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# TRANSVERSE EQUIVALENCE OF COMPLETE CONFORMAL FOLIATIONS

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*We study the problem of classification of complete non-Riemannian conformal foliations of codimension $q > 2$ with respect to transverse equivalence. It is proved that two such foliations are transversally equivalent if and only if their global holonomy groups are conjugate in the group of conformal transformations of the $q$-dimensional sphere* $\mathrm{Conf}\,(\mathbb{S}^q)$. *Moreover, any countable essential subgroup of the group* $\mathrm{Conf}\,(\mathbb{S}^q)$ *is realized as the global holonomy group of some non-Riemannian conformal foliation of codimension $q$. Bibliography*: 16 *titles.*

## 1 Introduction

A foliation $(M, F)$ is said to be *conformal* if it admits the transversal conformal structure (cf. an exact definition in Section 6). We recall that a subset of a foliated manifold is *saturated* if it can be represented as the union of leaves. By an *attractor* of a foliation $(M, F)$ we mean a nonempty closed saturated subset $\mathscr{M}$ of $M$ possessing an open saturated neighborhood $\mathscr{U}$ such that the closure $\overline{L}$ in $M$ of any leaf $L$ in $\mathscr{U} \setminus \mathscr{M}$ contains $\mathscr{M}$. The neighborhood $\mathscr{U}$ is called the *basin* of the attractor $\mathscr{M}$. If, in addition, $\mathscr{U} = M$, then the attractor $\mathscr{M}$ is said to be *global*.

As is proved in [1], any conformal foliation $(M, F)$ of codimension $q > 2$ either is Riemannian or has an attractor. In [1, Theorem 4], it is proved that any non-Riemannian conformal foliation of codimension $q > 2$ on a compact manifold is a transversally homogeneous foliation, which strengthens the results of [2, 3].

In this paper, we solve the classification problem for complete non-Riemannian conformal foliations of codimension $q > 2$ with respect to transverse equivalence.

We note that the group of conformal transformations of a $q$-dimensional Euclidean space $\mathbb{E}^q$ coincides with the similarity group $\mathrm{Sim}(\mathbb{E}^q)$ and is (canonically) isomorphic to the stationary subgroup of the Lie group $\mathrm{Conf}\,(\mathbb{S}^q)$ of all conformal transformations of the standard sphere $\mathbb{S}^q$. The group $\mathrm{Conf}\,(\mathbb{S}^q)$ can be identified with the group of Möbius transformations $\mathbb{M}ob(q)$ of the sphere $\mathbb{S}^q$ [4]. We recall that $(\mathrm{Sim}(\mathbb{E}^m), \mathbb{E}^m)$-foliations, $m \geqslant 1$, are referred to as *transversally similar foliations* in [5]

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It is known that any conformal foliation can be regarded as a Cartan foliation in the sense of Blumenthal [6] or (which is equivalent in this case) in the sense of [5]. A conformal foliation $(M, F)$ is said to be *complete* if it is a complete Cartan foliation in the sense of [5]. As is shown in [7, Theorem 3], the completeness of a conformal foliation is equivalent to the existence of Ehresmann connections in the sense of Blumenthal–Hebda [8] for this foliation.

As was proved in [7, Theorem 5], any complete non-Riemannian conformal foliation $(M, F)$ of codimension $q > 2$ is a $(\mathrm{Conf}(\mathbb{S}^q), \mathbb{S}^q)$-foliation and has a global attractor. Furthermore, if $f : \widetilde{M} \to M$ is the universal covering map for $M$, then the induced foliation $(\widetilde{M}, \widetilde{F})$ is formed by the fibres of a locally trivial fibration $r : \widetilde{M} \to N$, where $N = \mathbb{E}^q$ if the foliation $(M, F)$ is transversally similar; otherwise, $N = \mathbb{S}^q$. The group of desk transformations $\Gamma = \Gamma(f)$ induces the group $\Psi$ of conformal transformations of the conformal manifold $N$ which is called the *global holonomy group* of the foliation $(M, F)$. Thus, the foliation $(M, F)$ is covered by the locally trivial fibration $r : \widetilde{M} \to N$.

Let $\mathfrak{F}$ be the set of foliations covered by fibrations (in the sense of Definition 2.2 below). We denote by $\mathfrak{F}_{\mathfrak{C}}$ the subset of $\mathfrak{F}$ consisting of complete conformal non-Riemannian foliations of codimension $q > 2$ (cf. [7]).

For a foliation $(M, F)$ covered by a fibration we introduce a pair $(N, \Psi)$, where $N$ is the universal covering map for the base of this foliation and $\Psi$ is an at most countable group of diffeomorphisms of $N$, called the *global holonomy group* of this foliation (cf. Proposition 1.1). We use the notation $(N, \Psi) = \beta(M, F)$.

First of all, we prove the following assertion.

**Theorem 1.1.** *Let $N$ be an arbitrary simply connected manifold. Any countable subgroup of the group of diffeomorphisms of the manifold $N$ is realized as the global holonomy group of some foliation covered by a fibration.*

**Definition 1.1.** Foliations $(M, F)$ and $(M', F')$ are *transversally equivalent* if there exists a foliation $(\mathbb{M}, \mathbb{F})$ and surjective submersions $p : \mathbb{M} \to M$ and $p' : \mathbb{M} \to M'$, which are Serre fibrations, with connected fibers such that

$$\mathbb{F} = \{p^{-1}(L) \mid L \in F\} = \{p'^{-1}(L') \mid L' \in F'\}. \tag{1.1}$$

Unlike the notion of transverse equivalence due to Molino [9, p. 63], Definition 1.1 contains the additional requirement that the submersions $p : \mathbb{M} \to M$ and $p' : \mathbb{M} \to M'$ are Serre fibrations, i.e., possess the covering homotopy property (we recall the definition in Section 2). By this requirement, the transverse equivalence of foliations covered by fibrations is realized by a foliation covered by a fibration.

We say that two foliations $(M, F)$ and $(M', F')$ are *equivalent* if they are transversally equivalent in the sense of Definition 1.1. As will be shown below (Proposition 4.1), it is an equivalence relation. We denote by $[(M, F)]$ the equivalence class containing the foliation $(M, F)$.

We begin with the transverse equivalence of foliations covered by fibrations. The set of classes of transversally equivalent foliations in $\mathfrak{F}$ is denoted by $\widetilde{\mathfrak{F}}$. Thus, $\widetilde{\mathfrak{F}} = \{[(M, F)] \mid (M, F) \in \mathfrak{F}\}$.

We consider the category $\mathfrak{P}$ of pairs $(N, \Psi)$, where $N$ is an arbitrary simply connected manifold and $\Psi$ is an at most countable group of diffeomorphisms of the manifold $N$. Morphisms of two objects $(N, \Psi)$ and $(N', \Psi')$ in $\mathfrak{P}$ are pairs of maps $(d, \theta)$, where $d : N \to N'$ is a smooth map and $\theta : \Psi \to \Psi'$ is a group homomorphism such that $d \circ \psi = \theta(\psi) \circ d$ for all $\psi \in \Psi$. We denote by $[(N, \Psi)]$ the class of isomorphic objects of the category $\mathfrak{P}$ containing $(N, \Psi)$. Let $\widetilde{\mathfrak{P}} = \{[(N, \Psi)] \mid (N, \Psi) \in \mathfrak{P}\}$.

**Theorem 1.2.** *The map* $\mathscr{B} : \widetilde{\mathfrak{F}} \to \widetilde{\mathfrak{P}} : [(M,F)] \mapsto [\beta(M,F)] = [(N,\Psi)]$, $(M,F) \in \mathfrak{F}$, *is a bijection.*

We recall that the group $\Psi$ of conformal transformations of a Riemannian manifold $(N, g)$ with a Riemannian metric $g$ is called *inessential* if there is a smooth positive function $\lambda$ on $N$ such that $\Psi$ is the isometry group of a Riemannian manifold $(N, \lambda g)$. Otherwise, the subgroup $\Psi$ is called *essential.*

We denote by $\mathfrak{P}_{\mathfrak{C}}$ the subcategory of the category $\mathfrak{P}$ whose objects are pairs $(N,\Psi)$, where either $N = \mathbb{S}^q$ and $\Psi$ is an essential countable subgroup of the group $\mathrm{Conf}\,(\mathbb{S}^q)$ or $N = \mathbb{E}^q$ and $\Psi$ is an essential countable subgroup of the group $\mathrm{Sim}(\mathbb{E}^q)$. Morphisms in $\mathfrak{P}_{\mathfrak{C}}$ are isomorphisms $(d,\theta)$ of the category $\mathfrak{P}$, where $d : N \to N'$ is a conformal diffeomorphism. Moreover, either $N = N' = \mathbb{S}^q$ or $N = N' = \mathbb{E}^q$.

We set $\widetilde{\mathfrak{F}_{\mathfrak{C}}} := \{[(M,F)] \mid (M,F) \in \mathfrak{F}_{\mathfrak{C}}\}$ and $\widetilde{\mathfrak{P}_{\mathfrak{C}}} := \{[(N,\Psi)] \mid (N,\Psi) \in \mathfrak{P}_{\mathfrak{C}}\}$. Based on Theorem 1.2, we prove the following classification theorem.

