized minors $S_{ij}(z)$ introduced in the proof of Theorem 4.9: $S_{ij}(z):=\Delta_{E_jE_iv_{-\check{\omega}_i},v_{\check{\omega}_i}}=\langle E_jE_iv_{-\check{\omega}_i},g(z)v_{\check{\omega}_i}\rangle$. Then by Proposition 4.11 we have

$$\{Q_i(z),Q_j(u)\}_{\mathrm{trig}}=0, \tag{4.12}$$
$$\{Q_i(z),R_j(u)\}_{\mathrm{trig}}=-\check{d}_i\delta_{ij}\left(\frac{z+u}{2(z-u)}Q_i(z)R_j(u)-\frac{u}{z-u}R_i(z)Q_j(u)\right), \tag{4.13}$$
$$\begin{aligned}\{R_i(z),R_j(u)\}_{\mathrm{trig}}=&(1-\delta_{ij})((\check{\alpha}_i,\check{\alpha}_j)\frac{z+u}{2(z-u)}R_i(z)R_j(u)\\ &+\check{d}_i\check{d}_j\frac{z}{z-u}Q_i(z)S_{ji}(u)+\check{d}_i\check{d}_j\frac{u}{z-u}S_{ij}(z)Q_j(u)).\end{aligned} \tag{4.14}$$

The relation $\{w_{i,r}, w_{j,s}\}_{\mathrm{trig}} = 0$ is obvious from (4.12). Substituting $u = w_{j,s}$ to (4.13), we get $\{B_i, y_{j,s}\}_{\mathrm{trig}} = -\frac{\check{d}_i\delta_{ij}}{2} B_i y_{j,s}$ and $\{w_{i,r}, y_{j,s}\}_{\mathrm{trig}} = \check{d}_i\delta_{ij}\delta_{rs}w_{i,r}y_{j,s}$. Finally, substituting $z = w_{i,r}, u = w_{j,s}$ to (4.14), we get $\{y_{i,r}, y_{j,s}\}_{\mathrm{trig}} = (1 - \delta_{ij})(\check{\alpha}_i, \check{\alpha}_j)\frac{w_{i,r}+w_{j,s}}{2(w_{i,r}-w_{j,s})} y_{i,r}y_{j,s}$. □

Theorem 4.13 *The isomorphism* $s^{\lambda^*}_{\mu^*} \circ \mathrm{pr}\colon \mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*} \xrightarrow{\sim} {}^{\dagger}\mathring{Z}^\alpha$ *of Proposition 4.3 is a symplectomorphism.*

Proof Indeed, by Proposition 4.12 and Proposition 3.19 the Poisson brackets $\{\cdot,\cdot\}$ on ${}^{\dagger}\mathring{Z}^\alpha$ and $\{\cdot,\cdot\}_{\mathrm{trig}}$ on $\mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*}$ are given by the same formulas on coordinate functions $w_{i,r},\ y_{i,r}$. □

Remark 4.14 Note that the formulas (4.13) and (4.14) are different from (3.24) and (3.25), so the morphism $s^{\lambda^*}_{\mu^*} \circ \mathrm{pr}\colon \mathcal{F}\ell^{w_0\times\lambda^*}_{w_0\times\mu^*} \xrightarrow{\sim} {}^{\dagger}\mathring{Z}^\alpha$ *does not* extend to a Poisson morphism $\mathbf{Iw}_- \to \mathring{Y}^\alpha$.

5 A speculation on cluster structure

5.1 An affine Lie algebra

Let $\widehat{\mathfrak{g}}$ be the universal central extension of the polynomial loop algebra $\mathfrak{g}[z^{\pm 1}]$:

$$0 \to \mathbb{C} \to \widehat{\mathfrak{g}} \to \mathfrak{g}[z^{\pm 1}] \to 0 \tag{5.1}$$

Let $\mathfrak{g}_{\mathrm{aff}} = \widehat{\mathfrak{g}} \rtimes \mathbb{C}d$ be the semidirect product of $\widehat{\mathfrak{g}}$ with the degree operator. Then $\mathfrak{g}_{\mathrm{aff}}$ is an untwisted affine Kac-Moody Lie algebra. It has a triangular decomposition $\mathfrak{g}_{\mathrm{aff}} = \mathfrak{n}_- \oplus \mathfrak{t}_{\mathrm{aff}} \oplus \mathfrak{n}$ where $\mathfrak{n}_- = \mathrm{Lie}\,\mathbf{N}_- \cap \mathfrak{g}[z^{-1}],\ \mathfrak{n} = \mathrm{Lie}\,\mathbf{N} \cap \mathfrak{g}[z]$, and $\mathfrak{t}_{\mathrm{aff}}$ is the affine Cartan subalgebra. The fundamental weights will be denoted $\varpi_i,\ i \in I_a := I \sqcup \{i_0\}$. The corresponding fundamental integrable representations (where $\mathfrak{n}$ acts locally nilpotently) will be denoted V_{ϖ_i}, and their restricted duals (where $\mathfrak{n}_-$ acts locally nilpotently) will be denoted $V^*_{\varpi_i}$. We choose the highest weight vectors $v_{\varpi_i} \in V_{\varpi_i}$ and the lowest weight vectors $v_{-\varpi_i} \in V^*_{\varpi_i}$ such that $\langle v_{-\varpi_i}, v_{\varpi_i}\rangle = 1$. Note that the action of $\mathfrak{n}_-$ (resp. $\mathfrak{n}$) on $V^*_{\varpi_i}$ (resp. V_{ϖ_i}) integrates to the action of $\mathbf{N}_-$ (resp. $\mathbf{N}$). Given $w, y \in W_a$ and $i \in I_a$ we define the following regular function on $\mathbf{N}_-$ (a generalized minor):

