On Bose–Einstein condensation and superfluidity of trapped photons with coordinate-dependent mass and interactions

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The condensate density profile of trapped two-dimensional gas of photons in an optical microcavity, filled by a dye solution, is analyzed taking into account a coordinate-dependent effective mass of cavity photons and photon–photon coupling parameter. The profiles for the densities of the superfluid and normal phases of trapped photons in the different regions of the system at the fixed temperature are analyzed. The radial dependencies of local mean-field phase transition temperature \( T_{c0}(r) \) and local Kosterlitz–Thouless transition temperature \( T_r(r) \) for trapped microcavity photons are obtained. The coordinate dependence of cavity photon effective mass and photon–photon coupling parameter is important for the mirrors of smaller radius with the high trapping frequency, which provides Bose–Einstein condensation and superfluidity for smaller critical number of photons at the same temperature. We discuss a possibility of an experimental study of the density profiles for the normal and superfluid components in the system under consideration.

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1. INTRODUCTION

When a system of bosons is cooled to low temperatures, a substantial fraction of the particles spontaneously occupy the single lowest energy quantum state. This phenomenon is known as Bose–Einstein condensation (BEC) and it occurs in many-particle systems of bosons with masses \( m \) and temperature \( T \) when the de Broglie wavelength of the Bose particle exceeds the mean interparticle distance [1]. The most remarkable consequence of BEC is that there should be a temperature below which a finite fraction of all the bosons “condense” into the same one-particle state with macroscopic properties described by a single condensate wave function, promoting quantum physics to classical time and length scales.

Most recently, the observations at room temperature of the BEC of two-dimensional (2D) photon gas confined in an optical microcavity, formed by spherical mirrors and filled by a dye solution, were reported [2–5]. The interaction between microcavity photons is achieved through the interaction of the photons with the non-linear media of a microcavity, filled by a dye solution, and the interaction is non-contact interaction. It is known that BEC for bosons can exist without particle–particle interactions [6] (see [1] for the details), but at least the interactions with the surrounding media are necessary to achieve thermodynamical equilibrium. For photon BEC it can be achieved by interaction with incoherent phonons [7]. The influence of interactions on condensate-number fluctuations in a BEC of microcavity photons was studied in [8]. The kinetics of photon thermalization and condensation was analyzed in [9–11]. The kinetics of trapped photon gas in a microcavity, filled by a dye solution, was studied, and a crossover between driven-dissipative system laser dynamics and a thermalized Bose–Einstein condensation of photons was observed [12].

In previous theoretical studies the equation of motion for a BEC of photons confined by the axially symmetrical trap in a microcavity was obtained. It was assumed that the changes of the cavity width are much smaller than the width of the trap [13]. This assumption results in the coordinate-independent effective photon mass \( m_{\text{ph}} \) and photon–photon coupling parameter \( g \). In this paper, we study the local superfluid and normal density profiles for trapped two-dimensional gas of photons with the coordinate-dependent effective mass and photon–photon coupling parameter in an optical microcavity, filled by a dye solution. We propose the approach to study the local BEC and local superfluidity of cavity photon gas in the...
framework of local density approximation (LDA) in the traps of larger size without the assumption that total changes of the cavity width are much smaller than the size of the trap. In this case, we study the effects of coordinate-dependent effective mass and photon–photon coupling parameter on the superfluid and normal density profiles as well as the profiles of the local temperature of the phase transition for trapped cavity photons. Such an approach is useful for the mirrors of smaller radius with the high trapping frequency, which provide BEC and superfluidity for smaller critical number of photons at the same temperature.

The paper is organized in the following way. In Section 2, we obtain the condensate density profile for trapped microcavity photon BEC with locally variable mass and interactions. The expression for the number of particles in a condensate is analyzed in Section 3. In Section 4, the dependence of the condensate parameters on the geometry of the trap is discussed. In Section 5, we study the collective excitation spectrum and superfluidity of 2D weakly interacting Bose gas of cavity photons. The results of our calculations are discussed in Section 6. The proposed experiment for measuring the distribution of the local density of a photon BEC is described in Section 7. The conclusions follow in Section 8.

2. CONDENSATE DENSITY PROFILE

While at finite temperatures there is no true BEC in any infinite untrapped two-dimensional (2D) system, a true 2D BEC quantum phase transition can be obtained in the presence of a confining potential [14]. In an infinite translationally invariant two-dimensional system, without a trap, superfluidity occurs via a Kosterlitz–Thouless superfluid (KTS) phase transition [15]. While KTS phase transition occurs in systems, characterized by thermal equilibrium, it survives in a dissipative highly non-equilibrium system driven into a steady state [16].

