Finite plateau in spectral gap of polychromatic constrained random networks

V. Avetisov,[1,6] A. Gorsky,[2,3] S. Nechaev,[4,5] and O. Valba[1,6]

[1]*N.N. Semenov Institute of Chemical Physics RAS, 119991 Moscow, Russia*

[2]*Institute of Information Transmission Problems RAS, Moscow, Russia*

[3]*Moscow Institute of Physics and Technology, Dolgoprudny 141700 Russia*

[4]*Interdisciplinary Scientific Center Poncelet (CNRS UMI 2615), Moscow, Russia*

[5]*P.N. Lebedev Physical Institute RAS, 119991 Moscow, Russia*

[6]*Department of Applied Mathematics, National Research University Higher School of Economics, 101000 Moscow, Russia*

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We consider critical behavior in the ensemble of polychromatic Erdős-Rényi networks and regular random graphs, where network vertices are painted in different colors. The links can be randomly removed and added to the network subject to the condition of the vertex degree conservation. In these constrained graphs we run the Metropolis procedure, which favors the connected unicolor triads of nodes. Changing the chemical potential, μ, of such triads, for some wide region of μ, we find the formation of a finite plateau in the number of intercolor links, which exactly matches the finite plateau in the network algebraic connectivity (the value of the first nonvanishing eigenvalue of the Laplacian matrix, λ_2). We claim that at the plateau the spontaneously broken Z_2 symmetry is restored by the mechanism of modes collectivization in clusters of different colors. The phenomena of a finite plateau formation holds also for polychromatic networks with $M \geqslant 2$ colors. The behavior of polychromatic networks is analyzed via the spectral properties of their adjacency and Laplacian matrices.

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I. INTRODUCTION

Critical behavior in topological random networks has been under scrutiny since the mid-2000s. One of the most intriguing phenomena is the clusterization that occurs in dynamical (evolutionary) networks when some control parameters are changing. Typically, the network evolution consists of randomly adding and removing links via Metropolis dynamics with a demand of a global "optimization" of a network structure. Under "optimization," we can understand a network design of a special topology. The rearrangement of network links could be restricted by additional conservation laws, for example, by the condition of the vertex degree conservation in all nodes. In what follows, we consider in detail only such constrained networks. If the vertex degree is the same in all nodes, then such a network is known as the "random regular graph."

In unconstrained Erdős-Rényi (ER) graphs, links can be randomly added and removed to the system subject to the condition of their global conservation (but without the conservation of the vertex degree). The evolution of such networks, maximizing the number of connected closed triads of bonds, $n_\triangle$ (triangles), leads to the formation of one maximally dense cluster ("clique"). This clusterization occurs critically and is known as the *Strauss transition*. Often, it is convenient to work with the canonical ensemble of networks, in which the value $n_\triangle$ is coupled to the chemical potential, μ, playing the role of the "energy" assigned to every closed triad of connected bonds. At some critical value, $\mu_{\rm cr}$, the network falls into the "Strauss phase"—a single full subgraph with a maximally possible number of triangles (the "clique") [1]. This phenomenon has been studied and satisfactorily described in the frameworks of the random matrix model [2], as well as by using the mean-field arguments [3].

The analysis of the phase transition in constrained Erdős-Rényi networks (CERN) and in random regular graphs (RRG) of N vertices, carried out in Ref. [4], amounts to a picture which is essentially different from the Strauss transition. It has been found in Ref. [4] that the condition of maximization of a number of closed triangles, $n_\triangle$, forces the random network with a conserved vertex degree to form a multiclique ground state (contrary to a single-clique ground state in the Strauss model). The number of cliques for CERNs equals asymptotically $[1/p]$ (for $N \gg 1$), where p is the probability of a link formation and Np is the average vertex degree at the network preparation (at the first step of the network evolution). In RRGs (for $N \gg 1$), the maximal number of emerging cliques is $[N/k]$, where k is the vertex degree.

It is convenient to study the network evolution using its adjacency matrix, A, whose elements, a_{ij}, are defined as follows: $a_{ij} = 1$ if the vertices i and j are connected and $a_{ij} = 0$ otherwise. At the preparation condition, the spectral density of the matrix A is the Wigner semicircle for nonconstrained Erdős-Rényi networks and two-humped Kesten-McKay distribution for random regular graphs [5] (the case of CERNs, depending on p, interpolates between these two distributions). The decay of the network in clusters appears, in terms of the spectral density of the matrix A, as the splitting of $[1/p]$ eigenvalues from the main zone of the eigenvalues distribution, and formation of the second (non-perturbative) zone, separated from the main one by a finite gap [4]. Each eigenvalue, which tunnels from the main zone to this new one, corresponds to some dense cluster ("clique") in the network. The interaction between separated clusters is reflected in the transformation of the enveloping shape of the main zone, which acquires triangular form both for CERNs and RRGs. The decay of the network happens critically above some value, $\mu_{\rm cr}$.

In terms of the matrix model language, the formation of the second zone filled by discrete eigenvalues means breaking of the initial $U(N)$ symmetry of the reference state into the product of smaller symmetry groups corresponding to

cliques-blocks of the ground-state adjacency matrix. More information about the network structure can be obtained by studying the Laplacian matrix, L, instead of the adjacency one, A. Recall that the Laplacian matrix of the graph is related to the adjacency matrix as follows:

$$L = D - A, \tag{1}$$

where D is the degree matrix. We have checked that the second (nonperturbative) zone of L is formed by "soft" discrete eigenvalues (modes), which tunnel from the main zone, associated with the "hard" part of the spectrum. For RRGs, the spectra of L and A are identical up to the shift to some constant and mirror reflection—such that hard Laplacian modes get mapped onto soft adjacency modes and vice versa. The Laplacian matrix is positively defined and has the minimal eigenvalue $\lambda_1 = 0$ corresponding to the homogeneous eigenvector $\mathbf{v}_1 = (1, \ldots, 1)$. The degeneration of the Laplacian mode, λ_1 (i.e., the number of zero's eigenvalues in L) defines the number of disconnected components of the graph. The behavior of the second eigenvalue of the Laplacian, λ_2, in RRGs is the subject of the several mathematical studies [6–9] and has an important meaning, known as "the algebraic connectivity." In particular, if $\lambda_2 > 0$, then the graph is connected. The value of λ_2 plays an important role in the relaxation and transport properties of the network, since it defines the inverse diffusion time [10] and is crucial for determining the synchronization of multiplex networks [11]. The corresponding eigenvector (Fiedler vector or "algebraic connectivity") [12] establishes the bijection between the layers of the multilayer network.

Two important properties of spectral densities of CERN and RRG adjacency matrices beyond the critical point, μ_{cr}, have to be mentioned. First, it was found in Ref. [4] that the Laplacian spectra of each clique and of the whole network are very different. The spectrum of the clique is discrete, being typical for the sparselike graphs [13], while the spectrum of the whole network has two-zonal structure with the continuous triangle-like form of the first (main) zone. In Ref. [16] we have interpreted the presence of the second zone as the collectivization (or synchronization) between the eigenvalues in different clusters. The second property concerns the nonergodic behavior of modes in the main zone of the spectrum. It has been found in Ref. [17] that there is a memory on the initial (preparation) conditions in the continuum part of spectrum, which is a signature of the nonergodic behavior and existence of some hidden conservation laws. In the second (nonperturbative) zone, all modes are localized [17] both for CERNs and RRGs, while the modes in the continuous (central) zone remain delocalized. As the criterium of localization, we used the standard notion, widely exploited in random matrix theory, namely the "level spacing distribution" or the distribution between the nearest eigenvalues in the spectrum. The system is in the delocalized state if the level spacing shares the Wigner surmise, and is in the localized state if the distribution is Poissonian.

Many new phenomena were announced in Ref. [16] for "polychromatic" CERNs and RRGs, where it was implied that vertices are painted (once and for all) in M colors at the preparation conditions. Instead of maximizing the number of connected closed triads of bonds (as in unicolor networks), we maximize the number of connected unicolor triads (irrespective of whether they form triangles). In the canonical description, we associate the chemical potential, μ, to every unicolor connected triad [14]. It turns out that the initially Z_2-symmetric polychromatic network is absolutely unstable and at any value $\mu > 0$ of the unicolor trimer chemical potential, the network spontaneously breaks into weakly connected unicolor cliques. We have found in Ref. [16] that at some critical value, μ_{cr}, the number of cross-color links undergoes a phase transition and develops a wide, though *finite*, plateau in μ. Some possible interpretations of the plateau formation have been suggested in Ref. [16]; however, more involved analysis clarifying this issue is presented below.

In this paper we discuss the spectral properties of the adjacency and Laplacian matrices of polychromatic networks. Our aim is to determine the spectral density of the ground state of CERNs and RRGs, focusing on the statistics of the highest eigenvalues of the adjacency matrix and, correspondingly, the lowest eigenvalues of the Laplacian matrix. We separately discuss the models with chemical potentials associated with closed triangles of bonds (for colorless networks) and with unicolor trimers of nodes (for polychromatic networks). The case of chemical potential for unicolor triangles in polychromatic network can be considered as the generalization of the colorless triangles pattern. The behavior of polychromatic network in this case is similar to the monocromatic one—the network gets defragmented into unicolor cliques above some critical value of the chemical potential μ.

Specifically, our findings for polychromatic network are as follows:

(i) The dichromatic (polychromatic) network is absolutely unstable when the chemical potential, μ, of connected unicolor triads of nodes is introduced and immediately splits at any $\mu > 0$ into the collection of unicolor subgraphs-cliques, which can be interpreted as the spontaneously Z_2-symmetry breaking (correspondingly, Z_M-symmetry breaking for M-color networks).