$$\Delta_{w\varpi_i, y\varpi_i}(g) := \langle \overline{w}v_{-\varpi_i}, g\overline{y}v_{\varpi_i}\rangle \tag{5.2}$$

where $\overline{w}, \overline{y} \in G_{\mathrm{aff}}$ are the lifts of w, y defined similarly to [18, 2A].

5.2 An initial seed

B. Leclerc defines in [20] a cluster structure on the open Richardson varieties in the flag varieties of simple Lie algebras of types ADE, but presumably the construction can be extended to the affine Lie algebras of arbitrary types. Here we describe the initial seed for $\mathcal{F}\ell_{w_0}^{w_0\times\lambda}$ following [20, Sect. 5, 4.8.3, Corollary 4.4]. We choose a reduced expression in the affine Weyl group: $\lambda = s_{i_1}\dots s_{i_l}$ where $i_1,\dots,i_l \in I_a$, and $l = 2|\lambda|$ is the length of $\lambda \in \Lambda^+ \subset \Lambda_+ \subset \Lambda \subset W_a$ (for $\lambda = \sum_{i\in I} a_i\alpha_i$ we have $2|\lambda| = 2\sum_{i\in I} a_i$). Note that $i_1 = i_0$ (the affine simple reflection). Then the initial seed consists of all the (irreducible factors of the) generalized minors $\Delta_{w_0 s_{i_1}\cdots s_{i_r}\varpi_{i_r}, w_0\varpi_{i_r}}$, $1 \le r \le l$ (they are well defined as functions on $\mathcal{F}\ell_{w_0}^{w_0\times\lambda}$ according to *loc. cit.*). Among them, those which divide $\prod_{i\in I_a}\Delta_{(w_0\times\lambda)\varpi_i, w_0\varpi_i}$ are the frozen variables.

5.3 An exchange matrix

The rows of the exchange matrix B are numbered by $1 \le r \le l$, and the columns are numbered by those $1 \le s \le l$ for which there exists $r > s$ such that $i_r = i_s$ (the minimal among such r is denoted s^+). The matrix entries are as follows: $b_{s,s^+} = -b_{s^+,s} = -1$; and for $s < r < s^+$ such that for any $r < r' < s^+$ we have $i_r \ne i_{r'}$, the matrix entry $b_{s,r} = -C_{i_s,i_r}$, and $b_{r,s} = C_{i_r,i_s}$ (here $(C_{i,j})_{i,j\in I_a}$ is the Cartan matrix of $\widehat{\mathfrak{g}}$). All the other matrix entries are zero.

According to [20, Sect. 6], this cluster structure on $\mathcal{F}\ell_{w_0}^{w_0\times\lambda}$ is compatible with the symplectic structure of Sect. 4.10 on $\mathcal{F}\ell_{w_0}^{w_0\times\lambda}$ in the sense of [11, Sect. 4.1]. Taking $\mu = 0$ and $\alpha = \lambda$, and transferring the cluster structure via the isomorphism $s_0^{\lambda^*}\circ \mathrm{pr}\colon \mathcal{F}\ell_{w_0}^{w_0\times\lambda^*} \xrightarrow{\sim} {}^\dagger\overset{\circ}{Z}{}^\alpha$ we obtain a cluster structure on ${}^\dagger\overset{\circ}{Z}{}^\alpha$ compatible with its symplectic structure (see Theorem 4.13).

5.4 Destabilization

Let $\nu \in \Lambda^+$ be a dominant coweight. Then the open Richardson variety $\mathcal{F}\ell_{w_0\times\nu}^{w_0\times(\lambda+\nu)}$ also has a cluster structure with the initial cluster given by certain generalized minors, and with the same exchange matrix as in Sect. 5.3. However, the stabilization map (4.4) $\sigma_0^\nu\colon \mathcal{F}\ell_{w_0}^{w_0\times\lambda} \xrightarrow{\sim} \mathcal{F}\ell_{w_0\times\nu}^{w_0\times(\lambda+\nu)}$ *does not* take the initial seed of $\mathcal{F}\ell_{w_0}^{w_0\times\lambda}$ to the initial seed of $\mathcal{F}\ell_{w_0\times\nu}^{w_0\times(\lambda+\nu)}$ (already in the simplest example of 2-dimensional slices for $\mathfrak{g} = \mathfrak{sl}_2$ where both variables are frozen, cf. [18, Example 2.12]).

