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Topological Vector Spaces and Their Applications

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Contents

Preface

This book gives a concise exposition of the fundamentals of the theory of topological vector spaces, complemented by a survey of the most important results of a more subtle nature, which cannot be qualified as basic, but knowledge of which is useful for applications, and, finally, some of such applications connected with differential calculus in infinite-dimensional spaces and measure theory. Almost half of the book is devoted to these applications, which makes it very different from the whole series of known texts on topological vector spaces. Another notable difference between this book and known treatises like Bourbaki [**87**], Edwards [**150**], Grothendieck [**207**], Jarchow [**237**], Kelley, Namioka [**270**], Köthe [**292**], Narici, Beckenstein [**365**], Pérez Carreras, Bonet [**385**], Robertson, Robertson [**420**], Schaefer [**436**], Trèves [**530**], and Wilansky [**567**] is that we decided to include also some results without proofs (this does not concern the fundamentals, of course) with references instead, which enables us to inform the reader about many relatively recent achievements; some of them are disguised as exercises (with references to the literature), such exercises should not be confused with usual exercises marked by the symbol °. Thus, with respect to the presented information, our book is not covered by any other book on this subject (though, we cannot claim that it covers any such book).

Chapter 1 contains the fundamentals of the theory, including a large list of concrete examples, some general concepts (convex sets, seminorms, linear mappings) and a number of facts, the most important of which is the Hahn–Banach theorem on extensions of functionals in its diverse versions.

The main material of Chapter 2 is connected with projective and inductive limits (including strict inductive limits and inductive limits with compact embeddings, which is not sufficiently discussed in the existing literature), and also Grothendieck's method of constructing Banach spaces embedded into locally convex spaces.

Chapter 3 contains the classical material related to the so-called duality theory, i.e., introduction of different locally convex topologies on a given space giving the same set of continuous linear functionals. The central topics here are the Mackey–Arens theorem on topologies compatible with duality, the results on weak compactness, including the Eberlein–Šmulian and Krein–Šmulian theorems, and also some concepts and facts connected with completeness of locally convex spaces.

Chapter 4 is devoted to the fundamentals of the differentiation theory in locally convex spaces. It presents a general scheme of differentiability with respect to a system of sets (partial cases of which are Gâteaux, Hadamard and Fréchet differentiabilities) and a thorough discussion of important for applications differentiabilities with respect to systems of bounded and compact sets.

Chapter 5 gives a concise introduction to measure theory on locally convex spaces. Here we discuss extensions of cylindrical measures, the Fourier transform and conditions for the countable additivity in its terms (in particular, the Minlos and Sazonov theorems and their generalizations), covariance operators, measurable linear functionals and operators, measurable polynomials, and some important classes of measures (such as Gaussian, stable, and convex).

Each chapter opens with a brief synopsis of its content. All chapters contain many additional subsections with some more specialized information related to the main themes of the chapter, and also many exercises are given (more difficult ones are provided with hints or references). The book ends with the historic-bibliographic comments, the list of references (with indication of page numbers of citing the included works), and the author and subject indices.

The prerequisites for the first chapter of this book are just a grasp knowledge of calculus and linear algebra and some experience with basic concepts of topology, but for a thorough study it is advisable to be acquainted with a university course of functional analysis (following any text, e.g., Kolmogorov, Fomin [**284**] or Rudin [**425**]).

We are very grateful to T.O. Banakh, E.D. Kosov, I. Marshall, S.N. Popova, A.V. Shaposhnikov, A.S. Tregubov and E.V. Yurova for useful remarks and corrections.

Our work on this book began 25 years ago by the initiative of Vladimir Ivanovich Sobolev (1913–1995), the author of a series of widely known texts on functional analysis (including one of the first Russian texts, published as early as in 1951), and its completion is a tribute to the memory of this remarkable scientist and teacher.