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Comultiplication for shifted Yangians and quantum open Toda lattice



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To David Kazhdan on his 70th birthday, with admiration

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ABSTRACT

We study a coproduct in type A quantum open Toda lattice in terms of a coproduct in the shifted Yangian of \mathfrak{sl}_2 . At the classical level this corresponds to the multiplication of scattering matrices of euclidean $SU(2)$ monopoles. We also study coproducts for shifted Yangians for any simply-laced Lie algebra.

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*In my youth I have multiplied too many matrices,
so now I try to avoid it if there is a way around.*

(D. Kazhdan)

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1. Introduction

1.1. The Toda lattice

Let $G \supset B \supset T$ be a reductive group with a Borel subgroup and a Cartan subgroup; let U be the unipotent radical of B , and let \mathfrak{n} be the Lie algebra of U . Let $\psi: U(\mathfrak{n}) \rightarrow \mathbb{C}$ be a regular character. Let $\mathcal{D}(G)$ be the ring of differential operators on G . The action of \mathfrak{n} by the left-invariant (resp. right-invariant) vector fields on G gives rise to the homomorphism $U(\mathfrak{n}) \otimes U(\mathfrak{n}) \rightarrow \mathcal{D}(G)$. The ring $\mathcal{T}(G)$ is defined as the quantum hamiltonian reduction $\mathcal{D}(G) \parallel (U \times U; \psi, -\psi)$. It comes equipped with a homomorphism from the ring $ZU(\mathfrak{g})$ of biinvariant differential operators on G . The action of $U \times U$ on the big Bruhat cell $C_{w_0} = U \cdot T \cdot \dot{w}_0 \cdot U$ is free, and the quantum hamiltonian reduction $\mathcal{D}(C_{w_0}) \parallel (U \times U; \psi, -\psi)$ is isomorphic to the ring $\mathcal{D}(T)$ of differential operators on T . Thus we obtain a localization homomorphism $\mathcal{T}(G) \hookrightarrow \mathcal{D}(T)$, and the composed embedding $ZU(\mathfrak{g}) \hookrightarrow \mathcal{T}(G) \hookrightarrow \mathcal{D}(T)$. This is the classical construction of the quantum open Toda lattice due to Kazhdan–Kostant.

At the quasiclassical level, we denote by $\mathfrak{Z}(G)$ the symplectic variety obtained by the hamiltonian reduction of the cotangent bundle of G : $\mathfrak{Z}(G) = T^*G \parallel (U \times U; \psi, -\psi)$. It is equipped with a lagrangian projection onto $\text{Spec } ZU(\mathfrak{g}) = \mathfrak{h}^*/W$ where $ZU(\mathfrak{g})$ is the Harish-Chandra center, \mathfrak{h} is the Lie algebra of T , and W is the Weyl group of (G, T) . Furthermore, $\mathfrak{Z}(G)$ contains an open symplectic subvariety T^*T (the cotangent bundle to the torus T), and thus we obtain the composed lagrangian projection $\pi_G: T^*T \hookrightarrow \mathfrak{Z}(G) \rightarrow \mathfrak{h}^*/W$ (Poisson commuting Toda hamiltonians).

1.2. Multiplicative structure

In case $G = GL(n)$, there is the following explicit construction of the Toda hamiltonians (see e.g. [7, Section 2] and references therein). Let t_1, \dots, t_n be the diagonal matrix elements coordinates on the diagonal torus $T \subset GL(n)$. Let w_1, \dots, w_n be the corresponding coordinates on the dual Lie algebra \mathfrak{h}^* . For $r = 1, \dots, n$ we consider the local Lax matrix $L_r(z) = \begin{pmatrix} z - w_r & t_r \\ -t_r^{-1} & 0 \end{pmatrix} \in SL(2, \mathbb{C}[z])$, and form the complete monodromy matrix $L(z) = L_1(z) \cdots L_n(z) = \begin{pmatrix} Q(z) & R'(z) \\ R(z) & Q'(z) \end{pmatrix}$. Then the Toda hamiltonians $\pi_{GL(n)}(t_1, w_1, \dots, t_n, w_n)$ are nothing but the coefficients of the polynomial $Q(z)$.¹

Our note stems from a simple observation that the above multiplicative structure of type A Toda hamiltonians arises from the associative multiplication $\mathfrak{Z}(GL(k)) \times$

¹ Realizing $Sp(2n)$ as the folding of $GL(2n)$ and identifying the Siegel Levi subgroup with $GL(n)$, we deduce that the Toda hamiltonians $\pi_{Sp(2n)}(t_1, w_1, \dots, t_n, w_n)$ are nothing but the (even degree) coefficients of $Q(z)$ where $\begin{pmatrix} Q(z) & R'(z) \\ R(z) & Q'(z) \end{pmatrix} = L_1(z) \cdots L_n(z) L'_n(z) \cdots L'_1(z)$, and $L'_r(z) = \begin{pmatrix} z + w_r & t_r^{-1} \\ -t_r & 0 \end{pmatrix} \in SL(2, \mathbb{C}[z])$, $r = 1, \dots, n$.

$\mathfrak{Z}(GL(l)) \rightarrow \mathfrak{Z}(GL(k+l))$ that can be quantized to a coassociative comultiplication $\mathcal{T}(GL(k+l)) \rightarrow \mathcal{T}(GL(k)) \otimes \mathcal{T}(GL(l))$. More generally, for any pair of Levi subgroups $T \subset M \subset L \subset G$ we have a homomorphism $\mathcal{T}(L) \rightarrow \mathcal{T}(M)$ satisfying the obvious transitivity relations, see Section 2.2. Turning back to the type A case, note that $\mathfrak{Z}(GL(n))$ is isomorphic to the open zastava space \hat{Z}^n of degree n based maps from (\mathbb{P}^1, ∞) to (\mathbb{P}^1, ∞) , aka moduli space of euclidean $SU(2)$ -monopoles of topological charge n . Under this isomorphism, the above complete monodromy matrix goes to the scattering matrix, and the Toda multiplication goes to the zastava multiplication [4, 2(vi, xi, xii)] which we learned of from D. Gaiotto and T. Dimofte, see Theorem 2.8.

1.3. Shifted Yangians

According to [4, Appendix B], the quantization $\mathcal{T}(GL(n))$ of $\mathbb{C}[\mathfrak{Z}(GL(n))]$ is a certain explicit quotient of the shifted Yangian $Y_{-2n}(\mathfrak{sl}_2)$. One of our main results is that the above comultiplication $\mathcal{T}(GL(k+l)) \rightarrow \mathcal{T}(GL(k)) \otimes \mathcal{T}(GL(l))$ descends from a comultiplication $Y_{-2k-2l}(\mathfrak{sl}_2) \rightarrow Y_{-2k}(\mathfrak{sl}_2) \otimes Y_{-2l}(\mathfrak{sl}_2)$, see Theorem 6.12.

Much of this paper is concerned with the study of comultiplication for shifted Yangians (beyond \mathfrak{sl}_2). In Theorem 4.12, we establish the existence of such a coproduct for any simply-laced Lie algebra \mathfrak{g} . This generalizes the coproduct on shifted \mathfrak{gl}_n -Yangians defined in [5, Theorem 11.9], which is analogous to the dominant type A case of our construction (see Remark 4.13). We also study the corresponding multiplicative structure on the classical limit of shifted Yangians, which are the moduli spaces \mathcal{W}_μ which we introduced in [4], see Section 5.

We must admit that the identification of the quantum Toda for $GL(n)$ and the shifted Yangian for \mathfrak{sl}_2 (purely algebraic objects) goes through a topological medium: equivariant homology of the affine Grassmannian of $GL(n)$. According to [4, Appendix A], the latter convolution ring has a natural representation in the difference operators on \mathfrak{h}^* . As a bonus we obtain a bispectrality result in Proposition 2.15.

1.4. Acknowledgments

This note can be viewed as an appendix to an appendix of [4]. Its existence is due to the generous and patient explanations by the authors of [4], D. Gaiotto, T. Dimofte, B. Feigin, R. Bezrukavnikov, P. Etingof, P. Zinn-Justin, A. Marshakov, N. Guay, S. Gautam and A. Tsymbaliuk. We are very grateful to all of them. We thank H. Nakajima for pointing out a mistake in earlier versions of Theorem 4.8 and Remark 4.7. We also thank the referee for helpful comments. The study of L.R. has been funded by the Russian Academic Excellence Project ‘5-100’. The research of M.F. was supported by the grant RSF-DFG 16-41-01013. The research of J.K. was funded by NSERC. This research was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation.

2. Quantum Toda for Levi subgroups

2.1. Quantum Toda lattice

Let $G \supset B \supset T$ be a reductive group with a Borel and Cartan subgroup. Let $T \subset B_- \subset G$ be the opposite Borel subgroup; let U (resp. U_-) be the unipotent radical of B (resp. B_-). The Lie algebra of G (resp. U_-) will be denoted by \mathfrak{g} (resp. \mathfrak{n}_-). Let $U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{n}_-)$ stand for the \hbar -universal enveloping algebras of $\mathfrak{g}, \mathfrak{n}_-$. Let $\psi: U_{\hbar}(\mathfrak{n}_-) \rightarrow \mathbb{C}[\hbar]$ be a homomorphism such that $\psi(f_{\alpha}) = 1$ for any simple root α (we fix a root generator $f_{\alpha} \in \mathfrak{n}_- \subset U_{\hbar}(\mathfrak{n}_-)$). Let $\mathcal{D}_{\hbar}(G)$ stand for the global sections of the sheaf of \hbar -differential operators on G : it is the smash product of $U_{\hbar}(\mathfrak{g})$ and $\mathbb{C}[G]$. The action of \mathfrak{n}_- by the left-invariant (resp. right-invariant) vector fields on G gives rise to the homomorphism l (resp. r): $U_{\hbar}(\mathfrak{n}_-) \rightarrow \mathcal{D}_{\hbar}(G)$. Let $I_{\psi} \subset \mathcal{D}_{\hbar}(G)$ be the left ideal generated by the \hbar -differential operators of the sort $l(x_1) - \psi(x_1) + r(x_2) + \psi(x_2)$, $x_1, x_2 \in U_{\hbar}(\mathfrak{n}_-)$. We consider the quantum hamiltonian reduction

$$\mathcal{T}_{\hbar}(G) := (\mathcal{D}_{\hbar}(G)/I_{\psi})^{U_- \times U_-}$$

where the first (resp. second) copy of U_- acts on G (and hence on $\mathcal{D}_{\hbar}(G)$) by the left (resp. right) translations: $(u_1, u_2) \cdot g := u_1 g u_2^{-1}$. It is an algebra containing the center $ZU_{\hbar}(\mathfrak{g})$ via the embedding $ZU_{\hbar}(\mathfrak{g}) \hookrightarrow \mathcal{D}_{\hbar}(G)$ as both left- and right-invariant \hbar -differential operators. This is the classical Kazhdan–Kostant construction of the quantum Toda lattice, see [14].

2.2. Comparison with the Toda lattice for a Levi subgroup

For each element w of the Weyl group $W = N_G(T)/T$ we choose its lift \dot{w} into the normalizer $N_G(T)$. Let w_0 be the longest element of W . The action of $U_- \times U_-$ on the big Bruhat cell $C_{w_0} := U_- \cdot T \cdot \dot{w}_0 \cdot U_- = U_- \cdot T \cdot \dot{w}_0^{-1} \cdot U_- \subset G$ is free, and hence the quantum hamiltonian reduction of $\mathcal{D}_{\hbar}(C_{w_0})$ is isomorphic to $\mathcal{D}_{\hbar}(T) = \mathcal{T}_{\hbar}(T)$, a certain localization of $\mathcal{T}_{\hbar}(G)$. Thus we have an embedding $ZU_{\hbar}(\mathfrak{g}) \hookrightarrow \mathcal{T}_{\hbar}(G) \hookrightarrow \mathcal{D}_{\hbar}(T)$.²

More generally, let $T \subset L \subset P \supset B$ be a Levi subgroup of a parabolic subgroup of G ; let $P_- \supset B_-$ be the opposite parabolic subgroup. We denote by $\mathfrak{l}, \mathfrak{p}, \mathfrak{p}_-$ the Lie algebras of L, P, P_- . We denote by U^L (resp. U_-^L) the intersection $L \cap U$ (resp. $L \cap U_-$), and we denote by U^P (resp. U_-^P) the unipotent radical of P (resp. P_-). Finally, we denote by $\mathfrak{n}_-^L, \mathfrak{n}_-^P, \mathfrak{n}^P$ the Lie algebras of U_-^L, U_-^P, U^P . We will also need the subgroups $U_{w_0}^P := \dot{w}_0 U^P \dot{w}_0^{-1}$, $U_{w_0}^L := \dot{w}_0 U^L \dot{w}_0^{-1}$ and their Lie algebras $\mathfrak{n}_{w_0}^P, \mathfrak{n}_{w_0}^L$. The restriction of ψ to $U_{\hbar}(\mathfrak{n}_-^P)$ (resp. $U_{\hbar}(\mathfrak{n}_{w_0}^P)$, $U_{\hbar}(\mathfrak{n}_-^L)$, $U_{\hbar}(\mathfrak{n}_{w_0}^L)$) will be denoted by ψ^P (resp. $\psi_{w_0}^P$, ψ^L , $\psi_{w_0}^L$). Let \dot{w}_0^L be a lift of w_0^L (the longest element in the parabolic Weyl subgroup $W_L \subset W$) into the normalizer $N_G(T)$. Let $C_{W_L w_0} \subset G$ be an affine open subvariety equal to the

² Quite often the term *quantum Toda lattice* refers to the composite embedding $ZU_{\hbar}(\mathfrak{g}) \hookrightarrow \mathcal{D}_{\hbar}(T)$.

union of all the Bruhat cells $C_{w w_0}$ where $w \in W_L$ (it is the preimage of the big Bruhat cell $(B \cdot \dot{w}_0 P \dot{w}_0^{-1})/\dot{w}_0 P \dot{w}_0^{-1}$ in the partial flag variety $G/\dot{w}_0 P \dot{w}_0^{-1}$). The action of $U_-^P \times U_{w_0}^P$ on $C_{W_L w_0}$ is free, and a closed embedding $\iota_L^G: L \hookrightarrow C_{W_L w_0}$, $g \mapsto g \dot{w}_0^L \dot{w}_0^{-1}$, is a cross section of this action giving rise to an isomorphism

$$L \times U_-^P \times U_{w_0}^P \xrightarrow{\sim} C_{W_L w_0}, \quad (g, x_1, x_2) \mapsto x_1 g \dot{w}_0^L \dot{w}_0^{-1} x_2^{-1}, \quad (2.1)$$

and hence to $L \xrightarrow{\sim} C_{W_L w_0}/(U_-^P \times U_{w_0}^P)$. Hence

$$(\mathcal{D}_{\hbar}(C_{W_L w_0})/I_{\psi^P, \psi_{w_0}^P})^{U_-^P \times U_{w_0}^P} \xrightarrow{\sim} \mathcal{D}_{\hbar}(L),$$

where $I_{\psi^P, \psi_{w_0}^P} \subset \mathcal{D}_{\hbar}(C_{W_L w_0})$ is the left ideal generated by the \hbar -differential operators of the sort $l(x_1) - \psi^P(x_1) + r(x_2) + \psi_{w_0}^P(x_2)$, $x_1 \in U_{\hbar}(\mathfrak{n}_{-}^P)$, $x_2 \in U_{\hbar}(\mathfrak{n}_{w_0}^P)$. Indeed, (2.1) gives rise to an isomorphism $\mathcal{D}_{\hbar}(C_{W_L w_0}) \xrightarrow{\sim} \mathcal{D}_{\hbar}(U_-^P) \otimes \mathcal{D}_{\hbar}(L) \otimes \mathcal{D}_{\hbar}(U_{w_0}^P)$, but $(\mathcal{D}_{\hbar}(U_-^P)/I_{\psi^P})^{U_-^P} \simeq \mathbb{C}$, and $(\mathcal{D}_{\hbar}(U_{w_0}^P)/I_{\psi_{w_0}^P})^{U_{w_0}^P} \simeq \mathbb{C}$.

Composing the above isomorphism with the restriction to the open subset $\mathcal{D}_{\hbar}(G) \rightarrow \mathcal{D}_{\hbar}(C_{W_L w_0})$ we obtain a homomorphism

$$(\mathcal{D}_{\hbar}(G)/I_{\psi^P, \psi_{w_0}^P})^{U_-^P \times U_{w_0}^P} \rightarrow \mathcal{D}_{\hbar}(L).$$

Furthermore, we obtain a composed homomorphism

$$\begin{aligned} \tau_G^L: \mathcal{T}_{\hbar}(G) &= \left((\mathcal{D}_{\hbar}(G)/I_{\psi^P, \psi_{w_0}^P})^{U_-^P \times U_{w_0}^P} / I_{\psi^L, \psi_{w_0}^L} \right)^{U_-^L \times U_{w_0}^L} \\ &\rightarrow (\mathcal{D}_{\hbar}(L)/I_{\psi^L, \psi_{w_0}^L})^{U_-^L \times U_{w_0}^L} = \mathcal{T}_{\hbar}(L). \end{aligned}$$

For a pair of Levi subgroups $M \subset L \subset G$ we have $\iota_M^G = \iota_L^G \circ \iota_M^L$, and hence $\tau_G^M = \tau_G^L \circ \tau_L^M$.

2.3. Type A

For $G = GL(n)$ we will denote $\mathcal{T}_{\hbar}(GL(n))$ by \mathcal{T}_{\hbar}^n for short. We view $GL(n)$ as the group of invertible $n \times n$ -matrices. Let T be the diagonal subgroup, and let U (resp. U_-) be the subgroup of lower (resp. upper) triangular matrices with 1's on the main diagonal. We choose a lift $W \rightarrow N_G(T)$, $w \mapsto \dot{w}$ representing each $w \in W = \mathfrak{S}_n$ by the corresponding permutation matrix. In particular, for any Levi L , we have $(\dot{w}_0^L)^2 = 1$. Let L be the subgroup of block matrices with blocks of sizes k, l such that $k + l = n$. Then $\tau_{k,l} := \tau_G^L$ is a homomorphism from $\mathcal{T}_{\hbar}(G) = \mathcal{T}_{\hbar}^{k+l}$ to $\mathcal{T}_{\hbar}(L) = \mathcal{T}_{\hbar}^k \otimes \mathcal{T}_{\hbar}^l$. We obtained a coassociative comultiplication on $\bigoplus_n \mathcal{T}_{\hbar}^n$.

Let $V_{\varpi_1} = V = \mathbb{C}^n$ be the standard (tautological) representation of $GL(n)$ and $V_{\varpi_1}^*$ be its dual representation. Let $v_1, \dots, v_n \in V_{\varpi_1}$ be the standard basis (so that $v_n \in V_{\varpi_1}^U$, $v_1 \in V_{\varpi_1}^{U_-}$). Let $v_1^*, \dots, v_n^* \in V_{\varpi_1}^*$ be the dual basis (so that $v_1^* \in (V_{\varpi_1}^*)^U$, $v_n^* \in (V_{\varpi_1}^*)^{U_-}$). Then the functions $\Delta^l(g) := \langle g v_1, v_n^* \rangle$ and $\Delta(g) := \langle g^{-1} v_1, v_n^* \rangle$ on G are

$U_- \times U_-$ -invariant hence survive after Hamiltonian reduction. We denote their images in \mathcal{T}_h^n by the same symbols Δ' and Δ for brevity.

Any central element $C \in U_{\hbar}(\mathfrak{g})$ is naturally a bi-invariant \hbar -differential operator on G hence survives after Hamiltonian reduction as well. We denote by C_1 and C_2 the images in \mathcal{T}_h^n of the linear central element $\sum_{i=1}^n e_{ii}$ and the quadratic central element $\frac{1}{2} \sum_{i,j=1}^n (e_{ii}e_{jj} - e_{ij}e_{ji})$, respectively.

Proposition 2.4. *We have $\tau_{k,l}(\Delta') = 1 \otimes \Delta'$, $\tau_{k,l}(\Delta) = \Delta \otimes 1$, $\tau_{k,l}(C_1) = C_1 \otimes 1 + 1 \otimes C_1$, $\tau_{k,l}(C_2) = C_2 \otimes 1 + 1 \otimes C_2 + C_1 \otimes C_1 - \Delta' \otimes \Delta - \frac{\hbar}{2} C_1 \otimes 1 + \frac{\hbar}{2} 1 \otimes C_1$.*

Proof. Straightforward check. \square

2.5. Equivariant homology of an affine Grassmannian

Let G be as in Section 2.1, let G^\vee be its Langlands dual group, and let $\mathrm{Gr}_{G^\vee} = G_{\mathcal{K}}^\vee / G_{\mathcal{O}}^\vee$ be its affine Grassmannian; here $\mathcal{K} = \mathbb{C}((z)) \supset \mathbb{C}[[z]] = \mathcal{O}$. The affine Grassmannian is acted upon by a proalgebraic group $G_{\mathcal{O}}^\vee \rtimes \mathbb{C}^\times$ (the second factor acts by loop rotations). The equivariant homology $H_{\bullet}^{G_{\mathcal{O}}^\vee \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{G^\vee})$ forms a convolution algebra, and an isomorphism $\beta: H_{\bullet}^{G_{\mathcal{O}}^\vee \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{G^\vee}) \xrightarrow{\sim} \mathcal{T}_h(G)$ was constructed in [2, Theorem 3].³ In particular, $\beta^{-1}(\hbar)$ is a generator of $H_{\mathbb{C}^\times}^\bullet(pt)$, and $\beta^{-1}(ZU_{\hbar}(\mathfrak{g})) = H_{G_{\mathcal{O}}^\vee \rtimes \mathbb{C}^\times}^\bullet(pt) \subset H_{\bullet}^{G_{\mathcal{O}}^\vee \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{G^\vee})$.