We consider the following action of $\mathbb{Z}^I$ on ${}^\dagger\overset{\circ}{Z}{}^\alpha$: the generator $(0,\dots,0,1,0,\dots,0)$ (1 at the ith place) acts in the Plücker coordinates $(Q_j, R_j)_{j\in I}$ of Sect. 4.5 by an automorphism $\eta_i(Q_j, R_j) = (Q_j, z^{\delta_{ij}}R_j - \delta_{ij}r_{i,a_i-1}Q_j)$ (it is easy to check that this is indeed a biregular automorphism of ${}^\dagger\overset{\circ}{Z}{}^\alpha$). The frozen variables of the cluster structure on ${}^\dagger\overset{\circ}{Z}{}^\alpha$ are $(F_\alpha, q_{j,0})_{j\in I}$ where F_α is the equation of the boundary of zastava, see e.g. [4, Sect. 5]. Clearly, η_i takes $(F_\alpha, q_{j,0})_{j\in I}$ to $(F_\alpha q_{i,0}^{d_i}, q_{j,0})_{j\in I}$, i.e. does not preserve the frozen variables. However, it seems likely that η_i is an *almost cluster* transformation: a composition of a few mutations, and the above change of frozen variables. Furthermore, if we set $\eta_\nu := \prod_{i\in I}\eta_i^{n_i}$ for $\nu = \sum_{i\in I} n_i\alpha_i$, then the cluster structure transferred to ${}^\dagger\overset{\circ}{Z}{}^\lambda$ from the isomorphism $s_{\nu^*}^{\lambda^*+\nu^*}\circ\mathrm{pr}\colon \mathcal{F}\ell_{w_0\times\nu^*}^{w_0\times(\lambda^*+\nu^*)} \xrightarrow{\sim} {}^\dagger\overset{\circ}{Z}{}^\lambda$ differs from the reference one (transferred from the isomorphism $s_0^{\lambda^*}\circ\mathrm{pr}\colon \mathcal{F}\ell_{w_0}^{w_0\times\lambda^*} \xrightarrow{\sim} {}^\dagger\overset{\circ}{Z}{}^\lambda$) by the automorphism η_{ν^*}.

5.5 $\mathfrak{g} = \mathfrak{sl}_2$

For $\mathfrak{g} = \mathfrak{sl}_2$, a positive coroot α is but a positive integer a, and a cluster structure on ${}^\dagger\mathring{Z}^a$ was defined in [12, Sect. 5] (where ${}^\dagger\mathring{Z}^a$ is denoted $\mathcal{R}_a$). According to Theorem 5.7, this cluster structure is a particular case of the one of Sect. 5.4. In particular, the exchange matrix $B(\varepsilon)$, $(\varepsilon) = (2, 0, \dots, 0)$ of [12, (5.16)] coincides with the exchange matrix of Sect. 5.3. Note that the cluster variables of [12] are certain minors of a Hankel matrix composed of the coefficients of the formal series $R(z)/Q(z) \in \mathbb{C}[[z^{-1}]]$ (where R, Q are the Plücker coordinates of Sect. 4.5). It would be nice to have such an explicit formula for the cluster variables for general $\mathfrak{g}$. Also, the automorphism η of Sect. 5.4 is nothing but the *shift* of [12, Lemma 5.4.(i)] (a transformation from the type A_{a-1} Q-system, cf. the paragraph before [12, Remark 6.2]).

5.6 Gaiotto–Witten superpotential

Let $K_i(z)$, $i \in I$, be a collection of monic polynomials, $K_i(z) = z^{l_i} + \kappa_{i,l_i-1}z^{l_i-1} + \cdots + \kappa_{i,0}$. The data of $\{K_i(z)\}_{i\in I}$ is equivalent to the data of

(a) an ordered collection Λ of dominant coweights $\lambda_1, \dots, \lambda_N$;
(b) an ordered configuration $(z_1, \dots, z_N)$ of points in $\mathbb{A}^1$.

Namely, given the above data we set $K_i(z) := \prod_{1\le n\le N}(z - z_n)^{\langle\lambda_n,\check{\alpha}_i\rangle}$. We denote by $\mathring{\mathbb{A}}^\Lambda$ the moduli space of the above configurations of *distinct* points z_n.