**Theorem 1.3.** *The map* $\mathscr{B}_{\mathfrak{C}} = \mathscr{B}|_{\mathfrak{F}_{\mathfrak{C}}} : \widetilde{\mathfrak{F}_{\mathfrak{C}}} \to \widetilde{\mathfrak{P}_{\mathfrak{C}}} : [(M,F)] \mapsto [\beta(M,F)] = [(N,\Psi)]$, $(M,F) \in \mathfrak{F}_{\mathfrak{C}}$, *is a bijection.*

**Corollary 1.1.** 1. *Two complete conformal, but not transversally similar foliations* $(M_1, F_1)$ *and* $(M_2, F_2)$ *of codimension greater than* 2 *are transversally equivalent if and only if their global holonomy groups* $\Psi_1$ *and* $\Psi_2$ *coincide* (*up to conjugation in the group* $\mathrm{Conf}\,(\mathbb{S}^q)$).

2. *Two complete non-Riemannian transversally similar foliations* $(M_1, F_1)$ *and* $(M_2, \mathscr{F}_2)$ *of codimension at least* 1 *are transversally equivalent if and only if their global holonomy groups* $\Psi_1$ *and* $\Psi_2$ *coincide* (*up to conjugation in the group* $\mathrm{Sim}(\mathbb{E}^q)$).

**Remark 1.1.** By Corollary 1.1, the global holonomy group defined up to conjugation in the group $\mathrm{Conf}\,(\mathbb{S}^q)$ is a complete invariant of the class of transversally equivalent non-Riemannian conformal foliations of codimension greater than 2.

From the proof of Theorem 1.2 and [7, Theorem 7] we obtain the following assertion.

**Proposition 1.1.** *Every class of transversally equivalent foliations in* $\widetilde{\mathfrak{F}}$ *or* $\widetilde{\mathfrak{F}_{\mathfrak{C}}}$ *contains a two-dimensional suspended foliation*

In [5], for complete Cartan foliations $(M,F)$ and, in particular, conformal ones the structure Lie algebra $\mathfrak{g}_0 = \mathfrak{g}_0(M,F)$ was introduced in the case of Riemannian foliations on compact manifolds, it coincides with the structural algebra introduced in [9]. By [10, Theorem 7], it is possible to interpret this Lie algebra as follows.

**Theorem 1.4.** *The structure Lie algebra* $\mathfrak{g}_0$ *of a complete non-Riemannian conformal foliation* $(M,F)$ *of codimension* $q > 2$ *is isomorphic to the Lie algebra of the Lie group* $\overline{\Psi}$, *where* $\overline{\Psi}$ *is the closure of the global holonomy group* $\Psi$ *of this foliation in the Lie group* $\mathrm{Conf}\,(\mathbb{S}^q)$ *if* $(M,F)$ *is not a transversally similar foliation or in the Lie group* $\mathrm{Sim}(\mathbb{E}^q)$ *in the opposite case. In particular, the structure Lie algebra* $\mathfrak{g}_0$ *is equal to zero if and only if* $\Psi$ *is a Klein group.*

For any subgroup $\Psi$ of the group $\mathrm{Conf}\,(\mathbb{S}^q)$ the limit set is defined and is denoted by $\Lambda(\Psi)$, (cf. [11]). From Theorems 1.3 and 1.4 we obtain the following assertion.

**Proposition 1.2.** *The Lie group* $\overline{\Psi}$, *the structure Lie algebra* $\mathfrak{g}_0$, *and the limit set* $\Lambda(\Psi)$ *of the global holonomy group* $\Psi$ *of a complete non-Riemannian conformal foliation* $(M,F)$ *of codimension* $q > 2$ *are invariants with respect to transverse equivalence.*

From Theorem 1.3 and [7, Theorem 5] we obtain the following assertion.

**Theorem 1.5.** *Let $\xi$ be an arbitrary class of transversally equivalent non-Riemannian conformal foliations of codimension greater than* 2, *let* $(M,F)$ *be any foliation of* $\xi$, *and let* $[(N,\Psi)] = \mathscr{B}(\xi)$. *Then the induced foliation* $\widetilde{F} = f^*F$ *on the space* $\widetilde{M}$ *of the universal covering map* $f : \widetilde{M} \to N$ *is formed by the fibers of a locally trivial fibration* $r : \widetilde{M} \to N$, *where* $N$ *is either* $\mathbb{S}^q$ *or* $\mathbb{E}^q$; *there exists a global attractor* $\mathscr{M}$ *of the foliation* $(M,F)$; *moreover,* $\mathscr{M}$ *has structure of one of the following types*:

(i) *the attractor* $\mathscr{M}$ *is a nontrivial minimal set if and only if the global holonomy group* $\Psi$ *of the foliation* $(M,F)$ *is not elementary, and* $\mathscr{M} = f(r^{-1}(\Lambda(\Psi)))$, *where* $\Lambda(\Psi)$ *is the limit set of the group* $\Psi$,

(ii) *the attractor* $\mathscr{M}$ *is a single closed leaf or the union of two closed leaves if and only if* $\Psi$ *is an elementary subgroup of the group* $\mathrm{Conf}\,(\mathbb{S}^q)$ *or* $\mathrm{Sim}(\mathbb{E}^q)$ *respectively*; *moreover,* $\mathscr{M}$ *is the union of two leaves only for those foliations that are not transversally similar.*

Thus, by Theorem 1.5, all transversally equivalent conformal foliations in $\mathfrak{F}_{\mathfrak{C}}$ have global attractors with the same transversal structure determined by the structure of the global attractor $\Lambda(\Psi)$ of the global holonomy group $\Psi$.

Examples (cf. Section 7) show that the structure of global attractors of conformal foliations with the Klein global holonomy group, i.e., with the trivial structure Lie algebra can be rather complicated, unlike transversally similar foliations which are, in this case, proper and have a unique closed leaf which that is a global attractor.

## 2 Foliations Covered by Fibrations

**2.1. Preimage of a foliation under a submersion.** A continuous map $p : X \to Y$ possesses the *covering homotopy property* with respect to a topological space $K$ if for any continuous map $G_0 : K \to X$ and any homotopy $H_t : K \to Y$, $t \in [0,1]$, such that $p \circ G_0 = H_0$ there exists an extension of $G_0$ to a homotopy $G_t : K \to X$ satisfying the equality $p \circ G_t = H_t$.

We recall that a *Serre fibration* is a continuous surjective map having the covering homotopy property with respect to any finite polyhedron (cf., for example, [12]). It is known that for Serre fibrations it is possible to construct the exact homotopy sequence for a fibration. It is also known that any locally trivial fibration is a Serre fibration. If a Serre fibration is a submerion, then it is called a *smooth Serre fibration*.

Let $p : M \to N$ and $f : \widetilde{N} \to N$ be surjective submersions. We set $f^*M := \{(y,z) \in \widetilde{N} \times M \mid f(y) = p(z)\}$. Then $f^*M$ is a closed embedded submanifold of the product of manifolds $\widetilde{N} \times M$ which is the preimage of the diagonal $\Delta$ of $N \times N$ under the map $f \times p : \widetilde{N} \times M \to N \times N$. We introduce the canonical projections $\widetilde{p} : f^*M \to \widetilde{N} : (y,z) \to y$ and $\widetilde{f} : f^*M \to M : (y,z) \to z$, where $(y,z)$ is any point in $f^*M$.

We note that $f^*M \to p^*\widetilde{N} : (y,z) \mapsto (z,y)$, $(y,z) \in f^*M$, is a diffeomorphism of manifolds and the canonical projections $\widetilde{p}$ and $\widetilde{f}$ are equivalent.

**Lemma 2.1.** 1. *Let* $p : M \to N$ *and* $f : \widetilde{N} \to N$ *be surjective submersions. Then the canonical projections* $\widetilde{f} : f^*M \to \widetilde{N}$ *and* $\widetilde{p} : f^*M \to M$ *are also surjective submersions satisfying* *the commutative diagram*

$$\begin{array}{ccc} f^*M & \xrightarrow{\widetilde{f}} & M \\ \Big\downarrow{\scriptstyle \widetilde{p}} & & \Big\downarrow{\scriptstyle p} \\ \widetilde{N} & \xrightarrow{f} & N. \end{array} \tag{2.1}$$

*If $p$ ($f$ respectively) is a Serre fibration, then $\widetilde{p}$ ($\widetilde{f}$ respectively) is also a Serre fibration.*

2. *If $p : M \to N$ is a Serre fibration with connected fibers and $f : \widetilde{N} \to N$ is a covering map, then $\widetilde{p} : f^*M \to M$ is a Serre fibration with connected fibers which are diffeomorphically mapped to the corresponding fibres of the fibration $p : M \to N$ under the covering map $\widetilde{f} : f^*M \to M$.*

**Proof.** 1. The definition of $f^*M$ and canonical projections imply the commutativity of the diagram and the equalities $p \circ \widetilde{f} = pr_1 \circ (f \times p)$ and $\widetilde{p} \circ f = pr_2 \circ (f \times p)$, where $pr_i : N \times N \to N$, $i = 1, 2$, are the projections onto the first and second factors of the product respectively. Since $f \times p : \widetilde{N} \times M \to N \times N$ and $pr_i$ are surjective submersions, the above equalities imply that $\widetilde{f}$ and $\widetilde{p}$ are also surjective submersions.