The trap for the cavity photons can be formed by the concave spherical mirrors of the microcavity, which provide the axial symmetry for a trapped gas of photons. Thus the transverse (along the $xy$ plane of the cavity) confinement of photons can be achieved by using an optical microcavity with a variable width. Let us introduce the frame of reference, where the z-axis is directed along the axis of cavity mirrors, and the $(x,y)$ plane is perpendicular to this axis. The energy spectrum $E(k)$ for small wave vectors $k$ of photons, confined in the $z$ direction in an ideal microcavity with the coordinate-dependent width $L(r)$, is given by [2]

$$E(k) = \frac{\hbar c}{\sqrt{\varepsilon}} \sqrt{k_x^2 + k_z^2} \approx \frac{\hbar c n c}{\sqrt{\varepsilon} L(r)} + \frac{\hbar^2 k_z^2}{2 m_\text{ph}(r)}, \quad (1)$$

where $m_\text{ph}(r) = \frac{\hbar c n c}{\sqrt{\varepsilon} L(r)}$ is the transverse (along the $xy$ plane of the cavity) effective coordinate-dependent photon mass; $k_z$ is the wave vector component in the $z$ direction along the axis of the cavity; $k_x$ is the wave vector component in the $(x,y)$ plane, perpendicular to the axis of the cavity; $c$ is the speed of light in vacuum; $\varepsilon$ is the dielectric constant of the microcavity, and $n = 1, 2, \ldots$. Below we consider the lowest mode $n = 1$. Assuming the energy difference between the quantization levels $\tilde{n}$, caused by the boundaries of the cavity, is much larger than the second term in Eq. (1) for the photon energy, corresponding to a single quantum level, for small $k_z$, we treat our system as a quasi-two-dimensional system in the $(x,y)$ plane. Let us mention that Eq. (1) is valid only if the radius of the curvature of the mirrors is much larger than all other length scales in the system under consideration, i.e., the healing length $\xi$, etc., which it true for the parameters, used for our calculations, as it can be seen from the estimations below.

The Gross–Pitaevskii equation for the wave function of photon condensate in a weakly interacting (through a dye solution) photon gas can be obtained by following to the standard procedure for derivation of the Gross–Pitaevskii equation for any weakly interacting Bose gas [1] (see also [17]). Note that the Gross–Pitaevskii-like equation for the cavity photons can be obtained also from the Maxwell equations in a non-linear medium [13]. We generalize the Gross–Pitaevskii equation for cavity photons with the coordinate-dependent mass and photon–photon interaction. The energy functional of the trapped 2D photon BEC in an optical microcavity, filled by a dye solution, can be represented as

$$E[\psi] = \int \left[ -\frac{\hbar c}{2\pi \sqrt{\varepsilon}} \psi^*(r) \nabla \cdot \left( \sqrt{\varepsilon} \nabla \psi(r) \right) \right] + \left( \frac{\pi \hbar c}{\sqrt{\varepsilon} L(r)} - \mu \right) |\psi(r)|^2 + \frac{g(r)}{2} |\psi(r)|^4 \, dr, \quad (2)$$

where $\psi(r)$ is the wave function of the photon condensate; $\mu$ is the chemical potential of the cavity photons, determined by the laser pumping; $g(r)$ is the photon–photon coupling parameter, corresponding to the photon–photon interaction through the dye molecule. The form of the functional, presented in Eq. (2), implies essential dependence of the microcavity width $L(r)$ as a function of $r$. This leads to the effect of spatially variable effective mass $m_\text{ph}(r) = \frac{\hbar c n c}{\sqrt{\varepsilon} L(r)}$. In addition, the dependence $L(r)$ leads to the confining potential $V(r) = \frac{\pi \hbar c}{\sqrt{\varepsilon} L(r)}$, and also to the spatial dependence of the photon–photon coupling parameter $g(r)$. The energy functional similar to Eq. (2) was used for the particle with the coordinate-dependent mass in [17]. Let us mention that the energy functional, presented by Eq. (2), contains the coordinate-dependent coupling parameter $g(r)$, which was not under consideration in [18].

Let us mention that taking into account pumping and losses of photons in a microcavity and assuming that they are not very large, it is reasonable to expect that the spatial distribution of photons will not differ much (at least, qualitatively) from the spatial distribution of photons, obtained from the minimum of the energy functional (2), for the system of the photons in the thermodynamical equilibrium. As it is shown from the numerical solution of the generalized Gross–Pitaevskii equation with pumping and losses in [19,20], small pumping and losses for condensates of bosons lead to only non-essential quantitative change of the distribution of condensate profile in the trap. In the opposite case, the large pumping and losses exceeding some critical values result in quantum turbulence phenomena in the system, which manifest themselves in breaking continuous condensate density distribution and pattern formation [19,20].
The variation of the energy functional (2) with respect to \( \psi^*(\mathbf{r}) \) gives the following equation for the wave function \( \psi(\mathbf{r}) \) of the 2D photon condensate in microcavity traps:

\[
-\frac{\hbar c}{2\sqrt{\varepsilon}} \nabla \left[ \left( L(\mathbf{r}) \nabla \psi(\mathbf{r}) \right) \right] + \left( \frac{\pi \hbar c}{\sqrt{\varepsilon L(\mathbf{r})}} - \mu \right) \psi(\mathbf{r}) + \frac{g(\mathbf{r})}{2} |\psi(\mathbf{r})|^2 \psi(\mathbf{r}) = 0. \tag{3}
\]

Following the procedure of the derivation of the Gross–Pitaevskii equation for a cavity photon BEC, one obtains photon–photon coupling parameter \( g(\mathbf{r}) \). The coordinate-dependent photon–photon coupling parameter \( g(\mathbf{r}) \) is given by

\[
g(\mathbf{r}) = \frac{\pi \hbar c A}{\sqrt{\varepsilon L(\mathbf{r})}}, \tag{4}
\]

where the parameter \( A \) determines the strength of photon–photon coupling and depends on the properties of the medium, through which the photons interact (see also [13]).