(ii) In a wide interval, $\mu \in [\mu_{in}, \mu_{out}]$ the number of intercluster connections (i.e., cross-color bonds) remains unchanged. For dichromatic networks, this behavior is identical to the plateau formation in the second eigenvalue, $\lambda_2(\mu)$, of the corresponding Laplacian matrix: $\lambda_2(\mu) = \text{const}$ for $\mu \in [\mu_{in}, \mu_{out}]$.

(iii) The plateau is finite and is the synchronization region of unicolor subgraphs-cliques. Between the values, μ_{in} (at which the plateau begins) and μ_{out} (at which it ends), the cliques are synchronized and we can speak about the effective restoration of Z_2 symmetry for dichromatic networks (and Z_M symmetry for M-color networks).

The interesting question in statistics of networks consisting of many species deals with the influence of the interactions within a given class of species on the properties of the whole network. In our polychromatic networks, the dynamics (adding and removal of bonds) occurs in the whole network and is color insensitive; moreover, the condition of the vertex degree conservation also does not distinguish the node colors, and only the interaction between unicolor connected trimers is "in color." In contrast, in typical multilayer networks, the "in-layer" bonds rearrangement occurs independently in each layer, while the layers interact by imposing some collective "interlayer" contact interactions. The first striking example

of the dramatic effects enforced by interlayer interactions was found in the percolation context in Ref. [18]. Later it was recognized that local interactions in the network affect the diffusion properties of the whole system dramatically [19–21]. In Refs. [10,19], the authors found the formation of the semi-infinite plateau for the second eigenvalue λ_2 of the Laplacian matrix in a two-layer network with specific interlayer interaction. It was argued that, depending on the interaction strength, the network layers can behave coherently or individually. The corresponding property can be read off from the eigenvectors of lowest Laplacian eigenvalues.

To summarize, the key differences between the setups discussed in Refs. [10,20,21] and the model considered in our work are as follows:

(i) In our study the two-color (multicolor) network emerges dynamically from an initially homogeneous polychromatic network, due to the simplest color-sensitive trimer interaction, while in Ref. [10] the layers are handmade.

(ii) In our work the effect of the plateau formation is caused by two things: the conserved vertex degree of the network and the in-layer (i.e., the unicolor) trimer interactions, while in Ref. [10] the authors control the interlayer interactions.

(iii) In our work the plateau is finite with the well-specified entrance and exit points, while in Ref. [10] the plateau for λ_2 is semi-infinite and apparently cannot be made finite in the frameworks of the considered model.

However, despite the difference between polychromatic networks in the spontaneously Z_2-broken phase and the multilayer networks, these systems seem to have more common properties than distinctions.

The paper is organized as follows. In Sec. II we briefly note the setup for the models under consideration and describe the corresponding network ground states with the chemical potential μ of connected unicolor trimers. In Sec. III we consider the networks with positive μ and investigate the evolution of the spectrum yielding the plateau formation and termination for the number of cross-color bonds. It is argued that the key phenomena for the plateau formation is the rearrangement of a few lowest eigenvalues of the Laplacian matrix of the graph. In Sec. IV we discuss the localization properties of the spectrum of polychromatic networks with some preference for the formation of monochromatic triads of bonds. In Sec. V we compare the phenomena of the plateau formation in our study with previous considerations. The comments on the related issues and the open questions are summarized in the Discussion. Some mean-field arguments related to the plateau formation are presented in the Appendix.

II. DESCRIPTION OF THE MODEL

The conventional colorless Erdős-Rényi network is prepared by randomly joining any pair of N vertices with probability p. This produces the Poissonian distribution, $P(k)$, of the vertex degree, k, with the mean value $\langle k\rangle = pN$. For the ensemble of ER networks the density of eigenvalues of the adjacency matrix A of a network is defined in a standard way,

$$\rho(\lambda) = \lim_{N\to\infty} \frac{1}{N}\left\langle \sum_{i=1}^{N} \delta(\lambda-\lambda_i)\right\rangle, \tag{2}$$

where λ_i is the eigenvalue of the matrix A and $\langle ...\rangle$ means averaging over the ensemble of $2^{N^2/2}$ different realizations of such matrices. For $N\to\infty$ and finite p, the density $\rho_W(\lambda)$ tends to the Wigner semicircle:

$$\rho_W(\lambda) \to \frac{1}{2\pi N}\sqrt{4N-\lambda^2}, \quad -2\sqrt{N} \leqslant \lambda \leqslant 2\sqrt{N}. \tag{3}$$

The isolated largest eigenvalue $\lambda = pN-1$ corresponds to the homogeneous eigenvector $v = (1,\ldots,1)$.

The random regular graph is the network with the vertex degrees, k, identical in all network nodes. Locally, the RRG behaves as a Bethe tree; however, it differs from the infinite Bethe tree by the presence of cycles of finite lengths. The spectral density $\rho(\lambda)$ of the RRG adjacency matrix is given by the Kesten-McKay distribution [5,22] (see also Ref. [23] for the transparent derivation of $\rho_{\rm KM}(\lambda)$ from the return probability on the Bethe lattice)

$$\rho_{\rm KM}(\lambda) = \frac{k\sqrt{4(k-1)-\lambda^2}}{2\pi(k^2-\lambda^2)},$$
$$-2\sqrt{k-1} \leqslant \lambda \leqslant 2\sqrt{k-1}. \tag{4}$$

The full RRG spectral density has in addition one isolated eigenvalue located at $\lambda = k$ beyond the main zone of the spectral density. The density $\rho_{\rm KM}(\lambda)$ differs from the Wigner semicircle; however, it is similar to asymptotically at large k on proper scaling. The finite-size correction for the Kesten-McKay law has been found in Ref. [24].

The ER networks created with the probability p and the RRGs with the vertex degree $k = pN$ have many similar static properties—see, for example, Refs. [25–27] for a review. It particular, it was proved rigorously that the RRG can be "sandwiched" between two very close ER networks and shares their statistical characteristics, like cycle distributions and the chromatic numbers [28]. However, the stochastic dynamics on ER and on RRG (as, for instance, the redistribution of links) leads to essentially distinct final states in ER and RRG, because the stochastic dynamics is very sensitive to the absence or presence of the vertex degree conservation. Imposing the degree conservation condition in the ER network, one can make the dynamic behavior of the CERN very similar to that of the RRG.

Here we describe the model of evolving polychromatic (dichromatic) dynamic networks (both CERN and RRG), generalizing the setup of the colorless model considered in Ref. [4]. In general, our construction reproduces the one described in Ref. [16]; however, some details of the model are more specified, so it seems instructive to provide here the full definition of the system under consideration.

We have already mentioned in the Introduction that each monochromatic triad of nodes (the triadic "motif"), regardless of whether it is closed or open, is weighted with the chemical potential, μ. Thus, the grand-canonical partition function of the system can be written as follows:

$$Z(\mu) = \sum_{\{\rm states\}}{}' e^{-\mu n_{\triangle}}, \tag{5}$$

where the prime in Eq. (5) means that the summation runs over all possible configurations of links under the condition of fixed degree in each graph vertex, i ($i = 1,\ldots,N$).

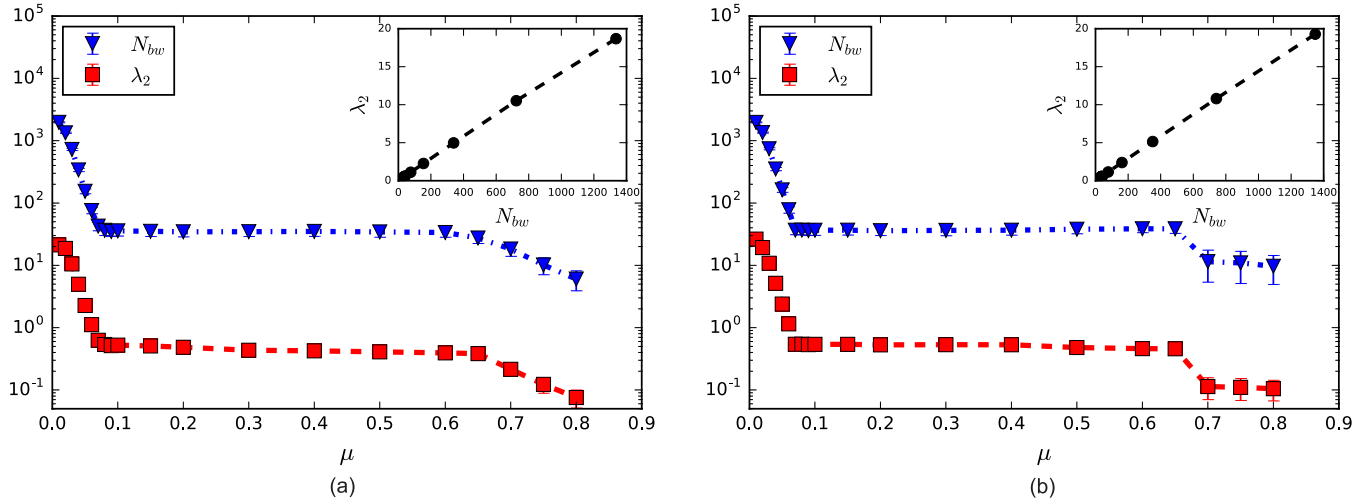

FIG. 1. Evolution of the spectral gap in the two-color CERN (a) and RRG (b) versus evolution of number of black-white links. Inserts: The linear dependence of the spectral gap on the number of black-white links. The numerical data are obtained by averaging over 500 randomly generated graphs of 256 vertices and with the probability $p = 0.15$ for CERN ensemble and $k = 38$ for RRG.