For a dominant coweight λ of G^\vee the corresponding $G_{\mathcal{O}}^\vee$ -orbit closure in Gr_{G^\vee} is denoted by $\overline{\mathrm{Gr}}_{G^\vee}^\lambda$, and its intersection cohomology sheaf is denoted by IC^λ . Let d_λ stand for $\dim \overline{\mathrm{Gr}}_{G^\vee}^\lambda$. We have a canonical morphism from the shifted constant sheaf on $\overline{\mathrm{Gr}}_{G^\vee}^\lambda$ to the intersection cohomology sheaf:

$$\mathbb{C}_{\overline{\mathrm{Gr}}_{G^\vee}^\lambda}[d_\lambda] \rightarrow \mathrm{IC}^\lambda, \quad (2.2)$$

and hence $H^{\bullet+d_\lambda}(\overline{\mathrm{Gr}}_{G^\vee}^\lambda) \rightarrow H^\bullet(\overline{\mathrm{Gr}}_{G^\vee}^\lambda, \mathrm{IC}^\lambda)$ inducing isomorphism in the top and lowest cohomology. The geometric Satake isomorphism is an identification $H^\bullet(\overline{\mathrm{Gr}}_{G^\vee}^\lambda, \mathrm{IC}^\lambda) \cong V_\lambda$ with an irreducible representation of G with highest weight λ . Note that V_λ comes equipped with a highest weight vector v_λ and a lowest weight vector $v_{w_0\lambda}$ arising from the top and lowest fundamental classes in the cohomology $H^{\bullet+d_\lambda}(\overline{\mathrm{Gr}}_{G^\vee}^\lambda)$. Hence the dual G -module V_λ^* comes equipped with a highest weight vector $v_{-w_0\lambda}^*$ and a lowest weight vector $v_{-\lambda}^*$.

We consider the fundamental cycle $[\overline{\mathrm{Gr}}_{G^\vee}^\lambda] \in H_{\bullet}^{G_{\mathcal{O}}^\vee \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{G^\vee})$. We want to describe $\beta[\overline{\mathrm{Gr}}_{G^\vee}^\lambda] \in \mathcal{T}_h(G)$. To this end note that the matrix coefficient $\langle gv_{w_0\lambda}, v_{-\lambda}^* \rangle \in \mathbb{C}[G]$ is $U_- \times U_-$ -invariant and hence gives rise to the same named element in $\mathcal{T}_h(G)$.

³ An isomorphism β was constructed in [2] for semisimple groups, but the argument works word for word for reductive groups.

Lemma 2.6. $\beta[\overline{\mathrm{Gr}}_{G^\vee}^\lambda] = \langle gv_{w_0\lambda}, v_{-\lambda}^* \rangle \in \mathcal{T}_h(G)$.

Proof. Dually to (2.2), we have a canonical morphism from the intersection cohomology sheaf to the shifted dualizing sheaf $\mathrm{IC}^\lambda \rightarrow \omega_{\overline{\mathrm{Gr}}_{G^\vee}^\lambda}[-d_\lambda]$ and hence

$$\begin{aligned} c: H_{G_0^\vee \rtimes \mathbb{C}^\times}^\bullet(\overline{\mathrm{Gr}}_{G^\vee}^\lambda, \mathrm{IC}^\lambda) &\rightarrow H_{G_0^\vee \rtimes \mathbb{C}^\times}^{\bullet+d_\lambda}(\overline{\mathrm{Gr}}_{G^\vee}^\lambda, \omega_{\overline{\mathrm{Gr}}_{G^\vee}^\lambda}) \rightarrow \\ &\rightarrow H_{G_0^\vee \rtimes \mathbb{C}^\times}^{\bullet+d_\lambda}(\mathrm{Gr}_{G^\vee}, \omega_{\mathrm{Gr}_{G^\vee}}) = H_{\bullet+d_\lambda}^{G_0^\vee \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{G^\vee}). \end{aligned} \quad (2.3)$$

According to [2, Section 3.2, Lemma 1], $H_{G_0^\vee \rtimes \mathbb{C}^\times}^\bullet(\overline{\mathrm{Gr}}_{G^\vee}^\lambda, \mathrm{IC}^\lambda)$ contains a $H_{G_0^\vee \rtimes \mathbb{C}^\times}^\bullet(pt)$ -submodule $H_{G_0^\vee \rtimes \mathbb{C}^\times}^\bullet(pt) \otimes \mathbb{C}v_{w_0\lambda}$, and $c(1 \otimes v_{w_0\lambda}) = [\overline{\mathrm{Gr}}_{G^\vee}^\lambda]$.

Recall the functor \mathfrak{F} of [2, 6.4] from the equivariant derived category $D_{G_0^\vee \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{G^\vee})$ to the category of asymptotic Harish-Chandra $U_h(\mathfrak{g})$ -bimodules. It takes IC^λ to the free bimodule $U_h(\mathfrak{g}) \otimes V_\lambda$, and $\omega_{\mathrm{Gr}_{G^\vee}}$ to the hamiltonian reduction $\mathcal{K} = \kappa_h(U_h(\mathfrak{g}) \otimes \mathbb{C}[G])$ (see [2, Proposition 4]). Furthermore, it takes the composed morphism $\mathrm{IC}^\lambda \rightarrow \omega_{\overline{\mathrm{Gr}}_{G^\vee}^\lambda}[-d_\lambda] \rightarrow \omega_{\mathrm{Gr}_{G^\vee}}[-d_\lambda]$ to the morphism $b: U_h(\mathfrak{g}) \otimes V_\lambda \rightarrow \mathcal{K}$ arising from $U_h(\mathfrak{g}) \otimes V_\lambda \ni u \otimes v \mapsto u \otimes v \otimes v_{-\lambda}^* \in U_h(\mathfrak{g}) \otimes V_\lambda \otimes V_\lambda^* \subset U_h(\mathfrak{g}) \otimes \mathbb{C}[G]$. Finally, it takes the induced morphism on the equivariant cohomology to $\kappa_h(b): \kappa_h(U_h(\mathfrak{g}) \otimes V_\lambda) \rightarrow \kappa_h(\mathcal{K})$. Hence $1 \otimes v_{w_0\lambda}$ goes to the image of $1 \otimes v_{w_0\lambda} \otimes v_{-\lambda}^*$ in $\kappa_h(\mathcal{K})$, that is to $\langle gv_{w_0\lambda}, v_{-\lambda}^* \rangle \in \mathcal{T}_h(G)$. \square

2.7. The classical limit

The quotient algebra $\mathcal{T}(G) := \mathcal{T}_h(G)/\hbar$ is a Poisson algebra containing a maximal Poisson commutative subalgebra $ZU_h(\mathfrak{g})/\hbar$ (classical Toda lattice). We denote the spectrum of the commutative algebra $\mathcal{T}(G)$ by $\mathfrak{Z}(G)$. The homomorphism τ_G^L of Section 2.2 reduced modulo \hbar gives rise the same named homomorphism $\tau_G^L: \mathcal{T}(G) \rightarrow \mathcal{T}(L)$ and a morphism $\mathfrak{z}_L^G: \mathfrak{Z}(L) \rightarrow \mathfrak{Z}(G)$. In the setup of Section 2.3 we have $\mathcal{T}^n = \mathcal{T}(GL(n))$, $\mathfrak{Z}^n = \mathrm{Spec} \mathcal{T}^n$, and we obtain a morphism $\mathfrak{z}_{k,l}^G: \mathfrak{Z}^k \times \mathfrak{Z}^l \rightarrow \mathfrak{Z}^{k+l}$.

The isomorphism β of Section 2.5 reduced modulo \hbar gives rise to the same named isomorphism $\beta: H_{\bullet}^{G^\circ}(\mathrm{Gr}_G) \xrightarrow{\sim} \mathcal{T}^n$ (cf. also [1, Theorem 2.12] and [19, Theorem 6.3]). We also have an isomorphism $\Xi: \mathbb{C}[\hat{\mathcal{Z}}^n] \xrightarrow{\sim} H_{\bullet}^{G^\circ}(\mathrm{Gr}_G)$ of [4, Theorem 3.1] where $\hat{\mathcal{Z}}^n$ stands for the open zastava space of degree n based maps from (\mathbb{P}^1, ∞) to the flag variety of $SL(2)$ with a marked point of the upper triangular Borel subgroup. Let $\iota: \hat{\mathcal{Z}}^n \xrightarrow{\sim} \hat{\mathcal{Z}}^n$ be the Cartan involution of [3, 1.4(3)]. We denote the composition of the above isomorphisms at the level of spectra by $\Upsilon = \iota \circ \mathrm{Spec} \Xi \circ \mathrm{Spec} \beta: \mathfrak{Z}^n \xrightarrow{\sim} \hat{\mathcal{Z}}^n$.

In elementary terms, $\hat{\mathcal{Z}}^n$ is the moduli space of pairs of relatively prime polynomials $(Q, R) \in \mathbb{C}[z]$ such that Q is monic of degree n , and $\deg(R) < n$. The embedding $\Psi: \hat{\mathcal{Z}}^n \hookrightarrow SL(2, \mathbb{C}[z])$ of [4, 2(xi), 2(xii)] takes (Q, R) to a unique matrix

$\begin{pmatrix} Q & R' \\ R & Q' \end{pmatrix}$ of determinant 1 such that $\deg R' < n$, $\deg Q' < n - 1$. The Cartan involution ι is intertwined by Ψ with the matrix transposition. The multiplication morphism $\mu: \hat{Z}^k \times \hat{Z}^l \rightarrow \hat{Z}^{k+l}$ of [4, 2(vi)] is intertwined by Ψ with the matrix multiplication: $\Psi(\mu((Q_1, R_1), (Q_2, R_2))) = \Psi(Q_1, R_1) \cdot \Psi(Q_2, R_2)$. We have the *factorization* projection $\pi: \hat{Z}^n \rightarrow \mathbb{A}^{(n)}$, $(Q, R) \mapsto Q$, to the configuration space of unordered roots of Q . The subalgebra $\mathbb{C}[\mathbb{A}^{(n)}] \subset \mathbb{C}[\hat{Z}^n]$ corresponds under Υ to the maximal Poisson commutative subalgebra $ZU(\mathfrak{gl}_n) \subset \mathcal{T}^n$. More precisely, let (e, h, f) be an \mathfrak{sl}_2 -triple in $\mathfrak{g} = \mathfrak{gl}_n$ such

that $e = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}$, and h is diagonal. Let $Z_{\mathfrak{g}}(f)$ be the centralizer of f in \mathfrak{g} ,

and let $\Sigma^n = e + Z_{\mathfrak{g}}(f) \subset \mathfrak{g}$ be a Kostant slice. Let $\mathfrak{Z}_{\mathfrak{g}}^G$ be the universal centralizer: the moduli space of pairs of commuting matrices (x, g) such that $x \in \Sigma^n$, and g is invertible. We have an isomorphism $\zeta: \mathfrak{Z}_{\mathfrak{g}}^G \xrightarrow{\sim} \hat{Z}^n$ taking a pair (x, g) to (Q, R) where Q is the characteristic polynomial of x , and R is a unique polynomial of degree less than n such that $R'(x) = g$. In particular,

$$R'(z) = g_{n1}z^{n-1} + g_{n-1,1}z^{n-1} + \dots + g_{21}z + g_{11}. \quad (2.4)$$

We also have an isomorphism $\eta: \mathfrak{Z}_{\mathfrak{g}}^G \xrightarrow{\sim} \mathfrak{Z}^n$ constructed as follows. The Killing form identifies $e \in \mathfrak{n}$ with $\psi \in \mathfrak{n}_-^*$. Under this identification, $\mathfrak{Z}_{\mathfrak{g}}^G \subset G \times \mathfrak{g} \simeq G \times \mathfrak{g}^* = T^*G$ (left-invariant identification) lies in the moment level $\mu^{-1}(\psi, -\psi)$ of the moment map $\mu: T^*G \rightarrow \mathfrak{n}_-^* \times \mathfrak{n}_-^*$. The composed projection $\mathfrak{Z}_{\mathfrak{g}}^G \hookrightarrow \mu^{-1}(\psi, -\psi) \twoheadrightarrow T^*G // (U_- \times U_-; \psi, -\psi) = \mathfrak{Z}^n$ is the desired isomorphism η . Finally, $\Upsilon = \zeta \circ \eta^{-1}: \mathfrak{Z}^n \xrightarrow{\sim} \hat{Z}^n$.

Theorem 2.8. *The following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{Z}^k \times \mathfrak{Z}^l & \xrightarrow{\beta_{k,l}} & \mathfrak{Z}^{k+l} \\ \downarrow \Upsilon \times \Upsilon & & \downarrow \Upsilon \\ \hat{Z}^k \times \hat{Z}^l & \xrightarrow{\mu} & \hat{Z}^{k+l} \end{array}.$$

Proof. The space \mathfrak{Z}^n is equipped with a Poisson structure by construction (in fact, it is symplectic). The space \hat{Z}^n is also equipped with a Poisson (symplectic) structure, see e.g. [8]. The isomorphism $\Upsilon: \mathfrak{Z}^n \xrightarrow{\sim} \hat{Z}^n$ is Poisson according to [4, Proposition 3.18] (more precisely, both $\text{Spec } \Xi$ and ι are anti-Poisson, and $\text{Spec } \beta$ is Poisson). The upper arrow in the diagram of Theorem 2.8 is a Poisson morphism by construction.

Lemma 2.9. μ is Poisson.

Proof. The rational R -matrix formula for the Poisson bracket $\{T_{ij}(u), T_{kl}(v)\} = \frac{1}{u-v}(T_{il}(u)T_{kj}(v) - T_{il}(v)T_{kj}(u))$ is well-defined on matrix-valued polynomials and compatible with the multiplication of matrices. Note that $\Psi(\hat{Z}^n) \subset SL(2, \mathbb{C}[z])$ forms a Poisson subvariety with respect to this bracket. Hence we get a well-defined Poisson structure on \hat{Z}^n compatible with μ . On the other hand this bracket on \hat{Z}^n is opposite to the standard one [8]. Indeed, it follows from [8] that $\{Q(u), Q(v)\} = 0$, $\{Q(u), R(v)\} = \frac{-1}{u-v}(Q(u)R(v) - Q(v)R(u))$ and $\{R(u), R(v)\} = 0$, hence on the coefficients of Q and R the Poisson brackets in question are opposite. On the other hand the field of rational functions on \hat{Z}^n is generated by the coefficients of Q and R . \square

Hence it suffices to check the commutativity of the diagram of Theorem 2.8 on an appropriate set of Poisson generators of the coordinate rings. Let $R(z) = \sum_{k=1}^n r_k z^{n-k}$, $R'(z) = \sum_{k=1}^n r'_k z^{n-k}$ and $Q(z) = z^n + \sum_{k=1}^n q_k z^{n-k}$. Clearly, the functions q_k, r_k, r'_k (for $k = 1, \dots, n-1$) generate $\mathbb{C}[\hat{Z}^n]$ as a commutative ring. We will also need the functions y_k, y'_k for $k = 1, 2, \dots$ defined as $\frac{R(z)}{Q(z)} = \sum_{k=1}^{\infty} y_k z^{-k}$ and $\frac{R'(z)}{Q(z)} = \sum_{k=1}^{\infty} y'_k z^{-k}$. Finally, we set $\frac{1}{Q^2(z)} = \sum_{k=1}^{\infty} x_k z^{-k}$; note that $x_1 = x_2 = \dots = x_{2n-1} = 0$, and $x_{2n} = 1$.

Lemma 2.10. *The elements r_1, r'_1, q_1, q_2 generate the coordinate ring $\mathbb{C}[\hat{Z}^n]$ as a Poisson algebra.*

Proof. It is easy to see that y_k, y'_k, x_k , $k = 1, 2, \dots$, generate $\mathbb{C}[\hat{Z}^n]$ as a commutative ring. We have $y_1 = r_1$, $y'_1 = r'_1$, $x_{2n+1} = -2q_1$, $x_{2n+2} = 3q_1^2 - 2q_2$. We have $\{x_{2n+2}, y_k\} = 2y_{k+1} + 2x_{2n+1}y_k$ and $\{x_{2n+2}, y'_k\} = -2y'_{k+1} - 2x_{2n+1}y'_k$, and hence all the functions y_k, y'_k are Poisson expressions of the above generators. Also, we have $\{y_m, y'_l\} = x_{m+l-1}$, and hence by induction we conclude that x_k are Poisson expressions of the above generators as well. \square

Now we can finish the proof of the theorem. By direct computation we have $\Upsilon^*(r_1) = -\Delta$, $\Upsilon^*(r'_1) = \Delta'$, $\Upsilon^*(q_1) = C_1$, $\Upsilon^*(q_2) = C_2$ (we denote the specializations of the elements from \mathcal{T}_h^n at $h = 0$ by the same symbols for brevity). Also, from the 2×2 -matrix multiplication we see that $\mu^*(r_1) = r_1 \otimes 1$, $\mu^*(r'_1) = 1 \otimes r'_1$, $\mu^*(q_1) = q_1 \otimes 1 + 1 \otimes q_1$, $\mu^*(q_2) = q_2 \otimes 1 + 1 \otimes q_2 + q_1 \otimes q_1 + r'_1 \otimes r_1$. So the theorem follows from Proposition 2.4. \square

Remark 2.11. We assume now G is semisimple, and $T \subset L \subset G$ is a Levi subgroup containing a Cartan torus. The classical Toda system is the projection $\mathfrak{Z}(L) \rightarrow \mathfrak{h}^*/W_L$ (where \mathfrak{h} is the Lie algebra of T). Let $\Gamma_L^G \subset \mathfrak{Z}(L) \times \mathfrak{Z}(G)$ be the graph of \mathfrak{z}_L^G . Then the projection $\text{pr}_{L,G}: \Gamma_L^G \rightarrow \mathfrak{h}^*/W_L \times \mathfrak{h}^*/W_G$ is generically finite. If $L \subset G =$

$SL(n)$ corresponds to a decomposition $n = k + l$, the degree of $\mathrm{pr}_{L,G}$ equals $\binom{n-2}{k-1}$ (D. Gaiotto, private communication). If $L = T \subset G = SL(n)$, the degree of $\mathrm{pr}_{T,SL(n)}$ equals 1, 1, 2, 4, 11, 33, 120, 470, 2107, 10189 for $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ respectively (E. Rains, private communication).

2.12. Some equivariant homology classes

We will need a special case of Lemma 2.6. Recall the setup of Section 2.5, and let $G = GL(n) \cong G^\vee$. Let $\mathrm{Gr}_G^{\varpi_1}$ (resp. $\mathrm{Gr}_G^{-\varpi_1}$) be the closed G_0 -orbit in Gr_G formed by all the lattices L such that $z\mathcal{O}^n \subset L \subset \mathcal{O}^n$ and $\dim_{\mathbb{C}} \mathcal{O}^n/L = 1$ (resp. $\mathcal{O}^n \subset L \subset z^{-1}\mathcal{O}^n$ and $\dim_{\mathbb{C}} L/\mathcal{O}^n = 1$). The fundamental class of $\mathrm{Gr}_G^{\varpi_1}$ (resp. $\mathrm{Gr}_G^{-\varpi_1}$) in $H_{\bullet}^{G_0 \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$ will be denoted $[\mathrm{Gr}_G^{\varpi_1}]$ (resp. $[\mathrm{Gr}_G^{-\varpi_1}]$).

Recall the elements $\Delta', \Delta \in \mathcal{T}_h$ introduced in Section 2.3.

Lemma 2.13. *We have $\beta[\mathrm{Gr}_G^{\varpi_1}] = \Delta'$ and $\beta[\mathrm{Gr}_G^{-\varpi_1}] = (-1)^{n-1}\Delta$.*

Proof. This follows from Lemma 2.6 up to a multiplicative constant. To determine the constants, recall that we have chosen the principal nilpotent $e \in \mathfrak{n}$ to be the sum of elementary matrices $e = \sum_{i=1}^n e_{i+1,i} \in \mathfrak{g} = \mathfrak{gl}_n$. The operator of multiplication by the first Chern class of the tautological G^\vee -bundle is identified with the action of the principal nilpotent $e \in \mathfrak{g}$ via the isomorphism $H_{G_0^\vee \rtimes \mathbb{C}^\times}^\bullet(\overline{\mathrm{Gr}}_{G^\vee}^\lambda, \mathrm{IC}^\lambda) = V_\lambda$. For $\lambda = \pm\varpi_1$ we have $\overline{\mathrm{Gr}}_{G^\vee}^\lambda \simeq \mathbb{P}^{n-1}$, hence the top-dimensional fundamental class of $\overline{\mathrm{Gr}}_{G^\vee}^\lambda$ gets identified with $e^{n-1}v_{w_0\lambda}$. The latter is v_n for $\lambda = \varpi_1$, and $(-1)^{n-1}v_1^*$ for $\lambda = -w_0\varpi_1$. \square

2.14. Bispectrality

This section is not used in what follows.