We assume that $p$ is a Serre fibration. We show that $\widetilde{p} : f^*M \to \widetilde{N}$ is also a Serre fibration. Let $K$ be an arbitrary finite complex, and let $H_t : K \to \widetilde{N}$, $t \in [0, 1]$, be a given homotopy. Consider a continuous map $G_0 : K \to f^*M$ such that $\widetilde{p} \circ G_0 = H_0$. Then $\Phi_t := f \circ H_t : K \to N$ is a homotopy in $N$ with $\Phi_0 = f \circ H_0$. Since $p \circ \widetilde{f} = f \circ \widetilde{p}$, the superposition $\widetilde{G_0} := \widetilde{f} \circ G_0 : K \to M$ lies over $\Phi_0$ with respect to $p$. Moreover, $G_0 = (H_0, \widetilde{G_0})$, which means $G_0(x) = (H_0(x), \widetilde{G_0}(x))$ for any $x \in K$. Since $p : M \to N$ is a Serre fibration, there exists a homotopy $\widetilde{\Phi_t} : K \to M$ covering $\Phi_t$ and such that $\widetilde{\Phi_0} = \widetilde{G_0}$. We note that $\widetilde{H_t} := (H_t, \widetilde{\Phi_t}) : K \to f^*M$ is a homotopy with $G_0 = (H_0, \widetilde{G_0})$ covering the homotopy $H_t$. This means that $\widetilde{p} : f^*M \to \widetilde{N}$ is a Serre fibration.

Because of the similarity of the canonical projections, a similar assertion holds for $f$ and $\widetilde{f}$.

2. Suppose that $p : M \to N$ is a Serre fibration with connected fibers and $f : \widetilde{N} \to N$ is a covering map. By definition, $f^*M = \{(y, z) \in \widetilde{N} \times M \mid f(y) = p(z)\}$ and $\widetilde{p} : f^*M \to \widetilde{N} : (y, z) \mapsto y$. Consequently, for any fixed point $y_0 \in \widetilde{N}$ the fiber $\widetilde{p}^{-1}(y_0)$ is diffeomorphically mapped to the fiber $p^{-1}(f(y_0))$ under the map $\widetilde{f}$ and, consequently, is connected.

Take any point $z \in M$. Then $x = p(z) \in N$. Since $f : \widetilde{N} \to N$ is a covering map, there exists a neighborhood $U$ of a point $x$ which is regularly covered by $f$. Moreover, $W = p^{-1}(U)$ is a neighborhood of the point $z$ in $M$ which is regularly covered by $\widetilde{f} : f^*M \to M$. Thus, $\widetilde{f} : f^*M \to M$ is a covering map. □

**Definition 2.1.** The above fibration $\widetilde{p} : f^*M \to \widetilde{N}$ is called the *preimage* of the fibration $p : M \to N$ under the submersion $f : \widetilde{N} \to N$.

**2.2. Equivalence of two definitions.** It is easy to prove the following assertion.

**Lemma 2.2.** *Let $p : \widehat{M} \to M$ be a covering map. If $(M, F)$ is a foliation, then there exists a foliation $(\widehat{M}, \widehat{F})$ such that the restriction $p|_{\widehat{L}} : \widehat{L} \to L$ of the projection $p$ onto an arbitrary leaf $\widehat{L}$ of $(\widehat{M}, \widehat{F})$ is a covering map on the corresponding leaf of $(M, F)$.*

The foliation $(\widehat{M}, \widehat{F})$ is said to be *induced* and is denoted by $\widehat{F} = p^*F$.

A foliation $(M, F)$ is *simple* if it is formed by the fibers of a submersion $p : M \to N$ [9].

**Lemma 2.3.** *Let $(M, F)$ be a simple foliation formed by a smooth Serre fibration $p : M \to N$, and let $k : \widetilde{M} \to M$ be a covering map. Then*

(i) *the induced foliation $(\widetilde{M}, \widetilde{F})$, $\widetilde{F} = k^*F$, is formed by the fibers of a smooth Serre fibration $\widetilde{p} : \widetilde{M} \to \widetilde{N}$ over the Hausdorff manifold $\widetilde{N}$,*

(ii) *there exists a covering map $\widetilde{k} : \widetilde{N} \to N$ such that following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{p}} & \widetilde{N} \\ \Big\downarrow{\scriptstyle k} & & \Big\downarrow{\scriptstyle \widetilde{k}} \\ M & \xrightarrow{p} & N \end{array} \tag{2.2}$$

*Furthermore if $k : \widetilde{M} \to M$ is the universal covering map, then $\widetilde{k} : \widetilde{N} \to N$ is also the universal covering map.*

**Proof.** We denote by $\widetilde{N} = \widetilde{M}/\widetilde{F}$ the space of leaves of the induced foliation $(\widetilde{M}, \widetilde{F})$. A leaf $L$, regarded as a point of the leaf space, is denoted by $[L]$. Let $\widetilde{p} : \widetilde{M} \to \widetilde{N} : \widetilde{L} \to [\widetilde{L}]$, $\widetilde{L} \in \widetilde{F}$, be the projection onto the leaf space. By Lemma 2.2, the image $L = k(\widetilde{L})$ of any leaf $\widetilde{L} \in \widetilde{F}$ is a leaf of the foliation $(M, F)$. Consequently, the mapping $\widetilde{k} : \widetilde{N} \to N$ satisfying the commutative diagram (2.2) is defined.

Consider any points $x \in N$ and $y \in p^{-1}(x)$. Since the foliation $(M, F)$ is simple, there exists a fibered neighborhood $U$ of the point $y$ such that every leaf of $(M, F)$ intersects $U$ along at most one connected subset, called a local leaf. Therefore, without loss of generality we assume that $U$ is a contractible coordinate neighborhood which is regularly covered by $k : \widetilde{M} \to M$. At any point $z \in k^{-1}(y)$, there exists a neighborhood $W$ such that $k|_W : W \to U$ is a diffeomorphism. Since $\widetilde{F} = k^*F$, every leaf of the foliation $(\widetilde{M}, \widetilde{F})$ intersecting the neighborhood $W$ intersects it over exactly one local leaf. Therefore, $(\widetilde{M}, \widetilde{F})$ is a regular foliation in the sense of Palais [13]. As is proved in [13], the space of leaves of a regular foliation $\widetilde{M}/\widetilde{F}$ is naturally equipped with the structure of a smooth (possibly, non-Hausdorff) manifold, relative to which the projection $\widetilde{p} : \widetilde{M} \to \widetilde{N}$ is a submersion.

Since the diagram (2.2) is commutative, $\widetilde{k} : \widetilde{N} \to N$ is a surjective submersion of manifolds of the same dimension. Hence $\widetilde{k}$ is a local diffeomorphism. Let us prove that $\widetilde{k} : \widetilde{N} \to N$ is a Serre fibration. Let $H_t : K \to N$, $t \in [0, 1]$, be a homotopy of a finite complex $K$ in $N$, started with $H_0$, and let $G_0 : K \to \widetilde{N}$ be a continuous map such that $\widetilde{k} \circ G_0 = H_0$. We consider any continuous map $\widetilde{G_0} : K \to \widetilde{M}$ such that $\widetilde{p} \circ \widetilde{G_0} = G_0$. The submersion $p \circ k$ is a Serre fibration since it can be regarded as a superposition of Serre fibrations. Therefore, for $H_t$ there exists a covering homotopy $\widetilde{G_t} : K \to \widetilde{M}$ started with $\widetilde{G_0}$. We set $\widetilde{H_t} := \widetilde{p} \circ \widetilde{G_t}$. Then $\widetilde{k} \circ \widetilde{H_t} = (\widetilde{k} \circ \widetilde{p}) \circ \widetilde{G_t} = (p \circ k)\widetilde{G_t} = H_t$. Consequently, $\widetilde{H_t} : K \to \widetilde{M}$ is a covering homotopy for $H_t : K \to N$; moreover, $\widetilde{H_0} = G_0$. Thus, $\widetilde{k} : \widetilde{N} \to N$ is a Serre fibration which is a local diffeomorphism. Therefore, $\widetilde{k} : \widetilde{N} \to N$ is a covering map.

We note that the manifold $\widetilde{N}$ is Hausdorff since it can be regarded as the covering space of a Hausdorff manifold $N$.

Similarly, it turns out that $\widetilde{p} : \widetilde{M} \to \widetilde{N}$ is a smooth Serre fibration.

Suppose that $k : \widetilde{M} \to M$ is the universal covering map. Since $\widetilde{p} : \widetilde{M} \to \widetilde{N}$ is a Serre fibration with simply connected fibre space and connected fibers, the exact homotopic sequence of this fibration implies the simple connectedness of its base $\widetilde{N}$. Consequently, $\widetilde{k} : \widetilde{N} \to N$ is the universal covering map. □

**Remark 2.1.** If, in the assumptions of Lemma 2.3, we omit the condition that a simple foliation $(M, F)$ is formed by the fibers of a Serre fibration, then the foliation $(\widetilde{M}, \widetilde{F})$ induced on the covering space $\widetilde{M}$ is not, generally speaking, simple, which is confirmed by Example 7.1.