To reach the ground state of the system, we run the stochastic evolution of the CERN or RRG (the discrete Langevin dynamics), starting from some initial configuration. The initial state of the network is prepared by connecting any randomly taken pair of vertices with the probability p (regardless the node color). Then, one randomly chooses two arbitrary links, say, between vertices i and j $(i-j)$ and between k and m $(k-m)$, and reconnects them, getting new links, $(i-k)$ and $(j-m)$. Such reconnection conserves the vertex degree [29]. Now one applies the standard Metropolis algorithm with the following rules: (i) If, after the reconnection the number of connected unicolor triads of nodes (both black and white) in the network is increased, a move is accepted, and (ii) if the number of connected unicolor triad of nodes is decreased by $\Delta n_{\triangle}$, or remains unchanged, then a move is accepted with the probability $e^{-\mu \Delta n_{\triangle}}$. Then the Metropolis algorithm runs repetitively for large set of randomly chosen pairs of links until it converges. In Refs. [30] it has been shown that the algorithm actually converges to the true ground state in the equilibrium ensemble of random undirected colorless Erdős-Renyi networks with fixed vertex degree. For polychromatic networks such a convergence has not yet been considered rigorously in the literature.

III. RESULTS

A. Finite spectral plateau in a network with advantage of connected unicolor trimers

Here we present the analysis of the ground state of $M =$ 2-color constrained Erdős-Rényi networks and of random regular graphs reached in the Metropolis dynamics under the condition of maximization of connected unicolor triads of nodes. In the canonical description, the chemical potential, μ, is attributed to every unicolor connected triad of nodes in a large N-vertex network with the conserved vertex degree. At any value $\mu > 0$, the M-color network undergoes the color separation into monochrome clusters (the case $M = 2$ was described for the first time in Ref. [16]). So one can say that after the color separation, the network spontaneously splits into an M-cluster graph with different weights for cross-color and in-color bonds. This splitting does not depend on the initial condition and the M-cluster state is the true ground state of the network.

The important feature of this model is the specific plateau formation in the number of cross-color links above some critical value $\mu_{\rm cr}$ of the chemical potential of connected monochrome trimers. The origin of this phenomena remains still hidden, though some plausible conjectures concerning the plateau formation have been made in Ref. [16]. Below we attack this question via spectral analysis, similarly to the unicolor case [4], focusing at the gap formation in the spectral density of the adjacency and Laplacian matrices of the networks.

For simplicity, we begin with the $M = 2$-color networks. The spectral density of the adjacency and Laplacian matrices for the CERN and RRG ensembles is insensitive to the network color structure, though is very sensitive to the cluster formation. We assume that the chemical potentials of black, μ_b, and white, μ_w, connected unicolor trimers are equal, i.e., $\mu_b = \mu_w$, and hence the Hamiltonian is Z_2 symmetric subject to the constraint provided by the fixed vertex degree. For a completely symmetric M-color system, the Hamiltonian is Z_M symmetric. At $\mu = 0$ the spectral density consists of the main zone (Wigner semicircle or Kesten-McKay form) and one single isolated eigenvalue, counting the number of isolated clusters. As soon as μ becomes infinitesimally positive, color splitting occurs, and one extra separated eigenvalue tunnels out of the main zone, thus breaking the Z_2 symmetry.

At some critical value, $\mu_{\rm cr}$, two effects happen simultaneously (see Fig. 1): (ii) The plateau begins in the number of black-white links, $N_{\rm bw}$, and (ii) the plateau begins in the spectral gap of the Laplacian matrix, measured by the algebraic connectivity, λ_2. The similarity in the behaviors of λ_2 and of $N_{\rm bw}$ is not occasional: The algebraic connectivity is linearly proportional to the number of links to be cut to get two disconnected components of the graph. Hence, for our network we have

$$\lambda_2(\mu) = cN_{\rm bw}(\mu), \tag{6}$$

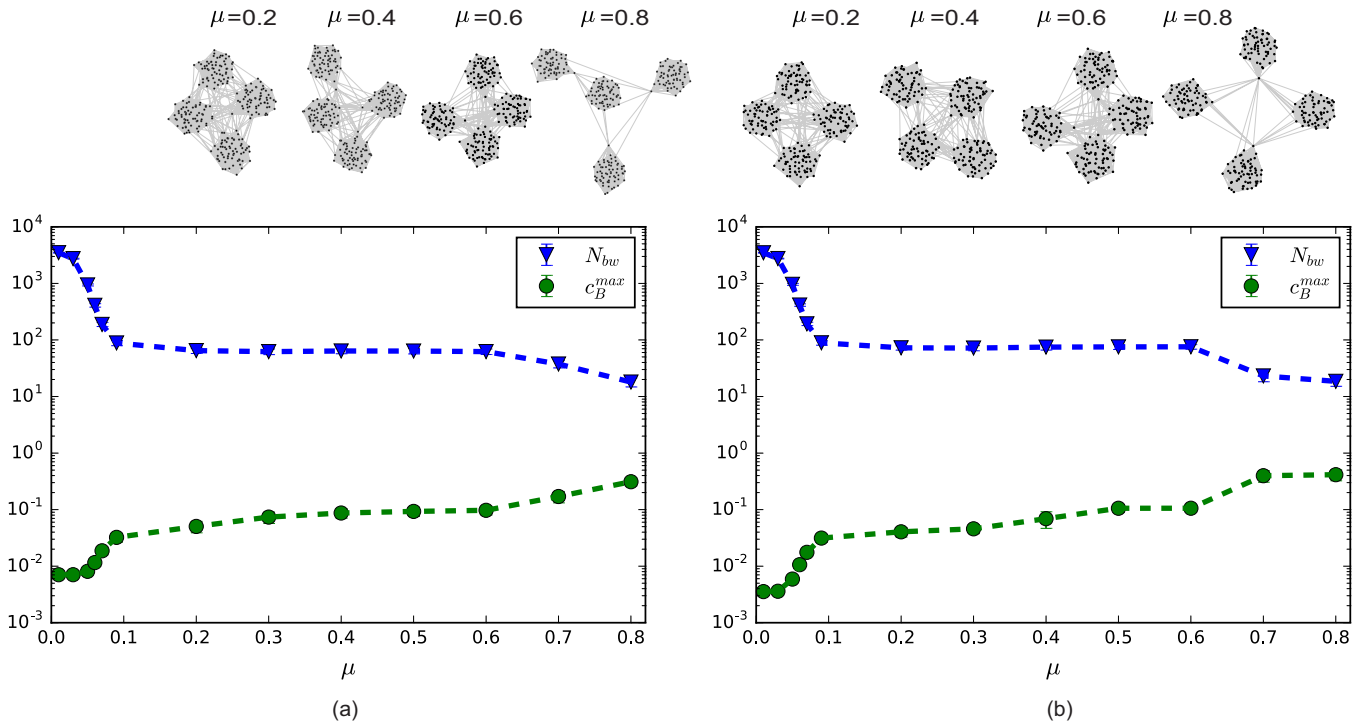

FIG. 2. Rearrangement of the cluster connection pattern along the plateau for two-color CERNs (a) and RRGs (b), Dependencies of the number of links between unicolor clusters, $N_b w$, and the evolution of the maximal betweenness centrality of a node $c_B^{\max}$ on μ.

where c is come constant. The plateau in the algebraic connectivity is the spectral counterpart of the phenomena described in Ref. [16]. The constraint of the degree conservation, i.e., the consideration of CERNs and RRGs, is crucial for the plateau formation phenomena: No plateau is formed if one considers the conventional evolution of Erdős-Rényi networks without the vertex degree conservation.

The phenomena of the exit from the plateau at the critical chemical potential, $\mu_{\rm cr}^-$, deserves some attention. We have investigated this question via visualization of the adjacency matrix evolution for ensembles of CERNs and RRGs. The key phenomena at the plateau exit are as follows. The number of polychromatic links at the plateau is stable; they connect vertices in clusters of distinct colors; and the "string" between clusters, measured in number of polychromatic links, has the finite width. When the value $\mu_{\rm cr}^-$ is approached, the spectral density and the geometry of connections change drastically; see Fig. 2. These changes can be easily detected by measuring the "betweenness centrality" [31] of network nodes. The betweenness centrality of a node v is given by the expression

$$c_B = \sum_{s\neq v\neq t} \frac{\sigma_{\rm sp}(v)}{\sigma_{\rm sp}}, \tag{7}$$

where $\sigma_{\rm sp}$ is the total number of shortest paths from node s to node t and $\sigma_{\rm sp}(v)$ is the number of those paths that go through v. Above $\mu_{\rm cr}^-$, the special "gates" (or "hubs") with a very high betweenness centrality get formed outside the clusters. It should be noted that *almost all* intercluster connections pass through these hubs. The formation of such hubs is accompanied by thinning of the string connecting different clusters at $\mu < \mu_{\rm cr}^-$. The dependencies of the number of links between unicolor clusters, $N_{\rm bw}$, and the evolution of the maximal betweenness centrality of a node $c_B^{\max}$ on μ are clearly seen in Fig. 2.