We will need the homomorphism $(\iota_*)^{-1}$ from $H_{\bullet}^{G_0 \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$ to a certain localization of $\left(H_{\bullet}^{T_0 \rtimes \mathbb{C}^\times}(\mathrm{Gr}_T)\right)^{\mathfrak{S}_n}$, see [4, A(i)]. The convolution algebra $\mathcal{A}_h(T, 0) := H_{\bullet}^{T_0 \rtimes \mathbb{C}^\times}(\mathrm{Gr}_T)$ is a $\mathbb{C}[\hbar]$ -algebra generated by $w_r, u_r^{\pm 1}$, $1 \leq r \leq n$, with relations $[u_r^{\pm 1}, w_s] = \pm \delta_{r,s} \hbar u_r^{\pm 1}$. It satisfies the Ore condition with respect to the set $\{w_r - w_s + m\hbar, 1 \leq r \neq s \leq n, m \in \mathbb{Z}\}$, and the corresponding localization is denoted $\tilde{\mathcal{A}}_h$. So we have the homomorphism $(\iota_*)^{-1}: H_{\bullet}^{G_0 \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G) \rightarrow (\tilde{\mathcal{A}}_h)^{\mathfrak{S}_n}$.

We define a certain space of formal functions on the diagonal torus $T \subset G$, containing Whittaker functions. Let t_1, \dots, t_n be the diagonal matrix elements considered as functions on the diagonal torus T . Let \mathfrak{h} (resp. \mathfrak{b}) stand for the Lie algebra of T (resp. B). Let $\mathfrak{R} := \mathbb{C}(\mathfrak{h}^* \times \mathbb{A}_{t_1}^1 \times \mathbb{A}_{\hbar}^1)[[t_1 t_2^{-1}, \dots, t_{n-1} t_n^{-1}]]$ be the ring of formal Taylor series in $t_1 t_2^{-1}, \dots, t_{n-1} t_n^{-1}$ with coefficients in the field of rational functions on the product of \mathfrak{h}^* and a line with coordinate t_1 and a line with coordinate \hbar . The ring $\mathcal{T}_h^n \subset \mathcal{D}_h(T)$ acts on \mathfrak{R} naturally. Also, the ring $\tilde{\mathcal{A}}_h$ acts on \mathfrak{R} via its action on $\mathbb{C}(\mathfrak{h}^* \times \mathbb{A}_{\hbar}^1)$ by the difference operators. Namely, let $w_1 = e_{11}, \dots, w_n = e_{nn}$ be the elementary matrices considered as

elements of the diagonal Cartan Lie algebra \mathfrak{h} , i.e. linear functions on \mathfrak{h}^* . Then we set $u_r^{\pm 1}(w_s) = w_s \pm \hbar \delta_{rs}$, $1 \leq r, s \leq n$. Let \mathcal{R} be a free rank one \mathfrak{A} -module with generator $t_1^{w_1/\hbar} \dots t_n^{w_n/\hbar}$. Then $\mathcal{T}_{\hbar}^n \subset \mathcal{D}_{\hbar}(T)$ and $\tilde{\mathcal{A}}_{\hbar}$ act on \mathcal{R} .

We consider the generic universal Verma module $\mathcal{M}_{\hbar}(-\rho) = U_{\hbar}(\mathfrak{g}) \otimes_{U_{\hbar}(\mathfrak{b})} \mathbb{C}(\mathfrak{h}^* \times \mathbb{A}_{\hbar}^1)(-\rho)$ where $\mathbb{C}(\mathfrak{h}^* \times \mathbb{A}_{\hbar}^1)(-\rho)$ is a $U_{\hbar}(\mathfrak{b})$ -module which factors through the $U_{\hbar}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^* \times \mathbb{A}_{\hbar}^1]$ -module where $x \in \mathfrak{h}$ acts by multiplication by $x - \hbar \rho(x)$ (and $\rho \in \mathfrak{h}^*$ is the halfsum of the positive roots). It is equipped with the $\mathbb{C}(\mathfrak{h}^* \times \mathbb{A}_{\hbar}^1)$ -valued Shapovalov form (\cdot, \cdot) . The $\mathbb{C}(\mathfrak{h}^* \times \mathbb{A}_{\hbar}^1)$ -vector space of $(U_{\hbar}(\mathfrak{n}_{-}), \psi)$ -coinvariants in $\mathcal{M}_{\hbar}(-\rho)$ is 1-dimensional, and any coinvariant is proportional to the Shapovalov scalar product with the Whittaker vector \mathfrak{w} in a completion of $\mathcal{M}_{\hbar}(-\rho)$. More precisely, let $\psi_{+}: U_{\hbar}(\mathfrak{n}) \rightarrow \mathbb{C}[\hbar]$ be a homomorphism such that $\psi_{+}(e_{i+1,i}) = 1$ for any $i = 1, \dots, n-1$. Then there is a unique vector $\mathfrak{w} = \sum_{\underline{d} \in \mathbb{N}^{n-1}} \mathfrak{w}_{\underline{d}} \in \hat{\mathcal{M}}_{\hbar}(-\rho)$ (an infinite sum of the weight components) such that the highest weight component $\mathfrak{w}_{\underline{0}}$ of \mathfrak{w} is $1 \in \mathbb{C}(\mathfrak{h}^* \times \mathbb{A}_{\hbar}^1)$, and $u\mathfrak{w} = \psi_{+}(u)\mathfrak{w}$ for any $u \in U_{\hbar}(\mathfrak{n})$. Finally, the Whittaker function \mathfrak{W} is defined as the Shapovalov scalar product $\mathfrak{W} := \prod_{r=1}^n t_r^{w_r/\hbar+r-1} \sum_{\underline{d} \in \mathbb{N}^{n-1}} (\mathfrak{w}_{\underline{d}}, \mathfrak{w}_{\underline{d}}) \prod_{s=1}^{n-1} (t_s/t_{s+1})^{d_s} \in \mathcal{R}$.

Proposition 2.15. *We have $\beta(h)\mathfrak{W} = (\iota_{*})^{-1}(h)\mathfrak{W}$ for any $h \in H_{\bullet}^{G_{\circ} \times \mathbb{C}^{\times}}(\mathrm{Gr}_G)$.*

Proof. We have $C_1\mathfrak{W} = (\sum_{r=1}^n w_r)\mathfrak{W}$, $C_2\mathfrak{W} = (\sum_{1 \leq r < s \leq n} w_r w_s + (\rho_n, \rho_n)\hbar^2)\mathfrak{W}$ (notations of Section 2.3; $\rho_n := (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2})$). Also, we have $\Delta'\mathfrak{W} = (\sum_{r=1}^n \prod_{s \neq r} (w_r - w_s)^{-1} u_r^{-1})\mathfrak{W}$, $\Delta\mathfrak{W} = (\sum_{r=1}^n \prod_{s \neq r} (w_r - w_s)^{-1} u_r^{-1})\mathfrak{W}$ according to [20, Theorem 3 and (6.7)]. According to [4, (A.3), (A.4)], $(\iota_{*})^{-1}[\mathrm{Gr}_G^{\varpi_1}] = \sum_{r=1}^n \prod_{s \neq r} (w_r - w_s)^{-1} u_r^{-1}$, $(\iota_{*})^{-1}[\mathrm{Gr}_G^{-\varpi_1}] = \sum_{r=1}^n \prod_{s \neq r} (w_s - w_r)^{-1} u_r^{-1}$. Since $C_1, C_2, \Delta', \Delta$ generate $\mathcal{T}_{\hbar}^n[\hbar^{-1}]$, the proposition follows from Lemma 2.13. \square

3. Shifted Yangians

In this section we consider the family of algebras known as shifted Yangians, following [4, Appendix B]. Our main goal is to prove a PBW theorem for these algebras.

Let \mathfrak{g} denote a simply-laced semisimple Lie algebra (of finite type) with simple roots $\{\alpha_i\}_{i \in I}$. We write $\alpha_i \cdot \alpha_j$ for the usual inner product of these simple roots.

Definition 3.1. The *Cartan doubled Yangian* $Y_{\infty} := Y_{\infty}(\mathfrak{g})$ is defined to be the \mathbb{C} -algebra with generators $E_i^{(q)}, F_i^{(q)}, H_i^{(p)}$ for $i \in I$, $q > 0$ and $p \in \mathbb{Z}$, with relations

$$\begin{aligned} [H_i^{(p)}, H_j^{(q)}] &= 0, \\ [E_i^{(p)}, F_j^{(q)}] &= \delta_{ij} H_i^{(p+q-1)}, \\ [H_i^{(p+1)}, E_j^{(q)}] - [H_i^{(p)}, E_j^{(q+1)}] &= \frac{\alpha_i \cdot \alpha_j}{2} (H_i^{(p)} E_j^{(q)} + E_j^{(q)} H_i^{(p)}), \\ [H_i^{(p+1)}, F_j^{(q)}] - [H_i^{(p)}, F_j^{(q+1)}] &= -\frac{\alpha_i \cdot \alpha_j}{2} (H_i^{(p)} F_j^{(q)} + F_j^{(q)} H_i^{(p)}), \\ [E_i^{(p+1)}, E_j^{(q)}] - [E_i^{(p)}, E_j^{(q+1)}] &= \frac{\alpha_i \cdot \alpha_j}{2} (E_i^{(p)} E_j^{(q)} + E_j^{(q)} E_i^{(p)}), \end{aligned}$$

$$\begin{aligned}
[F_i^{(p+1)}, F_j^{(q)}] - [F_i^{(p)}, F_j^{(q+1)}] &= -\frac{\alpha_i \cdot \alpha_j}{2} (F_i^{(p)} F_j^{(q)} + F_j^{(p)} F_i^{(q)}), \\
i \neq j, N = 1 - \alpha_i \cdot \alpha_j &\Rightarrow \text{sym}[E_i^{(p_1)}, [E_i^{(p_2)}, \dots [E_i^{(p_N)}, E_j^{(q)}] \dots]] = 0, \\
i \neq j, N = 1 - \alpha_i \cdot \alpha_j &\Rightarrow \text{sym}[F_i^{(p_1)}, [F_i^{(p_2)}, \dots [F_i^{(p_N)}, F_j^{(q)}] \dots]] = 0.
\end{aligned}$$

Remark 3.2. Although we have assumed \mathfrak{g} to be simply-laced and finite type, we expect that our results hold in greater generality (e.g. certain Kac–Moody algebras). Indeed, we make use of two results for the ordinary Yangian $Y = Y(\mathfrak{g})$: (1) the PBW theorem for Y , and (2) the existence of a coproduct $\Delta : Y \rightarrow Y \otimes Y$. If \mathfrak{g} is such that (1) and (2) are known to hold, then the results in this section and the next should hold. We will not pursue this direction further here.

We denote by $Y_\infty^>, Y_\infty^\geq$ the subalgebras of Y_∞ generated by the $E_i^{(q)}$ (resp. $E_i^{(q)}$ and $H_i^{(p)}$). Likewise we denote by $Y_\infty^<, Y_\infty^\leq$ the subalgebras generated by the $F_i^{(q)}$ (resp. $F_i^{(q)}, H_i^{(p)}$). Also denote by Y_∞^\equiv the subalgebra generated by the $H_i^{(p)}$. We will use similar notation for the various quotients of Y_∞ that we define below.

Remark 3.3. There is a surjective homomorphism $Y^> \twoheadrightarrow Y_\infty^>$, where $Y^>$ is the analogous subalgebra of the ordinary Yangian Y . This map is defined by $E_i^{(q)} \mapsto E_i^{(q)}$.

Remark 3.4. Consider a positive root β , and pick any decomposition $\beta = \alpha_{i_1} + \dots + \alpha_{i_\ell}$ into simple roots so that the element $[e_{i_1}, [e_{i_2}, \dots, [e_{i_{\ell-1}}, e_{i_\ell}] \dots]]$ is a non-zero element of the root space \mathfrak{g}_β . Consider also $q > 0$ and a decomposition $q + \ell - 1 = q_1 + \dots + q_\ell$ into positive integers. Then we define a corresponding element of Y_∞ :

$$E_\beta^{(q)} := [E_{i_1}^{(q_1)}, [E_{i_2}^{(q_2)}, \dots [E_{i_{\ell-1}}^{(q_{\ell-1})}, E_{i_\ell}^{(q_\ell)}] \dots]]. \quad (3.1)$$

This element, which we call a PBW variable, depends on the choices above. However, we will fix arbitrarily such a choice for each β and q .

Similarly, we define PBW variables $F_\beta^{(q)}$.

Definition 3.5. For any coweight μ , the shifted Yangian Y_μ is defined to be the quotient of Y_∞ by the relations $H_i^{(p)} = 0$ for all $p < -\langle \mu, \alpha_i \rangle$ and $H_i^{(-\langle \mu, \alpha_i \rangle + q)} = 1$.

Remark 3.6. The algebra $Y = Y_0$ is the usual Yangian with its standard Drinfeld presentation (except that upper indices are shifted by 1).

Remark 3.7. The homomorphism from Remark 3.3 can be extended to a surjection $Y^\geq \twoheadrightarrow Y_\mu^\geq$, defined by $E_i^{(q)} \mapsto E_i^{(q)}$ and $H_i^{(q)} \mapsto H_i^{(-\langle \mu, \alpha_i \rangle + q)}$ for $q > 0$. Similarly, there is a surjection $Y^\leq \twoheadrightarrow Y_\mu^\leq$.

There are natural “shift homomorphisms” between these algebras:

Proposition 3.8. *Let μ be a coweight, and μ_1, μ_2 be antidominant coweights. Then there exists a homomorphism $\iota_{\mu, \mu_1, \mu_2} : Y_\mu \longrightarrow Y_{\mu + \mu_1 + \mu_2}$ defined by*

$$H_i^{(r)} \mapsto H_i^{(r - \langle \mu_1 + \mu_2, \alpha_i \rangle)}, \quad E_i^{(r)} \mapsto E_i^{(r - \langle \mu_1, \alpha_i \rangle)}, \quad F_i^{(r)} \mapsto F_i^{(r - \langle \mu_2, \alpha_i \rangle)}. \quad (3.2)$$

Proof. Immediate from Definition 3.5. \square

Remark 3.9. Given our present conventions, for μ dominant, the shifted Yangian as defined in [12, Section 3.6] as a subalgebra of Y , can be identified with the image of the shift homomorphism $\iota_{\mu, 0, -\mu} : Y_\mu \longrightarrow Y_0 = Y$ (here we use Corollary 3.16 to see that the map is injective). On the other hand, when μ is not dominant, these shifted Yangians are not subalgebras of Y and their definition first appeared [4, Appendix B].

The definition of shifted Yangians was originally inspired by the work of Brundan–Kleshchev [5], who considered shifted Yangians inside the \mathfrak{gl}_n Yangian.

Remark 3.10. Set $S_i^{(-\langle \mu, \alpha_i \rangle + 1)} = H_i^{(-\langle \mu, \alpha_i \rangle + 1)}$ and

$$S_i^{(-\langle \mu, \alpha_i \rangle + 2)} = H_i^{(-\langle \mu, \alpha_i \rangle + 2)} - \frac{1}{2}(H_i^{(-\langle \mu, \alpha_i \rangle + 1)})^2 \quad (3.3)$$

For $r \geq 1$, it is not hard to check that

$$\begin{aligned} [S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, E_j^{(r)}] &= (\alpha_i \cdot \alpha_j) E_j^{(r+1)}, \\ [S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, F_j^{(r)}] &= -(\alpha_i \cdot \alpha_j) F_j^{(r+1)}. \end{aligned}$$

Lemma 3.11. *Let μ be an antidominant coweight. As a unital associative algebra, Y_μ is generated by $E_i^{(1)}, F_i^{(1)}, S_i^{(-\langle \mu, \alpha_i \rangle + 1)} = H_i^{(-\langle \mu, \alpha_i \rangle + 1)}$ and $S_i^{(-\langle \mu, \alpha_i \rangle + 2)} = H_i^{(-\langle \mu, \alpha_i \rangle + 2)} - \frac{1}{2}(H_i^{(-\langle \mu, \alpha_i \rangle + 1)})^2$. Alternatively, Y_μ is also generated by $E_i^{(1)}, F_i^{(1)}, H_i^{(-\langle \mu, \alpha_i \rangle + k)}$ ($k = 1, 2$). In particular, Y_μ is finitely generated.*

Proof. For the first assertion, it is enough to show that $E_i^{(r)}, F_i^{(r)} H_i^{(s)}$ lie in the subalgebra generated by $E_i^{(1)}, F_i^{(1)}, S_i^{(-\langle \mu, \alpha_i \rangle + k)}$ ($k = 1, 2$) for all $r \geq 1, s \geq -\langle \mu, \alpha_i \rangle + 1$. This is clear since $E_i^{(r)} = \frac{1}{2}[S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, E_i^{(r-1)}]$, $F_i^{(r)} = -\frac{1}{2}[S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, F_i^{(r-1)}]$ for all $r \geq 2$ and since $H_i^{(s)} = [E_i^{(1)}, F_i^{(s)}]$ for all $s \geq -\langle \mu, \alpha_i \rangle + 1$.

The second assertion follows immediately from the first since the subalgebra generated by $E_i^{(1)}, F_i^{(1)}, S_i^{(-\langle \mu, \alpha_i \rangle + k)}$ ($k = 1, 2$) is contained in the subalgebra generated by the $E_i^{(1)}, F_i^{(1)}, H_i^{(-\langle \mu, \alpha_i \rangle + k)}$ ($k = 1, 2$). \square

3.12. PBW theorem

In this section we will prove the PBW theorem for the algebras Y_μ . This generalizes the well-known case of the ordinary Yangian Y due to Levendorskii [16], as well as the case when μ is dominant [11, Proposition 3.11].

For each positive root β and $q > 0$, consider elements $E_\beta^{(q)}, F_\beta^{(q)} \in Y_\mu$ defined as images under $Y_\infty \rightarrow Y_\mu$ of those described in Remark 3.4. Choose a total order on the set of PBW variables

$$\left\{ E_\beta^{(q)} : \beta \in \Delta^+, q > 0 \right\} \cup \left\{ F_\beta^{(q)} : \beta \in \Delta^+, q > 0 \right\} \cup \left\{ H_i^{(p)} : i \in I, p > -\langle \mu, \alpha_i \rangle \right\} \quad (3.4)$$

If $\mu = 0$, then ordered monomials in these PBW variables form a basis of Y by [16].

For simplicity we will assume that we have chosen a block order with respect to the three subsets above, i.e. ordered monomials have the form EFH .

Proposition 3.13. *Y_μ is spanned by ordered monomials in the PBW variables.*

Proof. We first claim that Y_μ is spanned by elements of the form xy , with $x \in Y_\mu^>$ and $y \in Y_\mu^\leq$. Indeed, the relation $[E_i^{(p)}, F_j^{(q)}] = \delta_{ij} H_i^{(r+s-1)}$ allows us to reorder products of the generators $E_i^{(p)}$ and $F_j^{(q)}$, while the surjection $Y^\geq \rightarrow Y_\mu^\geq$ from Remark 3.7 shows that we may reorder products of the generators $E_i^{(p)}$ and $H_i^{(q)}$ (as this is true for the ordinary Yangian Y).

Next, we claim that $Y_\mu^>$ is spanned by ordered monomials in the elements $E_\beta^{(q)}$. To see this, we can again appeal to the surjection $Y^> \rightarrow Y_\mu^>$ together with the PBW theorem for $Y^>$. Similarly, Y_μ^\leq is spanned by ordered monomials in the elements $F_\beta^{(q)}$ and $H_i^{(p)}$. Altogether, this proves the proposition. \square

Theorem 3.14. *Let μ be antidominant. Then the set of ordered monomials in the PBW variables forms a basis for Y_μ over \mathbb{C} .*

The proof of the above theorem will be given in Section 3.21. For now we give two important corollaries:

Corollary 3.15. *For μ arbitrary, the set of ordered monomials in the PBW variables forms a basis for Y_μ over \mathbb{C} .*

Proof. We may choose a shift homomorphism $\iota: Y_\mu \rightarrow Y_{\mu'}$ from Proposition 3.8 such that μ' is antidominant. Under ι , the images of the elements $E_\beta^{(q)}, F_\beta^{(q)} \in Y_\mu$ have the form (3.1), i.e. we may consider these images as PBW variables for $Y_{\mu'}$. In particular, we may extend these images to a full set of PBW variables for $Y_{\mu'}$.

It follows from Theorem 3.14 that the set of ordered monomials in the PBW variables for Y_μ map bijectively under ι onto a subset of a basis for $Y_{\mu'}$, so in particular they are linearly independent in Y_μ . Combined with Proposition 3.13, this implies that the PBW monomials for Y_μ form a basis (and also that ι is injective). \square

Corollary 3.16. *For any μ and antidominant μ_1, μ_2 , the shift homomorphism $\iota_{\mu, \mu_1, \mu_2}: Y_\mu \rightarrow Y_{\mu+\mu_1+\mu_2}$ from Proposition 3.8 is injective.*

Proof. Analogous to the proof of the previous corollary. \square

3.17. The algebra \tilde{Y}

Definition 3.18. The algebra \tilde{Y} is defined to be the quotient of Y_∞ by the relations $H_i^{(p)} = 0$ for all $i \in I$ and $p < 0$.

To distinguish the generators of \tilde{Y} , we will denote them by $\tilde{E}_i^{(q)}, \tilde{F}_i^{(q)}, \tilde{H}_i^{(p)}$. The following result is a key tool in proving [Theorem 3.14](#):

Lemma 3.19.