**Definition 2.2.** We say that a foliation $(M, F)$ *is covered by a fibration* if there exists a covering map $f : M' \to M$ such that the induced foliation $f^*F$ formed by the fibers of a smooth Serre fibration $p : M' \to N$.

**Definition 2.3.** Let $(M, F)$ be an arbitrary smooth foliation, and let $k : \widetilde{M} \to M$ be the universal covering map. If the induced foliation $\widetilde{F} = k^*F$ on $\widetilde{M}$ is formed by the fibers of a smooth Serre fibration $f : \widetilde{M} \to N$, then we say that the *foliation* $(M, F)$ *is covered by a fibration.*

**Lemma 2.4.** *Definitions* 2.2 *and* 2.3 *are equivalent.*

**Proof.** Definition 2.3 implies Definition 2.2. Suppose that, in the sense of Definition 2.2, a foliation $(M, F)$ is covered by a fibration $p : M' \to N$ and $f : M' \to M$ is a covering mapping, i.e., the induced foliation $F' = f^*F$ is formed by the fibers of a smooth Serre fibration $p$. We consider the universal covering map $k : \widetilde{M} \to M'$. By Lemma 2.3, the induced foliation $\widetilde{F} = k^*F'$ is formed by the fibers of a smooth Serre fibration $\widetilde{p} : \widetilde{M} \to \widetilde{N}$. Since $f \circ k : \widetilde{M} \to M$ is the universal covering map and $(f \circ k)^*F = \widetilde{F}$ is the induced foliation on $\widetilde{M}$, then the foliation $(M, F)$ is covered by a fibration in the sense of Definition 2.3. $\square$

### 2.3. The global holonomy group.

**Proposition 2.1.** *Let $(M, F)$ be a foliation covered by a fibration $f : \widetilde{M} \to N$, where $k : \widetilde{M} \to M$ is the universal covering map. Let $\Phi$ be the group of desk transformations of this covering map. Then there exists a normal subgroup $\widetilde{\Phi}$ of the group $\Phi$ such that the action of the quotient group $\widehat{\Phi} = \Phi/\widetilde{\Phi}$ is induced on the quotient manifold $\widehat{M} = \widetilde{M}/\widetilde{\Phi}$; moreover, $M = \widehat{M}/\widehat{\Phi}$ and the following assertions hold*:

1) *the quotient maps $h : \widetilde{M} \to \widehat{M}$ and $s : \widehat{M} \to M$ are regular coverings with the groups of desk transformations $\widetilde{\Phi}$ and $\widehat{\Phi}$ respectively; moreover, $k = s \circ h$,*

2) *the induced foliation $\widehat{F}$ is formed by the fibers of a smooth Serre fibration $r : \widehat{M} \to N$; moreover, $f = r \circ h$ and the manifold $N$ is simply connected,*

3) *the group $\widehat{\Phi}$ induces on $N$ the group of diffeomorphisms $\Psi$ under a submersion $r : \widehat{M} \to N$; moreover, the natural projection $\chi : \widehat{\Phi} \to \Psi$ is a group isomorphism.*

**Proof.** Let $\Phi$ be the group of desk transformations of the universal covering map $k : \widetilde{M} \to M$. Since every $\varphi \in \Phi$ is an automorphism of the foliation $f : \widetilde{M} \to N$, it follows that $\varphi$ induces a diffeomorphism $\psi = \psi(\varphi)$ from the manifold $N$ onto itself satisfying the equality $f \circ \varphi = \psi(\varphi) \circ f$. The set of all such transformations form the group $\Psi = \{\psi(\varphi) \mid \varphi \in \Phi\}$, and the natural projection $\mu : \Phi \to \Psi : \varphi \mapsto \psi(\varphi)$ is a group epimorphism. Since $\widetilde{\Phi} := \operatorname{Ker} \mu$ is the normal subgroup of the group $\Phi$ acting on $\widetilde{M}$ freely and properly discontinuously, the quotient manifold $\widehat{M} := \widetilde{M}/\widetilde{\Phi}$ is defined and the action of the quotient group $\widehat{\Phi} := \Phi/\widetilde{\Phi}$ on $\widehat{M}$ is free and properly discontinuous; moreover, $M \cong \widehat{M}/\widehat{\Phi}$, the quotient map $h : \widetilde{M} \to \widehat{M} = \widetilde{M}/\widetilde{\Phi}$ satisfies the equality $k = s \circ h$, where $s : \widehat{M} \to M = \widehat{M}/\widehat{\Phi}$ is the projection of the group $\widehat{\Phi}$ on the orbit space. We note that $s : \widehat{M} \to M$ is a regular covering map with the group of desk transformations $\widehat{\Phi}$. By the definition of $\widehat{M}$, the foliation $\widehat{F} := s^*F$ is formed by the fibers of a submersion $r : \widehat{M} \to N$; moreover, $r \circ h = f$. Since $f$ and $h$ are Serre fibrations, the last equality implies that $r$ is also a Serre fibration. From the exact homotopy sequence of the Serre fibration $f : \widetilde{M} \to N$ it follows that the base $N$ is connected by the simple conectedness of $M$ and connectedness of fibers.

We note that the group of desk transformations $\widehat{\Phi}$ of a regular covering map $s : \widehat{M} \to M$ induces, under a submersion $r : \widehat{M} \to N$, the same group of diffeomorphisms $\Psi$ with the base $N$ as the group $\widehat{\Phi}$; moreover, the natural group epimorphism $\chi : \widehat{\Phi} \to \Psi$ is a group isomorphism. □

**Definition 2.4.** The group $\Psi$ in Proposition 2.1 is called the *global holonomy group of a foliation* $(M, F)$ covered by a fibration.

The global holonomy group $\Psi$ of a foliation $(M, F)$ is defined, up to an inner automorphism, as the group of desk transformations $\Phi$. To avoid the ambiguity, we assume that the points $x \in M$ and $y \in k^{-1}(x) \in \widetilde{M}$ defining the group $\Phi$ are fixed.

# 3 Proof of Theorem 1.1

**3.1. Suspended foliations.** The construction of a suspended foliation due to Haefliger consists in the following. Let $B$ and $N$ be smooth connected manifolds, and let $\rho : \pi_1(B, b) \to \mathrm{Diff}(N)$ be a group homomorphism. Suppose that $G := \pi_1(B, b)$ and $\Psi := \rho(G)$. We consider the universal covering map $\widehat{p} : \widehat{B} \to B$. We DEfine the right action of the group $G$ on the product of manifolds $\widehat{B} \times N$ as follows:

$$\Theta : \widehat{B} \times N \times G \to \widehat{B} \times N : (x, t, g) \to (g^{-1}(x),\ \rho(g^{-1})(t)),$$

where $\widehat{B} \to \widehat{B} : x \to g^{-1}(x)$ is the desk transformation induced by $g^{-1} \in G$.

The map $p : M := (\widehat{B} \times N)/G \to B = \widehat{B}/G$ determines a locally trivial fibration over $B$ with the standard fiber $N$ associated to the principal bundle $\widehat{p} : \widehat{B} \to B$ with the structure group $G$. Let $\Theta_g := \Theta|_{\widehat{B} \times \{t\} \times \{g\}}$.

Since $\Theta_g(\widehat{B} \times \{t\}) = \widehat{B} \times \{\rho(g^{-1})(t)\}$ for all $t \in N$, the action of the discrete group $G$ preserves the trivial foliation $F := \{\widehat{B} \times \{t\} \mid t \in N\}$ of the product $\widehat{B} \times N$. Consequently, the quotient map $f_0 : \widehat{B} \times N \to (\widehat{B} \times N)/G = M$ induces on $M$ a smooth foliation $F$ with leaves transversal to fibres of the fibration $p : M \to B$.

A pair $(M, F)$ is called a *suspended foliation* and is denoted by $\mathrm{Sus}\,(N, B, \rho)$. The group of diffeomorphisms $\Psi := \rho(G)$ of a manifold $N$ is the *global holonomy group* of the suspended foliation $(M, F)$.

**3.2. Proof of Theorem 1.1.** Let $N$ be any $q$-dimensional simply connected manifold, and let $\Psi$ be an arbitrary at most countable subgroup of the group of diffeomorphisms $\mathrm{Diff}(N)$ having a finite family of generators $\{\psi_1, \ldots, \psi_m\}$. We denote by $S^2_m$ the two-dimensional sphere with $m$ handles. The fundamental group $S^2_m$ is equal to $\{a_i, b_i, i = 1, \ldots, m \mid a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_m b_m a_m^{-1} b_m^{-1} = 1\}$. We set $B = S^2_m$ and define the group homomorphism $\rho : \pi_1(B, b) \to \mathrm{Diff}(N)$ by defining it on generators $\rho(a_i) := \psi_i$, $i = 1, \ldots, m$, where $\rho(b_i) := \mathrm{id}_{S^2_m}$ is the identity of the group $\Psi$. Then $(M, F) := \mathrm{Sus}\,(N, S^2_m, \rho)$ is a two-dimensional suspended foliation of codimension $q$. It is covered by the trivial fibration $\mathbb{R}^2 \times N \to N$ and has the global holonomy group $\Psi$.