Recall the Gaiotto–Witten superpotential $\mathcal{W}_-^{\Lambda,\alpha}$: a multivalued holomorphic function on $\mathfrak{h}^\vee \times \mathring{Z}^\alpha \times \mathring{\mathbb{A}}^\Lambda$ (see e.g. [4, 1.8]). We will denote by $\overline{\mathcal{W}}_-^{\Lambda,\alpha}$ the restriction of $\mathcal{W}_-^{\Lambda,\alpha}$ to $0 \times \mathring{Z}^\alpha \times \mathring{\mathbb{A}}^\Lambda$. In the coordinates $w_{i,r}$, $y_{i,r}$ of Sect. 4.5 we have

$$\overline{\mathcal{W}}_-^{\Lambda,\alpha}(\underline{w}, \underline{y}, \underline{K}) = \sum_{i,r} \frac{y_{i,r}K_i(w_{i,r})}{Q_i'(w_{i,r})} - \log F_\alpha + \sum_{1\le m<n\le N} \lambda_m \cdot \lambda_n \log(z_m - z_n) \quad (5.3)$$

where F_α is the equation of the zastava boundary $\partial Z^\alpha = Z^\alpha \setminus \mathring{Z}^\alpha$ (see e.g. [4, Sect. 5]). Let $\mathbb{C}[[z^{-1}]] \ni \frac{R_i(z)}{Q_i(z)} =: \sum_{p=0}^\infty h_{i,p}z^{-p-1}$. Then

$$\overline{\mathcal{W}}_-^{\Lambda,\alpha}(\underline{w}, \underline{y}, \underline{K}) = \sum_{i,p} h_{i,p}\kappa_{i,p} - \log F_\alpha + \sum_{1\le m<n\le N} \lambda_m \cdot \lambda_n \log(z_m - z_n) \quad (5.4)$$

In case $\mathfrak{g} = \mathfrak{sl}_2$, the boundary equation F_a is a frozen cluster variable of the cluster structure on ${}^\dagger\mathring{Z}^a$ of [12, Sect. 5], and all the coefficients h_p are cluster variables according to [12, Lemma 5.3, Proposition 5.4]. Hence $\overline{\mathcal{W}}_-^{\Lambda,\alpha}|_{K=K_0}$ is a constant $\ell(K_0) := \sum_{1\le m<n\le N} \lambda_m \cdot \lambda_n \log(z_{0,m} - z_{0,n})$ plus a totally positive function on ${}^\dagger\mathring{Z}^a$ for a monic polynomial K_0 with nonnegative coefficients κ_p. We expect a similar positivity for general $\mathfrak{g}$ (in particular, F_α is a frozen cluster variable). It would be

interesting to study its tropicalization and the corresponding set of positive integral tropical points, cf. [13].

Appendix: $G = SL_2$: identification with the cluster structure of [12] GALYNA DOBROVOLSKA

Recall that $\widehat{\mathfrak{sl}}_2$ has two fundamental representations, which we denote by V_{ϖ_1} and V_{ϖ_0} in accordance with the notation of Sect. 5.1. Recall the generalized minors $\Delta_{w_0 s_{i_1}\dots s_{i_r}\varpi_{i_r}, w_0\varpi_{i_r}}$ from Sect. 5.2. Since for $\mathfrak{g} = \mathfrak{sl}_2$ we have $w_0 = s_1$, these generalized minors are of the form $\Delta_{s_1(s_0s_1)^r\varpi_1, s_1\varpi_1}$ and $\Delta_{s_1(s_0s_1)^r\varpi_0, s_1\varpi_0}$.

Given a pair of polynomials $Q(z) = z^a + q_{a-1}z^{a-1} + \dots + q_0,\ R(z) = r_{a-1}z^{a-1} + r_{a-2}z^{a-2} + \dots + r_0$ representing a point of ${}^{\dagger}\mathring{Z}^a$, we can find a unique pair of polynomials $F(z) = z^a + f_{a-1}z^{a-1} + \dots + f_0,\ D(z) = d_{a-1}z^{a-1} + \dots + d_1 z + d_0$ such that $QF - RD = z^{2a}$. Then both the matrix $g(Q,R) := \begin{pmatrix} z^{-a}F(z) & z^{-a}D(z) \\ z^{-a}R(z) & z^{-a}Q(z) \end{pmatrix} \in \widehat{SL}_2$ and its inverse matrix $g(Q,R)^{-1} = \begin{pmatrix} z^{-a}Q(z) & -z^{-a}D(z) \\ -z^{-a}R(z) & z^{-a}F(z) \end{pmatrix} \in \widehat{SL}_2$ actually lie in $\mathbf{N}_- \subset \widehat{SL}_2$ (notations of Sect. 5.1). Moreover, according to Proposition 4.6, we have $s_0^{\lambda} \circ \mathrm{pr}(\overline{g(Q,R)}) = (Q,R)$. Here $\lambda = a\alpha$ is a multiple of the simple coroot of $\mathfrak{sl}_2$, and $\overline{g(Q,R)} \in \mathcal{F}\ell_{s_1}^{s_1 \times a\alpha} \subset \mathbf{Fl}_{s_1} = \mathbf{N}_- \cdot s_1$ is the point $g(Q,R) \cdot s_1$.