The behavior of M-color CERNs and RRGs with equal chemical potentials for all colors is similar to that of the two-color case. Again, we introduce the chemical potential, μ, for connected unicolor trimers of vertices. At an infinitesimally positive value of chemical potential, the network gets immediately defragmented into M unicolor clusters, which implies the spontaneous breaking of the Z_M discrete symmetry. The unicolor clusters, $1,\ldots,M$, are connected by some number of "polychromatic links" which can be considered as the order parameters of the model. We denote by N_{ik} the number of polychromatic links between the clusters i and k. At some value, $\mu_{\rm cr}$, the plateaus in the dependencies $N_{ik}(\mu)$ get formed simultaneously for any pair (i,k). From the viewpoint of the spectrum of the Laplacian matrices of CERN and RRG, M-smallest eigenvalues develop plateaus for each pair (λ_i,λ_k), where $(i,k) = 1,\ldots,M$ $(i \neq k)$.

In principle, two scenarios could be realized: (i) The isolated eigenvalues corresponding to each color arrange their "own" nonperturbative zone or (ii) the common second zone involving the eigenvalues of all colors gets formed. The result of the numerical simulations for the case of equal chemical potentials for all colors in the $M = 4$-color network is presented in Fig. 3. For small μ there are the central zone plus one isolated eigenvalue $\lambda_1 = 0$. With μ increasing, the eigenvalues corresponding to four monochrome clusters split off from the central zone, forming the second nonperturbative zone in the spectral density of the ensemble of Laplacian matrices. Figure 3 demonstrates that the *common* two-zonal

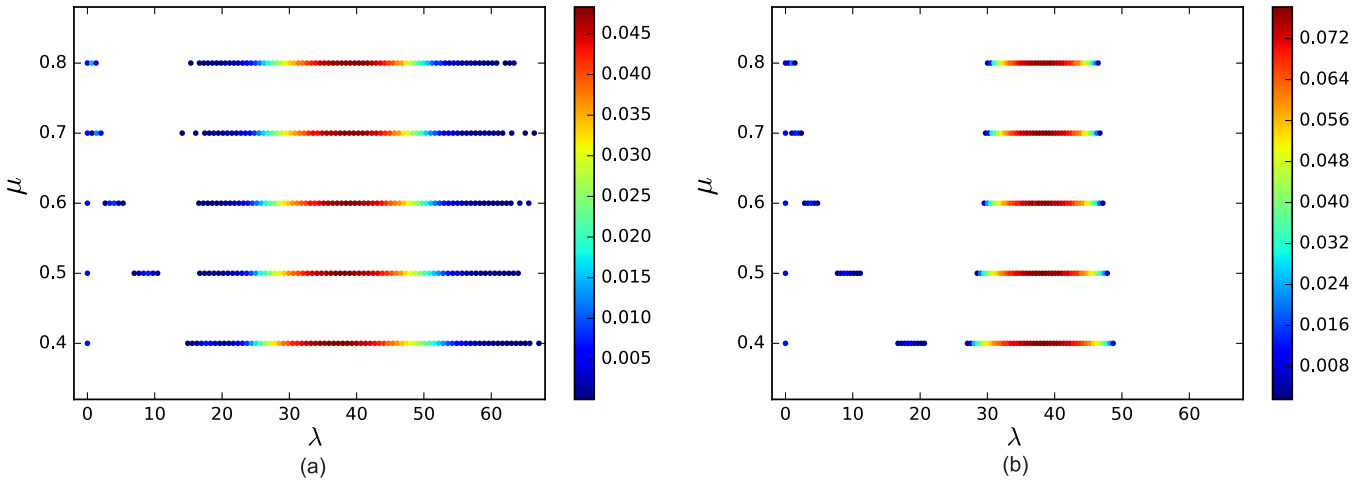

FIG. 3. Formation of the two-zonal structure in the spectral densities of Laplacian matrices in ensembles of $M = 4$-color CERN (a) and RRG (b) on the color separation. The numerical data are obtained for ensembles of 250 randomly generated graphs of 256 vertices and with the probability $p = 0.15$ for CERN and $k = 38$ for RRG. The color bar indicates the spectral density.

spectral structure plus one isolated eigenvalue, $\lambda_1 = 0$, get formed.

B. Semianalytic discussion of the plateau formation

We can provide simple estimates for the value of $\mu_{\rm cr}^{+}$ at which the plateau begins. Consider one black-white bond. This bond does not contribute to the energy, determined through the number of connected unicolor triads, but has an entropy contribution, which can be estimated as $\sim \ln N$, where N is the number of graph vertices. If we switch this link to a single-color (for example, black) cluster, then the energy contribution associated with such a switch becomes $2Np\,\mu_{\rm cr}$, because one link (ij) increases the number of triples by $k_i + k_j - 2$ (k_i is the degree of the vertex i). At the plateau, the entropy contribution to free energy from this bond becomes comparable with the energy contribution in the cluster, and hence we obtain the estimate for $\mu_{\rm cr}$ for the plateau beginning, as

$$\mu_{\rm cr} \sim \frac{\ln N}{2Np}. \tag{8}$$

Equation (8) confirms the observation that $\mu_{\rm cr}$ depends on p as p^{-1} and indicates that in the thermodynamic limit $N \to \infty$ the plateau begins at $\mu_{\rm cr} = 0$, which is consistent with the mean-field arguments presented in the Appendix.

Now we turn to the qualitative explanation of the plateau formation phenomena. Consider the Laplacian matrix, L, for the dichromatic case, for which the matrix L has the block structure:

$$L = \begin{pmatrix} L_1 & C \\ C^{\top} & L_2 \end{pmatrix}, \tag{9}$$

where L_1 and L_2 designate unicolor clusters and the matrix C includes links between nodes of different colors. Using the standard expression for the determinant of the block matrix, we can write:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\ \det(D - CA^{-1}B). \tag{10}$$

We are interested in the μ dependence of the first nonzero eigenvalue, λ_2, of the Laplacian matrix, L:

$$\det \begin{pmatrix} L_1 - \lambda & C \\ C^{\top} & L_2 - \lambda \end{pmatrix} = \det(L_1 - \lambda) \times \det(L_2 - C^{\top}(L_1 - \lambda)^{-1}C) = 0. \tag{11}$$

The explicit form of matrices L_1, L_2, C and their μ dependence are unknown, although some claims concerning the plateau formation can be done. There are two sets of eigenvalues coming from the first and the second determinants, respectively. There is the competition between the second eigenvalue, $\lambda_2^{(L_1)}$, of block L_1 [note that $\lambda_1^{(L_1)} = 0$] and the lowest eigenvalue of the second determinant. There are two possible options: (i) The eigenvalue $\lambda_2^{(L_1)}$ is the lowest nonvanishing eigenvalue of the whole network and (ii) the eigenvalue, $\lambda_3^{(L)}$, of the second determinant plays the role of the algebraic connectivity. In our case, the color block matrix, $L_1(\mu)$, has the nontrivial μ dependence, while the off-diagonal block, C, has no explicit μ dependence in the matrix elements since there is no chemical potential for polychromatic links. However, the number of nonvanishing elements in C is μ dependent.

We claim that the extremities of the plateau correspond to points of the intersection of the cluster eigenvalue, $\lambda_2^{L_1}(\mu)$, and the whole network eigenvalue, $\lambda_3^{(L)}$. The number of snapshots of eigenvalue "trajectories" are depicted in Fig. 4, from which it becomes clear that $\lambda_3^{(L)}$ is almost μ independent. Since $\lambda_3^{(L)}$ is the lowest nonvanishing eigenvalue of the whole network Laplacian matrix, we conclude that there is no μ dependence in the corresponding eigenvalue. We identify this regime with the plateau. The positions for the plateau entrance, $\mu_{\rm cr}^{+}$, and exit, $\mu_{\rm cr}^{-}$, are the solutions of the equation

$$\lambda_2^{(L_1)}(\mu) = \lambda_3^{(L)}. \tag{12}$$

Note that λ_2 at the plateau coincides with $\lambda_3^{(L)}$, whose value is close to the boundary of the main spectral zone.

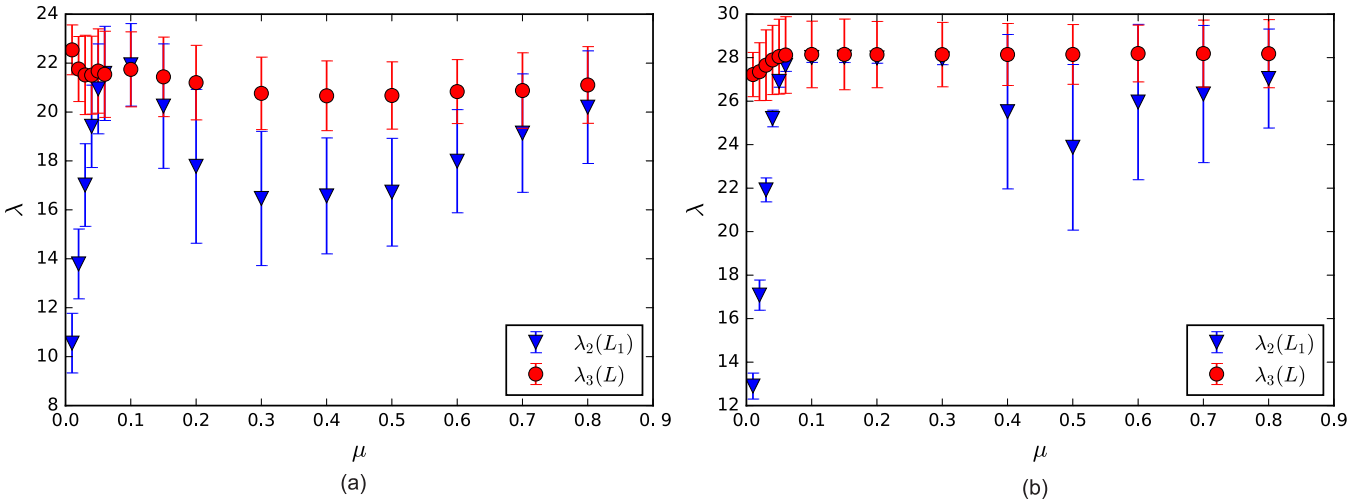

FIG. 4. Dependencies of $\lambda_2^{(L_1)}(\mu)$ and $\lambda_3^{(L)}(\mu)$ in a wide range of μ.