Due to the mirrors’ axial symmetry, the wave function \( \psi(\mathbf{r}) \) of the 2D photon condensate has the axial symmetry. Since in the stationary non-rotating BEC the angular momentum equals to zero, one can rewrite Eq. (3) in the polar coordinates as

\[
-\frac{\hbar c}{2\sqrt{\varepsilon}} \left( L(r) \frac{d^2}{dr^2} + \frac{dL(r)}{dr} \frac{d}{dr} + \frac{L(r)}{r} \frac{d}{dr} \right) \psi(r) + \left( \frac{\pi \hbar c}{\sqrt{\varepsilon L(r)}} - \mu \right) \psi(r) + \frac{g(r)}{2} |\psi(r)|^2 \psi(r) = 0, \tag{5}
\]

where \( \psi(r) \) is the radial component of the condensate wave function \( \psi(\mathbf{r}) \).

We consider a trapped Bose gas with a fixed number of particles in a condensate. In general case, the density profile for photon BEC \( n(r) = |\psi(r)|^2 \) can be obtained by numerical solution of Eq. (5) for a given function \( L(r) \). In this paper, we focus on obtaining an analytical expression for the profile of the microcavity photon BEC. For a large number of condensate photons [8], one can use for this profile the Thomas–Fermi approximation [1] analogously to applicability of the Thomas–Fermi approximation for other physical realizations for bosons. The conditions of applicability of Thomas–Fermi approximation are discussed in Appendix A. In the Thomas–Fermi approximation [1], neglecting the gradient terms acting on the condensate wave function in Eq. (5), and, assuming the slowly varying width of the cavity, for the chemical potential \( \mu \), satisfying to the condition

\[
\mu > \frac{\pi \hbar c}{\sqrt{\varepsilon L(r)}}, \tag{6}
\]

one gets

\[
\frac{\pi \hbar c}{\sqrt{\varepsilon L(r)}} - \mu + \frac{g(r)}{2} n^2(r) = 0, \tag{7}
\]

where \( n(r) = |\psi(\mathbf{r})|^2 \) is the condensate density. As follows from Eq. (7), \( n(r) \) is a slowly varying function, defined by the slowly varying function \( L(\mathbf{r}) \):

\[
n(r) = \frac{2(\mu - \pi \hbar c/\sqrt{\varepsilon L(r)})}{g(r)} = \frac{2\mu - 2m_{ph}(r) c^2/\varepsilon}{g(r)}. \tag{8}
\]

Therefore, the photonic BEC exists only if the laser pumping provides the chemical potential \( \mu \) that satisfies the condition given by Eq. (6). For an axially symmetrical trap the maximal radius of the BEC spot \( r_0 \) is defined by \( \mu = \pi \hbar c/\sqrt{\varepsilon L(r_0)} \). The corresponding value of \( r_0 \) is valid at \( r_0 < R \), where \( R \) is the radius of the trap, which is defined by the shape of the mirrors.

### 3. NUMBER OF PARTICLES IN A CONDENSATE

We assume that the axially symmetrical trap has a harmonic shape. As it is shown in [3], when the distance from the axis of the microcavity to the mirror is essentially smaller than the radius of the spherical mirror, the harmonic approximation for the shape of mirrors forming a microcavity is valid. Let us derive the expression for the chemical potential \( \mu \) for the harmonic trap. For the harmonic trap \( V_0 + \gamma r^2/2 = \pi \hbar c/\sqrt{\varepsilon L(r)} \) we have \( L(r) = \pi \hbar c/\sqrt{\varepsilon (V_0 + \gamma r^2/2)} \), where \( V_0 = \pi \hbar c/\sqrt{\varepsilon L(0)} \), and \( \gamma \) is the constant, determining the curvature of the harmonic trap. In the approximation, applied for the harmonic trap, \( \gamma \) is defined as \( \gamma = m_1 \Omega^2 \), where \( m_1 = 6.7 \times 10^{-36} \text{ kg} \) is the cavity effective photon mass at \( r = 0 \), given in [5], and \( \Omega \) is the frequency of the harmonic trap, defined as \( \Omega = \sqrt{2/(L(0) R_m)} \) [2]. The radius of the curvature of the mirrors \( R_m \) is related to the trapping frequency \( \Omega \) as [2]

\[
R_m = \frac{2c^2}{L(0) \Omega^2}. \tag{9}
\]