We assume that $\Psi \subset \mathrm{Diff}(N)$ has a countable family of generators $\{\psi_i | i \in \mathbb{N}\}$. Let $B := \mathbb{R}^2 \setminus A$ be the plane without the discrete subset $A = \{(i, 0) \in R^2 | i \in \mathbb{N}\}$, and let $b = (1, 1) \in B$. Then

$G = \pi_1(B, b) = \{g_i \mid i \in \mathbb{N}\}$ is a free group with a countable family of generators. The equalities $\rho_\infty(g_i) := \psi_i, i \in \mathbb{N}$, defines a group epimorphism $\rho_\infty : \pi_1(B, b) \to \Psi$. The suspended foliation $(M, F) := \mathrm{Sus}\,(N, B, \rho_\infty)$ has codimension $q$ and is covered by the trivial fibration $\mathbb{R}^2 \times N \to N$. The global holonomy group of the foliation $(M, F)$ is equal to $\Psi$. □

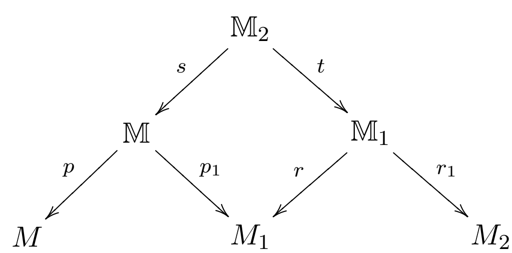
## 4 Transverse Equivalence of Foliations Covered by Fibrations

We say that two foliations are *transversally equivalent* if they are transversally equivalent in the sense of Definition 1.1.

**Proposition 4.1.** *The transverse equivalence of foliations in the sense of Definition* 1.1 *is an equivalence relation.*

**Proof.** It is obvious that the reflexivity and symmetry properties are satisfied.

Let $(\mathbb{M}, \mathbb{F})$ and submersions $p : \mathbb{M} \to M$, $p_1 : \mathbb{M} \to M_1$ realize the transverse equivalence of foliations $(M, F)$ and $(M_1, F_1)$, whereas $(\mathbb{M}_1, \mathbb{F}_1)$ and submersions $r : \mathbb{M}_1 \to M_1$, $r_1 : \mathbb{M} \to M_2$ realize the transverse equivalence of foliations $(M_1, F_1)$ and $(M_2, F_2)$. Let $\mathbb{M}_2 = \{(y, z) \in \mathbb{M} \times \mathbb{M}_1 \mid p_1(y) = r(z)\}$. We define $s : \mathbb{M}_2 \to \mathbb{M}$ and $t : \mathbb{M}_2 \to \mathbb{M}_1$ by setting $s(y, z) := y$ and $t(y, z) := z$ for all $(y, z) \in \mathbb{M}_2$. We set $q := p \circ s$ and $q_1 := r_1 \circ t$. Then $q : \mathbb{M}_2 \to M$ and $q_1 : \mathbb{M}_2 \to M_2$ are surjective submersions with connected fibers; moreover, $\mathbb{F}_2 = \{q^{-1}(L) | L \in F\} = \{q_1^{-1}(L') | L' \in F_2\}$ is a foliation over $\mathbb{M}_2$, i.e., the relations (1.1) in Definition 1.1 hold and the following diagram is commutative:




(4.1)

We note that $\mathbb{M}_2 = {p_1}^*\mathbb{M}_1$ is the preimage of the Serre fibration $r : \mathbb{M}_1 \to M_1$ under the submersion $p_1 : \mathbb{M} \to M_1$; moreover, $p_1 : \mathbb{M} \to M_1$ is also a Serre fibration. By the first assertion of Lemma 2.1, the canonical projections $s : \mathbb{M}_2 \to \mathbb{M}$ and $t : \mathbb{M}_2 \to \mathbb{M}_1$ are Serre fibrations. Thus, the projections $q : \mathbb{M}_2 \to M$ and $q_1 : \mathbb{M}_2 \to M_1$ are Serre fibrations because they are the superpositions of Serre fibrations. Consequently, the foliations $(M, F)$ and $(M_2, F_2)$ are transversally equivalent in the sense of Definition 1.1. □

## 5 Proof of Theorem 1.2

Throughout the section, $i = 1, 2$. Let $(M_i, F_i)$ be a foliation covered by a fibration $r_i : \widetilde{M_i} \to N_i$, where $k_i : \widetilde{M_i} \to M_i$ is the universal covering map. We assume that the foliations $(M_1, F_1)$ and $(M_2, F_2)$ are transversally equivalent. By Definition 1.1, there exists a foliation $(\mathbb{M}, \mathbb{F})$ that, together with submersions $p_1 : \mathbb{M} \to M_1$ and $p_2 : \mathbb{M} \to M_2$, realize the transverse equivalence of $(M_1, F_1)$ and $(M_2, F_2)$; moreover, $p_1$ and $p_2$ are Serre fibrations.

We consider the preimage ${k_i}^*\mathbb{M}$, $i = 1, 2$, of the Serre fibration $p_i : \mathbb{M} \to M_i$ under the submersion $k_i : \widetilde{M_i} \to M_i$. Denote by $\tau_i : {k_i}^*\mathbb{M} \to \mathbb{M}$ and $q_i : {k_i}^*\mathbb{M} \to \widetilde{M_i}$ the canonical submersions. By the first assertion of Lemma 2.1, they satisfy the equality $p_i \circ \tau_i = k_i \circ q_i$; moreover, $q_i : {k_i}^*\mathbb{M} \to \widetilde{M_i}$ is a Serre fibration and $\tau_i : {k_i}^*\mathbb{M} \to \mathbb{M}$ is a covering map.

We denote by $\tau : \widetilde{\mathbb{M}} \to \mathbb{M}$ the universal covering map for the manifold $\mathbb{M}$. By universality, there exists a covering map $\delta_i : \widetilde{\mathbb{M}} \to {k_i}^*\mathbb{M}$ such that $\tau = \tau_i \circ \delta_i$. The map $f_i = q_i \circ \delta_i : \widetilde{\mathbb{M}} \to \widetilde{M_i}$, being the superposition of Serre fibrations, is a Serre fibration and satisfies the equality $k_i \circ f_i = p_i \circ \tau$. Consequently, the induced foliation $(\widetilde{\mathbb{M}}, \widetilde{\mathbb{F}})$, $\widetilde{\mathbb{F}} = \tau^*\mathbb{F}$, together with the projections $f_i : \widetilde{\mathbb{M}} \to \widetilde{M_i}$, realizes the transverse equivalence of simple foliations $(\widetilde{M_1}, \widetilde{F_1})$ and $(\widetilde{M_2}, \widetilde{F_2})$. We denote by $N$ the space of leaves of the foliation $(\widetilde{\mathbb{M}}, \widetilde{\mathbb{F}})$. Let $r : \widetilde{\mathbb{M}} \to N$ be the quotient map on the leaf space. Then we can define the map $d_i : N \to N_i : [\mathbb{L}] \mapsto [f_i(\mathbb{L})]$ which is a homeomorphism such that $d_i \circ r = r_i \circ f_i$. Consequently, the following diagram is commutative:

$$\begin{array}{ccccc} N_1 & \xleftarrow{d_1} & N & \xrightarrow{d_2} & N_2 \\ \uparrow{\scriptstyle r_1} & & \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle r_2} \\ \widetilde{M_1} & \xleftarrow{f_1} & \widetilde{\mathbb{M}} & \xrightarrow{f_2} & \widetilde{M_2} \\ \downarrow{\scriptstyle k_1} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle k_2} \\ M_1 & \xleftarrow{p_1} & \mathbb{M} & \xrightarrow{p_2} & M_2. \end{array} \tag{5.1}$$

In the same way as in [9, Lemma 2.6], one can prove that the commutativity of the diagram (5.1) implies that $d := d_2 \circ d_1^{-1} : N_1 \to N_2$ is a diffeomorphism. On $N$, we introduce a smooth structure relative to which $d_i : N \to N_i$ is a diffeomorphism and the quotient map $r : \widetilde{\mathbb{M}} \to N$ is a submersion. As in Lemma 2.3, we verify that $r$ is a Serre fibration.

Thus, the foliation $(\mathbb{M}, \mathbb{F})$ realizing the transverse equivalence of foliations covered by a fibration is a foliation itself.

Let $\Gamma$ and $\Gamma_i$ be the groups of desk transformations of the covering maps $\tau$ and $k_i$ corresponding to fixed points $v \in \widetilde{\mathbb{M}}$ and $v_i = f_i(v) \in \widetilde{M_i}$. The continuous map $p_i : \mathbb{M} \to M_i$ induces a homomorphism of the fundamental groups $p_{i*} : \pi_1(\mathbb{M}, y) \to \pi_1(M_i, y_i)$, where $y = \tau(v)$, $y_i = p_i(y)$. We identify the corresponding isomorphisms of groups: $\Gamma$ with $\pi_1(\mathbb{M}, y)$ and $\Gamma_i$ with $\pi_1(M_i, y_i)$. Moreover, $\nu_i := p_{i*} : \Gamma \to \Gamma_i$ is a group homomorphism. We emphasize that $\gamma_i = \nu_i(\gamma)$ if and only if $f_i \circ \gamma = \gamma_i \circ f_i$.