The discussion of the plateau termination is more subtle. At the plateau exit, the intercluster interactions proceed through hubs, which means that the matrix C has a nondiagonal structure. After the exit from the plateau, one sees again the nontrivial μ dependence, which signals that the eigenvalue $\lambda_2^{(L_1)}(\mu)$ dominates again. The key question concerns the coherence between the unicolor clusters at the plateau. The condition for the entrance at the plateau means that main parts of the spectral zone of each cluster and of the whole network have same supports. Thus, at the plateau we have complete synchronization of clusters, similar to the one in Ref. [32] in multilayer networks, which can be seen from the behavior of corresponding eigenvectors.

Similarly to Ref. [32], we can perform the search for the algebraic connectivity by looking at the ground state of the following Hamiltonian:

$$H = \langle v^\top | L | v \rangle + g v^\top v + f \langle v | 1 \rangle, \tag{13}$$

where v is the eigenfunction of the network Laplacian, while g and f are Lagrangian multipliers for the conditions of eigenvectors normalization and their orthogonality to the homogeneous $\lambda = 0$ state. The Hamiltonian involves the matrix Laplacian matrix of the CERN (or RRG), hence the averaging over ensembles of networks is implied. Note that this Hamiltonian in the continuum has a lot in common with the $O(N)$-sigma model interacting with two-dimensional gravity. Indeed, the first term yields the kinetic contribution in the random geometry, while rest of the terms are familiar constraints in the sigma model. The phase transition in the continuum denotes the change of the space dependence of the ground-state wave function.

IV. LOCALIZATION IN POLYCHROMATIC NETWORKS

It was found in Ref. [17] that the transport properties of topologically perturbed CERN and RRG are nontrivial. The analysis of the level spacing distribution (the distribution between nearest-neighboring eigenvalues), $P(s)$, performed in Ref. [17], shows that the states in the central continuum zone are delocalized, while the level spacings in the second nonperturbative zone share the Poissonian distribution, meaning that these states are localized. Define

$$P_{\rm del}(s) = A\, s\, e^{-Bs^2} \quad \text{below mobility edge, } \lambda_m, \text{ for GOE}$$
$$P_l(s) = e^{-s} \quad \text{above mobility edge, } \lambda_m, \text{ for GOE}, \tag{14}$$

where GOE denotes the Gaussian orthogonal ensemble and A, B are some model-dependent coefficients. Since delocalized modes in the main zone correspond to spectra inside the clusters, we figuratively associate such clusters with the metallic phase, while the entire network is a kind of insulator. Moreover, it was found in Ref. [17] that the spectral density in the main zone strongly depends on the network preparation conditions. This observation has led us to the conclusion that corresponding delocalized states can be thought as nonergodic.

It is worthwhile to compare the above findings with other related studies. The transport properties of the unperturbed RRG have been analyzed in Ref. [26], where the delocalization of all modes was reported. The transport properties of perturbed RRGs with the on-site Anderson-like disorder become much more interesting. At some critical strength of the disorder, there is the Anderson localization transition [33–35] for which the ergodic and non-ergodic behavior of delocalized states is under intensive discussion. The renewed interest in the one-particle localization in RRGs deals with the attempt to treat RRG as a model of the Fock space for some interacting many-body system [36]. The one-particle localization in RRGs presumably is related to the phenomena of many-body localization in the real space, where initial degrees of freedom are identified with the localized states in the one-particle model on the graph [37,38].

In this section we provide results of similar study for polychromatic RRGs. We are focused at the level spacing distribution as a criterion of the localization. The number of colors, M, is large enough to make the second zone sufficiently wide. The results are shown in Fig. 5 for $M = 8$. It is seen from (14) and Fig. 5 that the states in the main zone are delocalized, while the ones in the second zone (which is the

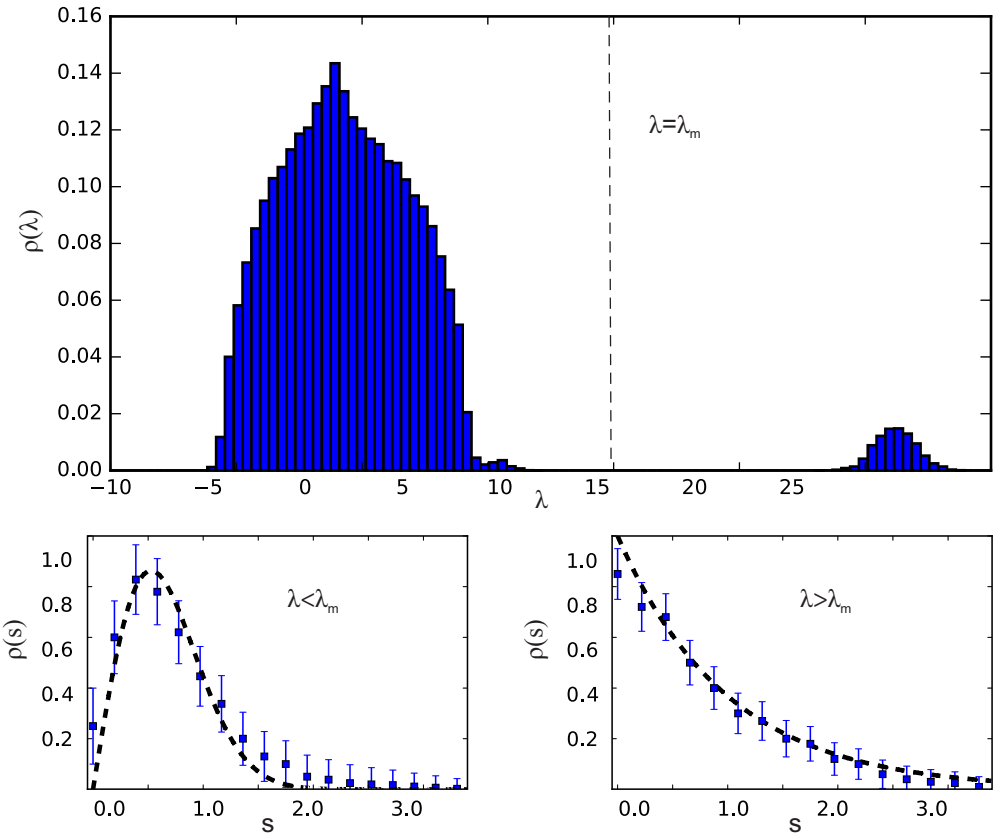

FIG. 5. Level-spacing distribution for eight-color RRG. The regions $\lambda < \lambda_m$ (Wigner-like theory for level spacing) and $\lambda > \lambda_m$ (Poissonian-like distribution for level spacing) correspond to delocalized and localized parts of the graph. Numerical results are obtained for an ensemble of 250 randomly generated graphs of 256 vertices with the degree $k = 20$.

collection of eigenvalues splitted from the main zone), are localized. We can speculate that the polychromatic RRG serves as the Fock space for some interacting many-body system. Then the one-particle localization in the polychromatic RRG is mapped onto the many-body localization in the real space and the degrees of freedom in the real space are the localized states in the Fock space. Accepting this, we conclude that degrees of freedom in the real space are clusters of different colors.

Let us highlight the features specific for polychromatic networks, which are absent in monochromatic ones.

(i) In colorless networks, the effects of graph topology are crucial (for example, the network clusterisation is forced by increasing the number of closed triangles in the network). In contrast, in polychromatic networks just the unicolor triples of connected nodes play the key role.

(ii) Since the Fock space of a multicolor network is also polychromatic, one should identify the color with the specific quantum number in some many-body system embedded into the real space (for instance, the color could mark the representation of some global group).

It is worthwhile to attribute the physical meaning to the chemical potential of unicolor triads of nodes in terms of the system in the real space. We can conjecture that since the cycles are not involved in the Fock space, the connected unicolor triad of vertices could be attributed with the perturbation mixing three energy levels. Certainly this issue deserves deeper consideration.

V. PLATEAU FORMATION AND EXIT: COMPARISON WITH OTHER MODELS

In Ref. [16] we have conjectured that the plateau formation reflects the phase coexistence in the perturbed network for the chemical potential, μ, lying in the inteval $[\mu_{\rm cr}^{+},\mu_{\rm cr}^{-}]$. Below we provide the qualitative arguments supporting this conjecture and compare it with other models. The dynamical behavior of networks, which resembles the one discussed in this work, has been considered in Refs. [10,21,32] in a different setting. The authors considered the two-layer network constructed "by hand" from the very beginning. They have introduced the parameter p, controlling the interlayer interactions, and have analyzed the dependence $\lambda_2(p)$. At $p = 0$ the interaction is absent and two layers are completely disconnected, while at nonvanishing p there are two regimes in the $\lambda_2(p)$ dependence. At small p it has been found that $\lambda_2(p) \propto p$, while at $p > p^*$ the eigenvalue λ_2 was independent on p. The found plateau was semi-infinite since no other critical behavior in p is observed. It was also noted in Ref. [39] that the number of hubs used to connect two layers decreases sharply at the plateau.