Therefore, for the harmonic trap, one obtains

\[
m_{ph}(r) = \frac{\pi \hbar c}{\sqrt{\varepsilon}} \left( V_0 + \frac{\gamma r^2}{2} \right), \tag{10}
\]

and

\[
g(r) = A \left( V_0 + \frac{\gamma r^2}{2} \right). \tag{11}
\]

While for the density profile of the photonic BEC at \( r < r_0 \) from Eq. (8) one gets

\[
n(r) = \frac{2}{A} \left( \frac{\mu}{V_0 + \gamma r^2/2} - 1 \right), \tag{12}
\]

the normalization condition for the density profile of the photonic BEC \( N_{\text{BEC}} = \int n(r) d\mathbf{r} \) and Eq. (12) lead to

\[
N_{\text{BEC}} = \frac{4\pi \mu}{A \gamma} \ln \left[ 1 + \frac{\gamma r_0^2}{2V_0} - \frac{2\pi r_0^2}{A} \right]. \tag{13}
\]

We find the equation for \( r_0 \) using Eq. (12) from the condition \( n(r_0) = 0 \):

\[
\mu = V_0 + \frac{\gamma r_0^2}{2}, \tag{14}
\]

which results in

\[
.N_{\text{BEC}} = \frac{4\pi \mu}{A \gamma} \ln \left[ 1 + \frac{\gamma r_0^2}{2V_0} - \frac{2\pi r_0^2}{A} \right]. \tag{13}
\]
\[
    r_0 = \sqrt{\frac{2(\mu - V_0)}{\gamma}}.
\]  

(15)

For the harmonic trap, the chemical potential \(\mu\), corresponding to the existence of the BEC, has to satisfy to the condition, which follows from Eq. (6):

\[
    \mu \geq V_0 + \frac{\gamma^2}{2},
\]  

(16)

where \(r \leq r_0\) [see Eq. (14)].

Substituting Eqs. (15) into (13), we get the following expression, which connects the number of photons in BEC \(N_{\text{BEC}}\) with the chemical potential \(\mu\) and other parameters of the system:

\[
    N_{\text{BEC}} = \frac{4\pi\mu}{\gamma} \ln \left[ \frac{\mu}{V_0} + \frac{4\pi(\mu - V_0)}{\gamma} \right] - \frac{4\pi\mu(\mu - V_0)}{\gamma}.
\]  

(17)

Equation (17) will be used to calculate the spatial condensate density profile, applying Eq. (12) (see Section 6).

4. DEPENDENCE OF THE CONDENSATE PARAMETERS ON THE GEOMETRY OF THE TRAP

The radius of the BEC spot \(r_0\) is determined by the number of photons in BEC \(N_{\text{BEC}}\) and the parameter of strength of photon–photon coupling \(A\). However, since the photon–photon interaction strength is currently not well known, we obtain the parameter of strength of photon–photon coupling \(A\), using the experimental results [2]. The experiment [2] was maintained at the finite temperature \(T = 300\) K, when the radius of the photon spot is different from the radius of the BEC spot, while at \(T = 0\) K, assuming almost all photons belong to BEC, the radii of the photon spot and BEC spot are equal. For our calculations in the framework of the Thomas–Fermi approximation at \(T = 0\) K, we assume that the radius of the photon spot \(r_0\) corresponds by the order of magnitude to the one reported in the experiment [21]. The parameter of strength of photon–photon coupling \(A\) was estimated by substituting \(r_0 = 20\) \(\mu\)m, and \(L(0) = 1.7 \times 10^{-6}\) m, \(\gamma = 7.929 \times 10^{-13}\) J/m², \(N_{\text{BEC}} = 1.7 \times 10^5\), \(\varepsilon = 2.045\) from [5] into Eqs. (14) and (17). Then, one obtains \(A = 2.87 \times 10^{-5}\) \(\mu\)m².

According to the experiment, at \(T = 300\) K BEC exists at \(N_{\text{BEC}} > N_c\), where \(N_c = 8.5 \times 10^4\) [21]. The experiments have been performed for \(N_{\text{BEC}}\) in the range from \(3 \times 10^4\) up to \(5.5 \times 10^5\) [5]. For our calculations we use \(N_{\text{BEC}} = 1.7 \times 10^5\). While for the experimental parameters [5], implying \(\Omega = 2\pi \times 36.5\) GHz and \(\gamma = 3.524 \times 10^{-13}\) J/m², the Thomas–Fermi approximation is not applicable, we use for our calculations \(\Omega = 2\pi \times 54.75\) GHz and \(\gamma = 7.929 \times 10^{-13}\) J/m², where the Thomas–Fermi approximation is valid, as it is demonstrated in Appendix A.