On the other hand, following [12], we consider the Taylor expansion at $\infty \in \mathbb{P}^1$ of $\frac{R(z)}{Q(z)} = \frac{c_0}{z} + \frac{c_1}{z^2} + \dots + \frac{c_j}{z^j} + \cdots$. We form the corresponding Hankel matrix using the elements $c_0, \dots, c_{2a-2}$, namely the $a \times a$ matrix $[c_{j+k}]_{j,k=0}^{a-1}$. We consider two kinds of minors of this matrix, the principal minors $\mathsf{C}_1, \dots, \mathsf{C}_a$ of sizes $1, \dots, a$, and the minors $\mathsf{D}_1, \dots, \mathsf{D}_{a-1}$ of sizes $1, \dots, a-1$ which are obtained from the principal minors of the same size by a shift of all entries by one unit to the right (or, equivalently, by a shift of all entries by one unit down). We will also denote these minors by $\mathsf{C}_r(Q,R)$, $\mathsf{D}_r(Q,R)$ when we want to stress the dependence on Q, R. These Hankel minors (also called Hankel determinants) arise as cluster variables in the cluster corresponding to $(\varepsilon) = (2, 0, \dots, 0)$ in [12, 5.2]. See also the survey [14] for general properties of Hankel matrices.

In this appendix we prove the following theorem:

Theorem 5.7 *(a) The generalized minor $\Delta_{s_1(s_0s_1)^r\varpi_1, s_1\varpi_1}(g(Q,R))$ is equal (up to a change of sign) to the Hankel minor $\mathsf{C}_r(Q,R)$.*

(b) The generalized minor $\Delta_{s_1(s_0s_1)^r\varpi_0, s_1\varpi_0}(g(Q,R))$ is equal (up to a change of sign) to the Hankel minor $\mathsf{D}_r(Q,R)$.

Before starting the proof of Theorem 5.7, we will recall a theorem of Kronecker which we will use.

First, for two polynomials $Q(z) = z^a + q_{a-1}z^{a-1} + \dots + q_0,\ R(z) = r_{a-1}z^{a-1} + r_{a-2}z^{a-2} + \dots + r_0$ we will write the $(2a-1) \times (2a-1)$ Sylvester matrix (the determinant of which computes the resultant of Q and R) in the following form:

$$\begin{bmatrix} 1 & q_{a-1} & \dots & \dots & q_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & q_{a-1} & \dots & q_1 & q_0 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & & & & & \ddots & \\ 0 & 0 & \dots & 1 & q_{a-1} & \dots & & \dots & q_1 & q_0 \\ 0 & 0 & \dots & 0 & r_{a-1} & r_{a-2} & \dots & \dots & r_1 & r_0 \\ 0 & 0 & \dots & r_{a-1} & r_{a-2} & \dots & & \dots & r_0 & 0 \\ \vdots & & & \dots & & & & & & \vdots \\ 0 & r_{a-1} & \dots & \dots & r_0 & 0 & \dots & & \dots & 0 \\ r_{a-1} & r_{a-2} & \dots & r_0 & 0 & 0 & \dots & & \dots & 0 \end{bmatrix}$$

Next we define for each i an odd sub-resultant R_i (which coincides with the notion of sub-resultant in [25, (2.6)]) to be the minor of the Sylvester matrix which is obtained by removing the same number i of columns and rows at the top, the bottom, the right, and the left. We also define an even sub-resultant S_i to be the minor of the Sylvester matrix obtained by removing the middle row, the same number i of rows at the top and the bottom, and removing i columns at the left, and $i+1$ columns at the right.

Now we can state the following formula of Kronecker [19] (cf. [25, Corollary 3.2] for a modern reference) expressing sub-resultants in terms of Hankel determinants of the Taylor expansion of the ratio of two polynomials:

Proposition 5.8 *(L. Kronecker)* $\mathsf{R}_i = \mathsf{C}_{a-i}$.

We will also recall some facts from the theory of infinite-dimensional Lie algebras which we will need in the course of the proof (we will follow the exposition in [17] and use the notation of *loc. cit.*).