Let us check that $\nu_i : \Gamma \to \Gamma_i$ is an epimorphism. Let $\varphi \in \Gamma_i$. We recall that the inclusion $\varphi \in \Gamma_i$ is equivalent to the conditions $\varphi \in \mathrm{Diff}(\widetilde{M_i})$ and $k_i \circ \varphi = \varphi$. We define $\widetilde{\varphi} : {k_i}^*\mathbb{M} \to {k_i}^*\mathbb{M}$ by setting $\widetilde{\varphi}((y,z)) = (y, \varphi(z))$ for any $(y,z) \in {k_i}^*\mathbb{M}$. Since $p_i(y) = k_i(\varphi(z)) = k_i(z)$ for all $(y,z) \in {k_i}^*\mathbb{M}$, the map $\widetilde{\varphi}$ is indeed defined. It is easy to see that $\widetilde{\varphi}$ is a diffeomorphism of the manifold ${k_i}^*\mathbb{M}$; moreover, $\tau_i \circ \widetilde{\varphi} = \tau_i$, where $\tau_i : {k_i}^*\mathbb{M} \to \mathbb{M} : (y,z) \mapsto y$ is the canonical projection. This means that $\widetilde{\varphi}$ is the desk transformation of the covering map $\tau_i : {k_i}^*\mathbb{M} \to \mathbb{M}$. Since $\tau = \tau_i \circ \delta_i : \widetilde{\mathbb{M}} \to {k_i}^*\mathbb{M}$ is the universal covering map with the group of desk transformations $\Gamma$, there exists $\gamma \in \Gamma$ lying over $\widetilde{\varphi}$ with respect to $\delta_i$, i.e., satisfying the equality $\delta_i \circ \gamma = \widetilde{\varphi} \circ \delta_i$. The following chain of equalities holds:

$$\begin{aligned} f_i \circ \gamma = (q_i \circ \delta_i) \circ \gamma = q_i \circ (\delta_i \circ \gamma) = q_i \circ (\widetilde{\varphi} \circ \delta_i) = (q_i \circ \widetilde{\varphi}) \circ \delta_i \\ = (\varphi \circ q_i) \circ \delta_i = \varphi \circ (q_i \circ \delta_i) = \varphi \circ f_i \end{aligned}$$

which implies that $\gamma$ lies over $\varphi$ relative to the submersion $f_i : \widetilde{\mathbb{M}} \to {k_i}^*\mathbb{M}$. Therefore, $\nu_i(\gamma) = \varphi$, which completes the proof.

We note that the groups of desk transformations $\Gamma$ and $\Gamma_i$ preserve simple induced foliations $(\widetilde{\mathbb{M}}, \widetilde{\mathbb{F}})$ and $(\widetilde{M_i}, \widetilde{F_i})$ respectively. Therefore, they induce the group of diffeomorphisms $\Psi$ and $\Psi_i$ of manifolds of leaves $N$ and $N_i$ of the above foliations. Let $\mu : \Gamma \to \Psi$ and $\mu_i : \Gamma_i \to \Psi_i$ be the corresponding group epimorphism.

For any $\gamma \in \mathrm{Ker}\,(\mu)$ we have $\gamma(\widetilde{\mathbb{L}}) = \widetilde{\mathbb{L}}$ for all $\widetilde{\mathbb{L}} \in \widetilde{\mathbb{F}}$. Let $\gamma_i = \nu_i(\gamma)$, which is equivalent to $f_i \circ \gamma = \gamma_i \circ f_i$. For an arbitrary leaf $\widetilde{L} \in \widetilde{F_i}$ there is a leaf $\widetilde{\mathbb{L}} \in \widetilde{\mathbb{F}}$ such that $\widetilde{L} = f_i(\widetilde{\mathbb{L}})$. By the chain of equalities

$$\gamma_i(\widetilde{L}) = \gamma_i(f_i(\widetilde{\mathbb{L}})) = (\gamma_i \circ f_i)(\widetilde{\mathbb{L}}) = (f_i \circ \gamma)(\widetilde{\mathbb{L}}) = f_i(\gamma(\widetilde{\mathbb{L}})) = f_i(\widetilde{\mathbb{L}}) = \widetilde{L},$$

every leaf $\widetilde{L}$ of the foliation $\widetilde{F_i}$ is invariant under the diffeomorphism $\gamma_i$. Therefore, $\gamma_i \in \mathrm{Ker}\,(\mu_i)$. Thus, the kernels of epimorphisms $\mu$ and $\mu_i$ satisfy the inclusion $\nu_i(\mathrm{Ker}\,(\mu)) \subset \mathrm{Ker}\,(\mu_i)$. Hence we defined the group epimorphism $\theta_i : \Psi \to \Psi_i$ for which the following diagram is commutative:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\nu_i} & \Gamma_i \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu_i} \\ \Psi & \xrightarrow{\theta_i} & \Psi_i. \end{array} \tag{5.2}$$

To show that $\theta_i : \Psi \to \Psi_i$ is a group isomorphism, we consider arbitrary elements $\psi \in \mathrm{Ker}\,(\theta_i)$ and $\gamma \in \mu^{-1}(\psi)$. We assume that $\gamma \notin \mathrm{Ker}\,(\mu)$. Then there is a leaf $\widetilde{\mathbb{L}}$ of the foliation $(\widetilde{\mathbb{M}}, \widetilde{\mathbb{F}})$ such that $\gamma(\widetilde{\mathbb{L}}) = \widetilde{\mathbb{L}}' \neq \widetilde{\mathbb{L}}$. Since the diagram (5.1) is commutative, we have $\widetilde{L} = f_i(\widetilde{\mathbb{L}}) \neq f_i(\widetilde{\mathbb{L}}') = \widetilde{L}'$. Hence $\gamma_i = \nu_i(\gamma) \notin \mathrm{Ker}\,(\mu_i)$, which contradicts the commutativity of the diagram (5.2). Consequently, $\gamma \in \mathrm{Ker}\,(\mu)$ and $\psi$ is the identity of the group $\Psi$. Thus, $\theta = \theta_2 \circ \theta_i^{-1} : \Psi_1 \to \Psi_2$ is a group isomorphism. Furthermore, the commutativity of the diagram (5.1) and the definition of the groups $\Psi$ and $\Psi_i$ imply the equality $d \circ \varphi_1 = \theta(\varphi_1) \circ d$ for any $\varphi_1 \in \Psi_1$. Consequently, the pairs $(N_1, \Psi_1)$ and $(N_2, \Psi_2)$ are isomorphic in the category of pairs $\mathfrak{P}$ defined in Section 1. By definition, $\Psi_i$ is the global holonomy group of the foliation $(M_i, F_i)$, i.e., $\beta(M_i, F_i) = (N_i, \Psi_i)$.

Thus, we defined the map $\mathscr{B} : \widetilde{\mathfrak{F}} \to \widetilde{\mathfrak{P}} : [(M, F)] \mapsto [\beta(M, F)] = [(N, \Psi)]$ for all $(M, F) \in \mathfrak{F}$, associating the class of transversally equivalent foliations $[(M, F)]$ covered by fibrations with the class of equivalent pairs $[(N, \Psi)]$, where $\Psi$ is the global holonomy group of the foliation $(M, F)$ acting on the simply connected manifold $N$.

We prove that $\mathscr{B} : \widetilde{\mathfrak{F}} \to \widetilde{\mathfrak{P}}$ is an injection. We assume that $\mathscr{B}([(M_1, F_1)]) = \mathscr{B}([(M_2, F_2)])$. Then $[(N_1, \Psi_1)] = [(N_2, \Psi_2)]$, where $\Psi_i$ is the global holonomy group of the foliation $(M_i, F_i)$ acting on the simply connected manifold $N_i$. By the condition $[(N_1, \Psi_1)] = [(N_2, \Psi_2)]$, there exists a group isomorphism $\theta : \Psi_1 \to \Psi_2$ and a diffeomorphism $d : N_1 \to N_2$ such that $d \circ \psi_1 = \theta(\psi_1) \circ d$ for all $\psi_1 \in \Psi_1$. For the sake of simplicity we identify the manifolds $N_1$ and $N_2$ by the diffeomorphism $d$. Moreover, the group $\Psi_1$ is identified with the group $\Psi_2$. By the identification, we write $N$ and $\Psi$ instead of $N_i$ and $\Psi_i$. Since $(M_i, F_i)$ is a foliation covered by a fibration, Proposition 2.1 implies the existence of a regular covering map $s_i : \widehat{M_i} \to M_i$ such that

1) the induced foliation $(\widehat{M_i}, \widehat{F_i})$, $\widehat{F_i} = s_i^* F_i$, is formed by the fibers of a smooth Serre fibration $r_i : \widehat{M_i} \to N$ over the simply connected base $N$,

2) the group of desk transformations $\Gamma_i$ of the covering map $s_i : \widehat{M_i} \to M_i$ by means $r_i : \widehat{M_i} \to N$ induces on $N$ the group of diffeomorphisms $\Psi$, called the *global holonomy group* of this foliation; moreover, the natural projection $\mu_i : \Gamma_i \to \Psi$ is a group isomorphism.