(a) *There is an embedding of algebras $\tilde{Y} \hookrightarrow Y \otimes_{\mathbb{C}} \mathbb{C}[H_i^{(0)} : i \in I]$, defined by*

$$\tilde{E}_i^{(q)} \mapsto E_i^{(q)} \otimes H_i^{(0)}, \quad \tilde{F}_i^{(q)} \mapsto F_i^{(q)} \otimes 1, \quad \tilde{H}_i^{(p)} \mapsto H_i^{(p)} \otimes H_i^{(0)}. \quad (3.5)$$

(b) *Ordered monomials in the elements of the set*

$$\left\{ \tilde{E}_\beta^{(q)} : \beta \in \Delta^+, q > 0 \right\} \cup \left\{ \tilde{F}_\beta^{(q)} : \beta \in \Delta^+, q > 0 \right\} \cup \left\{ \tilde{H}_i^{(p)} : i \in I, p \geq 0 \right\} \quad (3.6)$$

form a basis for \tilde{Y} over \mathbb{C} .

Proof. Using the relations for \tilde{Y} as inherited from Y_∞ , one can verify that (3.5) is a homomorphism.

The remainder of the proof is analogous to that of [Corollary 3.15](#): first one shows that PBW monomials span \tilde{Y} by using the relations among its generators, and second one observes that these monomials map bijectively under (3.5) onto a subset of a basis for $Y \otimes_{\mathbb{C}} \mathbb{C}[H_i^{(0)}]$ (using the PBW theorem for Y). \square

Corollary 3.20. *Let μ be an antidominant coweight. Then \tilde{Y} is free as a right module over the polynomial ring*

$$\mathbb{C}[\tilde{H}_i^{(s)} : i \in I, 0 \leq s \leq -\langle \mu, \alpha_i \rangle], \quad (3.7)$$

with basis consisting of ordered monomials in the set

$$\left\{ \tilde{E}_\gamma^{(r)} : \gamma \in \Delta_+, r \geq 1 \right\} \cup \left\{ \tilde{F}_\gamma^{(r)} : \gamma \in \Delta_+, r \geq 1 \right\} \cup \left\{ \tilde{H}_i^{(s)} : i \in I, s > -\langle \mu, \alpha_i \rangle \right\}. \quad (3.8)$$

3.21. The PBW theorem in the antidominant case

Let μ be antidominant. The following result is immediate from the definitions of Y_μ and \tilde{Y} as quotients of Y_∞ :

Lemma 3.22. *There is a surjective homomorphism $\tilde{Y} \rightarrow Y_\mu$ defined by $\tilde{E}_i^{(q)} \mapsto E_i^{(q)}$, $\tilde{F}_i^{(q)} \mapsto F_i^{(q)}$ and $\tilde{H}_i^{(p)} \mapsto H_i^{(p)}$. The kernel of this homomorphism is the ideal*

$$I_\mu := \left\langle \tilde{H}_i^{(s)} - \delta_{s, -\langle \mu, \alpha_i \rangle} : i \in I, 0 \leq s \leq -\langle \mu, \alpha_i \rangle \right\rangle_{\text{two-sided}}. \quad (3.9)$$

Lemma 3.23. *I_μ is equal to the left ideal*

$$\left\langle \tilde{H}_i^{(s)} - \delta_{s, -\langle \mu, \alpha_i \rangle} : i \in I, 0 \leq s \leq -\langle \mu, \alpha_i \rangle \right\rangle_{\text{left}}. \quad (3.10)$$

Proof. Denote this left ideal by I_μ^{left} . We will prove the claim by showing that I_μ^{left} is also a right ideal.

In \tilde{Y} , we have the relations

$$[\tilde{H}_i^{(0)}, \tilde{E}_j^{(r)}] = 0, \quad [\tilde{H}_i^{(1)}, \tilde{E}_j^{(r)}] = (\alpha_i, \alpha_j) \tilde{E}_j^{(r)} \tilde{H}_i^{(0)}, \quad (3.11)$$

$$[\tilde{H}_i^{(s+1)}, \tilde{E}_j^{(r)}] = [\tilde{H}_i^{(s)}, \tilde{E}_j^{(r+1)}] + \frac{(\alpha_i, \alpha_j)}{2} [\tilde{H}_i^{(s)}, \tilde{E}_i^{(r)}] + (\alpha_i, \alpha_j) \tilde{E}_j^{(r)} \tilde{H}_i^{(s)}. \quad (3.12)$$

By induction on s , it follows that for any $r \geq 1$,

$$[\tilde{H}_i^{(s)}, \tilde{E}_j^{(r)}] \in \left\langle \tilde{H}_i^{(0)}, \dots, \tilde{H}_i^{(s-1)} \right\rangle_{\text{left}}. \quad (3.13)$$

For $0 \leq s \leq -\langle \mu, \alpha_i \rangle$, we therefore have

$$(\tilde{H}_i^{(s)} - \delta_{s, -\langle \mu, \alpha_i \rangle}) \tilde{E}_j^{(r)} \in \tilde{E}_j^{(r)} (\tilde{H}_i^{(s)} - \delta_{s, -\langle \mu, \alpha_i \rangle}) + I_\mu^{\text{left}} = I_\mu^{\text{left}}. \quad (3.14)$$

It follows that right multiplication by $\tilde{E}_j^{(r)}$ preserves I_μ^{left} . Similarly for $\tilde{F}_j^{(r)}$, while for $\tilde{H}_j^{(r)}$ this is clear. These elements generate \tilde{Y} , so I_μ^{left} is a right ideal. \square

Proof of Theorem 3.14. Consider the homomorphism

$$\mathbb{C}[\tilde{H}_i^{(s)} : i \in I, 0 \leq s \leq -\langle \mu, \alpha_i \rangle] \longrightarrow \mathbb{C}, \quad (3.15)$$

defined by $\tilde{H}_i^{(s)} \mapsto \delta_{s, -\langle \mu, \alpha_i \rangle}$. By Lemma 3.23, we see that Y_μ is the base change of the right module \tilde{Y} with respect to the map (3.15):

$$Y_\mu = \tilde{Y} \otimes_{\mathbb{C}[\tilde{H}_i^{(s)} : i \in I, 0 \leq s \leq -\langle \mu, \alpha_i \rangle]} \mathbb{C}. \quad (3.16)$$

Recall that \tilde{Y} is a free right module over $\mathbb{C}[\tilde{H}_i^{(s)} : i \in I, 0 \leq s \leq -\langle \mu, \alpha_i \rangle]$, so the basis from Corollary 3.20 yields a basis for Y_μ over \mathbb{C} . This completes the proof of Theorem 3.14. \square

4. Coproduct

We continue with \mathfrak{g} a simply-laced semisimple Lie algebra.

In this section we will describe a family of coproducts for shifted Yangians. For any splitting $\mu = \mu_1 + \mu_2$, in [Theorem 4.12](#) we will establish the existence of a homomorphism

$$\Delta_{\mu_1, \mu_2} : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2} \quad (4.1)$$

This generalizes the coproduct for the ordinary Yangian $Y = Y_0$.

4.1. A new presentation

Let μ be an antidominant coweight. We will follow [\[15\]](#) and define another presentation for Y_μ .

Fix a decomposition $\mu = \mu_1 + \mu_2$ where the μ_i 's are antidominant coweights.

Denote by Y_{μ_1, μ_2} the algebra generated by: $S_i^{(-\langle \mu, \alpha_i \rangle + 1)}$, $S_i^{(-\langle \mu, \alpha_i \rangle + 2)}$, $E_i^{(r)}$ ($1 \leq r \leq -\langle \mu_1, \alpha_i \rangle + 2$), $F_i^{(r)}$ ($1 \leq r \leq -\langle \mu_2, \alpha_i \rangle + 2$) for all $i \in I$, with the following relations:

$$[S_i^{(k)}, S_j^{(l)}] = 0; \quad (4.2)$$

$$[S_i^{(-\langle \mu, \alpha_i \rangle + 1)}, E_j^{(r)}] = (\alpha_i \cdot \alpha_j) E_j^{(r)}, \quad 1 \leq r \leq \langle \mu_1, \alpha_j \rangle + 1; \quad (4.3)$$

$$[S_i^{(-\langle \mu, \alpha_i \rangle + 1)}, F_j^{(r)}] = -(\alpha_i \cdot \alpha_j) F_j^{(r)}, \quad 1 \leq r \leq \langle \mu_2, \alpha_j \rangle + 1; \quad (4.4)$$

$$[S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, E_j^{(r)}] = (\alpha_i \cdot \alpha_j) E_j^{(r+1)}, \quad 1 \leq r \leq \langle \mu_1, \alpha_j \rangle + 1; \quad (4.5)$$

$$[S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, F_j^{(r)}] = -(\alpha_i \cdot \alpha_j) F_j^{(r+1)}, \quad 1 \leq r \leq \langle \mu_2, \alpha_j \rangle + 1; \quad (4.6)$$

$$[E_i^{(r)}, F_j^{(s)}] = \begin{cases} 0 & i \neq j \\ 0 & i = j, r + s < -\langle \mu, \alpha_i \rangle + 1 \\ 1 & i = j, r + s = -\langle \mu, \alpha_i \rangle + 1 \\ S_i^{(-\langle \mu, \alpha_i \rangle + 1)} & i = j, r + s = -\langle \mu, \alpha_i \rangle + 2 \\ S_i^{(-\langle \mu, \alpha_i \rangle + 2)} + \frac{1}{2} (S_i^{(-\langle \mu, \alpha_i \rangle + 1)})^2 & i = j, r + s = -\langle \mu, \alpha_i \rangle + 3 \end{cases} \quad (4.7)$$

$$[E_i^{(r+1)}, E_j^{(s)}] = [E_i^{(r)}, E_j^{(s+1)}] + \frac{\alpha_i \cdot \alpha_j}{2} (E_i^{(r)} E_j^{(s)} + E_j^{(s)} E_i^{(r)}); \quad (4.8)$$

$$[F_i^{(r+1)}, F_j^{(s)}] = [F_i^{(r)}, F_j^{(s+1)}] - \frac{\alpha_i \cdot \alpha_j}{2} (F_i^{(r)} F_j^{(s)} + F_j^{(s)} F_i^{(r)}); \quad (4.9)$$

$$\text{ad}(E_i^{(1)})^{1-(\alpha_i \cdot \alpha_j)}(E_j^{(1)}) = 0; \quad (4.10)$$

$$\text{ad}(F_i^{(1)})^{1-(\alpha_i \cdot \alpha_j)}(F_j^{(1)}) = 0; \quad (4.11)$$

$$[S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, [E_i^{(-\langle \mu_1, \alpha_i \rangle + 2)}, F_i^{(-\langle \mu_2, \alpha_i \rangle + 2)}]] = 0. \quad (4.12)$$

For $r \geq 2$ and $s \geq 1$, set

$$\begin{aligned} E_i^{(r)} &= \frac{1}{2}[S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, E_i^{(r-1)}]; \\ F_i^{(r)} &= -\frac{1}{2}[S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, F_i^{(r-1)}]; \\ H_i^{(s)} &= [E_i^{(1)}, F_i^{(s)}]. \end{aligned}$$

Remark 4.2. Note that $H_i^{(s)} = 0$ if $s < -\langle \mu, \alpha_i \rangle$ and $H_i^{(-\langle \mu, \alpha_i \rangle)} = 1$.

Next, we have the following theorem, whose proof is almost exactly the same as in [15].

Theorem 4.3. *There exists an isomorphism $Y_\mu \longrightarrow Y_{\mu_1, \mu_2}$ of unital associative algebras given by*

$$E_i^{(r)} \mapsto E_i^{(r)}, F_i^{(r)} \mapsto F_i^{(r)}, H_i^{(s)} \mapsto H_i^{(s)},$$

for $r \geq 1$ and $s \geq -\langle \mu, \alpha_i \rangle + 1$.

4.4. The coproduct for the ordinary Yangian

Recall the following theorem going back to Drinfeld [6]:

Theorem 4.5. *There is a homomorphism $\Delta: Y \longrightarrow Y \otimes Y$, defined on the generators (see Lemma 3.11) by*

$$\begin{aligned} \Delta(X_i^{(1)}) &= X_i^{(1)} \otimes 1 + 1 \otimes X_i^{(1)}, \quad \text{for } X = E, F, S, \\ \Delta(S_i^{(2)}) &= S_i^{(2)} \otimes 1 + 1 \otimes S_i^{(2)} - \sum_{\gamma > 0} \langle \alpha_i, \gamma \rangle F_\gamma^{(1)} \otimes E_\gamma^{(1)} \end{aligned}$$

This formula for the coproduct was given without a proof in [13, (2.8)–(2.11)]. The proof is given in a recent paper of Guay–Nakajima–Wendlandt [10, Theorem 4.1].

4.6. The coproduct in the antidominant case

Let μ, μ_1, μ_2 be antidominant coweights with $\mu = \mu_1 + \mu_2$. We wish to define a homomorphism $\Delta_{\mu_1, \mu_2}: Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$ (we will denote $\Delta = \Delta_{\mu_1, \mu_2}$ when the algebras involved are clear). To do so, we define it on generators as follows:

$$\begin{aligned} \Delta(E_i^{(r)}) &= E_i^{(r)} \otimes 1, \quad 1 \leq r \leq -\langle \mu_1, \alpha_i \rangle; \\ \Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 1)}) &= E_i^{(-\langle \mu_1, \alpha_i \rangle + 1)} \otimes 1 + 1 \otimes E_i^{(1)}; \\ \Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 2)}) &= E_i^{(-\langle \mu_1, \alpha_i \rangle + 2)} \otimes 1 + 1 \otimes E_i^{(2)} + S_i^{(-\langle \mu_1, \alpha_i \rangle + 1)} \otimes E_i^{(1)} \end{aligned}$$

$$\begin{aligned}
& - \sum_{\gamma > 0} F_{\gamma}^{(1)} \otimes [E_i^{(1)}, E_{\gamma}^{(1)}]; \\
\Delta(F_i^{(r)}) &= 1 \otimes F_i^{(r)}, \quad 1 \leq r \leq -\langle \mu_2, \alpha_i \rangle; \\
\Delta(F_i^{(-\langle \mu_2, \alpha_i \rangle + 1)}) &= 1 \otimes F_i^{(-\langle \mu_2, \alpha_i \rangle + 1)} + F_i^{(1)} \otimes 1; \\
\Delta(F_i^{(-\langle \mu_2, \alpha_i \rangle + 2)}) &= 1 \otimes F_i^{(-\langle \mu_2, \alpha_i \rangle + 2)} + F_i^{(2)} \otimes 1 + F_i^{(1)} \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle + 1)} \\
& \quad + \sum_{\gamma > 0} [F_i^{(1)}, F_{\gamma}^{(1)}] \otimes E_{\gamma}^{(1)}; \\
\Delta(S_i^{(-\langle \mu, \alpha_i \rangle + 1)}) &= S_i^{(-\langle \mu_1, \alpha_i \rangle + 1)} \otimes 1 + 1 \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle + 1)}; \\
\Delta(S_i^{(-\langle \mu, \alpha_i \rangle + 2)}) &= S_i^{(-\langle \mu_1, \alpha_i \rangle + 2)} \otimes 1 + 1 \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle + 2)} - \sum_{\gamma > 0} \langle \alpha_i, \gamma \rangle F_{\gamma}^{(1)} \otimes E_{\gamma}^{(1)}.
\end{aligned}$$

Remark 4.7. When $\mu = \mu_1 = \mu_2 = 0$, it is not hard to see that $\Delta_{0,0}$ agrees with the coproduct from [Theorem 4.5](#), and hence is well-defined.

Recall that there are shift maps $\iota_{0,\mu_1,0}$ and $\iota_{0,0,\mu_2}$, by [Proposition 3.8](#). It is not hard to see that, for $k = 1, 2$,

$$\begin{aligned}
\Delta(S_i^{(-\langle \mu, \alpha_i \rangle + k)}) &= (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(S_i^{(k)}), \\
\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + k)}) &= (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(E_i^{(k)}), \\
\Delta(F_i^{(-\langle \mu_2, \alpha_i \rangle + k)}) &= (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(F_i^{(k)}).
\end{aligned}$$

Theorem 4.8. $\Delta: Y_{\mu} \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$ is a well-defined map.

Proof. We have to check that Δ preserves the defining relations. By [Theorem 4.3](#) it suffices to check the relations [\(4.2\)–\(4.12\)](#).

First, we check relation [\(4.2\)](#). For $1 \leq k, l \leq 2$,

$$\begin{aligned}
& [\Delta(S_i^{(-\langle \mu, \alpha_i \rangle + k)}), \Delta(S_j^{(-\langle \mu, \alpha_j \rangle + l)})] = \\
& = [(\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(S_i^{(k)}), (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(S_j^{(l)})] = 0.
\end{aligned}$$

We check relation [\(4.3\)](#). For $1 \leq r \leq -\langle \mu_1, \alpha_j \rangle$,

$$[\Delta(S_i^{(-\langle \mu, \alpha_i \rangle + 1)}), \Delta(E_j^{(r)})] = [S_i^{(-\langle \mu_1, \alpha_i \rangle + 1)}, E_j^{(r)}] \otimes 1 = (\alpha_i \cdot \alpha_j) \Delta(E_j^{(r)}).$$

For $r = -\langle \mu_1, \alpha_j \rangle + 1$,

$$\begin{aligned}
[\Delta(S_i^{(-\langle \mu, \alpha_i \rangle + 1)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)})] &= (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}([S_i^{(1)}, E_j^{(1)}]) \\
&= (\alpha_i \cdot \alpha_j) (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(E_j^{(1)}) \\
&= (\alpha_i \cdot \alpha_j) \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}).
\end{aligned}$$

The proof for relation (4.4) is similar to that of relation (4.3).

We check relation (4.5). For $1 \leq r \leq -\langle \mu_1, \alpha_j \rangle$,

$$\begin{aligned} [\Delta(S_i^{(-\langle \mu, \alpha_i \rangle + 2)}), \Delta(E_j^{(r)})] &= [S_i^{(-\langle \mu_1, \alpha_i \rangle + 2)}, E_j^{(r)}] \otimes 1 + \sum_{\gamma > 0} \langle \alpha_i, \gamma \rangle [E_j^{(r)}, F_\gamma^{(1)}] \otimes E_\gamma^{(1)} \\ &= (\alpha_i \cdot \alpha_j) E_j^{(r+1)} \otimes 1 + \sum_{\gamma > 0} \langle \alpha_i, \gamma \rangle [E_j^{(r)}, F_\gamma^{(1)}] \otimes E_\gamma^{(1)}. \end{aligned}$$

Note that if $r < -\langle \mu_1, \alpha_i \rangle$, then $[E_j^{(r)}, F_l^{(1)}] = 0$ for all l . Then, by induction, $[E_j^{(r)}, F_\gamma^{(1)}] = 0$ for all $\gamma > 0$. The result follows in this case. If $r = -\langle \mu_1, \alpha_i \rangle$, then $[E_j^{(r)}, F_i^{(1)}] = \delta_{ij} 1$. Then, by induction, $[E_j^{(r)}, F_\gamma^{(1)}] = 0$ for all γ of height greater than or equal to 2. The second summand becomes $(\alpha_i \cdot \alpha_j) 1 \otimes E_j^{(1)}$. Hence, the result follows.

For $r = -\langle \mu_1, \alpha_j \rangle + 1$,

$$\begin{aligned} [\Delta(S_i^{(-\langle \mu, \alpha_i \rangle + 2)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)})] &= (\iota_{0, \mu_1, 0} \otimes \iota_{0, 0, \mu_2}) \Delta_{0,0}([S_i^{(2)}, E_j^{(1)}]) \\ &= (\alpha_i \cdot \alpha_j) (\iota_{0, \mu_1, 0} \otimes \iota_{0, 0, \mu_2}) \Delta_{0,0}(E_j^{(2)}) \\ &= (\alpha_i \cdot \alpha_j) \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 2)}). \end{aligned}$$

Similarly, Δ preserves relation (4.6).

Next, we check relation (4.7). If $1 \leq r \leq -\langle \mu_1, \alpha_i \rangle$ and $1 \leq s \leq -\langle \mu_2, \alpha_j \rangle$, then

$$[\Delta(E_i^{(r)}), \Delta(F_j^{(s)})] = [E_i^{(r)} \otimes 1, 1 \otimes F_j^{(s)}] = 0.$$

For $r = -\langle \mu_1, \alpha_i \rangle + 1$ and $1 \leq s \leq -\langle \mu_2, \alpha_j \rangle$,

$$[\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 1)}), \Delta(F_j^{(s)})] = 1 \otimes [E_i^{(1)}, F_j^{(s)}] = \delta_{ij} 1 \otimes H_i^{(s)}.$$

The result follows for this case.

The case where $r \leq -\langle \mu_1, \alpha_i \rangle$ and $s = -\langle \mu_2, \alpha_j \rangle + 1$ is similar.

Consider the case where $r = -\langle \mu_1, \alpha_i \rangle + 2$ and $1 \leq s \leq -\langle \mu_2, \alpha_j \rangle$,

$$\begin{aligned} [\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 2)}), \Delta(F_j^{(s)})] &= \\ &= 1 \otimes [E_i^{(2)}, F_j^{(s)}] + S_i^{(-\langle \mu_1, \alpha_i \rangle + 1)} \otimes [E_i^{(1)}, F_j^{(s)}] - \sum_{\gamma > 0} F_\gamma^{(1)} \otimes [[E_i^{(1)}, E_\gamma^{(1)}], F_j^{(s)}] \\ &= \delta_{ij} 1 \otimes H_i^{(s+1)} + \delta_{ij} S_i^{(-\langle \mu_1, \alpha_i \rangle + 1)} \otimes H_i^{(s)} - \sum_{\gamma > 0} F_\gamma^{(1)} \otimes [[E_i^{(1)}, E_\gamma^{(1)}], F_j^{(s)}]. \end{aligned}$$

Note that

$$[[E_i^{(1)}, E_\gamma^{(1)}], F_j^{(s)}] = [E_i^{(1)}, [E_\gamma^{(1)}, F_j^{(s)}]] + [E_\gamma^{(1)}, [F_j^{(s)}, E_i^{(1)}]].$$

Since $s \leq -\langle \mu_2, \alpha_j \rangle$, by induction, $[E_\gamma^{(1)}, F_j^{(s)}] \in \mathbb{C}1$. Hence, $[E_i^{(1)}, [E_\gamma^{(1)}, F_j^{(s)}]] = 0$. Again, since $s \leq -\langle \mu_2, \alpha_j \rangle$, $[F_j^{(s)}, E_i^{(1)}] = \delta_{ij} H_j^{(s)} \in \mathbb{C}1$. So, $[E_\gamma^{(1)}, [F_j^{(s)}, E_i^{(1)}]] = 0$. Hence, the last sum is 0. Moreover, it is straightforward to check that the first two summands are consistent with the relation.