We start with an infinite-dimensional vector space vector space $V = \oplus_j \mathbb{C} v_j$. For each $m \in \mathbb{Z}$ we have the infinite-dimensional vector space $F^{(m)}$ with a vacuum vector $\psi_m = v_m \wedge v_{m-1} \wedge \cdots$ and a basis given by $v_{i_0} \wedge v_{i_{-1}} \wedge \cdots$ (such that $i_0 > i_{-1} > \cdots$ and $i_k = k+m$ for $k \ll 0$). We define a group GL_∞ as the group of invertible infinite matrices with entries $a_{i,j}$ ($i, j \in \mathbb{Z}$) such that all but finitely many of $a_{i,j} - \delta_{i,j}$ are zero. The group GL_∞ acts in $F^{(m)}$ as follows: $A(v_{i_0} \wedge v_{i_{-1}} \wedge \dots) = Av_{i_0} \wedge Av_{i_{-1}} \wedge \dots = \sum_{j_0 > j_{-1} > \dots} \det A^{i_0, i_{-1}, \dots}_{j_0, j_{-1}, \dots} v_{j_0} \wedge v_{j_{-1}} \wedge \dots$, where $A^{i_0, i_{-1}, \dots}_{j_0, j_{-1}, \dots}$ denotes the matrix located on the intersection of the rows $j_0, j_{-1}, \dots$ and columns $i_0, i_{-1}, \dots$ of the matrix A.

Consider the standard n-dimensional representation U of $\mathfrak{sl}_n$ with basis $u_1, u_2, \dots, u_n$ (in this appendix we will only use $n = 2$). Note that according to [17, (9.8)], one can define an action of $\widehat{\mathfrak{sl}}_n$ on V in the following way, where an element $A = \sum_i A_i z^i \in \widehat{\mathfrak{sl}}_n$ acts as the infinite matrix below (note that this action is obtained from the representation $U[z^{\pm 1}]$ of $\widehat{\mathfrak{sl}}_n$ by identifying a basis element v_j of V with $z^k \cdot u_r$ where $j = kn + r$, and $r \in \{1, 2, \dots, n\}$):

$$\begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & A_{-1} & A_0 & A_1 & \dots & \dots \\ \dots & \dots & A_{-1} & A_0 & A_1 & \dots \\ \dots & \dots & \dots & A_{-1} & A_0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Note that the images of the matrices in $\widehat{\mathfrak{sl}}_n$ obtained in this way have finitely many non-zero diagonals. Therefore by [17, Sect. 4.4] the action of $\widehat{\mathfrak{sl}}_n$ in $F^{(m)}$ is given by the same formula as for GL_∞, namely $A(v_{i_0} \wedge v_{i_{-1}} \wedge \cdots) = \sum_{j_0 > j_{-1} > \ldots} \det A^{i_0, i_{-1}, \ldots}_{j_0, j_{-1}, \ldots} v_{j_0} \wedge v_{j_{-1}} \wedge \cdots$ for $A \in \widehat{\mathfrak{sl}}_n$. This way for $m = 0, 1, 2, \ldots, n-1$ we obtain all the fundamental representations V_{ϖ_m}, where V_{ϖ_m} is the irreducible sub-representation of $\widehat{\mathfrak{sl}}_n$ in $F^{(m)}$ which is generated by the vacuum vector $\psi_m = v_m \wedge v_{m-1} \wedge \cdots$.

Finally, note that the action of $\mathfrak{n}_-$ in $F^{(m)}$ is not integrable (in general, $g v_{i_0} \wedge v_{i_{-1}} \wedge \cdots$ is an infinite sum for an element $g \in \mathbf{N}_-$). However, for any basis element $v_{j_0} \wedge v_{j_{-1}} \wedge \cdots$ of $F^{(m)}$ its coefficient in the (infinite) expansion of $g v_{i_0} \wedge v_{i_{-1}} \wedge \cdots$ in the elements of the basis of $F^{(m)}$ is well-defined and can be computed by the same formula as for $g \in GL_\infty$, namely it is equal to $\det A^{i_0, i_{-1}, \ldots}_{j_0, j_{-1}, \ldots}$. Note that here we caculate $\det A^{i_0, i_{-1}, \ldots}_{j_0, j_{-1}, \ldots}$ in the following way. By definition there exists N such that for $k < N$ we have $j_k = i_k = k + m$; then $\det A^{i_0, i_{-1}, \ldots}_{j_0, j_{-1}, \ldots} = \det A^{i_0, i_{-1}, \ldots, j_N}_{j_0, j_{-1}, \ldots, j_N}$, which is a finite determinant. The justification for this formula is that the infinite matrix with rows $j_0, j_{-1}, \ldots$ and columns $i_0, i_{-1}, \ldots$ can be divided into four blocks, where the two diagonal blocks are the finite block with rows $j_0, j_{-1}, \ldots, j_N$ and columns $i_0, i_{-1}, \ldots, i_N$ and the inifinite lower-triangular block with 1's on the diagonal with rows $j_{N-1}, j_{N-2}, \ldots$ and columns $i_{N-1}, i_{N-2}, \ldots$.

Now we are ready to prove our theorem.

Proof of Theorem 5.7 Using exterior powers, the computation of generalized minors reduces to the computation of finite minors because the infinite matrices we use are upper triangular up to a finite portion. As a result we obtain finite minors which stand in certain rows and columns of the infinite matrix, depending on the element of the Weyl group which appears in the definition of a particular generalized minor.