Let us mention that using \(\Omega = 2\pi \times 54.75\) GHz larger than the value used in [2] corresponds to slightly smaller radius \(R_m\) of the curvature of the mirrors than in [2]. Thus, for the parameters used for our calculations, we have \(R_m = 0.444\) m. In [5], the mirrors of the radius of curvature \(R_m = 1\) m have been used. The advantage of using the mirrors of smaller radius with higher trapping frequencies is the increase of the constant \(\gamma\), which results in higher critical temperature of BEC for the same number of photons, because the critical temperature \(T_{\text{BEC}}^{(0)}\) of BEC for a non-interacting Bose gas can be qualitatively estimated as (see, e.g., [14])

\[
    T_{\text{BEC}}^{(0)} \sim \frac{\hbar}{\pi k_B} \left( \frac{6\pi N_{\text{BEC}}}{m_{\text{ph}}(0)} \right)^{1/2},
\]  

(18)

where \(k_B\) is the Boltzmann constant. Thus at fixed temperature \(T\) the critical number of photons, corresponding to the BEC transition, is inversely proportional to the constant \(\gamma\). Using Eq. (18), one obtains at \(T = 300\) K, for the experimental constant \(\gamma = 3.524 \times 10^{-13}\) J/m², the estimation for the critical number of photons for the BEC transitions as \(N_c \sim 6.694 \times 10^5\). At the same parameters, for used in our calculations the constant \(\gamma = 7.929 \times 10^{-13}\) J/m², the critical number of photons for the BEC transitions can be estimated as \(N_c \sim 2.975 \times 10^5\). The latter demonstrates that for the microcavity with a smaller radius of the mirrors—implying according to Eq. (9) larger \(\Omega\), and, therefore, larger constant \(\gamma\)—BEC can be achieved for a smaller critical number of photons at the same temperature. Let us mention that using the mirrors of the radius \(R_m = 0.444\) m, which corresponds to that used in our calculations, the constant \(\gamma = 7.929 \times 10^{-13}\) J/m² does not break the validity of Eq. (1), since this radius of the curvature of the mirrors is much larger than all other length scales in the system under consideration.

Let us mention that the effect of taking into account the spatial dependence of the cavity effective photon mass \(m_{\text{ph}}(r)\) and the photon–photon coupling parameter \(g(r)\) can be illustrated by the following ratios, calculated with the parameters introduced above: \(m_{\text{ph}}(r_0)/m_{\text{ph}}(0) = g(r_0)/g(0) = 1.004\). At the location of the condensate the change of the cavity width can be illustrated by the ratio \(L(r_0)/L(0) = 0.996\). For the relatively small radius of the mirror \(R_m = 0.035\) m, implying \(\Omega = 2\pi \times 277.9\) GHz and \(\gamma = 2.043 \times 10^{-13}\) J/m², one obtains \(m_{\text{ph}}(r_0)/m_{\text{ph}}(0) = g(r_0)/g(0) = 1.1\). In this case of the mirrors of such small radius, at the location of the condensate the change of the cavity width can be illustrated by the ratio \(L(r_0)/L(0) = 0.909\). Therefore, for the mirrors of the smaller radius, the coordinate dependence of the effective photon mass and the photon–photon coupling parameter is stronger. Note that formation of the traps for the cavity photons, located in the convexities of this small radius on a plane mirror, seems to be possible. In this paper we are interested in a small radius of the mirrors, because it corresponds to a smaller critical number of photons for BEC at fixed temperature and microscopical traps can be used for quantum technology applications.

According to Eq. (17), the radius of the BEC spot \(r_0\) depends on the parameter of strength of photon–photon coupling \(A\) and the number of photons in BEC \(N_{\text{BEC}}\). We study the photon–photon interaction strength, assuming the substitution of the experimental values \(r_0\) and \(N_{\text{BEC}}\) into Eq. (17). The parameter of strength of photon–photon coupling \(A\), required to achieve the certain radius of the BEC spot \(r_0\) at the defined total number of photons in the BEC \(N_{\text{BEC}}\), is shown in Fig. 1 for various \(r_0\) and \(N_{\text{BEC}}\). According to Fig. 1 at the fixed \(N_{\text{BEC}}\)
larger $A$ is required to achieve larger $r_0$, and for larger $N_{\text{BEC}}$ smaller $A$ is required to achieve the fixed spot radius $r_0$.

The justification of the Thomas–Fermi approximation and slowly varying cavity width approximation for the parameters used in our calculations is discussed in Appendix A.

5. SUPERFLUIDITY OF THE MICROCavitY PHOTONS

Below we study the collective excitation spectrum and superfluidity of 2D weakly interacting Bose gas of cavity photons. While at zero temperature the entire system is superfluid, at the non-zero temperatures below the KTS phase transition temperature in a 2D superfluid the normal component appears in the cores of the vortices, with the superfluid circulating around these cores [15,22]. Below we consider the axially symmetrical trap, where the size of the condensate is essentially larger than the average distance between two vortices. The maximal density of the vortices is estimated as $n_v \lesssim r_v^{-2}$, where $r_v$ is the size of the core of a vortex. The average distance $\xi_n$ between the vortices cannot be smaller than the size of the core of a vortex $r_v$; $\xi_n \lesssim r_v$ (see, e.g., [23]). Since the size of the core of a vortex is of the order of the magnitude of the healing length $r_v \approx \xi$ [24], the size of the condensate is larger than the average distance between two vortices when the inequality $\xi < r_0$ holds, which does not contradict the condition of validity of the Thomas–Fermi approximation, presented by Eq. (A2). Therefore, one can estimate the local temperature of Kosterlitz–Thouless phase transition using the parameters obtained from the Thomas–Fermi approximation.