The case where $1 \leq r \leq -\langle \mu_1, \alpha_i \rangle$ and $s = -\langle \mu_2, \alpha_j \rangle + 2$ is similar.

Next, for $1 \leq k, l \leq 2$ not both equal to 2, we have that

$$\begin{aligned} [\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + k)}), \Delta(F_j^{(-\langle \mu_2, \alpha_j \rangle + l)})] &= (\iota_{0, \mu_1, 0} \otimes \iota_{0, 0, \mu_2}) \Delta_{0,0}([E_i^{(k)}, F_j^{(l)}]) \\ &= \delta_{ij} (\iota_{0, \mu_1, 0} \otimes \iota_{0, 0, \mu_2}) \Delta_{0,0}(H_i^{(k+l-1)}) \\ &= \delta_{ij} \Delta(H_i^{(-\langle \mu, \alpha_i \rangle + k+l-1)}). \end{aligned}$$

Next, we check relation (4.8).

First, consider the case $1 \leq r < -\langle \mu_1, \alpha_i \rangle$ and $1 \leq s < -\langle \mu_1, \alpha_j \rangle$. Then, we have

$$\begin{aligned} [\Delta(E_i^{(r+1)}), \Delta(E_j^{(s)})] &= [E_i^{(r+1)}, E_j^{(s)}] \otimes 1 = \\ &= \left([E_i^{(r)}, E_j^{(s+1)}] + \frac{\alpha_i \cdot \alpha_j}{2} (E_i^{(r)} E_j^{(s)} + E_j^{(s)} E_i^{(r)}) \right) \otimes 1 \\ &= [E_i^{(r)} \otimes 1, E_j^{(s+1)} \otimes 1] + \frac{\alpha_i \cdot \alpha_j}{2} ((E_i^{(r)} \otimes 1)(E_j^{(s)} \otimes 1) + (E_j^{(s)} \otimes 1)(E_i^{(r)} \otimes 1)). \end{aligned}$$

Consider the case $1 \leq r < -\langle \mu_1, \alpha_i \rangle$ and $s = -\langle \mu_1, \alpha_j \rangle$.

$$\begin{aligned} [\Delta(E_i^{(r+1)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle)})] &= [\Delta(E_i^{(r)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)})] \\ &= ([E_i^{(r+1)}, E_j^{(-\langle \mu_1, \alpha_j \rangle)}] - [E_i^{(r)}, E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}]) \otimes 1 \\ &= \frac{\alpha_i \cdot \alpha_j}{2} ((E_i^{(r)} \otimes 1)(E_j^{(-\langle \mu_1, \alpha_j \rangle)} \otimes 1) + (E_j^{(-\langle \mu_1, \alpha_j \rangle)} \otimes 1)(E_i^{(r)} \otimes 1)). \end{aligned}$$

The case where $r = -\langle \mu_1, \alpha_i \rangle$ and $1 \leq s < -\langle \mu_1, \alpha_j \rangle$ is similar.

Next, consider the case $1 \leq r < -\langle \mu_1, \alpha_i \rangle$ and $s = -\langle \mu_1, \alpha_j \rangle + 1$.

$$\begin{aligned} [\Delta(E_i^{(r+1)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)})] &= [\Delta(E_i^{(r)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 2)})] \\ &= [E_i^{(r+1)}, E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}] \otimes 1 - [E_i^{(r)}, E_j^{(-\langle \mu_1, \alpha_j \rangle + 2)}] \otimes 1 - [E_i^{(r)}, S_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}] \otimes E_j^{(1)} \\ &\quad + \sum_{\gamma > 0} [E_i^{(r)}, F_\gamma^{(1)}] \otimes [E_j^{(1)}, E_\gamma^{(1)}] \\ &= \frac{\alpha_i \cdot \alpha_j}{2} (E_i^{(r)} E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)} + E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)} E_i^{(r)}) \otimes 1 + (\alpha_i \cdot \alpha_j) E_i^{(r)} \otimes E_j^{(1)} \\ &\quad + \sum_{\gamma > 0} [E_i^{(r)}, F_\gamma^{(1)}] \otimes [E_j^{(1)}, E_\gamma^{(1)}]. \end{aligned}$$

Since $r < -\langle \mu_1, \alpha_i \rangle$, by induction, $[E_i^{(r)}, F_\gamma^{(1)}] = 0$ for all $\gamma > 0$. The current case follows. The proof for $r = -\langle \mu_1, \alpha_i \rangle + 1$ and $s < -\langle \mu_1, \alpha_j \rangle$ is similar.

Next, let us look at the case $r = -\langle \mu_1, \alpha_i \rangle$ and $s = -\langle \mu_1, \alpha_j \rangle$.

$$\begin{aligned} & [\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 1)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle)})] - [\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)})] \\ &= ([E_i^{(-\langle \mu_1, \alpha_i \rangle)}, E_j^{(-\langle \mu_1, \alpha_j \rangle)}] - [E_i^{(-\langle \mu_1, \alpha_i \rangle)}, E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}]) \otimes 1 \\ &= \frac{\alpha_i \cdot \alpha_j}{2} ((E_i^{(-\langle \mu_1, \alpha_i \rangle)} \otimes 1)(E_j^{(-\langle \mu_1, \alpha_j \rangle)} \otimes 1) + (E_j^{(-\langle \mu_1, \alpha_j \rangle)} \otimes 1)(E_i^{(-\langle \mu_1, \alpha_i \rangle)} \otimes 1)). \end{aligned}$$

Next, for $r = -\langle \mu_1, \alpha_i \rangle$ and $s = -\langle \mu_1, \alpha_j \rangle + 1$.

$$\begin{aligned} & [\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 1)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)})] - [\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 2)})] = \\ &= [E_i^{(-\langle \mu_1, \alpha_i \rangle + 1)}, E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}] \otimes 1 + 1 \otimes [E_i^{(1)}, E_j^{(1)}] - [E_i^{(-\langle \mu_1, \alpha_i \rangle)}, E_j^{(-\langle \mu_1, \alpha_j \rangle + 2)}] \otimes 1 \\ &- [E_i^{(-\langle \mu_1, \alpha_i \rangle)}, S_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}] \otimes E_j^{(1)} + \sum_{\gamma > 0} [E_i^{(-\langle \mu_1, \alpha_i \rangle)}, F_\gamma^{(1)}] \otimes [E_j^{(1)}, E_\gamma^{(1)}] \\ &= \frac{\alpha_i \cdot \alpha_j}{2} (E_i^{(-\langle \mu_1, \alpha_i \rangle)} E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)} \otimes 1 + E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)} E_i^{(-\langle \mu_1, \alpha_i \rangle + 1)} \otimes 1 \\ &+ 2E_i^{(-\langle \mu_1, \alpha_i \rangle)} \otimes E_j^{(1)}) + 1 \otimes [E_i^{(1)}, E_j^{(1)}] + \sum_{\gamma > 0} [E_i^{(-\langle \mu_1, \alpha_i \rangle)}, F_\gamma^{(1)}] \otimes [E_j^{(1)}, E_\gamma^{(1)}]. \end{aligned}$$

Note that $[E_i^{(-\langle \mu_1, \alpha_i \rangle)}, F_l^{(1)}] \in \mathbb{C}1$. So, if γ is of height greater than or equal to 2, then $[E_i^{(-\langle \mu_1, \alpha_i \rangle)}, F_\gamma^{(1)}] = 0$ by induction. Hence, the only term that survives in the last summand is $1 \otimes [E_j^{(1)}, E_i^{(1)}]$ and we are done. The case $r = -\langle \mu_1, \alpha_i \rangle + 1$ and $s = -\langle \mu_1, \alpha_j \rangle$ is totally analogous.

Lastly, consider the case $r = -\langle \mu_1, \alpha_i \rangle + 1$ and $s = -\langle \mu_1, \alpha_j \rangle + 1$.

$$\begin{aligned} & [\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 2)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)})] - [\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 1)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 2)})] = \\ &= (\iota_{0, \mu_1, 0} \otimes \iota_{0, 0, \mu_2}) \Delta_{0, 0} ([E_i^{(2)}, E_j^{(1)}] - [E_i^{(1)} E_j^{(2)}]) \\ &= \frac{\alpha_i \cdot \alpha_j}{2} (\iota_{0, \mu_1, 0} \otimes \iota_{0, 0, \mu_2}) \Delta_{0, 0} (E_i^{(1)} E_j^{(1)} + E_j^{(1)} E_i^{(1)}) \\ &= \frac{\alpha_i \cdot \alpha_j}{2} (\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 1)}) \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}) + \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}) \Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 1)})). \end{aligned}$$

Relation (4.9) can be checked in the same fashion.

We now check relation (4.10). Set $N = 1 - \alpha_i \cdot \alpha_j$. First, if $1 \leq -\langle \mu_1, \alpha_i \rangle$ and $1 \leq -\langle \mu_1, \alpha_j \rangle$, then

$$\text{ad}(\Delta(E_i^{(1)}))^N (\Delta(E_j^{(1)})) = \text{ad}(E_i^{(1)} \otimes 1)^N (E_j^{(1)} \otimes 1) = ((\text{ad } E_i^{(1)})^N (E_j^{(1)})) \otimes 1 = 0.$$

For $1 \leq -\langle \mu_1, \alpha_i \rangle$ and $1 = -\langle \mu_1, \alpha_j \rangle + 1$.

$$\begin{aligned} \text{ad}(\Delta(E_i^{(1)}))^N (\Delta(E_j^{(1)})) &= \text{ad}(E_i^{(1)} \otimes 1)^N (E_j^{(1)} \otimes 1 + 1 \otimes E_j^{(1)}) \\ &= ((\text{ad } E_i^{(1)})^N (E_j^{(1)})) \otimes 1 = 0, \end{aligned}$$

since $E_i^{(1)} \otimes 1$ commutes with $1 \otimes E_j^{(1)}$.

Next, suppose $1 = -\langle \mu_1, \alpha_i \rangle + 1$. Since $[E_i^{(1)} \otimes 1, 1 \otimes E_i^{(1)}] = 0$,

$$\begin{aligned} (\text{ad}(\Delta(E_i^{(1)})))^N &= (\text{ad}(E_i^{(1)} \otimes 1) + \text{ad}(1 \otimes E_i^{(1)}))^N \\ &= \sum_{l=0}^N \binom{N}{l} \text{ad}(E_i^{(1)} \otimes 1)^l \text{ad}(1 \otimes E_i^{(1)})^{N-l}. \end{aligned}$$

Now, if $1 \leq -\langle \mu_1, \alpha_j \rangle$, then

$$\begin{aligned} (\text{ad}(\Delta(E_i^{(1)})))^N (\Delta(E_j^{(1)})) &= \sum_{l=0}^N \binom{N}{l} \text{ad}(E_i^{(1)} \otimes 1)^l \text{ad}(1 \otimes E_i^{(1)})^{N-l} (E_j^{(1)} \otimes 1) \\ &= \text{ad}(E_i^{(1)})^N (E_j^{(1)}) \otimes 1 = 0. \end{aligned}$$

If $1 = -\langle \mu_1, \alpha_j \rangle + 1$, then

$$\begin{aligned} \text{ad}(\Delta(E_i^{(1)}))^N (\Delta(E_j^{(1)})) &= \sum_{l=0}^N \binom{N}{l} \text{ad}(E_i^{(1)} \otimes 1)^l \text{ad}(1 \otimes E_i^{(1)})^{N-l} (E_j^{(1)} \otimes 1 + 1 \otimes E_j^{(1)}) \\ &= \text{ad}(E_i^{(1)})^N (E_j^{(1)}) \otimes 1 + 1 \otimes \text{ad}(E_i^{(1)})^N (E_j^{(1)}) = 0. \end{aligned}$$

The proof for (4.11) is similar to that of (4.10).

Finally, we check relation (4.12).

$$\begin{aligned} &[\Delta(S_i^{(-\langle \mu, \alpha_i \rangle + 2)}), [\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 2)}), \Delta(F_i^{(-\langle \mu_2, \alpha_i \rangle + 2)})]] \\ &= (\iota_{0, \mu_1, 0} \otimes \iota_{0, 0, \mu_2}) \Delta_{0, 0}([S_i^{(2)}, [E_i^{(2)}, F_i^{(2)}]]) = 0. \end{aligned}$$

This proves that Δ is well-defined. \square

By Lemma 3.11, we have the following:

Lemma 4.9. *The coproduct $\Delta: Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$ is uniquely determined by*

$$\begin{aligned} \Delta(E_i^{(1)}) &= E_i^{(1)} \otimes 1 + \delta_{\langle \mu_1, \alpha_i \rangle, 0} 1 \otimes E_i^{(1)}; \\ \Delta(F_i^{(1)}) &= \delta_{\langle \mu_2, \alpha_i \rangle, 0} F_i^{(1)} \otimes 1 + 1 \otimes F_i^{(1)}; \\ \Delta(S_i^{(-\langle \mu, \alpha_i \rangle + 1)}) &= S_i^{(-\langle \mu_1, \alpha_i \rangle + 1)} \otimes 1 + 1 \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle + 1)}; \\ \Delta(S_i^{(-\langle \mu, \alpha_i \rangle + 2)}) &= S_i^{(-\langle \mu_1, \alpha_i \rangle + 2)} \otimes 1 + 1 \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle + 2)} - \sum_{\gamma > 0} \langle \alpha_i, \gamma \rangle F_\gamma^{(1)} \otimes E_\gamma^{(1)}. \end{aligned}$$

Proposition 4.10. *Let $\mu = \mu_1 + \mu_2 + \mu_3$ where the μ_i 's are antidominant coweights. Then, we have the following commutative diagram*

$$\begin{array}{ccc}
Y_\mu & \xrightarrow{\Delta_{\mu_1, \mu_2 + \mu_3}} & Y_{\mu_1} \otimes Y_{\mu_2 + \mu_3} \\
\Delta_{\mu_1 + \mu_2, \mu_3} \downarrow & & \downarrow 1 \otimes \Delta_{\mu_2, \mu_3} \\
Y_{\mu_1 + \mu_2} \otimes Y_{\mu_3} & \xrightarrow{\Delta_{\mu_1, \mu_2} \otimes 1} & Y_{\mu_1} \otimes Y_{\mu_2} \otimes Y_{\mu_3}
\end{array}$$

Proof. By Lemma 4.9, it is enough to check for $S_i^{(-\langle \mu, \alpha_i \rangle + k)}$ ($k = 1, 2$), $E_i^{(1)}$ and $F_i^{(1)}$.

$$\begin{aligned}
(1 \otimes \Delta_{\mu_2, \mu_3}) \Delta_{\mu_1, \mu_2 + \mu_3}(E_i^{(1)}) &= E_i^{(1)} \otimes 1 \otimes 1 + \delta_{\langle \mu_1, \alpha_i \rangle, 0} 1 \otimes E_i^{(1)} \otimes 1 \\
&\quad + \delta_{\langle \mu_1, \alpha_i \rangle, 0} \delta_{\langle \mu_2, \alpha_i \rangle, 0} 1 \otimes 1 \otimes E_i^{(1)}, \\
(\Delta_{\mu_1, \mu_2} \otimes 1) \Delta_{\mu_1 + \mu_2, \mu_3}(E_i^{(1)}) &= E_i^{(1)} \otimes 1 \otimes 1 + \delta_{\langle \mu_1, \alpha_i \rangle, 0} 1 \otimes E_i^{(1)} \otimes 1 \\
&\quad + \delta_{\langle \mu_1 + \mu_2, \alpha_i \rangle} 1 \otimes 1 \otimes E_i^{(1)}.
\end{aligned}$$

The result follows for $E_i^{(1)}$ since $\delta_{\langle \mu_1 + \mu_2, \alpha_i \rangle, 0} = \delta_{\langle \mu_1, \alpha_i \rangle, 0} \delta_{\langle \mu_2, \alpha_i \rangle, 0}$. The computation for $F_i^{(1)}$ is totally analogous. The computation $S_i^{(-\langle \mu, \alpha_i \rangle + 1)}$ is straightforward.

Finally, we have that

$$\begin{aligned}
&(1 \otimes \Delta_{\mu_2, \mu_3}) \Delta_{\mu_1, \mu_2 + \mu_3}(S_i^{(-\langle \mu, \alpha_i \rangle + 2)}) = \\
&= 1 \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle + 2)} \otimes 1 + 1 \otimes 1 \otimes S_i^{(-\langle \mu_3, \alpha_i \rangle + 2)} - \sum_{\beta > 0} \langle \alpha_i, \beta \rangle 1 \otimes F_\beta^{(1)} \otimes E_\beta^{(1)} \\
&+ S_i^{(-\langle \mu_1, \alpha_i \rangle + 2)} \otimes 1 \otimes 1 - \sum_{\gamma > 0} \langle \alpha_i, \gamma \rangle F_\gamma^{(1)} \otimes \Delta_{\mu_2, \mu_3}(E_\gamma^{(1)}), \\
&(\Delta_{\mu_1, \mu_2} \otimes 1) \Delta_{\mu_1 + \mu_2, \mu_3}(S_i^{(-\langle \mu, \alpha_i \rangle + 2)}) = \\
&= S_i^{(-\langle \mu_1, \alpha_i \rangle + 2)} \otimes 1 \otimes 1 + 1 \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle + 2)} \otimes 1 - \sum_{\beta > 0} \langle \alpha_i, \beta \rangle F_\beta^{(1)} \otimes E_\beta^{(1)} \otimes 1 \\
&+ 1 \otimes 1 \otimes S_i^{(-\langle \mu_3, \alpha_i \rangle + 2)} - \sum_{\gamma > 0} \langle \alpha_i, \gamma \rangle \Delta_{\mu_1, \mu_2}(F_\gamma^{(1)}) \otimes E_\gamma^{(1)}.
\end{aligned}$$

For a positive root $\gamma = \sum_i n_i \alpha_i$, by a simply induction, we can show that $\Delta_{\mu_2, \mu_3}(E_\gamma^{(1)}) = E_\gamma^{(1)} \otimes 1 + C_\gamma 1 \otimes E_\gamma^{(1)}$ and that $\Delta_{\mu_1, \mu_2}(F_\gamma^{(1)}) = 1 \otimes F_\gamma^{(1)} + C_\gamma F_\gamma^{(1)} \otimes 1$ where $C_\gamma = \prod_i \delta_{\langle \mu_2, \alpha_i \rangle, 0}^{n_i}$. The result follows. \square

4.11. The coproduct in the general case

Theorem 4.12. Let $\mu = \mu_1 + \mu_2$ where μ, μ_1, μ_2 are arbitrary coweights. There exists a coproduct $\Delta_{\mu_1, \mu_2}: Y_\mu \rightarrow Y_{\mu_1} \otimes Y_{\mu_2}$ such that, for all antidominant coweights η_1, η_2 , the following diagram is commutative

$$\begin{array}{ccc}
 Y_\mu & \xrightarrow{\Delta_{\mu_1, \mu_2}} & Y_{\mu_1} \otimes Y_{\mu_2} \\
 \downarrow \iota_{\mu, \eta_1, \eta_2} & & \downarrow (\iota_{\mu_1, \eta_1, 0}) \otimes (\iota_{\mu_2, 0, \eta_2}) \\
 Y_{\mu + \eta_1 + \eta_2} & \xrightarrow{\Delta_{\mu_1 + \eta_1, \mu_2 + \eta_2}} & Y_{\mu_1 + \eta_1} \otimes Y_{\mu_2 + \eta_2}
 \end{array}$$

Proof. First, we need to define the map Δ_{μ_1, μ_2} . Let η_1, η_2 be antidominant coweights such that $\mu_1 + \eta_1$ and $\mu_2 + \eta_2$ are also antidominant. We see that $\mu + \eta_1 + \eta_2$ is also antidominant.

Consider the diagram

$$\begin{array}{ccc}
 Y_\mu & & Y_{\mu_1} \otimes Y_{\mu_2} \\
 \downarrow \iota_{\mu, \eta_1, \eta_2} & & \downarrow (\iota_{\mu_1, \eta_1, 0}) \otimes (\iota_{\mu_2, 0, \eta_2}) \\
 Y_{\mu + \eta_1 + \eta_2} & \xrightarrow{\Delta = \Delta_{\mu_1 + \eta_1, \mu_2 + \eta_2}} & Y_{\mu_1 + \eta_1} \otimes Y_{\mu_2 + \eta_2}
 \end{array}$$

In order to define Δ_{μ_1, μ_2} , we need to show that

$$\Delta(\iota_{\mu, \eta_1, \eta_2}(Y_\mu)) \subseteq (\iota_{\mu_1, \eta_1, 0} \otimes \iota_{\mu_2, 0, \eta_2})(Y_{\mu_1} \otimes Y_{\mu_2}).$$

Note that $Y_{\mu_1 + \eta_1}^{\leq} \otimes Y_{\mu_2 + \eta_2}^{\geq} \subseteq \iota_{\mu_1, \eta_1, 0} \otimes \iota_{\mu_2, 0, \eta_2}(Y_{\mu_1} \otimes Y_{\mu_2})$.