For example, for the element of the Weyl group given by $s_0 s_1 s_0 s_1$ and the fundamental representation V_{ϖ_1}, we obtain the following minors:

$$
\begin{array}{c|cccccccccccccc}
 & -5 & -4 & \mathbf{-3} & \mathbf{-2} & -1 & \mathbf{0} & 1 & \mathbf{2} & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
-5 & 1 & & & & & & & & & & & & & \\
-4 & 0 & 1 & & & & & & & & & & & & \\
-3 & f_{a-1} & d_{a-1} & 1 & & & & & & & & & & & \\
\mathbf{-2} & r_{a-1} & q_{a-1} & \mathbf{0} & \mathbf{1} & & & & & & & & & & \\
-1 & f_{a-2} & d_{a-2} & f_{a-1} & d_{a-1} & 1 & & & & & & & & & \\
\mathbf{0} & r_{a-2} & q_{a-2} & \mathbf{r_{a-1}} & \mathbf{q_{a-1}} & \mathbf{0} & \mathbf{1} & & & & & & & & \\
1 & f_{a-3} & d_{a-3} & f_{a-2} & d_{a-2} & f_{a-1} & d_{a-1} & 1 & & & & & & & \\
\mathbf{2} & r_{a-3} & q_{a-3} & \mathbf{r_{a-2}} & \mathbf{q_{a-2}} & \mathbf{r_{a-1}} & \mathbf{q_{a-1}} & 0 & \mathbf{1} & & & & & & \\
3 & f_{a-4} & d_{a-4} & f_{a-3} & d_{a-3} & f_{a-2} & d_{a-2} & f_{a-1} & d_{a-1} & 1 & & & & & \\
\mathbf{4} & r_{a-4} & q_{a-4} & \mathbf{r_{a-3}} & \mathbf{q_{a-3}} & \mathbf{r_{a-2}} & \mathbf{q_{a-2}} & r_{a-1} & \mathbf{q_{a-1}} & 0 & 1 & & & & \\
5 & f_{a-5} & d_{a-5} & f_{a-4} & d_{a-4} & f_{a-3} & d_{a-3} & f_{a-2} & d_{a-2} & f_{a-1} & d_{a-1} & 1 & & & \\
\mathbf{6} & r_{a-5} & q_{a-5} & \mathbf{r_{a-4}} & \mathbf{q_{a-4}} & \mathbf{r_{a-3}} & \mathbf{q_{a-3}} & r_{a-2} & \mathbf{q_{a-2}} & r_{a-1} & q_{a-1} & 0 & 1 & & \\
7 & f_{a-6} & d_{a-6} & f_{a-5} & d_{a-5} & f_{a-4} & d_{a-4} & f_{a-3} & d_{a-3} & f_{a-2} & d_{a-2} & f_{a-1} & d_{a-1} & 1 & \\
8 & r_{a-6} & q_{a-6} & r_{a-5} & q_{a-5} & r_{a-4} & q_{a-4} & r_{a-3} & q_{a-3} & r_{a-2} & q_{a-2} & r_{a-1} & q_{a-1} & 0 & 1
\end{array}
$$

After we collect the entries at the intersections of the marked rows and columns, we obtain the following submatrix (which after transposing it and exchanging the order and the signs of some the rows will be exactly the $(a-3)$-th sub-resultant R_{a-3} for the polynomials Q and R):

$$\begin{bmatrix} 0 & 1 & & & \\ r_{a-1} & q_{a-1} & 0 & 1 & \\ r_{a-2} & q_{a-2} & r_{a-1} & q_{a-1} & 1 \\ r_{a-3} & q_{a-3} & r_{a-2} & q_{a-2} & q_{a-1} \\ r_{a-4} & q_{a-4} & r_{a-3} & q_{a-3} & q_{a-2} \end{bmatrix}$$

We see that the finite minors we obtain up to permutation of rows and transposition are exactly the sub-resultants in the case of odd number of rows. Indeed, for the element of the Weyl group given by $(s_0 s_1)^r$ and the fundamental representation V_{ϖ_1} we obtain that the finite minor consists of $2r+1$ rows numbered $2r+2, 2r, \dots, -2r+2$ and $2r+1$ columns numbered $2, 0, -1, \dots, -2r+2$. If we permute the columns so that the odd rows stand on the left in the same order as they were before and even rows stand on the right in the same order as they were before, we obtain exactly an odd sub-resultant as defined above. Now we use the above theorem of Kronecker to conclude that the generalized minors for the fundamental representation V_{ϖ_1} of $\widehat{\mathfrak{sl}}_2$ are equal to the corresponding principal Hankel minors.