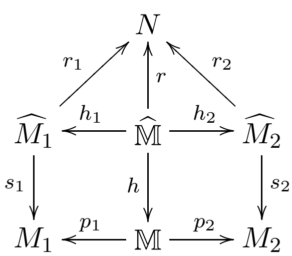
Let $\widehat{\mathbb{M}} := \{(x_1, x_2) \in \widehat{M_1} \times \widehat{M_2} \mid r_1(x_1) = r_2(x_2)\}$ be the preimage of the fibration $r_2 : \widehat{M_2} \to N$ relative to $r_1 : \widehat{M_1} \to N$, and let $h_i : \widehat{\mathbb{M}} \to \widehat{M_i}$ be its canonical projections. By Lemma 2.1, $h_i$

is the projection of a Serre fibration. There is a submersion $r = r_i \circ h_i : \widehat{\mathbb{M}} \to N$ with connected fibers. The fibres of this submersion form a simple foliation $(\widehat{\mathbb{M}}, \widehat{\mathbb{F}})$ which, together with the projections $h_i$, form an isomorphism between simple foliations $(\widehat{M_1}, \widehat{F_1})$ and $(\widehat{M_2}, \widehat{F_2})$.

From the definition of the group $\Psi$ it follows that $r_i \circ \gamma_i = \mu_i(\gamma_i) \circ r_i = \psi \circ r_i$ for $\psi = \mu_i(\gamma_i)$. Therefore, $r_i(x_i) = c \in N$ implies $r_i(\gamma_i(x_i)) = \psi(r_i(x_i)) = \psi(c)$, where $x_i \in \widehat{N}_i$. Consequently, for $\gamma_i = \mu_i(\psi)$ the equality

$$\psi(x_1, x_2) := (\gamma_1(x_1), \gamma_2(x_2)) \quad \forall \psi \in \Psi,\ \forall (x_1, x_2) \in \mathbb{M},$$

defines the action of the group $\Psi$ on $\widehat{\mathbb{M}}$. Since $\Psi$ acts on $\widehat{\mathbb{M}}$ by means of the group of desk transformations $\Gamma_i$, this action is free and properly discontinuous. Therefore, the quotient manifold $\mathbb{M} := \widehat{\mathbb{M}}/\Psi$ is defined and the quotient map $h : \widehat{\mathbb{M}} \to \mathbb{M}$ is a regular covering with the group of desk transformations, isomorphic to the group $\Psi$. We note that there are submersions $p_i : \mathbb{M} \to M_i : \Psi \cdot (x_1, x_2) \mapsto \Gamma_i \cdot (x_i)$, $(x_1, x_2) \in \mathbb{M}$, sending the orbit $\Psi \cdot (x_1, x_2)$ to the orbit $\Gamma_i \cdot (x_i)$ and satisfying the commutative diagram



(5.3)

As was shown above, $r_i(x_i) = c$ implies $r_i(\gamma_i(x_i)) = \psi(c)$. Therefore, for any $(x_1, x_2) \in \widehat{\mathbb{L}} = r^{-1}(c)$ we have $r(\psi(x_1, x_2) = r(\gamma_1(x_1), \gamma_2(x_2)) = r_i(\gamma_i(x_i)) = \psi(c)$, i.e., $\psi(r^{-1}(c)) = r^{-1}(\psi(c))$ for any $c \in N$. Thus, the action of the group $\Psi$ on $\widehat{\mathbb{M}}$ preserves the foliation $(\widehat{\mathbb{M}}, \widehat{\mathbb{F}})$. Therefore, the projection $h : \widehat{\mathbb{M}} \to \mathbb{M}$ induces the foliation $(\mathbb{M}, \mathbb{F})$. Using the commutativity of the diagram (5.3), it is easy to verify that

$$\mathbb{F} = \{p_1^{-1}(L) \mid L \in F_1\} = \{p_2^{-1}(L') \mid L' \in F_2\}.$$

Since $h$, $h_i$, and $s_i$ are Serre fibrations, from the commutativity of the diagram (5.3) it follows that $p_i : \mathbb{M} \to M_i$ is also a Serre fibration.

Thus, the foliation $(\mathbb{M}, \mathbb{F})$ with submersions $p_i$ realize the transverse equivalence of $(M_1, F_1)$ and $(M_2, F_2)$. Consequently, $\mathscr{B} : \widetilde{\mathfrak{F}} \to \widetilde{\mathfrak{P}}$ is injective.

By Theorem 1.1, for any pair $(N, \Psi)$, where $N$ is an arbitrary simply connected manifold and $\Psi$ is an at most countable group of diffeomorphisms of the manifold $N$, there exists a foliation $(M, F)$ covered by a fibration for which $\Psi$ is the global holonomy group, i.e., $\beta(M, F) = (N, \Psi)$. Therefore, for any $[(N, \Psi)] \in \widetilde{\mathfrak{P}}$ there exists a class of transversally equivalent foliations $[(M, F)]$ such that $\mathscr{B}[(M, F)] = [(N, \Psi)]$, i.e., $\mathscr{B}$ is surjective. Thus, $\mathscr{B} : \widetilde{\mathfrak{F}} \to \widetilde{\mathfrak{P}}$ is a bijection. □

## 6 Transverse Equivalence of Complete Conformal Foliations

**6.1. Conformal manifolds.** Let $(N_1, g_1)$ and $(N_2, g_2)$ be Riemannian manifolds, where $g_1$ and $g_2$ are Riemannian metrics. A diffeomorphism $f : N_1 \to N_2$ is *conformal* if there exists a smooth function $\lambda$ on $N_1$ such that $f^* g_2 = \lambda g_1$. If $\mathrm{id}_N$ is a conformal diffeomorphism of

Riemannian manifolds $(N, g_1)$ and $(N, g_2)$, then the Riemannian metrics $g_1$ and $g_2$ on $N$ are said to be *conformally equivalent.* The class $[g]$ of conformally equivalent Riemannian metrics on $N$ is called the *conformal structure* on $N$, and the pair $(N, [g])$ is referred to as a *conformal manifold.* Thus, on any Riemannian manifold $(N, g)$, the conformal structure $[g]$ is uniquely defined. A conformal diffeomorphism $f$ from a Riemannian manifold $(N, g)$ onto itself is called a *conformal transformation.*

**6.2. Conformal foliations.** Let a smooth foliation $(M, F)$ of codimension $q$ be given by an $N$-cocycle $\zeta = \{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ (cf., for example, [14]). This means that we a given

1) a (possibly, disconnected) smooth $q$-dimensional manifold $N$,

2) a locally finite open cover $\{U_i \mid i \in J\}$ of $M$ with respect to compact subsets $U_i$,

3) submersions with connected fibers $f_i : U_i \to V_i$ on $V_i \subset N$,

4) for $U_i \cap U_j \neq \varnothing$ the diffeomorphism $\gamma_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ of open subsets of the manifold $N$ satisfying the equality $f_i = \gamma_{ij} \circ f_j$ on $U_i \cap U_j$.

If $U_i \cap U_j \cap U_k \neq \varnothing$, then $\gamma_{ik} = \gamma_{ij} \circ \gamma_{jk}$; moreover, $\gamma_{ii} = \mathrm{id}\,|_{U_i}$.

Every maximal, with respect to inclusion, $N$-cocycle $\widehat{\zeta}$ possessing the above-listed properties defines a new topology on $M$ with the base consisting of the fibers of all submersions $f_i$. This topology is called the *leaf topology* and is denoted by $\tau_F$. The linear connected components of the topological space $(M, \tau_F)$ form a partition of $M$, denoted by $F = \{L_\alpha \mid \alpha \in A\}$ and called the *foliation with leaves $L_\alpha$ given by the $N$-cocycle $\widehat{\zeta}$.* Since any $N$-cocycle is contained in a unique maximal $N$-cocycle, to determine the foliation $(M, F)$ it suffice to define some $N$-cocycle $\zeta$ with the above properties.

A smooth foliation $(M, F)$ of codimension $q$ is *conformal* if it is defined by an $N$-cocylce $\zeta = \{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ and the manifold $N$ is equipped with a Riemannian metric $g$ such that all $\{\gamma_{ij}\}$ are local conformal diffeomorphisms of the Riemannian submanifolds induced on the corresponding open subsets of the Riemannian manifold $(N, g)$.

**6.3. Proof of Theorem 1.3.** As was proved in [7, Theorem 5], for any complete non-Riemannian conformal foliation $(M, F)$ of codimension $q > 2$ there exists a regular covering map $f : \widehat{M} \to M$ such that the induced foliation $(\widehat{M}, \widehat{F})$, $\widehat{F} = f^*F$, is formed by the fibers of a locally trivial fibration $r : \widehat{M} \to N$. Moreover, either $N = \mathbb{E}^q$ or $N = \mathbb{S}^q$. Since any locally trivial fibration is a Serre fibration, the foliation $(M, F)$ is covered by a fibration in the sense of Definition 2.3. Therefore, $(M, F) \in \mathfrak{F}_{\mathfrak{C}}$. Furthermore, the global holonomy group $\Psi$ of the foliation $(M, F)$ is a countable essential subgroup of the Lie group $\mathrm{Conf}\,(\mathbb{S}^q)$ if $(M, F)$ is not transversally similar or $\Psi$ is a countable essential subgroup of the Lie group $\mathrm{Sim}(\mathbb{E}^q)$ in the opposite case. Consequently, $(N, \Psi) \in \mathfrak{P}_{\mathfrak{C}}$.