First, for $r \geq 1$, we claim that

$$\begin{aligned}
 \Delta(E_i^{(r)}) &\in E_i^{(r)} \otimes 1 + Y_{\mu_1 + \eta_1}^{\leq} \otimes Y_{\mu_2 + \eta_2}^{\geq}, \\
 \Delta(F_i^{(r)}) &\in 1 \otimes F_i^{(r)} + Y_{\mu_1 + \eta_1}^{\leq} \otimes Y_{\mu_2 + \eta_2}^{\geq}.
 \end{aligned}$$

We prove the claim for E , the proof for F is similar. We proceed by induction.

If $1 \leq -\langle \mu_1 + \eta_1, \alpha_i \rangle$, then it is clear since $\Delta(E_i^{(1)}) = E_i^{(1)} \otimes 1$.

If $0 = \langle \mu_1 + \eta_1, \alpha_i \rangle$, then it is also clear since $\Delta(E_i^{(1)}) = E_i^{(1)} \otimes 1 + 1 \otimes E_i^{(1)}$ and since $1 \otimes E_i^{(1)} \in Y_{\mu_1 + \eta_1}^{\leq} \otimes Y_{\mu_2 + \eta_2}^{\geq}$.

The induction step follows from the fact that Δ is a homomorphism and the fact that $[S_i^{(-\langle \mu + \eta_1 + \eta_2, \alpha_i \rangle + 2)}, E_i^{(r)}] = 2E_i^{(r+1)}$. This proves the claim.

Note that $\iota_{\mu, \eta_1, \eta_2}(Y_\mu)$ is generated by $E_i^{(r)}(r > -\langle \eta_1, \alpha_i \rangle)$, $F_i^{(s)}(s > -\langle \eta_2, \alpha_i \rangle)$ and $H_i^{(t)}(t > -\langle \mu + \eta_1 + \eta_2, \alpha_i \rangle)$.

Applying the claim for $r > -\langle \eta_1, \alpha_i \rangle$, we get $\Delta(E_i^{(r)}) \in (\iota_{\mu_1, \eta_1, 0} \otimes \iota_{\mu_2, 0, \eta_2})(Y_{\mu_1} \otimes Y_{\mu_2})$ since $E_i^{(r)} \otimes 1 = (\iota_{\mu_1, \eta_1, 0} \otimes \iota_{\mu_2, 0, \eta_2})(E_i^{(r + \langle \eta_1, \alpha_i \rangle)} \otimes 1)$.

Similarly, we obtain $\Delta(F_i^{(r)}) \in (\iota_{\mu_1, \eta_1, 0} \otimes \iota_{\mu_2, 0, \eta_2})(Y_{\mu_1} \otimes Y_{\mu_2})$ for $s > -\langle \eta_2, \alpha_i \rangle$.

Finally, for $t > -\langle \mu + \eta_1 + \eta_2, \alpha_i \rangle$,

$$\begin{aligned} \Delta(H_i^{(t)}) &= [\Delta(E_i^{(t)}), \Delta(F_i^{(1)})] \\ &\in [E_i^{(t)} \otimes 1, Y_{\mu_1+\eta_1}^{\leq} \otimes Y_{\mu_2+\eta_2}^{\geq}] + [Y_{\mu_1+\eta_1}^{\leq} \otimes Y_{\mu_2+\eta_2}^{\geq}, 1 \otimes F_i^{(1)}] \\ &\subseteq Y_{\mu_1+\eta_1}^{\leq} \otimes Y_{\mu_2+\eta_2}^{\geq}. \end{aligned}$$

Therefore, we have a coproduct $\Delta_{\mu_1, \mu_2} : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$.

Next, we show that Δ_{μ_1, μ_2} is independent of the choice of η_1, η_2 , i.e., for all η_1, η_2 such that $\mu_1 + \eta_1, \mu_2 + \eta_2$ are antidominant, the diagram in the statement of the theorem is commutative. To see this, let η'_1, η'_2 be another such pair of coweights. Consider the diagram

$$\begin{array}{ccc} Y_\mu & & Y_{\mu_1} \otimes Y_{\mu_2} \\ \downarrow \iota_{\mu, \eta_1, \eta_2} & & \downarrow (\iota_{\mu_1, \eta_1, 0}) \otimes (\iota_{\mu_2, 0, \eta_2}) \\ Y_{\mu+\eta} & \xrightarrow{\Delta_{\mu_1+\eta_1, \mu_2+\eta_2}} & Y_{\mu_1+\eta_1} \otimes Y_{\mu_2+\eta_2} \\ \downarrow \iota_{\mu+\eta, \eta'_1, \eta'_2} & & \downarrow (\iota_{\mu_1+\eta_1, \eta'_1, 0}) \otimes (\iota_{\mu_2+\eta_2, 0, \eta'_2}) \\ Y_{\mu+\eta+\eta'} & \xrightarrow{\Delta_{\mu_1+\eta_1+\eta'_1, \mu_2+\eta_2+\eta'_2}} & Y_{\mu_1+\eta_1+\eta'_1} \otimes Y_{\mu_2+\eta_2+\eta'_2} \end{array}$$

We see that $\iota_{\mu+\eta, \eta'_1, \eta'_2} \circ \iota_{\mu, \eta_1, \eta_2} = \iota_{\mu, \eta_1+\eta'_1, \eta_2+\eta'_2}$ and $\iota_{\mu_1+\eta_1, \eta'_1, 0} \circ \iota_{\mu_1, \eta_1, 0} = \iota_{\mu, \eta_1+\eta'_1, 0}$ and $\iota_{\mu_2+\eta_2, 0, \eta'_2} \circ \iota_{\mu_2, 0, \eta_2} = \iota_{\mu_2, 0, \eta_2+\eta'_2}$. Moreover, it is not hard to check on generators that the lower square commutes.

Therefore, the choice of Δ_{μ_1, μ_2} is the same for the pairs of coweights (η_1, η_2) and $(\eta_1 + \eta'_1, \eta_2 + \eta'_2)$. By swapping the roles of η and η' in the above, the choice of Δ_{μ_1, μ_2} is also the same for the pairs (η'_1, η'_2) and $(\eta_1 + \eta'_1, \eta_2 + \eta'_2)$.

Finally, we check that the diagram in the statement of the theorem commutes for any pair of antidominant coweights η_1, η_2 . Let η'_1, η'_2 be antidominant coweights such that $\mu_k + \eta_k + \eta'_k$ ($k = 1, 2$) are antidominant. Consider the diagram

$$\begin{array}{ccc} Y_\mu & \xrightarrow{\Delta_{\mu_1, \mu_2}} & Y_{\mu_1} \otimes Y_{\mu_2} \\ \downarrow \iota_{\mu, \eta_1, \eta_2} & & \downarrow (\iota_{\mu_1, \eta_1, 0}) \otimes (\iota_{\mu_2, 0, \eta_2}) \\ Y_{\mu+\eta} & \xrightarrow{\Delta_{\mu_1+\eta_1, \mu_2+\eta_2}} & Y_{\mu_1+\eta_1} \otimes Y_{\mu_2+\eta_2} \\ \downarrow \iota_{\mu+\eta, \eta'_1, \eta'_2} & & \downarrow (\iota_{\mu_1+\eta_1, \eta'_1, 0}) \otimes (\iota_{\mu_2+\eta_2, 0, \eta'_2}) \\ Y_{\mu+\eta+\eta'} & \xrightarrow{\Delta_{\mu_1+\eta_1+\eta'_1, \mu_2+\eta_2+\eta'_2}} & Y_{\mu_1+\eta_1+\eta'_1} \otimes Y_{\mu_2+\eta_2+\eta'_2} \end{array}$$

Since $\iota_{\mu+\eta, \eta'_1, \eta'_2} \circ \iota_{\mu, \eta_1, \eta_2} = \iota_{\mu, \eta_1+\eta'_1, \eta_2+\eta'_2}$ and $\iota_{\mu_1+\eta_1, \eta'_1, 0} \circ \iota_{\mu_1, \eta_1, 0} = \iota_{\mu, \eta_1+\eta'_1, 0}$ and $\iota_{\mu_2+\eta_2, 0, \eta'_2} \circ \iota_{\mu_2, 0, \eta_2} = \iota_{\mu_2, 0, \eta_2+\eta'_2}$, the outer square and the lower square are commu-

tative. Since $\iota_{\mu_1+\eta_1,\eta'_1,0} \otimes \iota_{\mu_2+\eta_2,0,\eta'_2}$ is injective, we see that the upper square is also commutative. \square

Remark 4.13. Brundan and Kleshchev define a coproduct for shifted \mathfrak{gl}_n -Yangians in [5, Theorem 11.9], which is analogous to our coproduct in the \mathfrak{sl}_n case when $\mu = \mu_1 + \mu_2$ are all dominant. Namely, form the associated lower (resp. upper) triangular shift matrices σ' , σ'' by extending $s'_{i+1,i} = \mu_{1,i}$ and $s''_{i,i+1} = \mu_{2,i}$, and take $\sigma = \sigma' + \sigma''$. Then Brundan and Kleshchev's coproduct $Y_n(\sigma) \rightarrow Y_n(\sigma') \otimes Y_n(\sigma'')$ is defined by embedding into $Y(\mathfrak{gl}_n) \rightarrow Y(\mathfrak{gl}_n) \otimes Y(\mathfrak{gl}_n)$. However, the standard inclusion of Hopf algebras $Y(\mathfrak{sl}_n) \hookrightarrow Y(\mathfrak{gl}_n)$ is *not* compatible with our shift map $\iota_{\mu,-\mu_1,-\mu_2}$.

Proposition 4.14. Suppose that $\mu = \mu_1 + \mu_2 + \mu_3$ where μ_2 is antidominant. Then, the following diagram is commutative:

$$\begin{array}{ccc} Y_\mu & \xrightarrow{\Delta_{\mu_1,\mu_2+\mu_3}} & Y_{\mu_1} \otimes Y_{\mu_2+\mu_3} \\ \Delta_{\mu_1+\mu_2,\mu_3} \downarrow & & \downarrow 1 \otimes \Delta_{\mu_2,\mu_3} \\ Y_{\mu_1+\mu_2} \otimes Y_{\mu_3} & \xrightarrow{\Delta_{\mu_1,\mu_2} \otimes 1} & Y_{\mu_1} \otimes Y_{\mu_2} \otimes Y_{\mu_3} \end{array}$$

Proof. Let η_1, η_3 be antidominant coweights such that $\mu'_1 = \mu_1 + \eta_1$ and $\mu'_3 = \mu_3 + \eta_3$ are antidominant. Consider the diagram

$$\begin{array}{ccccc} & & Y_{\mu'_1+\mu_2+\mu'_3} & \xrightarrow{\Delta} & Y_{\mu'_1} \otimes Y_{\mu_2+\mu'_3} \\ & \nearrow \iota_{\mu,\eta_1,\eta_3} & \downarrow & \nearrow \iota_{\mu_1,\eta_1,0} \otimes \iota_{\mu_2+\mu_3,0,\eta_3} & \downarrow 1 \otimes \Delta \\ Y_\mu & \xrightarrow{\Delta} & Y_{\mu_1} \otimes Y_{\mu_2+\mu_3} & & \\ \Delta \downarrow & & \downarrow & & \\ & \nearrow \iota_{\mu_1+\mu_2,\eta_1,0} \otimes \iota_{\mu_3,0,\eta_3} & Y_{\mu'_1+\mu_2} \otimes Y_{\mu'_3} & \xrightarrow{\Delta \otimes 1} & Y_{\mu'_1} \otimes Y_{\mu_2} \otimes Y_{\mu'_3} \\ & & \downarrow & \nearrow \iota_{\mu_1,\eta_1,0} \otimes 1 \otimes \iota_{\mu_3,0,\eta_3} & \\ Y_{\mu_1+\mu_2} \otimes Y_{\mu_3} & \xrightarrow{\Delta \otimes 1} & Y_{\mu_1} \otimes Y_{\mu_2} \otimes Y_{\mu_3} & & \end{array}$$

We have the commutativity of all faces of this cube except for that of

$$\begin{array}{ccc} Y_\mu & \xrightarrow{\Delta_{\mu_1,\mu_2+\mu_3}} & Y_{\mu_1} \otimes Y_{\mu_2+\mu_3} \\ \Delta_{\mu_1+\mu_2,\mu_3} \downarrow & & \downarrow 1 \otimes \Delta_{\mu_2,\mu_3} \\ Y_{\mu_1+\mu_2} \otimes Y_{\mu_3} & \xrightarrow{\Delta_{\mu_1,\mu_2} \otimes 1} & Y_{\mu_1} \otimes Y_{\mu_2} \otimes Y_{\mu_3} \end{array}$$

Using the commutativity of the other faces and injectivity of shift maps, we see that the above square also commutes. \square

Remark 4.15. In general, the coproducts are not coassociative. More precisely, when μ_2 is not antidominant, the diagram from Proposition 4.14 does not commute. This can already be seen for $\mathfrak{g} = \mathfrak{sl}_2$, $\mu_1 = \mu_3 = 0$, $\mu_2 = 2$.

5. Classical limit

In this section, we continue with \mathfrak{g} as simply-laced semisimple Lie algebra and we let G be the semisimple complex group of adjoint type whose Lie algebra is \mathfrak{g} .

5.1. Generalities on filtrations

Let A be an algebra and let $F^\bullet A = \cdots \subseteq F^{-1}A \subseteq F^0A \subseteq F^1A \subseteq \cdots$ be a separated and exhaustive filtration, meaning that $\cap_k F^k A = 0$ and $\cup_k F^k A = A$. We assume that this filtration is compatible with the algebra structure in the sense that $F^k A \cdot F^l A \subseteq F^{k+l} A$ and $1 \in F^0 A$.

In this case, we define the *Rees algebra* of A to be the graded $\mathbb{C}[\hbar]$ -algebra $\text{Rees}^F A := \bigoplus_k \hbar^k F^k A$, viewed as a subalgebra of $A[\hbar, \hbar^{-1}]$. We also define the associated graded of A to be the graded algebra $\text{gr}^F A := \bigoplus F^k A / F^{k-1} A$. Note that we have a canonical isomorphism of graded algebras $\text{Rees}^F A / \hbar \text{Rees}^F A \cong \text{gr}^F A$.

We say that the filtered algebra A is *almost commutative* if $\text{gr}^F A$ is commutative.

Now suppose that our algebra A is also graded, $A = \bigoplus_n A_n$ and define $F^k A_n := F^k A \cap A_n$. Assume that for each k , $F^k A = \bigoplus_n F^k A_n$. Define a new filtration G on A by setting $G^k A = \bigoplus_{n+r=k} F^r A_n$.

Lemma 5.2. *With the above setup, we have canonical algebra isomorphisms $\text{Rees}^F A \cong \text{Rees}^G A$ and $\text{gr}^F A \cong \text{gr}^G A$.*

Proof. We prove the isomorphism for the associated graded (the Rees one is similar). Define $B_{k,n} = F^k A_n / F^{k-1} A_n$. Then $\text{gr}^F A = \bigoplus_{k,n} B_{k,n}$. Now $B_{k,n} = G^{k+n} A_n / G^{k+n-1} A_n$. Thus we also see that $\text{gr}^G A = \bigoplus B_{k,n}$. This gives us the isomorphism of vector spaces $\text{gr}^F A \rightarrow \text{gr}^G A$ which is easily seen to be an algebra isomorphism as well. \square

Remark 5.3. We say that the filtration $F^\bullet A$ admits an *expansion* as a filtered \mathbb{C} -vector space, if there exists a filtered vector space isomorphism $\text{gr}^F A \rightarrow A$ (this condition is always satisfied if the filtration is bounded below). In this case, it is easy to see that $\text{Rees}^F A$ is a free $\mathbb{C}[\hbar]$ -module.

Moreover, suppose we have two filtered algebras $F^\bullet A$ and $F^\bullet B$. We can define a filtration on $A \otimes B$ by $F^n(A \otimes B) = \bigoplus_{k+l=n} F^k A \otimes F^l B$. If the filtrations of A and B both admit expansions, then we have $\text{Rees}(A \otimes B) \cong \text{Rees} A \otimes \text{Rees} B$ and $\text{gr}(A \otimes B) \cong \text{gr} A \otimes \text{gr} B$.

5.4. Filtrations on the shifted Yangian

Let μ be any coweight.

Given any pair of coweights ν_1, ν_2 such that $\nu_1 + \nu_2 = \mu$, we define a filtration $F_{\nu_1, \nu_2} Y_\mu$ by defining degrees on the PBW variables as follows

$$\deg E_\alpha^{(q)} = \langle \nu_1, \alpha \rangle + q, \deg F_\beta^{(q)} = \langle \nu_2, \beta \rangle + q, \deg H_i^{(p)} = \langle \mu, \alpha_i \rangle + p \quad (5.1)$$

More precisely, we define $F_{\nu_1, \nu_2}^k Y_\mu$ to be the span of all *ordered* monomials in the PBW variables whose total degree is at most k . A priori it is not clear that this filtration is independent of the choice of PBW variables, nor that it is independent of the order used to form the monomials, nor that it is even an algebra filtration. We establish these properties in Proposition 5.7 below.

Our goal now is to prove that $\text{gr}^{F_{\nu_1, \nu_2}} Y_\mu$ is commutative and to construct an isomorphism between this ring and the coordinate ring of a certain infinite type affine variety.

Now, assume that μ is antidominant. Define a filtration F_μ on $Y \otimes \mathbb{C}[H_i^{(0)} : i \in I]$ by taking a tensor product of the filtration $F_{0,0} Y$ with the filtration on $\mathbb{C}[H_i^{(0)} : i \in I]$ given by setting $\deg H_i^{(0)} = \langle \mu, \alpha_i \rangle$.

Define a filtration F_μ on \tilde{Y} by defining degrees on the PBW variables as follows

$$\deg \tilde{E}_\alpha^{(q)} = \langle \mu, \alpha \rangle + q, \deg \tilde{F}_\beta^{(q)} = q, \deg \tilde{H}_i^{(p)} = \langle \mu, \alpha_i \rangle + p$$

As above, the filtered piece $F_\mu^k \tilde{Y}$ is defined to be the span of those ordered monomials in the PBW variables whose total degree is at most k .

Lemma 5.5. *The inclusion $\tilde{Y} \hookrightarrow Y \otimes_{\mathbb{C}} \mathbb{C}[H_i^{(0)} : i \in I]$ is compatible with the filtrations F_μ on both algebras. Moreover, this inclusion is strict, i.e. for each k ,*

$$F_\mu^k(\tilde{Y}) = F_\mu^k(Y \otimes \mathbb{C}[H_i^{(0)} : i \in I]) \cap \tilde{Y}.$$

Thus the resulting map

$$\text{gr}^{F_\mu} \tilde{Y} \rightarrow \text{gr}^{F_\mu} (Y \otimes \mathbb{C}[H_i^{(0)} : i \in I])$$

is injective.

Proof. Both filtrations are defined by the degrees of monomials, and therefore it suffices to verify that the degree of a monomial from \tilde{Y} is equal to the degree of its image in $Y \otimes \mathbb{C}[H_i^{(0)}]$. \square

Corollary 5.6. *The filtration $F_\mu \tilde{Y}$ is an algebra filtration, and \tilde{Y} is almost commutative. Moreover, $F_\mu \tilde{Y}$ is independent of the choice of PBW variables and is also independent of the order used to form the monomials.*

Proof. Since $\tilde{Y} \hookrightarrow Y \otimes \mathbb{C}[H_i^{(0)}]$ is an inclusion of algebras, it follows immediately from Lemma 5.5 that $F_\mu \tilde{Y}$ is an algebra filtration. We know from [11] that $\mathrm{gr}^{F_{0,0}} Y$ is commutative. (In fact, it is isomorphic to $\mathbb{C}[G_1[[z^{-1}]]]$.) Thus $\mathrm{gr}^{F_\mu}(Y \otimes \mathbb{C}[H_i^{(0)} : i \in I])$ is commutative, so Lemma 5.5 implies that $\mathrm{gr}^{F_\mu} \tilde{Y}$ is commutative. Finally, independence of choice of PBW monomials also follows for the corresponding property for $F_{0,0} Y$. \square

Now, we show that Y_μ is almost commutative.

Proposition 5.7. *The filtration $F_{\mu,0} Y_\mu$ is an algebra filtration, and Y_μ is almost commutative. Moreover, $F_{\mu,0} Y_\mu$ is independent of the choice of PBW variables and is also independent of the order used to form the monomials.*

Proof. First consider the case of μ antidominant. We then have a surjective map of algebras $\tilde{Y} \rightarrow Y_\mu$, under which $F_{\mu,0} Y_\mu$ is the quotient filtration of $F_\mu \tilde{Y}$. All three properties follow from Corollary 5.6.