Now we will analyze the spectrum of the collective excitations in the superfluid of microcavity photons. For small momenta ($P \equiv \hbar k$) $P \ll \sqrt{2m_{ph}(r)g(r)n(r)}$ and small temperatures, the energy spectrum of the quasi-particles $\epsilon(P, r)$ is given by [25] $\epsilon(P, r) \approx c_s(r)P$, where $c_s(r)$ is the sound velocity in the Popov approximation [26]:

$$c_s(r) = \sqrt{\frac{g(r)n(r)}{m_{ph}(r)}}$$  

(19)

For the harmonic trap, substituting Eqs. (10) and (11) into Eq. (19), one obtains

$$c_s(r) = c_s \sqrt{\frac{A n(r)}{e}} = \frac{c_s}{\sqrt{\epsilon}} \sqrt{2 \left( \frac{\mu}{V_0 + \gamma r^2/2} - 1 \right)}.$$  

(20)

The dilute photon gas in an optical microcavity filled by a dye solution forms a 2D weakly interacting gas of bosons with the pair short-range repulsion, caused by the photon–photon interaction through the dye molecule. Since the spectrum of a weakly interacting gas of the cavity photons is a linear sound spectrum, satisfying the Landau criterion of superfluidity [27], superfluidity of the cavity photons can be observed in the trap. Therefore, at small temperatures there are two components in the trapped gas of cavity photons: the normal component and the superfluid component. We obtain the number of photons in the superfluid component as a function of temperature applying the procedure similar to the one used for the microcavity exciton polaritons in a 2D trap [28]. We define the total number of particles in the superfluid component $N_s \equiv N - N_n$, where $N_n$ is a total number of particles in the normal component. $N_n$ is defined analogously to the procedure applied for the definition of the density of the normal component in the infinite system $n_n$ [27] using the isotropy of the trapped cavity photonic gas instead of the translational symmetry for an infinite system. According to the Landau theory of quasi-particles, at finite temperatures the non-interacting quasi-particles, contributing to the normal component, are characterized by the same energy spectrum as the weakly interacting particles at the zero temperature [27]. The Landau theory of quasi-particles is valid at low temperatures, when the number of particles in the normal component is much less than the total number of particles: $N_n \ll N$. The temperatures when our approach is applicable must be much smaller than the critical phase transition temperatures. Therefore, the Landau theory of quasi-particles is valid at these temperatures. Our estimations using Eq. (18) show that the possible transition temperatures can even exceed room temperatures for realistic experimental parameters. Assuming an axially symmetric 2D trap for microcavity photons, we imagine that a “gas of quasi-particles” rotates in the liquid in the plane perpendicular to the axis of the trap with some small macroscopic angular velocity $\nu$. In this case, the distribution function of a gas of quasi-particles can be obtained from the distribution function of a gas at rest by substituting for the energy spectrum of the quasi-particles $\epsilon(P) - M\nu$, where $M = r \times P$ is the angular momentum of the particle. Assuming $Pr/\hbar \gg 1$, we apply the quasi-classical approximation for the angular momentum: $M \approx Pr$ and $\epsilon(M, r) = c_sP = r^{-1}c_s(r)M$. The total angular momentum in a trap per unit of area $M_{tot}(r)$ is given by

$$M_{tot}(r) = \int \frac{d^2 M}{(2\pi\hbar)^2} M_{\text{ele}}(r, L) - M\nu,$$  

(21)

where we assume that at small temperatures the quasi-particles are non-interacting, and they are described by the Bose–Einstein distribution function $n_{\text{B}}(\epsilon) = (\exp[\epsilon/(k_B T)] - 1)^{-1}$. For small angular velocities, $n_{\text{B}}(\epsilon - M\nu)$ can be expanded in terms of $M\nu$. Then in the linear approximation we get

\[ n_{\text{B}}(\epsilon - M\nu) \approx 2\nu M\nu ^{-1} \int d\epsilon \exp[\epsilon-(k_B T)/2]g(\epsilon) - M\nu \epsilon. \]
\[
M_{\text{tot}}(r) = - \int \frac{d^2M}{(2\pi r)^2} M(M) \frac{\partial L(r)}{\partial c}. \tag{22}
\]

Assuming that only quasi-particles contribute to the total angular momentum, we define the density of the normal component \( n_n(r, T) \) by \( M_{\text{ns}}(r) = n_n(r, T) M_n(r) \), where \( M_n(r) = m_{ph}(r) r \nu \) is the angular momentum of one quasi-particle. From Eq. (22), the local coordinate-dependent density of the normal component is obtained as

\[
n_n(r, T) = \frac{3\zeta(3) k_B^3 T^3}{2 \pi \hbar^2 c_l^2(r) m_{ph}(r)} = \frac{3\zeta(3) k_B^3 L(r) c T^3}{4 \pi^2 \hbar^3 \sqrt{c_l^2(r)}}. \tag{23}
\]

Let us mention that the density of the normal component \( n_n(r, T) \) does not depend on the angular velocity of rotation \( \nu \), because \( n_n(r, T) \) is a linear response of the total angular momentum in a trap per unit of area on the external angular velocity. Hence, \( n_n(r, T) \) is determined only by the equilibrium properties of the system.