The linear connection between layers is simple enough to admit an exact solution based on the properties of block diagonal matrices with uniformly weighted off-diagonal elements.

It was argued in Ref. [32] that the spectrum of the Laplacian matrix yielding $\lambda_2(p)$ has two regimes corresponding to two solutions of the simple equation for the wave function overlap if one considers the Laplacian as the Hamiltonian of the system. Denote by $\langle v_a|v_b\rangle$ and by $\langle v_a|1\rangle$ the overlaps of parts of the wave function $v=(v_a,v_b)$ (corresponding to λ_2) with the other part and with the ground-state wave function, respectively. It was shown that these variables undergo the sharp transition at some $p=p^*$. In the analysis of Refs. [20,21,32], the off-diagonal block of the Laplacian matrix is chosen in the simplified form as $C=pI$, while the block A is p independent and I is the identity matrix. To obtain the eigenvalues, consider the determinant identity for the block matrix:

$$\det(A-\lambda+pI)\ \det\left(A-\lambda+pI-pI\frac{1}{A-\lambda+pI}pI\right)=0. \tag{15}$$

Looking at the second determinant and selecting the eigenvector of A with zero eigenvalue, we immediately obtain the equation $(\lambda-p)^2=p^2$, which implies the competition between the value $2p$ and the second eigenvalue, $\lambda_2^{(A)}$, of the matrix A. The plateau starts when

$$\lambda_2^{(A)}=2p. \tag{16}$$

It was shown in Ref. [32] that exactly at this point the third eigenvalue, $\lambda_3^{(L)}$, hits the second one. Therefore, at small values of p, one has $\lambda_2^{(L)}=p$, while at plateau, $\lambda_3^{(L)}=\lambda_2^{(A)}$.

Before the plateau gets formed, the system can be considered an interacting bilayer system, whose matrix A has zero's eigenvalue with the homogenous eigenvector. This allows us to identify the eigenvector of the Laplacian corresponding to the eigenvalue $2p$, which is

$$V=\begin{pmatrix}1\\-1\end{pmatrix}. \tag{17}$$

At the plateau the layers are coherent and practically indistinguishable. A more general phase structure with different synchronization patterns between two layers has been analyzed in Ref. [21]. The behavior of the system depends on the ratio of the mean values of in-layer and interlayer links.

The model developed in Ref. [32] has some similarities and differences with the one considered in our work. In our dichromatic network the separation in unicolor clusters at any positive μ can be regarded as a two-layer system. These two layers emerge dynamically from the spontaneous symmetry breaking and are not created artificially ("by hand") from the very beginning. We have introduced the chemical potential not for interlayer links but for the connected unicolor (in-layer) trimers of nodes. Still, our weight, $e^{-\mu}$, plays the same role as p of Ref. [32] since both quantities measure the relative strengths of the in-layer and interlayer links. The networks in Refs. [10,21,32] evolve by independent dynamics in each layer, whereas in our network the dynamics is "cross-layered" (i.e., common for all clusters-layers).

To compare the dependencies $\lambda_2(p)$ of Ref. [32] and $\lambda_2(\mu)$ of our work, note that the small-p linear regime in Ref. [32] corresponds to the Arrenius regime after the exit from the plateau in our case. Thus, the plateau entrance in Ref. [32] gets mapped to the plateau exit in our case. The most striking difference between $\lambda_2(p)$ and $\lambda_2(\mu)$ is the seemingly infiniteness of the plateau in Ref. [32] and its finiteness in our work. Another remark concerns the role of hubs for the interlayer links. The exit from the plateau in our case indicates a sharp increase of the number of hubs which looks similar to the decrease of the number of hubs at the plateau entrance discussed in Ref. [39].

Let us emphasize that we observe the collectivization of the network clusters due to effective asymmetry of the relative in-layer and interlayer interactions caused by advantage of connected triads of unicolor links. We claim that the spontaneous segregation of clusters can be regarded as dynamical exudation of hidden layers in the initially homogenous system. However, the rigorous microscopic description of this phenomena is still absent. The attempt to construct the mean-field theory of the dichromatic model is proposed in the Appendix.

We would like to pay attention to some similarity of the plateau emergence with the so-called Griffiths phase known in the theory of the phase transitions in disordered systems (see Ref. [40] for review). The Griffiths phase emerges when, due to fluctuations, the spontaneous droplets are formed, in which the phase transition occurs earlier than in the whole system. To complete the phase transition in the whole system it takes some "time" to collectivize the degrees of freedom in formed clusters due to the long-range interactions. In our polychromatic networks we have an ideologically similar situation. At some critical value, μ, we have a kind of phase transition in the system which yields the restoration of the Z_2 symmetry due to the collectivization of modes at the plateau above $\mu_{\rm cr}^+$. However, due to the interclusters connections, the phase transition in the whole system is incomplete and the value of the control parameter should be increased up to $\mu_{\rm cr}^-$ to get the new phase in the whole system. This happens on the plateau exit where the clusters become completely disconnected.

It is worth asking what the continuum counterpart of the phenomena we have found in the network is. Since in the continuum the network becomes the membrane and the Laplacian of the network becomes the membrane Laplacian, the natural counterpart of our network study is the interplay between the wave functions of two membrane "bubbles" associated with clusters. Indeed, the effects of collectivization of the modes in two bubbles (finite reservoirs or clusters) takes place depending on the length and the diameter of the tube linking these bubbles [41].

Another analogy in the continuum is as follows. Consider the quantum spectrum of the particle in the double-well potential from the phase-space viewpoint. If the potential is a slightly skewed, then we have two minima at the classical level and two disconnected closed phase-space trajectories in the (p,x) plane. Quantum tunneling provides the effective connection (via instantons) between two classically disconnected regions of the phase space. From the textbooks we know what happens when the potential is Z_2 symmetric. Naively, we expect doubly degenerated ground state and breaking Z_2 symmetry; however, due to the nonperturbative effects, the sum over instanton–anti-instanton pairs amounts to the level splitting, and the difference between the two lowest energy levels is nonanalytic in the coupling constant. From the phase-space viewpoint, we could consider the two regions in the phase

space connected by the "nonperturbative tube." It would be interesting to pursue this analog further.

VI. DISCUSSION

In this work we have described the generic critical phenomena of spontaneous structure formation (clusterization) with subsequent synchronization between emerged clusters, which can occur in disordered distributed systems assembled from elementary species ("graph nodes") of various types. To demonstrate such a behavior, the evolution of the complex system should satisfy two basic requirements: (i) the stochastic dynamics affects all nodes and has some global conservation law in every node (like the vertex degree conservation), (ii) the stochastic dynamics happens in presence of a "driving force" which is controlled by the energy μ of small collectives of nodes of a given type (like connected triads of unicolor vertices).

At the critical value of the control parameter, μ, the network splits via the eigenvalue tunneling mechanism into the collection of weakly connected extended objects (clusters, bubbles, vesicles, etc.). The clusterization is accompanied by the emergence of a nonperturbative zone of soft modes around zero's eigenvalue of the Laplacian matrix of the whole system. The formation of these soft modes is the result of collectivization of cross-cluster interactions. The hard modes live inside the clusters only and are collectivized as well.

The networks are absolutely unstable with respect to the color separation. From this point of view, the clusterization that occurs at any positive μ can be regarded as a mechanism of revealing of hidden layers. Our study, supported by Ref. [32], clearly demonstrates the existence of a generic phenomena of the "layer confinement" accompanied by the collectivization of their degrees of freedom and formation of a "confined" ground state of the system. The mechanism of the ground-state confinement formation is entropic and is based on an asymmetry of the interlayer and in-layer interactions. In Ref. [32] the interlayer links play the key role, while in our study the interaction of connected in-layer trimers results in the plateau formation. Moreover, introducing the chemical potential μ for one color only (say, only to "black" connected trimers in dichromatic network), we still see the formation of two layers and their synchronization at the plateau. This clearly demonstrates that interactions in one layer only induce the effective interactions in the second one. It is yet unclear which types of network motifs could induce the confinement of layers and which could not. In particular, we see that presence of the chemical potential of unicolor *dimers* or *closed* unicolor triads is not sufficient to produce the clusterization.

Recently, another model involving triangles and links has been discussed in Ref. [42]. Instead of introducing the chemical potential for the number of triangles, the authors fixed the *numbers* of triangles and links. The authors in Ref. [42] have demonstrated the existence of several infinite series of phase transitions with different symmetry breaking patterns. It would be interesting to identify these transitions with the phase transition corresponding to a particular symmetry breaking found in Ref. [4]. Note that in our study there are quite complicated symmetry-breaking states at the intermediate stages of the stochastic evolution. However, the symmetry breaking in the ground state in perturbed CERN and RRG is unique. This suggests that possibly many phases found in the microcanonical ensemble in Ref. [42] correspond to the intermediate metastable states in our mixed ensemble.