Next, consider the case of general μ . Choose μ_1 antidominant such that $\mu + \mu_1$ is antidominant, and consider the shift homomorphism $\iota_{\mu,\mu_1,0} : Y_\mu \rightarrow Y_{\mu+\mu_1}$. This map is injective by Corollary 3.16, and it is compatible with the filtrations $F_{\mu,0} Y_\mu \rightarrow F_{\mu+\mu_1,0} Y_{\mu+\mu_1}$. Moreover it is *strict*, by the same reasoning as Lemma 5.5. We now reason as in the proof of Corollary 5.6, proving the claim. \square

Now, let ν_2 be any coweight and let $\nu_1 = \mu - \nu_2$. Define a grading on Y_μ by setting the graded degree of the generators as follows

$$\deg E_i^{(q)} = \langle -\nu_2, \alpha_i \rangle, \quad \deg F_i^{(q)} = \langle \nu_2, \alpha_i \rangle, \quad \deg H_i^{(p)} = 0 \quad (5.2)$$

This is easily seen to be a grading on Y_μ . (This grading is the eigenspaces of the adjoint action of the element $\sum_i \langle -\nu_2, \omega_i \rangle H_i^{(-\langle \mu, \alpha_i \rangle + 1)}$ where ω_i is a fundamental weight.)

The filtration $F_{\nu_1, \nu_2} Y_\mu$ comes from the filtration $F_{\mu,0} Y_\mu$ and the above grading using the construction given in Section 5.1. Thus, we get a canonical isomorphism $\mathrm{gr}^{F_{\nu_1, \nu_2}} Y_\mu \cong \mathrm{gr}^{F_{\mu,0}} Y_\mu$ by Lemma 5.2 and in particular the former is commutative. Since all $\mathrm{gr}^{F_{\nu_1, \nu_2}} Y_\mu$ are canonically isomorphic (as algebras, but not as graded algebras), we will write $\mathrm{gr} Y_\mu$ to denote any one of them, when we are not concerned with the grading. Similarly, all Rees algebras $\mathrm{Rees}^{F_{\nu_1, \nu_2}} Y_\mu$ are all canonically isomorphic and we will write $\mathbf{Y}_\mu := \mathrm{Rees} Y_\mu$.

Corollary 5.8. *The algebra $\mathrm{gr} Y_\mu$ is a polynomial ring in the PBW variables.*

5.9. The variety \mathcal{W}_μ

For any algebraic group H , we write $H_1[[z^{-1}]]$ for the kernel of the evaluation map $H[[z^{-1}]] \rightarrow H$.

We define the (infinite type) scheme

$$\mathcal{W}_\mu := U_1[[z^{-1}]]T_1[[z^{-1}]]z^\mu U_{-,1}[[z^{-1}]] \subset G((z^{-1})) \quad (5.3)$$

We will also need a different description of this scheme. The inclusion $U_1[[z^{-1}]] \rightarrow U((z^{-1}))$ gives rise to an isomorphism $U_1[[z^{-1}]] \cong U((z^{-1}))/U[z]$. Thus we can identify \mathcal{W}_μ with the quotient $U[z] \backslash U((z^{-1}))T_1[[z^{-1}]]z^\mu U_-((z^{-1}))/U_-[z]$ and we write π for this isomorphism.

The scheme \mathcal{W}_μ is the moduli space of the following data (cf. [4, 2(ii)]): (a) a G -bundle \mathcal{P} on \mathbb{P}^1 ; (b) a trivialization $\sigma: \mathcal{P}|_{\mathbb{P}^1_\infty} \xrightarrow{\sim} \mathcal{P}|_{\mathbb{P}^1_\infty}$ in the formal neighborhood of $\infty \in \mathbb{P}^1$; (c) a B -structure ϕ on \mathcal{P} of degree $w_0\mu$ having fiber $B_- \subset G$ at $\infty \in \mathbb{P}^1$ (with respect to the trivialization σ of \mathcal{P} at $\infty \in \mathbb{P}^1$). This is explained in [4, 2(xi)]. In particular, \mathcal{W}_μ contains the finite dimensional affine varieties $\overline{\mathcal{W}}_\mu^\lambda$ (generalized slices) for the dominant coweights λ , and the closed subvariety $\overline{\mathcal{W}}_\mu^\lambda \subset \mathcal{W}_\mu$ is cut out by the condition that σ extends as a rational trivialization with a unique pole at $0 \in \mathbb{P}^1$, and the order of the pole of σ at $0 \in \mathbb{P}^1$ is $\leq \lambda$.

For μ_1, μ_2 antidominant, we define shift maps $\iota_{\mu, \mu_1, \mu_2}: \mathcal{W}_{\mu+\mu_1+\mu_2} \rightarrow \mathcal{W}_\mu$ by $g \mapsto \pi(z^{-\mu_1}gz^{-\mu_2})$.

Remark 5.10. Note that \mathcal{W}_0 is the group $G_1[[z^{-1}]]$. Moreover, for μ dominant, we can identify \mathcal{W}_μ with the $G_1[[z^{-1}]]$ orbit of z^μ in the thick affine Grassmannian $G((z^{-1}))/G[z]$. In this case, the shift map $\iota_{\mu, 0, -\mu}: \mathcal{W}_0 \rightarrow \mathcal{W}_\mu$ is exactly the action map. In fact, $\mathcal{W}_\mu = \mathcal{G}_\mu z^\mu$, where \mathcal{G}_μ is the subgroup of $G_1[[z^{-1}]]$ defined in [4, B(viii)(a)].

Recall the multiplication morphisms $m_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}: \overline{\mathcal{W}}_{\mu_1}^{\lambda_1} \times \overline{\mathcal{W}}_{\mu_2}^{\lambda_2} \rightarrow \overline{\mathcal{W}}_{\mu_1+\mu_2}^{\lambda_1+\lambda_2}$ constructed in [4, 2(vi)]. We define the multiplication morphism $m_{\mu_1, \mu_2}: \mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \rightarrow \mathcal{W}_{\mu_1+\mu_2}$ by the formula $m_{\mu_1, \mu_2}(g_1, g_2) = \pi(g_1g_2)$. Comparing the constructions of [4, 2(vi) and 2(xi)], we see that $m_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}$ is the restriction of m_{μ_1, μ_2} .

Lemma 5.11. *The shift maps and multiplication maps are compatible. More precisely, let μ_1, μ_2 be any coweights and let ν_1, ν_2 be antidominant coweights. The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{W}_{\mu_1+\nu_1} \times \mathcal{W}_{\mu_2+\nu_2} & \longrightarrow & \mathcal{W}_{\mu_1+\mu_2+\nu_1+\nu_2} \\ \downarrow \iota_{\mu_1, \nu_1, 0} \times \iota_{\mu_2, 0, \nu_2} & & \downarrow \iota_{\mu_1+\mu_2, \nu_1, \nu_2} \\ \mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} & \longrightarrow & \mathcal{W}_{\mu_1+\mu_2} \end{array}$$

Proof. A simple diagram chase show that for $(g_1, g_2) \in \mathcal{W}_{\mu_1+\nu_1} \times \mathcal{W}_{\mu_2+\nu_2}$, both paths are computed by $\pi(z^{-\nu_1}g_1g_2z^{-\nu_2})$. (Here we use that if $u \in U[z]$, then $z^{-\nu_1}uz^{\nu_1} \in U[z]$.) \square

For $s \in \mathbb{C}^\times$, and $g(z) \in G((z^{-1}))$, we define $\kappa_s(g(z)) = g(sz)$. This loop rotation action does not preserve $\mathcal{W}_\mu \subset G((z^{-1}))$. But given a pair of coweights ν_1, ν_2 such that $\nu_1 + \nu_2 = \mu$, we define an action κ^{ν_1, ν_2} of \mathbb{C}^\times on \mathcal{W}_μ by

$$\kappa_s^{\nu_1, \nu_2}(g) = s^{-\nu_1} \kappa_s(g) s^{-\nu_2}$$

5.12. Classical limit

Let $p_i: U \rightarrow \mathbb{C}$, $p_i^-: U_- \rightarrow \mathbb{C}$, $\mathbf{p}_i: T \rightarrow \mathbb{C}^\times$ be the projections according to a simple root α_i . Then we get maps $p_i^{(r)}: U_1[[z^{-1}]] \rightarrow \mathbb{C}$ given by taking the coefficient of z^{-r} in p_i . Similarly, we get functions $p_i^{- (r)}$ and $\mathbf{p}_i^{(r)}$. Using the Gauss decomposition of an element $uhz^\mu u_- \in \mathcal{W}_\mu$, we get functions $p_i^{(r)}, p_i^{- (r)}$ and $\mathbf{p}_i^{(r)}$ on \mathcal{W}_μ by

$$p_i^{(r)}(g) := p_i^{(r)}(u), \quad p_i^{- (r)}(g) := p_i^{- (r)}(u_-), \quad \mathbf{p}_i^{(r)}(g) := \mathbf{p}_i^{(r)}(hz^\mu) \quad (5.4)$$

These functions can also be described using generalized minors (i.e. matrix coefficients), analogously to [11].

As described in [11], $\mathcal{W}_0 = G_1[[z^{-1}]]$ can be given the structure of a Poisson–Lie group, corresponding to Yang’s Manin triple. The ring of functions $\mathbb{C}[\mathcal{W}_0]$ is graded via the loop rotation action. The following result is a reformulation of [11, Theorem 3.9]:

Theorem 5.13. *There is an isomorphism of graded Poisson–Hopf algebras $\mathrm{gr}^{F_{0,0}} Y \cong \mathbb{C}[\mathcal{W}_0]$, such that*

$$H_i^{(r)} \mapsto \mathbf{p}_i^{(r)}, \quad E_i^{(r)} \mapsto p_i^{(r)}, \quad F_i^{(r)} \mapsto p_i^{- (r)} \quad (5.5)$$

Proposition 5.14.

- (a) *For any coweight ν , there is an isomorphism of graded Poisson algebras $\mathrm{gr}^{F_{\nu, -\nu}} Y \cong \mathbb{C}[\mathcal{W}_0]$ such that (5.5) holds, and where the grading on $\mathbb{C}[\mathcal{W}_0]$ comes from the $\kappa^{\nu, -\nu}$ action.*
- (b) *This restricts to a graded isomorphism*

$$\begin{array}{ccc} \mathrm{gr}^{F_{\nu, -\nu}} Y^> & \xrightarrow{\sim} & \mathbb{C}[U_1[[z^{-1}]]] \\ \downarrow & & \downarrow \\ \mathrm{gr}^{F_{\nu, -\nu}} Y & \xrightarrow{\sim} & \mathbb{C}[\mathcal{W}_0] \end{array}$$

where the right-hand vertical arrow corresponds to the map $\mathcal{W}_0 \rightarrow U_1[[t^{-1}]]$, $uhu_- \mapsto u$. Similarly, there are isomorphisms

$$\mathrm{gr}^{F_{\nu, -\nu}} Y^= \cong \mathbb{C}[T_1[[z^{-1}]]], \quad \mathrm{gr}^{F_{\nu, -\nu}} Y^< \cong \mathbb{C}[U_{-,1}[[z^{-1}]]]$$

Part (b) reflects the triangular decomposition Y on the one hand, and the Gauss decomposition of W_0 on the other.

Proof. For (a), it suffices to show that the grading (5.2) corresponds to the \mathbb{C}^\times -action on W_0 given by $s \cdot g = s^{-\nu} g s^\nu$. First, we claim that the latter is a Poisson action: this follows from the explicit formula [11, Proposition 2.13] for the Poisson bracket in terms of generalized minors. Now, under the corresponding grading on $\mathbb{C}[W_0]$ the degrees of generators $p_i^{(r)}, p_i^{-(r)}, \mathfrak{p}_i^{(r)}$ agree with the grading (5.2). Since both are Poisson gradings, and these are Poisson generators of $\mathbb{C}[W_0]$, the two gradings agree.

We now prove (b), in the case of $Y^>$. Recall that $G_1[[z^{-1}]]$ is a Poisson algebraic group, so $z^{-1}\mathfrak{g}[[z^{-1}]]$ is a Lie bialgebra (see [11, 2C]). Under its cobracket, we have

$$\delta(z^{-1}\mathfrak{b}^-[[z^{-1}]]) \subset (z^{-1}\mathfrak{b}^-[[z^{-1}]]) \otimes (z^{-1}\mathfrak{g}[[z^{-1}]]) + (z^{-1}\mathfrak{g}[[z^{-1}]]) \otimes (z^{-1}\mathfrak{b}^-[[z^{-1}]])$$

By [18, Theorem 6], this implies that there is an induced structure of Poisson homogeneous space on $G_1[[z^{-1}]]/B_1^-[[z^{-1}]]$. In other words, $\mathbb{C}[G_1[[z^{-1}]]]^{B_1^-[[z^{-1}]]}$ is a Poisson subalgebra of $\mathbb{C}[G_1[[z^{-1}]]]$.

The map $G_1[[z^{-1}]] \rightarrow U_1[[z^{-1}]]$, $uhu_- \mapsto u$ identifies $\mathbb{C}[U_1[[z^{-1}]]] \cong \mathbb{C}[G_1[[z^{-1}]]]^{B_1^-[[z^{-1}]]}$. Since the functions $p_i^{(r)}$ lie in this subalgebra, $\mathbb{C}[U_1[[z^{-1}]]]$ contains the Poisson subalgebra that they generate. Therefore we can identify $\mathrm{gr}^{F_{\nu,-\nu}} Y^> \subset \mathbb{C}[U_1[[z^{-1}]]]$. We will prove equality by a dimension count. It suffices to consider the filtration $F^{0,0} Y^>$ and the loop rotation action $\kappa^{0,0}$. By the PBW theorem, $\mathrm{gr}^{F_{0,0}} Y^>$ has Hilbert series

$$\prod_{i=1}^{\infty} \frac{1}{(1 - q^i)^{\dim \mathfrak{n}}} \quad (5.6)$$

Since $U_1[[z^{-1}]]$ is a pro-unipotent group, the Hilbert series of $\mathbb{C}[U_1[[z^{-1}]]]$ for the loop rotation is the same as that of $\mathrm{Sym}(z^{-1}\mathfrak{n}[[z^{-1}]])$. This is also given by (5.6), proving the claim. \square

Consider a coweight μ . By Remark 3.7, there is a surjection of algebras $Y^> \twoheadrightarrow Y_\mu^>$ defined by $E_i^{(p)} \mapsto E_i^{(p)}$. By the PBW theorem for Y_μ , it follows that this map is an isomorphism. Moreover, for any coweights ν_1, ν_2 such that $\nu_1 + \nu_2 = \mu$, we see from (5.1) that it is an isomorphism of filtered algebras $F^{\nu_1, -\nu_1} Y^> \xrightarrow{\sim} F^{\nu_1, \nu_2} Y_\mu^>$, where these filtrations are inherited as subspaces of Y and Y_μ , respectively.

By the Gauss decomposition there is a projection map $W_\mu \twoheadrightarrow U_1[[z^{-1}]]$, $uhz^\mu u_- \mapsto u$. This provides an embedding $\mathbb{C}[U_1[[z^{-1}]]] \hookrightarrow \mathbb{C}[W_\mu]$. Consider the composition of maps:

$$\mathrm{gr}^{F_{\nu_1, \nu_2}} Y_\mu^> \xrightarrow{\sim} \mathrm{gr}^{F_{\nu_1, -\nu_1}} Y^> \xrightarrow{\sim} \mathbb{C}[U_1[[z^{-1}]]] \hookrightarrow \mathbb{C}[W_\mu] \quad (5.7)$$

where the second map comes from Proposition 5.14. Note that under this composition, $E_i^{(r)} \mapsto p_i^{(r)}$. This map is graded, where $\mathbb{C}[W_\mu]$ is graded by the action κ^{ν_1, ν_2} .

Analogously, there are compositions

$$\mathrm{gr}^{F_{\nu_1, \nu_2}} Y_{\mu}^{<} \xrightarrow{\sim} \mathrm{gr}^{F_{-\nu_2, \nu_2}} Y^{<} \xrightarrow{\sim} \mathbb{C}[U_{-,1}[[z^{-1}]]] \hookrightarrow \mathbb{C}[\mathcal{W}_{\mu}], \quad (5.8)$$

$$\mathrm{gr}^{F_{\nu_1, \nu_2}} Y_{\mu}^{=} \xrightarrow{\sim} \mathrm{gr}^{F_{0,0}} Y^{=} \xrightarrow{\sim} \mathbb{C}[T_1[[z^{-1}]]] \hookrightarrow \mathbb{C}[\mathcal{W}_{\mu}] \quad (5.9)$$

which take $F_i^{(r)} \mapsto p_i^{-(r)}$ and $H_i^{(r)} \mapsto p_i^{(r)}$.

From Corollary 5.8, there is a triangular decomposition (of algebras)

$$\mathrm{gr}^{F_{\nu_1, \nu_2}} Y_{\mu} \cong (\mathrm{gr}^{F_{\nu_1, \nu_2}} Y_{\mu}^{>}) \otimes (\mathrm{gr}^{F_{\nu_1, \nu_2}} Y_{\mu}^{=}) \otimes (\mathrm{gr}^{F_{\nu_1, \nu_2}} Y_{\mu}^{<})$$

By the Gauss decomposition \mathcal{W}_{μ} , we get:

Theorem 5.15. *For any coweights ν_1, ν_2 such that $\nu_1 + \nu_2 = \mu$, the tensor product of the maps (5.7), (5.8) and (5.9) yields an isomorphism of graded algebras*

$$\mathrm{gr}^{F_{\nu_1, \nu_2}} Y_{\mu} \xrightarrow{\sim} \mathbb{C}[\mathcal{W}_{\mu}] \quad (5.10)$$

Here the grading on $\mathbb{C}[\mathcal{W}_{\mu}]$ comes from the κ^{ν_1, ν_2} action. Moreover, the isomorphism is compatible with the shift maps $\iota_{\mu, \mu_1, \mu_2}$ on both sides.

Proof. The only thing left to prove is the compatibility of the shift maps $\iota_{\mu, \mu_1, \mu_2} : Y_{\mu} \rightarrow Y_{\mu + \mu_1 + \mu_2}$ and $\iota_{\mu, \mu_1, \mu_2} : \mathcal{W}_{\mu + \mu_1 + \mu_2} \rightarrow \mathcal{W}_{\mu}$ with the above isomorphism $\mathrm{gr} Y_{\mu} \cong \mathbb{C}[\mathcal{W}_{\mu}]$. Since this isomorphism is constructed using the Gauss decomposition, it suffices to prove the compatibility on each piece separately. For $Y_{\mu}^{=}$ and $T_1[[z^{-1}]]$ it follows from the construction of (5.9). Now it suffices to check the compatibility with the isomorphisms (5.7) and (5.8). Since these are similar, we will just concentrate on the isomorphism (5.8).

For any η antidominant, define a map $\psi_{\eta} : U_{-,1}[[z^{-1}]] \rightarrow U_{-,1}[[z^{-1}]]$ by $\psi_{\eta}(u) = \pi(z^{\eta} u z^{-\eta})$, where as usual π denotes the projection $\pi : U_{-}((z^{-1})) \rightarrow U_{-}((z^{-1}))/U_{-}[z] \cong U_{-,1}[[z^{-1}]]$.

For any coweight μ and any antidominant μ_1, μ_2 we have the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{W}_{\mu + \mu_1 + \mu_2} & \longrightarrow & U_{-,1}[[z^{-1}]] \\ \downarrow \iota_{\mu, \mu_1, \mu_2} & & \downarrow \psi_{\mu_2} \\ \mathcal{W}_{\mu} & \longrightarrow & U_{-,1}[[z^{-1}]] \end{array}$$

On the other hand, consider the shift map $\psi_{\eta} : Y^{<} \rightarrow Y^{<}$ given by $F_i^{(q)} \mapsto F_i^{(q - \langle \eta, \alpha_i \rangle)}$. Then ψ_{μ_2} is the restriction of the shift map $\iota_{\mu, \mu_1, \mu_2}$ to $Y^{<} \cong Y_{\mu}^{<}$.

Thus, in order to show that (5.8) is compatible with the shift $\iota_{\mu, \mu_1, \mu_2}$, it suffices to show that the isomorphism $\mathrm{gr} Y^{<} \cong \mathbb{C}[U_{-,1}[[z^{-1}]]]$ is compatible with the two ψ_{η} maps, for any antidominant η .

Since $-\eta$ is dominant, it follows from [11, Theorem 3.12], that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{gr}Y_{-\eta} & \longrightarrow & \mathbb{C}[\mathcal{W}_{-\eta}] \\ \downarrow \iota_{-\eta,0,\eta} & & \downarrow \\ \mathrm{gr}Y & \longrightarrow & \mathbb{C}[G_1[[z^{-1}]]] \end{array}$$

where the right vertical arrow is just (dual to) the action map $g \mapsto gz^{-\eta}$. This action map is compatible with the shift map ψ_η on $U_{-,1}[[z^{-1}]]$. Thus we deduce that the isomorphism $\mathrm{gr}Y^< \cong \mathbb{C}[U_{-,1}[[z^{-1}]]]$ is compatible with the two shift maps ψ_η and this completes the proof. \square

Remark 5.16. The above theorem provides \mathcal{W}_μ with a Poisson structure. It is compatible with the Poisson structure on $\overline{\mathcal{W}}_\mu^\lambda$ constructed in [4]. This is because the Poisson structure on $\overline{\mathcal{W}}_\mu^\lambda$ comes from its quantization (the quantized Coulomb branch) and we have a surjective map from $\mathrm{Rees}Y_\mu$ to this quantized Coloumb branch provided by [4, Theorem B.18].