The temperature dependence of the local density of the superfluid component \( n_s(r, T) \) is given by

\[
n_s(r, T) = n(r) - n_n(r, T), \tag{24}
\]

where \( n(r) \) is the profile of the total photon density, which almost does not change at low temperatures. Assuming that at low temperatures the majority of photons belong to BEC, and substituting Eqs. (8) and (23) into Eq. (24), we obtain the temperature dependence of the local coordinate-dependent density of the superfluid component:

\[
n_s(r, T) = \frac{2(\mu - \pi \hbar c/\sqrt{c_l(r)}) - 3\zeta(3) k_B^3 L(r) T^3}{3 \pi \hbar^3 \sqrt{c_l^2(r)}}. \tag{25}
\]

For the total number of photons in the normal component we obtain

\[
N_n(T) = 2 \pi \int_0^\infty n_n(r, T) r dr = \int_0^\infty \frac{3\zeta(3) k_B^3 T^3}{\hbar^3 c_l^2(r) m_{ph}(r)} r dr,
\]

where \( \zeta(\varepsilon) \) is the Riemann zeta function \( (\zeta(3) \approx 1.202) \) and \( k_B \) is the Boltzmann constant, and we assume that at low temperatures almost all photons are in the condensate.

For the total number of photons in the superfluid component we get

\[
N_s(T) = N - N_n(T) = N - \int_0^\infty \frac{3\zeta(3) k_B^3 T^3}{\hbar^3 c_l^2(r) m_{ph}(r)} r dr. \tag{27}
\]

We assume that the width of the cavity \( L(r) \) very slowly depends on the coordinate on scales of the order of the mean separation between vortices (but the total change of \( L(r) \) in the trap is essential). The superfluid-normal phase transition in the 2D system is the Kosterlitz–Thouless transition [15], and the local coordinate-dependent temperature of this transition \( T_s \) in a two-dimensional microcavity photon system is determined by the equation [15]

\[
T_s(r) = \frac{\hbar^2 n_s(r, T_s(r))}{2 k_B m_{ph}(r)}. \tag{28}
\]

We can use Eq. (28) only in the framework of the quasi-local approximation, assuming very slow changes of \( L(r) \), such that the characteristic length of the changes in \( L(r) \) is much less than the average distance between the vortices in the superfluid.

Substituting Eq. (25) for the density \( n_n(r, T) \) of the superfluid component into Eq. (28), we obtain an equation for the local Kosterlitz–Thouless transition temperature \( T_s(r) \). The solution of this equation is

\[
T_s(r) = \left[ 1 + \left( \frac{32}{27} \frac{m_{ph}(r) k_B T^3}{\pi \hbar^2 n(r)} \right)^{1/3} + 1 \right]^{1/3} \tag{29}
\]

where \( T^0(r) \) is the local temperature at which the superfluid density vanishes in the mean-field approximation at the points with the coordinate vector \( r \) (i.e., \( n_n(r, T^0(r)) = 0 \),

\[
T^0(r) = \frac{1}{k_B} \frac{2 \pi \hbar^2 n(r) c_l(r) m_{ph}(r)}{3 \zeta(3)}. \tag{30}
\]

Equations (29) and (30) generalize the results of [29] for the coordinate-dependent particle mass and photon–photon interaction.