Understanding the striking similarity of the color separation in polychromatic networks and the QCD chiral symmetry breaking seems to be very challenging. In the QCD case, the instanton–anti-instanton ensemble is considered and the vertex degree which counts the number of the instanton's zero modes remains unchanged, being topologically fixed. The overlap matrix for zero modes plays the role of the Laplacian matrix for the network. Playing with some control parameter, we can get the formation of extended objects built from collection of instantons. Since the Dirac operator on the graph can be considered, roughly speaking, as the square root of the graph Laplacian, L, the behavior of the spectrum of the Laplacian matrix near vanishing eigenvalue can tell us about the corresponding behavior of the spectrum of the Dirac operator. The latter is relevant for the chiral condensate formation. The existence of the nonperturbative soft zone around $\lambda = 0$, which we have found for the Laplacian, yields the condensate emergence via the Casher-Banks relation for the Dirac operator spectrum $\langle\bar{\Psi}\Psi\rangle = -\pi\rho(0)$. We have found in this paper that states in the soft part of the spectrum are localized and the appearance of the nonperturbative zone corresponds to the Anderson transition. This correlates with the interpretation of the chiral phase transition in QCD as the Anderson transition [43]. It would be very interesting to develop this line of reasoning further.

The spectral gap can been considered as an order parameter in the statistical systems. Recently, it was argued that the collision of two largest eigenvalues of the entanglement Hamiltonian for two parts of the whole system in the Schmidt decomposition corresponds exactly to the point of the phase transition in the whole system. Moreover, the spectrum of the entanglement Hamiltonian defines the entanglement entropy [44]. In the context of the tensor networks the entanglement entropy of two subnetworks is closely related to the number of links to be cut to make two subnetworks disconnected. This certainly has parallels with our study and we hope to discuss the spectrum of the entanglement Hamiltonian in dichromatic and polychromatic models and entanglement entropy of clusters emerging in polychromatic networks in a separate publication.

To summarize, the novel criticality found in polychromatic topological networks is a quite general phenomenon and seems to have various practical applications in the structure formation of evolving distributed systems related to life science, such as social networks, electrical networks, and neural networks. For example,a polychromatic network, such as the famous Schelling segregation pattern of social behavior [45], could describe a society in which the number of connections of each individual is approximately constant, although it may vary from one individual to the other. Each "color" designates some typical habit to which the individual belongs. Imposing the condition that three (or more) individuals with the same habit have some preference, μ, we discuss the existence of the "stability plateau" on which the number of intercolor connections is insensitive to μ in Ref. [46]. At the plateau exit, which occurs at sufficiently large μ, we see the formation of spontaneously induced hubs associated with the leadership.

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APPENDIX: PARTITION FUNCTION OF DICHROMATIC NETWORK IN THE MEAN-FIELD APPROXIMATION

Let us provide the mean-field-like consideration of the dichromatic network with fixed fugacity of unicolor trimers (irrespective of their topology). We introduce $s_{ij} = 0,1,-1$, the "spin" variable corresponding to the bond ij (the matrix element a_{ij} in the adjacency matrix A):

$$s_{ij} = \begin{cases} +1 & \text{bond between same colors} \\ -1 & \text{bond between different colors.} \\ 0 & \text{absence of a bond} \end{cases} \tag{A1}$$

The Hamiltonian H_{ijk} of the interactions between unicolor triples (ijk) of network bonds reads:

$$H_{ijk} = \frac{\mu}{4}\left(s_{ij}s_{jk} + s_{ij}s_{jk}^2 + s_{ij}^2 s_{jk} + s_{ij}^2 s_{jk}^2\right). \tag{A2}$$

The Hamiltonian H_{ijk} is equal to μ if and only if the neighboring values of s_{ij},s_{jk} correspond to the same color. If they correspond to different colors, or are absent, then $H_{ijk} = 0$.

The partition function depending on the concentration of black-white bonds, $c_{\rm bw}$, can be written as follows:

$$Z(c_{\rm bw}) = \sum_{\{s_{ij}=0,\pm1\}} \exp\left(\sum_{ijk}^{N} H_{ijk}\right)\delta\left(\sum_{ijk}^{N}\left(s_{ij} - s_{ij}^2 + 2c_{\rm bw}\right)\right) \times \prod_{j=1}^{N}\delta\left(\sum_{i=1}^{N} s_{ij}^2 - d_j\right), \tag{A3}$$

where d_j is the vertex degree of the graph (if $d_j = d$ for all $j = 1,\ldots,N$, then we have a regular random graph). The partition function (A3) is the exact expression for requested partition function of two-color network.

Let us transform the terms in the Hamiltonian H_{ijk} in the following way:

$$\begin{aligned}
\frac{\mu}{4}s_{ij}s_{jk} &= \frac{\mu}{8}(s_{ij}+s_{jk})^2 - \frac{\mu}{8}\left(s_{ij}^2 + s_{jk}^2\right) \\
\frac{\mu}{4}s_{ij}s_{jk}^2 &= \frac{\mu}{8}\left(s_{ij}+s_{jk}^2\right)^2 - \frac{\mu}{8}\left(s_{ij}^2 + s_{jk}^4\right) \\
\frac{\mu}{4}s_{ij}^2s_{jk} &= \frac{\mu}{8}\left(s_{ij}^2+s_{jk}\right)^2 - \frac{\mu}{8}\left(s_{ij}^4 + s_{jk}^2\right) \\
\frac{\mu}{4}s_{ij}^2s_{jk}^2 &= \frac{\mu}{8}\left(s_{ij}^2+s_{jk}^2\right)^2 - \frac{\mu}{8}\left(s_{ij}^4 + s_{jk}^4\right).
\end{aligned} \tag{A4}$$

Now we can introduce the auxiliary Gaussian fields which allow us to decouple interacting terms in (A4):

$$\begin{aligned}
e^{\frac{\mu}{8}(s_{ij}+s_{jk})^2} &= \sqrt{\frac{2}{\mu\pi}}\int_{-\infty}^{\infty} d\varphi_{ijk}\; e^{-\frac{2}{\mu}\varphi_{ijk}^2 + (s_{ij}+s_{jk})\varphi_{ijk}} \\
e^{\frac{\mu}{8}(s_{ij}+s_{jk}^2)^2} &= \sqrt{\frac{2}{\mu\pi}}\int_{-\infty}^{\infty} d\chi_{ijk}\; e^{-\frac{2}{\mu}\chi_{ijk}^2 + (s_{ij}+s_{jk}^2)\chi_{ijk}} \\
e^{\frac{\mu}{8}(s_{ij}^2+s_{jk})^2} &= \sqrt{\frac{2}{\mu\pi}}\int_{-\infty}^{\infty} d\omega_{ijk}\; e^{-\frac{2}{\mu}\omega_{ijk}^2 + (s_{ij}^2+s_{jk})\omega_{ijk}} \\
e^{\frac{\mu}{8}(s_{ij}^2+s_{jk}^2)^2} &= \sqrt{\frac{2}{\mu\pi}}\int_{-\infty}^{\infty} d\psi_{ijk}\; e^{-\frac{2}{\mu}\psi_{ijk}^2 + (s_{ij}^2+s_{jk}^2)\psi_{ijk}}.
\end{aligned} \tag{A5}$$

To each triad of vertices ijk, we identify for independent Gaussian fields $\varphi_{ijk},\chi_{ijk},\omega_{ijk},\psi_{ijk}$. Having such a representation (in fact, this is a type of Hubbard-Stratonovich transform), we decouple adjacent spins s_{ij} and s_{jk} interacting in the common graph vertex j. Exponentiating all the Kronecker δ functions in (A3),

$$\delta(x) = \frac{1}{2\pi i}\oint \frac{d\lambda}{\lambda^{x+1}} = \begin{cases} 1 & \text{for} x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{A6}$$

and using (A5), we may rewrite the partition function (A3) as follows:

$$\begin{aligned}
Z(c_{\rm bw},d_1,\ldots,d_N) = \left(\frac{2}{\mu\pi}\right)^{6N} \frac{1}{(2\pi i)^{N+1}}\oint d\lambda\; \lambda^{-2c_{\rm bw}-1}\prod_{j=1}^{N}\oint d\xi_j\; \xi_j^{d_j-1} \\
\times \prod_{ijk}^{N}\int_{-\infty}^{\infty} d\varphi_{ijk}\; e^{-\frac{2}{\mu}\varphi_{ijk}^2}\int_{-\infty}^{\infty} d\chi_{ijk}\; e^{-\frac{2}{\mu}\chi_{ijk}^2}\int_{-\infty}^{\infty} d\omega_{ijk}\; e^{-\frac{2}{\mu}\omega_{ijk}^2}\int_{-\infty}^{\infty} d\psi_{ijk}\; e^{-\frac{2}{\mu}\psi_{ijk}^2} \\
\times \sum_{\{s_{ij}=0,\pm1\}} \exp\left\{\sum_{ijk}^{N}\left[(s_{ij}+s_{jk})\varphi_{ijk} + (s_{ij}+s_{jk}^2)\chi_{ijk} + (s_{ij}^2+s_{jk})\omega_{ijk} + (s_{ij}^2+s_{jk}^2)\psi_{ijk}\right.\right. \\
\left.\left. - \frac{\mu}{4}(s_{ij}^2+s_{jk}^2) - \frac{\mu}{4}(s_{ij}^4+s_{jk}^4) - (s_{ij}-s_{ij}^2)\ln\lambda - s_{ij}^2\ln\xi_j\right]\right\}.
\end{aligned} \tag{A7}$$

In (A7) all spins s_{ij} sitting on bonds are decoupled and hence we can first perform summation over spins $\{s_{ij}=0,\pm 1\}$ to get an effective action for Gaussian fields φ_{ijk} and ψ_{ijk}. We have used the following identities (since the summation is carried out over the whole network and all bonds are independent):

$$\prod_{ijk}\exp[(s_{ij}+s_{jk})\varphi_{ijk}]\equiv\prod_{ijk}\exp[2s_{ij}\,\varphi_{ijk}]=\prod_{ij}\exp\left[2s_{ij}\sum_{k=1}^{N}\varphi_{ijk}\right]$$
$$\prod_{ijk}\exp\left[\left(s_{ij}+s_{jk}^2\right)\chi_{ijk}\right]\equiv\prod_{ijk}\exp\left[\left(s_{ij}+s_{ij}^2\right)\chi_{ijk}\right]=\prod_{ij}\exp\left[\left(s_{ij}+s_{ij}^2\right)\sum_{k=1}^{N}\chi_{ijk}\right] \tag{A8}$$

and similar identities for the fields ω_{ijk} and ψ_{ijk}.