Lemma 5.17. *Let μ be an antidominant coweight. Then the classical shifted Yangian $\mathrm{gr}^FY_\mu \cong \mathbb{C}[\mathcal{W}_\mu]$ is generated by $E_i^{(1)} = p_i^{(1)}$, $F_i^{(1)} = p_i^{-(1)}$, $H_i^{(-\langle\mu,\alpha_i\rangle+1)} = p_i^{(1)}$ and $H_i^{(-\langle\mu,\alpha_i\rangle+2)} = p_i^{(2)}$ as a Poisson algebra.*

Proof. From the PBW theorem, gr^FY_μ is generated by the PBW variables. These variables are all constructed from the generators $E_i^{(r)}, F_i^{(r)}, H_i^{(r)}$ using Poisson brackets. So it suffices to show that we can construct these generators.

Indeed, $\{H_i^{(-\langle\mu,\alpha_i\rangle+2)}, E_i^{(r)}\}$ is $2E_i^{(r+1)}$ plus some expression in $H_i^{(-\langle\mu,\alpha_i\rangle+1)}$ and $E_i^{(r)}$, and the same holds for $F_i^{(r)}$, hence the algebra generated by the above elements contains $E_i^{(r)}, F_i^{(r)}$ for all $r \in \mathbb{Z}_{>0}$. Every $H_i^{(s)}$ with positive s is a bracket of some E and F , hence we have $H_i^{(-\langle\mu,\alpha_i\rangle+r)}$ for all $r \in \mathbb{Z}_{>0}$. \square

5.18. Classical multiplication and the coproduct

Let μ_1, μ_2 be coweights.

The multiplication map $m: \mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \rightarrow \mathcal{W}_{\mu_1+\mu_2}$ gives us an algebra map

$$\Delta_{\mu_1,\mu_2}^1: \mathbb{C}[\mathcal{W}_{\mu_1+\mu_2}] \rightarrow \mathbb{C}[\mathcal{W}_{\mu_1}] \otimes \mathbb{C}[\mathcal{W}_{\mu_2}]$$

On the other hand, the coproduct

$$Y_{\mu_1+\mu_2} \rightarrow Y_{\mu_1} \otimes Y_{\mu_2}$$

is compatible with the filtrations $F_{\mu_1, \mu_2} Y_{\mu_1 + \mu_2}$, $F_{\mu_1, 0} Y_{\mu_1}$, $F_{0, \mu_2} Y_{\mu_2}$ and thus gives rise to a map

$$\Delta_{\mu_1, \mu_2}^2 : \operatorname{gr} Y_{\mu_1 + \mu_2} \rightarrow \operatorname{gr} Y_{\mu_1} \otimes \operatorname{gr} Y_{\mu_2}$$

Under the isomorphism [Theorem 5.15](#), this gives us another map $\mathbb{C}[\mathcal{W}_{\mu_1 + \mu_2}] \rightarrow \mathbb{C}[\mathcal{W}_{\mu_1}] \otimes \mathbb{C}[\mathcal{W}_{\mu_2}]$.

When $\mu_1 = \mu_2 = 0$, we have $\mathcal{W}_{\mu_1} = \mathcal{W}_{\mu_2} = \mathcal{W}_{\mu_1 + \mu_2} = G_1[[z^{-1}]]$ and the multiplication map is just the ordinary multiplication map on $G_1[[z^{-1}]]$. On the other hand, the Drinfeld–Gavarini duality (also called Quantum Duality Principle; see [\[11\]](#)) shows us that the coproduct on $\operatorname{gr} Y$ is just the usual coproduct on $\mathbb{C}[G_1[[z^{-1}]]]$. Thus we conclude that $\Delta_{0,0}^1 = \Delta_{0,0}^2$. So it is natural to expect that for all μ_1, μ_2 , we have $\Delta_{\mu_1, \mu_2}^1 = \Delta_{\mu_1, \mu_2}^2$. (In [Corollary 6.5](#), we will show that this holds when $\mathfrak{g} = \mathfrak{sl}_2$ and μ_1, μ_2 are antidominant.)

Proposition 5.19. *If μ_1, μ_2 are antidominant, then the Δ_{μ_1, μ_2}^1 and Δ_{μ_1, μ_2}^2 agree on the Poisson generators $p_i^{(1)}, p_i^{- (1)}, p_i^{(1)}, p_i^{(2)}$.*

Proof. Let μ_1 and μ_2 be both antidominant. Then the multiplication map $\mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \rightarrow \mathcal{W}_{\mu_1 + \mu_2}$ is given just by multiplication in $G((z^{-1}))$. Let $u^k h^k z^{\mu_k} u_-^k$, $k = 1, 2$ be any elements of \mathcal{W}_{μ_k} . Then the product is

$$u^1 z^{\mu_1} h^1 u_-^1 u^2 h^2 z^{\mu_2} u_-^2.$$

We take the Gaussian decomposition of the middle part of this expression, i.e. we write

$$h^1 u_-^1 u^2 h^2 = u' h' u_-',$$

where $u' \in U_1[[z^{-1}]]$, $u_-' \in U_{-,1}[[z^{-1}]]$, $h' \in T_1[[z^{-1}]]$. Then the product is

$$u^1 (z^{\mu_1} u' z^{-\mu_1}) h' z^{\mu_1 + \mu_2} (z^{-\mu_2} u_-' z^{\mu_2}) u_-^2.$$

The first and second Fourier coefficients of u', u_-' and h' are easy to compute. To write the answer we need to define the functions $p_\gamma^{(r)}, p_\gamma^{- (r)}$ for any positive root γ . Fix an isomorphism $c: \mathfrak{n} \rightarrow U$ between the formal neighborhood of $0 \in \mathfrak{n}$ and the formal neighborhood of $e \in U$ such that $d_0 c = \operatorname{Id}: \mathfrak{n} \rightarrow \mathfrak{n}$ (e.g. $c = \exp: \mathfrak{n} \rightarrow U$). Let $P_\gamma: \mathfrak{n} \rightarrow \mathfrak{n}_\gamma = \mathbb{C}$ be the projection to the corresponding root space. Define $p_\gamma^{(r)}$ as the coefficient of z^{-r} in the composite map $P_\gamma \circ c^{-1}: U_1[[z^{-1}]] \rightarrow \mathbb{C}$, and similarly for $p_\gamma^{- (r)}$. Note that $p_\gamma^{(1)}$ and $p_\gamma^{- (1)}$ do not depend on the choice of c . We get the following formulas

$$\Delta^1(p_i^{(1)}) = p_i^{(1)} \otimes 1 + 1 \otimes p_i^{(1)}, \quad (5.11)$$

$$\Delta^1(p_i^{(2)}) = p_i^{(2)} \otimes 1 + 1 \otimes p_i^{(2)} + p_i^{(1)} \otimes p_i^{(1)} - \sum_{\gamma > 0} \langle \alpha_i, \gamma \rangle p_\gamma^{-(1)} \otimes p_\gamma^{(1)}, \quad (5.12)$$

$$\Delta^1(p_i^{(1)}) = p_i^{(1)} \otimes 1 + \delta_{\langle \mu_1, \alpha_i \rangle, 0} 1 \otimes p_i^{(1)}, \quad (5.13)$$

$$\Delta^1(p_i^{- (1)}) = \delta_{\langle \mu_2, \alpha_i \rangle, 0} p_i^{- (1)} \otimes 1 + 1 \otimes p_i^{- (1)}. \quad (5.14)$$

Note that $p_\gamma^{(1)}$ is the image of $E_\gamma^{(1)}$ under the isomorphism from [Theorem 5.15](#) (and similarly $p_\gamma^{- (1)}$).

Comparing with [Lemma 4.9](#) gives the desired result. \square

Conjecture 5.20. *The multiplication map $\mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \rightarrow \mathcal{W}_{\mu_1 + \mu_2}$ is Poisson.*

We know that it is true when $\mu_1 = \mu_2 = 0$, since in this case it is just the usual multiplication in the Poisson group $G_1[[z^{-1}]]$.

Proposition 5.21. *If [Conjecture 5.20](#) holds, then the two maps Δ^1 and Δ^2 agree.*

Proof. If μ_1, μ_2 are antidominant, then [Proposition 5.19](#) shows that Δ_{μ_1, μ_2}^1 and Δ_{μ_1, μ_2}^2 agree on the Poisson generators for the algebra $\mathbb{C}[\mathcal{W}_{\mu_1 + \mu_2}]$. Since both maps are Poisson, they must agree.

Now, suppose that μ_1, μ_2 are arbitrary. As in the proof of [Theorem 4.12](#), we can embed $\mathbb{C}[\mathcal{W}_{\mu_1 + \mu_2}]$ into an antidominant situation. Both Δ^1 and Δ^2 are compatible with this embedding. For Δ^1 , this follows from [Lemma 5.11](#), while for Δ^2 this follows from the construction in [Theorem 4.12](#). Thus the result follows. \square

6. Toda and comultiplication

Throughout this section we work with shifted Yangians of \mathfrak{sl}_2 and the Toda lattice for $GL(n)$.

6.1. A presentation of \mathfrak{sl}_2 shifted Yangians

Following [\[17, Definition 2.24\]](#), we can write down the defining relations of the shifted Yangian $\mathbf{Y}_m(\mathfrak{sl}_2)$ of \mathfrak{sl}_2 in current form. In this case $\mu = m \in \mathbb{Z}$, and from now on we assume $m \leq 0$, i.e. our Yangian is antidominantly shifted. We introduce the series $E(u) := \sum_{p=1}^{\infty} E^{(p)} u^{-p}$, $F(u) := \sum_{p=1}^{\infty} F^{(p)} u^{-p}$, $H(u) := u^m + \sum_{p=-m+1}^{\infty} H^{(p)} u^{-p}$. Then the defining relations can be written in the following form:

$$[H(u), H(v)] = 0, \quad (6.1)$$

$$[E(u), F(v)] = -\hbar \frac{H(u) - H(v)}{u - v}, \quad (6.2)$$

$$[E(u), E(v)] = -\hbar \frac{(E(u) - E(v))^2}{u - v}, \quad (6.3)$$

$$[F(u), F(v)] = \hbar \frac{(F(u) - F(v))^2}{u - v}, \quad (6.4)$$

$$[H(u), E(v)] = -\hbar \frac{H(u)(E(u) - E(v)) + (E(u) - E(v))H(u)}{u - v}, \quad (6.5)$$

$$[H(u), F(v)] = \hbar \frac{H(u)(F(u) - F(v)) + (F(u) - F(v))H(u)}{u - v}. \quad (6.6)$$

6.2. Some automorphisms of $\mathbf{Y}_m(\mathfrak{sl}_2)$

It is clear from the formulas above that the additive shifts of the variable u act on the shifted Yangian \mathbf{Y}_m by automorphisms. We denote the corresponding automorphisms by $T_\varepsilon: E(u) \mapsto E(u - \varepsilon), F(u) \mapsto F(u - \varepsilon), H(u) \mapsto H(u - \varepsilon)$.

6.3. Coproduct on $\mathbf{Y}_m(\mathfrak{sl}_2)$ for $m \leq 0$

The following formulas define the coproduct Δ on the usual Yangian $\mathbf{Y}(\mathfrak{sl}_2) = \mathbf{Y}_0(\mathfrak{sl}_2)$ (see [17, Definition 2.24]).

$$\Delta: E(u) \mapsto E(u) \otimes 1 + \sum_{j=0}^{\infty} (-1)^j F(u + \hbar)^j H(u) \otimes E(u)^{j+1}; \quad (6.7)$$

$$\Delta: F(u) \mapsto 1 \otimes F(u) + \sum_{j=0}^{\infty} (-1)^j F(u)^{j+1} \otimes H(u) E(u + \hbar)^j; \quad (6.8)$$

$$\Delta: H(u) \mapsto \sum_{j=0}^{\infty} (-1)^j (j+1) F(u + \hbar)^j H(u) \otimes H(u) E(u + \hbar)^j. \quad (6.9)$$

Proposition 6.4. *Let $l, k \leq 0$ and $m = l + k$. Then the coproduct $\Delta: \mathbf{Y}_m(\mathfrak{sl}_2) \rightarrow \mathbf{Y}_l(\mathfrak{sl}_2) \otimes \mathbf{Y}_k(\mathfrak{sl}_2)$ is also given by the formulas (6.7)–(6.9), where by abuse of notation $E(u), F(u), H(u)$ denote the generating series for each respective algebra.*

Proof. We make use of the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Y}_0 & \xrightarrow{\Delta} & \mathbf{Y}_0 \otimes \mathbf{Y}_0 \\ \downarrow \iota_{0,l,k} & & \downarrow \iota_{0,l,0} \otimes \iota_{0,0,k} \\ \mathbf{Y}_m & \xrightarrow{\Delta} & \mathbf{Y}_l \otimes \mathbf{Y}_k \end{array} \quad (6.10)$$

from the statement of Theorem 4.12. By commutativity, we may compute the coproduct of any elements in the image of $\iota_{0,l,k}: \mathbf{Y}_0 \rightarrow \mathbf{Y}_m$ by passing around the top of the diagram. Modulo accounting for the shift homomorphisms involved, this is given by Molev's formulas (6.7)–(6.9).

Note that the homomorphism $\iota_{0,l,k}: \mathbf{Y}_0 \rightarrow \mathbf{Y}_m$ is not surjective: the generators $E^{(r)}, F^{(s)}$ are not in its image for $1 \leq r \leq l$ and $1 \leq s \leq k$. However, the coproducts of these elements were explicitly described in Section 4.6. Piecing these coproducts together with those computed above, the claim follows. \square

Corollary 6.5. *In this case, the comultiplication $\Delta: \mathbf{Y}_m(\mathfrak{sl}_2) \rightarrow \mathbf{Y}_l(\mathfrak{sl}_2) \otimes \mathbf{Y}_k(\mathfrak{sl}_2)$ quantizes the multiplication map $\mathcal{W}_k \times \mathcal{W}_l \rightarrow \mathcal{W}_m$. (In other words, in the notation of Section 5.18, we have $\Delta_{k,l}^1 = \Delta_{k,l}^2$.)*

Proof. For the ordinary Yangian $Y(\mathfrak{sl}_2)$, the classical limit of the formulas (6.7)–(6.9) corresponds to multiplication in the group $(PGL_2)_1[[z^{-1}]]$, written with respect to Gauss decompositions. Explicitly, any element of $(PGL_2)_1[[z^{-1}]]$ can be written uniquely in the form

$$g = \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \quad (6.11)$$

with $e, f \in z^{-1}\mathbb{C}[[z^{-1}]]$ and $h \in 1 + z^{-1}\mathbb{C}[[z^{-1}]]$. The product of two such elements, rewritten in the above form, is

$$g_1 g_2 = \begin{pmatrix} 1 & 0 \\ e_1 + \frac{h_1 e_2}{1+f_1 e_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{h_1 h_2}{(1+f_1 e_2)^2} \end{pmatrix} \begin{pmatrix} 1 & \frac{f_1 h_2}{1+f_1 e_2} + f_2 \\ 0 & 1 \end{pmatrix} \quad (6.12)$$

On the level of coordinate rings, this corresponds precisely to (6.7)–(6.9) with $\hbar = 0$.

Any $g \in \mathcal{W}_n$ can also be written uniquely in the form (6.11), but where now we take $h \in z^n + z^{n-1}\mathbb{C}[[z^{-1}]]$. When $k, l \leq 0$ the multiplication map $\mathcal{W}_k \times \mathcal{W}_l \rightarrow \mathcal{W}_m$ is given by matrix multiplication, and (6.12) generalizes immediately. This proves the claim. \square

Remark 6.6. Assuming one has explicit formulas for the coproduct $\Delta: Y_0(\mathfrak{g}) \rightarrow Y_0(\mathfrak{g}) \otimes Y_0(\mathfrak{g})$, a similar logic to Proposition 6.4 gives explicit formulas for the coproduct $\Delta: Y_\mu(\mathfrak{g}) \rightarrow Y_{\mu_1}(\mathfrak{g}) \otimes Y_{\mu_2}(\mathfrak{g})$ in the case when μ, μ_1, μ_2 are all antidominant.

6.7. Shifted Yangian of \mathfrak{sl}_2 and Toda

According to [4, Theorem B.18], for a simple simply laced \mathfrak{g} , there is a homomorphism from the shifted Yangian $\mathbf{Y}_\mu(\mathfrak{g})$ to a quantized Coulomb branch. Let us describe it in the simplest case $\mathfrak{g} = \mathfrak{sl}_2$, $\mu = -2n$, $\lambda = 0$ where n is a positive integer.

Proposition 6.8. ([4, Theorem B.18]) *There is a homomorphism $\overline{\Phi}_{-2n}^0: \mathbf{Y}_{-2n}(\mathfrak{sl}_2) \rightarrow H_{\bullet}^{G_{\circ}^{\vee} \times \mathbb{C}^{\times}}(\mathrm{Gr}_{G^{\vee}})$ for $G = G^{\vee} = GL_n$. We have*

$$\begin{aligned} \overline{\Phi}_{-2n}^0(A^{(p)}) &= e_p \in H_{G_{\circ}^{\vee} \times \mathbb{C}^{\times}}^{\bullet}(pt) \subset H_{\bullet}^{G_{\circ}^{\vee} \times \mathbb{C}^{\times}}(\mathrm{Gr}_{G^{\vee}}), \\ \overline{\Phi}_{-2n}^0(F^{(1)}) &= [\mathrm{Gr}_{G^{\vee}}^{\varpi_1}], \quad \overline{\Phi}_{-2n}^0(E^{(1)}) = (-1)^n [\mathrm{Gr}_{G^{\vee}}^{\varpi_1}] \end{aligned}$$

Note that according to Lemma 2.10 and the paragraph following it, the homomorphism $\overline{\Phi}_{-2n}^0$ is surjective.

The ring $H_{G_{\circ}^{\vee} \times \mathbb{C}^{\times}}^{\bullet}(pt)$ gets identified with the center of the universal enveloping algebra $ZU_{\hbar}(\mathfrak{g})$ via the Satake correspondence. The Harish-Chandra homomorphism

identifies the center $ZU_{\hbar}(\mathfrak{g})$ with the algebra $\mathbb{C}[\hbar^*]^W$ of W -invariant polynomials with respect to the W -action shifted by $-\hbar\rho$. Here we write e_p for the p -th elementary symmetric function in $\mathbb{C}[\hbar^*]^W$ (shifted by $-\hbar\rho_n$ where $\rho_n := (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{-n+1}{2})$). So we can compute the images of $A^{(1)}$ and $A^{(2)}$ as elements of the center of the universal enveloping algebra $ZU_{\hbar}(\mathfrak{g})$. Combining it with $\beta: H_{\bullet}^{G_{\circ}^{\vee} \rtimes \mathbb{C}^{\times}}(\mathrm{Gr}_{G^{\vee}}) \xrightarrow{\sim} \mathcal{T}_{\hbar}^n$ we get

Proposition 6.9. *There is a surjective homomorphism $\beta \circ \overline{\Phi}_{-2n}^0: \mathbf{Y}_{-2n}(\mathfrak{sl}_2) \rightarrow \mathcal{T}_{\hbar}^n$ which takes the subalgebra generated by the $A^{(p)}$ to $ZU_{\hbar}(\mathfrak{g})$. In particular, we can see that the homomorphism takes $A^{(1)}$ to C_1 , $A^{(2)}$ to $C_2 - (\rho_n, \rho_n)\hbar^2$, $E^{(1)}$ to $-\Delta$ and $F^{(1)}$ to Δ' .*

Remark 6.10. In section 6 of [9], the authors construct certain elements $A_n(\lambda), B_n(\lambda), C_n(\lambda) \in \mathcal{T}_{\hbar}^n[[\lambda]]$. They observe that these elements satisfy some (but not all) of the relations of the \mathfrak{sl}_2 Yangian.

It is easy to see that the elements $A_n(\lambda), B_n(\lambda), C_n(\lambda)$ defined in [9] are the images of the same named elements of $\mathbf{Y}_{-2n}(\mathfrak{sl}_2)[[\lambda]]$ under the homomorphism $\beta \circ \overline{\Phi}_{-2n}^0$. This explains why these elements satisfy the relations from [9, (6.5)].

Moreover, the formulas (6.7) from [9] are special cases of the formulas from [4, Corollary B.17].

6.11. Compatibility of the coproducts

According to Proposition 6.9 there is a homomorphism $\beta \circ \overline{\Phi}_{-2n}^0: \mathbf{Y}_{-2n}(\mathfrak{sl}_2) \rightarrow \mathcal{T}_{\hbar}^n$. Twisting by the additive shift automorphisms T_{ε} (notations of Section 6.2) gives a family of homomorphisms $\beta \circ \overline{\Phi}_{-2n}^0[\varepsilon] := \beta \circ \overline{\Phi}_{-2n}^0 \circ T_{\varepsilon}: \mathbf{Y}_{-2n}(\mathfrak{sl}_2) \rightarrow \mathcal{T}_{\hbar}^n$.

We have the following quantum version of Theorem 2.8:

Theorem 6.12. *The following diagram commutes:*

$$\begin{array}{ccc} \mathbf{Y}_{-2k-2l}(\mathfrak{sl}_2) & \xrightarrow{\Delta} & \mathbf{Y}_{-2k}(\mathfrak{sl}_2) \otimes \mathbf{Y}_{-2l}(\mathfrak{sl}_2) \\ \downarrow \beta \circ \overline{\Phi}_{-2k-2l}^0 & & \downarrow \beta \circ \overline{\Phi}_{-2k}^0[\frac{l\hbar}{2}] \otimes \beta \circ \overline{\Phi}_{-2l}^0[-\frac{k\hbar}{2}] \\ \mathcal{T}_{\hbar}^{k+l} & \xrightarrow{\tau_{k,l}} & \mathcal{T}_{\hbar}^k \otimes \mathcal{T}_{\hbar}^l \end{array}$$

Proof. Follows from Proposition 2.4 and Lemma 4.9. \square

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