We consider an arbitrary pair of transversally equivalent foliations $(M_i, F_i) \in \mathfrak{F}_{\mathfrak{C}}$, $i = 1, 2$. Let $\beta(M_i, F_i) = (N_i, \Psi_i)$. By Theorem 1.2, there exists a group isomorphism $\theta : \Psi_1 \to \Psi_2$ and a diffeomorphism of manifolds $d : N_1 \to N_2$ which are connected by the relation $d \circ \psi_1 = \theta(\psi_1) \circ d$ for all $\psi_1 \in \Psi_1$. Therefore, if $N_1 = \mathbb{S}^q$, then it is necessary $N_2 = \mathbb{S}^q$. Let $g$ be the standard Riemannian metric on the sphere $\mathbb{S}^q$, and let $[g]$ be the class of conformally equivalent metrics which contains $g$. Since $\Psi_i$, $i = 1, 2$, is the group of automorphisms of the conformal structure $(\mathbb{S}^q, [g])$, we see that $\Psi_1$ is the essential group of automorphisms of the conformal structure $(\mathbb{S}^q, [d^*g])$. As is known [15], among Riemannian manifolds of dimension $q > 2$, only the standard sphere $\mathbb{S}^q$ and the Euclidean space $\mathbb{E}^q$ have the essential group of conformal transformations. Therefore, it is necessary $[g] = [d^*g]$. Consequently, $d$ is a conformal diffeomorphism of the sphere $\mathbb{S}^q$. Thus, it is necessary that $\Psi_1$ and $\Psi_2$ be conjugate subgroups of the Lie group $\mathrm{Conf}\,(\mathbb{S}^q)$.

If $N_1 = \mathbb{E}^q$, then it is necessary $N_2 = \mathbb{E}^q$. In this case, as above, it is proved that $\Psi_1$ and $\Psi_2$ are conjugate subgroups of the Lie group $\mathrm{Sim}(\mathbb{E}^q)$.

Thus, in both cases, the pairs $(N_1, \Psi_1)$ and $(N_2, \Psi_2)$ are isomorphic in the category $\mathfrak{P}_{\mathfrak{C}}$.

From [7, Theorems 3 and 7] (cf. also Theorem 1.1) it follows that any countable essential subgroup $\Psi$ of the Lie groups $\mathrm{Conf}\,(\mathbb{S}^q)$ and $\mathrm{Sim}(\mathbb{E}^q)$ is realized as the global holonomy group of a complete conformal foliation of codimension $q$.

Therefore, by Theorem 1.2, $\mathscr{B}_{\mathfrak{C}} = \mathscr{B}|_{\mathfrak{F}_{\mathfrak{C}}} : \widetilde{\mathfrak{F}}_{\mathfrak{C}} \to \widetilde{\mathfrak{P}}_{\mathfrak{C}}$ is a bijection from the set of classes of transversally equivalent conformal foliations $\widetilde{\mathfrak{F}}_{\mathfrak{C}}$ to the set $\widetilde{\mathfrak{P}}_{\mathfrak{C}}$ of classes of equivalence of their global holonomy groups. □

**6.4. Proof of Theorem 1.4.** By [10, Theorem 7], the structure Lie algebra of a foliation $(M, F)$ with transverse rigid geometry covered by the locally trivial fibration over the simply connected base $N$, is isomorphic to the Lie algebra of the Lie group $\overline{\Psi}$, equal to the closure of the global holonomy group $\Psi$ of this foliation in the Lie group of all automorphisms of the transverse rigid geometry on the manifold $N$. Since the conformal foliation $(M, F)$ of codimension $q > 2$ is Cartan and is modeled on the effective Cartan geometry, $(M, F)$ can be regarded as a complete foliation with transverse rigid geometry in the sense of [10]. As was noted in the proof of Theorem 1.3, the conformal foliation $(M, \mathscr{F})$ is covered by a locally trivial fibration. Consequently, the required assertions follow from [10, Theorem 7]. □

## 7 Examples

**Example 7.1.** We consider the universal covering map $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1 \times \mathbb{S}^1 : (x, y) \mapsto (x, e^{2\pi y i})$, where $(x, y) \in \mathbb{R}^1 \times \mathbb{R}^1$, of the cylinder $C = \mathbb{R}^1 \times \mathbb{S}^1$ by the plane. Let $p_C : C \to \mathbb{R}^1$ be the canonical projection on the first factor. We consider a manifold $M = C \setminus \{a\}$, where $a = f((0, 0))$, and a submersion $p = p_C|_M : M \to \mathbb{R}^1$. We denote by $(M, F)$ the simple foliation formed by the fibers of the submersion $p$. We emphasize that this submersion is not a Serre fibration. In the opposite case, the exact homotopic sequence for the fibration $p : M \to \mathbb{R}^1$ implies an isomorphism between the fundamental group of any fibre and the group of integers $\mathbb{Z}$, which contradicts the contractibility of $p^{-1}(0)$.

Let $\widehat{M} = \mathbb{R}^2 \setminus A$, where $A = \{(0, m) \mid m \in \mathbb{Z}\}$. We note that $k := f_{\widehat{M}} : \widehat{M} \to M$ is a regular covering map for $M$ with the group of desk transformations isomorphic to the group of integers $\mathbb{Z}$. We denote by $pr_2 : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1 : (x, y) \mapsto y$ the canonical projection on the first factor. The induced foliation $(\widehat{M}, \widehat{F})$, where $\widehat{F} = k^*F$, is formed by the connected components of the fibers of a submersion $pr_2|_{\widehat{M}} : \widehat{M} \to N$, where $N$ is a non-Hausdorff line containing a countable family of inseparable points instead of zero.

Thus, the lift of a simple foliation to a covering space is not, generally speaking, a simple foliation. Therefore, the requirement that a simple foliation $(M, F)$ is formed by the fibers of a Serre fibration is essential in Lemma 2.3.

**Example 7.2.** For $k \geqslant 6$ and $m \geqslant 3$ we consider the group $\Gamma_{km} = \langle s_i, i = 1, \dots, k \mid s_i^m = 1, [s_i, s_{i+1}] = 1 \rangle$. As was shown in [16], if $\sin \pi/k < 1/\sqrt{m}$, then there exists an exact representation $\alpha_{km} : \Gamma_{km} \to \mathrm{Conf}\,(\mathbb{S}^{2m-2})$; moreover, $\Psi_{km} = \alpha_{km}(\Gamma_{km})$ is a Fuchsian group (i.e., a discrete subgroup of $\mathrm{Conf}\,(\mathbb{S}^{2m-2})$ with finitely many generators) with the limit set $\Lambda(\Psi_{km})$

homeomorphic to a Menger curve. Moreover, $\psi_i = \alpha(s_i)$ are generators of the group $\Psi_{km}$.

Since $\sin\pi/k < \pi/k$ for all $k \geqslant 6$, for $\pi/k < 1/\sqrt{m} \Leftrightarrow k > \pi\sqrt{m}$ the initial inequality is satisfied. Let $k(m) = [\pi\sqrt{m}]$ denote the integer part of $\pi\sqrt{m}$. Then for any fixed natural number $m \geqslant 3$ there exists a countable family of Fuchsian groups $\Psi_{km}$, $k > k(m)$, such that $\Lambda(\Psi_{km})$ is homeomorphic to a Menger curve. For example, $k(3) = 5$, $k(4) = 6$, $k(5) = k(6) = 7$, $k(7) = k(8) = 8$, $k(9) = k(10) = 9$.

As above, let $S^2_k$ be the two-dimensional sphere with $k$ handles. The fundamental group of $S^2_k$ is $\{a_i, b_i, i = 1, \dots, k \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_k b_k a_k^{-1} b_k^{-1} = 1\}$. We set $B = S^2_k$ and define a group homomorphism $\rho_{km} : \pi_1(B, b) \to \mathrm{Diff}(\mathbb{S}^{2m-2})$ by setting $\rho_{km}(a_i) := \psi_i$, $i = 1, \dots, k$, on the generators, where $\rho_{km}(b_i) := \mathrm{id}_{S^2_k}$ is the identity of the group $\Psi$. Then $(M_{km}, F_{km}) := \mathrm{Sus}\,(\mathbb{S}^{2m-2}, S^2_m, \rho_{km})$ is a two-dimensional suspended foliation of codimension $2m-2$. The universal covering map for $M_{km}$ has the form $f_{km} : \mathbb{R}^2 \times \mathbb{S}^{2m-2} \to M_{km}$, and the fibration induced on the universal cover is formed by the fibers of the canonical projection $r_m : \mathbb{R}^2 \times \mathbb{S}^{2m-2} \to \mathbb{S}^{2m-2}$. The foliation $(M_{km}, F_{km})$ has the global attractor $\mathscr{M}_{km} = f(r_m^{-1}(\Lambda(\Psi_{km})))$.

We recall that the minimal set $\mathscr{M}$ of a foliation is said to be *exotic* if it has no interior points and the intersection $\mathscr{M} \cap T$ for any local transversal $T$ is not totally disconnected. Since the Menger curve is a connected one-dimensional fractal, by Theorem 1.5 any foliation of class $[(M_{km}, F_{km})]$ has an attractor that is an exotic minimal set whose transversal structure locally coincides with the structure of the Menger curve $\Lambda(\Psi_{km})$.

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