For summation over spins on bonds we suppose that each link could take three independent values: $s_{ij}=0,+1,-1$. Performing summation, we arrive at the following expression for the effective partition function:

$$\begin{aligned}Z(c_{\rm bw},d_1,\ldots,d_N)=&\left(\frac{2}{\mu\pi}\right)^{2N}\frac{1}{2\pi i}\oint d\lambda\;\lambda^{-2c_{\rm bw}-1}\prod_{j=1}^{N}\frac{1}{2\pi i}\oint d\xi_j\;\xi_j^{d_j-1}\\&\times\prod_{ijk}^{N}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}d\varphi_{ijk}d\chi_{ijk}d\omega_{ijk}d\psi_{ijk}\;e^{-\frac{2}{\mu}(\varphi_{ijk}^2+\chi_{ijk}^2+\omega_{ijk}^2+\psi_{ijk}^2)}\\&\times\prod_{ijk}\left\{1+e^{-\mu N-\ln\xi_j}\left[\exp\left(2\sum_{k=1}^{N}(\varphi_{ijk}+\chi_{ijk}+\omega_{ijk}+\psi_{ijk})\right)\right.\right.\\&\left.\left.+\exp\left(-2\sum_{k=1}^{N}(\varphi_{ijk}-\psi_{ijk})-2\ln\lambda\right)\right]\right\}.\end{aligned} \tag{A9}$$

The effective Hamiltonian, H, reads

$$\begin{aligned}H=&-\frac{2}{\mu}\sum_{ijk}\left(\varphi_{ijk}^2+\chi_{ijk}^2+\omega_{ijk}^2+\psi_{ijk}^2\right)+\sum_{ij}\ln\left\{1+e^{-\mu N-\ln\xi_j}\left[\exp\left(2\sum_{k=1}^{N}\left(\varphi_{ijk}+\chi_{ijk}+\omega_{ijk}+\psi_{ijk}\right)\right)\right.\right.\\&\left.\left.+\exp\left(-2\sum_{k=1}^{N}(\varphi_{ijk}-\psi_{ijk})-2\ln\lambda\right)\right]\right\}.\end{aligned} \tag{A10}$$

The minimization of H over all fields leads to the following system of equations:

$$\begin{aligned}&\frac{\partial H}{\partial\varphi_{ijk}}=0\\&\frac{\partial H}{\partial\chi_{ijk}}=0\\&\frac{\partial H}{\partial\omega_{ijk}}=0\\&\frac{\partial H}{\partial\psi_{ijk}}=0\end{aligned}\quad\Rightarrow\quad\begin{aligned}&\frac{4}{\mu}\varphi_{ijk}-\frac{e^{-\mu N-\ln\xi_j}[e^{2\sum_k(\varphi_{ijk}+\chi_{ijk}+\omega_{ijk}+\psi_{ijk})}-e^{-2\sum_k(\varphi_{ijk}-\psi_{ijk})-2\ln\lambda}]}{A_{ij}}=0\\&\frac{4}{\mu}\chi_{ijk}-\frac{e^{-\mu N-\ln\xi_j}\;e^{2\sum_k(\varphi_{ijk}+\chi_{ijk}+\omega_{ijk}+\psi_{ijk})}}{A_{ij}}=0\\&\frac{4}{\mu}\omega_{ijk}-\frac{e^{-\mu N-\ln\xi_j}\;e^{2\sum_k(\varphi_{ijk}+\chi_{ijk}+\omega_{ijk}+\psi_{ijk})}}{A_{ij}}=0\\&\frac{4}{\mu}\psi_{ijk}-\frac{e^{-\mu N-\ln\xi_j}[e^{2\sum_k(\varphi_{ijk}+\chi_{ijk}+\omega_{ijk}+\psi_{ijk})}+e^{-2\sum_k(\varphi_{ijk}-\psi_{ijk})-2\ln\lambda}]}{A_{ij}}=0,\end{aligned} \tag{A11}$$

where

$$A_{ij}=1+e^{-\mu N-\ln\xi_j}\left[\exp\left(2\sum_{k=1}^{N}\left(\varphi_{ijk}+\chi_{ijk}+\omega_{ijk}+\psi_{ijk}\right)\right)+\exp\left(-2\sum_{k=1}^{N}\left(\varphi_{ijk}-\psi_{ijk}\right)-2\ln\lambda\right)\right]. \tag{A12}$$

From (A11) we see that $\chi_{ijk}=\omega_{ijk}$. Let us introduce now two new composite fields,

$$\begin{aligned}u_{ij}&=\sum_k(\varphi_{ijk}+\chi_{ijk}+\omega_{ijk}+\psi_{ijk})\\v_{ij}&=\sum_k(\varphi_{ijk}-\psi_{ijk}).\end{aligned} \tag{A13}$$

In terms of u_{ij} and v_{ij} we can rewrite (A11) and (A12) as a system of two independent equations:

$$\begin{aligned}&\frac{1}{\mu N}u_{ij}-\frac{e^{-\mu N-\ln\xi_j}e^{2u_{ij}}}{1+e^{-\mu N-\ln\xi_j}(e^{2u_{ij}}+e^{-2v_{ij}-2\ln\lambda})}=0\\&\frac{2}{\mu N}v_{ij}+\frac{e^{-\mu N-\ln\xi_j-2\ln\lambda-2v_{ij}}}{1+e^{-\mu N-\ln\xi_j}(e^{2u_{ij}}+e^{-2v_{ij}-2\ln\lambda})}=0.\end{aligned} \tag{A14}$$

In the mean-field approximation we suggest that $u_{ij} \equiv u$, $v_{ij} \equiv v$, and $\xi_j \equiv \xi$ and obtain the closed system of two equations on two fields u, v:

$$\frac{1}{\mu'}u - \frac{e^{-\mu'-\ln\xi}e^{2u}}{1+e^{-\mu'-\ln\xi}(e^{2u}+e^{-2v-2\ln\lambda})} = 0$$

$$\frac{2}{\mu'}v + \frac{e^{-\mu'-\ln\xi-2\ln\lambda-2v}}{1+e^{-\mu'-\ln\xi}(e^{2u}+e^{-2v-2\ln\lambda})} = 0, \quad \text{(A15)}$$

where we have introduced the normalized chemical potential of trimers, $\mu' = \mu N$.

It turns out that the system (A15) can be converted to the single transcendental equation for one unknown function, u (respectively, v). The equation for u reads

$$u - \lambda^2[2\mu' - u(\xi e^{\mu'-2u}+1)]\exp(3u + u\xi e^{\mu'-2u} - 2\mu') = 0. \quad \text{(A16)}$$

The (A16) can be analyzed in the regime when $|u| \ll 1$. Expanding (A16) up to quadratic terms in u and solving corresponding algebraic equation in a form $u = u(\xi,\lambda,\mu')$, we can identify the regime when the u does no depend (or very weakly depends) on μ'. This regime would correspond to the emergence of the plateau. In Fig. 6 we have shown the dependence $u(\mu')$ for few fixed values of ξ and λ.

For rather broad interval of ξ and λ, we see that $u(\mu') \approx$ const, which could be regarded as strong evidence of plateau existence, since the dependence $u(\mu')$ can be straightforwardly translated to the dependence of the concentration of black-white bonds on μ'. The proposed analysis is very preliminary and deserves more involved consideration; however, first indications that the effect of the plateau formation can be catched within the mean-field consideration make us very optimistic.

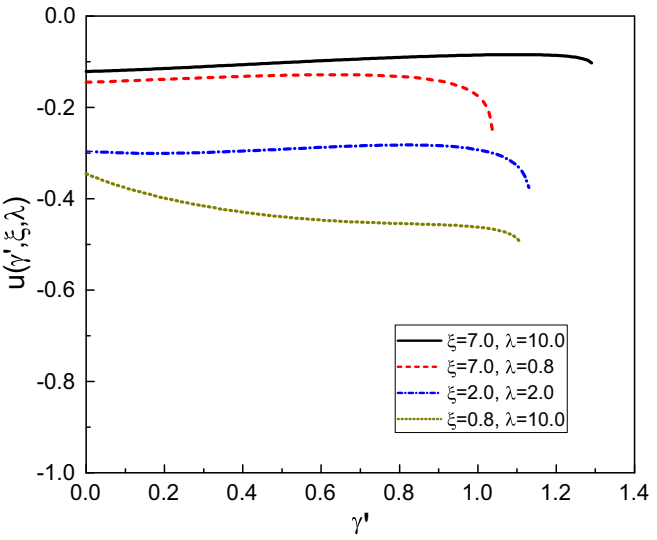

FIG. 6. Dependence $u(\mu')$ for few values of fixed parameters, ξ and λ. In a broad interval of these parameters, the value u is almost independent on g.

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