Now we turn to the generalized minors of the basic representation V_{ϖ_0}. For example, for the element $s_0 s_1 s_0 s_1$ of the Weyl group and the fundamental representation V_{ϖ_0} we have the following submatrix:

$$\begin{array}{r|cccccccccccccc|}
 & -5 & -4 & \mathbf{-3} & \mathbf{-2} & \mathbf{-1} & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
-5 & 1 & & & & & & & & & & & & & \\
-4 & 0 & 1 & & & & & & & & & & & & \\
-3 & f_{a-1} & d_{a-1} & 1 & & & & & & & & & & & \\
\mathbf{-2} & r_{a-1} & q_{a-1} & \mathbf{0} & \mathbf{1} & & & & & & & & & & \\
-1 & f_{a-2} & d_{a-2} & f_{a-1} & d_{a-1} & 1 & & & & & & & & & \\
\mathbf{0} & r_{a-2} & q_{a-2} & \mathbf{r_{a-1}} & \mathbf{q_{a-1}} & \mathbf{0} & \mathbf{1} & & & & & & & & \\
1 & f_{a-3} & d_{a-3} & f_{a-2} & d_{a-2} & f_{a-1} & d_{a-1} & 1 & & & & & & & \\
\mathbf{2} & r_{a-3} & q_{a-3} & \mathbf{r_{a-2}} & \mathbf{q_{a-2}} & \mathbf{r_{a-1}} & \mathbf{q_{a-1}} & 0 & 1 & & & & & & \\
3 & f_{a-4} & d_{a-4} & f_{a-3} & d_{a-3} & f_{a-2} & d_{a-2} & f_{a-1} & d_{a-1} & 1 & & & & & \\
\mathbf{4} & r_{a-4} & q_{a-4} & \mathbf{r_{a-3}} & \mathbf{q_{a-3}} & \mathbf{r_{a-2}} & \mathbf{q_{a-2}} & r_{a-1} & q_{a-1} & 0 & 1 & & & & \\
5 & f_{a-5} & d_{a-5} & f_{a-4} & d_{a-4} & f_{a-3} & d_{a-3} & f_{a-2} & d_{a-2} & f_{a-1} & d_{a-1} & 1 & & & \\
6 & r_{a-5} & q_{a-5} & r_{a-4} & q_{a-4} & r_{a-3} & q_{a-3} & r_{a-2} & q_{a-2} & r_{a-1} & q_{a-1} & 0 & 1 & & \\
7 & f_{a-6} & d_{a-6} & f_{a-5} & d_{a-5} & f_{a-4} & d_{a-4} & f_{a-3} & d_{a-3} & f_{a-2} & d_{a-2} & f_{a-1} & d_{a-1} & 1 & \\
8 & r_{a-6} & q_{a-6} & r_{a-5} & q_{a-5} & r_{a-4} & q_{a-4} & r_{a-3} & q_{a-3} & r_{a-2} & q_{a-2} & r_{a-1} & q_{a-1} & 0 & 1
\end{array}$$

After we collect the entries at the intersections of the marked rows and columns we obtain the following submatrix (which after transposing and exchanging the order of the rows is the $(a-3)$-th sub-resultant S_{a-3} for the polynomials Q and R):

$$\begin{bmatrix} 0 & 1 & & \\ r_{a-1} & q_{a-1} & 0 & 1 \\ r_{a-2} & q_{a-2} & r_{a-1} & q_{a-1} \\ r_{a-3} & q_{a-3} & r_{a-2} & q_{a-2} \\ r_{a-4} & q_{a-4} & r_{a-3} & q_{a-3} \end{bmatrix}$$

More generally, for the element of the Weyl group given by $(s_0 s_1)^r$ and the fundamental representation V_{ϖ_0} we obtain that the finite minor consists of $2r+1$ rows numbered $2r, 2r-2, \dots, -2r+2$ and $2r$ rows numbered $0, -1, \dots, -2r+1$. If we permute the columns so that the odd rows stand on the left in the same order as they were before and even rows stand on the right in the same order as they were before, we obtain exactly an even sub-resultant as defined above.

Finally, we can reduce the case of even number of rows to the case of odd number of rows in the following way. Note that the even sub-resultant S_i of the polynomials $R(z)$ and $Q(z)$ is equal to a usual sub-resultant (cf. [25, (2.6)]) of the polynomials $R(z)$ and $zQ(z)$. We claim that this (usual) sub-resultant of the polynomials $R(z)$ and $zQ(z)$ is equal to the determinant of the original Hankel minor D_{a-i-1}. Indeed, by the theorem of Kronecker mentioned above, the usual sub-resultants of $R(z)$ and $zQ(z)$ equal the corresponding (principal) Hankel minors for $R(z)$ and $zQ(z)$. To conclude, we notice that the equality $R(z)/Q(z) = z(R(z)/(zQ(z)))$ implies that the principal Hankel minors for $R(z)$, $zQ(z)$ are equal to the Hankel minors for $R(z)$, $Q(z)$ which are obtained by shifting the corresponding principal Hankel minors by one unit to the right. □

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