6. RESULTS AND DISCUSSION

Since \( T_s(r) \) depends on the coordinate \( r \), at fixed finite temperatures \( T \) above the minimal possible critical temperature \( T_{\text{crit}} = 0 \) at the edge of the BEC and below the maximal possible critical temperature \( T_{\text{max}} \), e.g., \( 0 < T < T_{\text{max}} \), there is the superfluid (S) phase in the region of the system where \( T < T_s(r) \), (in S phase the superfluid component coexists with the normal component), and the normal (N) phase in the other regions of the system where \( T > T_s(r) \) (with only the normal component). At the zero temperature the entire system is superfluid, and at the temperatures above \( T_{\text{crit}} \) the entire system is normal. Since \( T_s(r) \) is a decreasing function of \( m_{ph}(r) \), and \( m_{ph}(r) \) is a decreasing function of the width of the microcavity \( L(r) \), \( T_s(r) \) increases if \( L(r) \) increases. If we consider the axially symmetrical trap, where \( L(r) \) is a decreasing function of \( r \), then \( T_s(r) \) decreases when \( r \) increases. For the axially symmetrical trap we have \( T_{\text{max}} = T_s(0) \). If we consider the temperature \( T_1 \) in the range \( 0 < T_1 < T_s(0) \), then we have \( T_s(r_1) = T_1 \), where \( T_s(r_1) \) is the critical temperature, corresponding to the width of the cavity \( L(r_1) \), which can be found from the solution of Eq. (29) with respect to \( L(r) \), substituting \( T_s = T_1 \). The corresponding \( r_1 \) is the radius of the spot with the superfluid and normal components inside and only the normal component outside, filling the ring with the width \( r_0 - r_1 \) (see the inset in Fig. 2). While the total density and superfluid density are monotonically decreasing functions of \( r \) due to the increase of the effective photonic mass with the increase of \( r \), the normal density is a non-monotonic function of \( r \). At \( 0 < r < r_1 \), the normal density \( n_n(T) = n(r) - n_n(r) \) increases with the increase of \( r \) due to the decrease of \( n_n(r) \), but at \( r_1 \leq r \leq r_0 \), the normal density \( n_n(T) = n(r) \) is a decreasing function of \( r \) in the range \( 0 < r < r_1 \), while the normal density \( n(r) \) is a decreasing function of \( r \) in the range \( r_1 < r < r_0 \). The profiles for the total concentration \( n(r) \), the concentrations of the normal \( n_n(r) \), and superfluid...
The distance from the center of the trap and $T_c$ tons escaping the optical microcavity. The fiber photodetector, measurements are based on the observation of local distribution of photonic superfluidity of trapped microcavity photons. These experiments were designed to observe the superfluidity of microcavity polaritons to observe the superfluid flow does not experience any scattering at the obstacles; and (2) by observing the quantized vortices in the microcavity quality factor would be negligible. The gradient of the colors reflects the local concentrations of the superfluid and normal components.

According to Eqs. (29) and (30), the local mean-field phase transition temperature $T_c^0(r)$ and local Kosterlitz–Thouless transition temperature $T_c^1(r)$ decrease with the increase of the distance from the center of the trap $r$, and both $T_c^0(r)$ and $T_c^1(r)$ vanish at the edge of the trap. Everywhere inside a trap, the local mean-field phase transition temperature $T_c^0(r)$ is greater than the local Kosterlitz–Thouless transition temperature $T_c^1(r)$, and the difference between $T_c^0(r)$ and $T_c^1(r)$ decreases with the increase of the distance from the center of the trap $r$.

The superfluidity of cavity photons can be observed experimentally analogously to the system of microcavity polaritons: (1) by observing the photon condensate flow induced by initial gradient of density through the obstacles impurities, where the superfluid flow does not experience any scattering at the obstacles; and (2) by observing the quantized vortices in the system of cavity photons. The experimental evidence for superfluid motion of exciton polaritons in a semiconductor microcavity was reported in [30]. The superfluidity of microcavity exciton polaritons was studied in terms of the Landau criterion and manifested itself as the suppression of scattering from defects when the flow velocity was slower than the speed of sound in the fluid [30]. We suggest to generalize the methods used to observe the superfluidity of microcavity polaritons to observe the superfluidity of the photons in an optical cavity, filled with molecular medium, that are excited by laser light.

7. PROPOSED EXPERIMENT FOR MEASURING THE DISTRIBUTION OF THE LOCAL DENSITY OF A PHOTON BEC

We propose the following experiment relevant to BEC and superfluidity of trapped microcavity photons. These experiments are based on the observation of local distribution of photons escaping the optical microcavity. The fiber photodetector, formed by a single fiber probe and attached to a piezo scanner, can be moved above different regions of the mirrors at the distance from the mirrors about several micrometers, which is much less than the size of the BEC. These photodetectors can register the local intensity of the lines of the angular distribution of light, which is proportional to the number of the photons, escaped from the nearest to the detector region of the microcavity with the given angle $\alpha$ between the momentum of photons escaping the optical microcavity and the normal to the microcavity. In the absence of photon flow, the average angle between the momentum of photons escaping the optical microcavity and the normal to the microcavity is $\bar{\alpha} = 0$, because the angular distribution is symmetrical. The photons escaping from the BEC inside the circle of the radius $r_1$ form the sharp bright spot with very narrow line, registered by the fiber photodetector, because the photons from BEC are characterized by (almost) zero momentum component $P = 0$ in the $(x,y)$ plane, normal to the axis of the cavity. Hence, these photons from the BEC escape in the direction normal to the plane of the microcavity. The photons escape from the non-condensate with various $P$, characterized by various angles $\alpha$ between the momentum and the normal to the microcavity. Hence, photons escape from the non-condensate, forming line broadening. Therefore, photons escaping from the BEC region inside the circle of radius $r_1$ will form very narrow line of very high intensity, corresponding to the BEC, and this narrow line will be surrounded by the broad lower intensity line, corresponding to the non-condensate. The photons escaping from the ring of the inner radius $r_1$ and the outer radius $r_0$ form only the broad lower intensity line, because there is no BEC inside this ring. The scheme of this possible experiment is presented in Fig. 3. A fiber-based detector located near the mirror can be scanned along the surface to register the spatial distribution of photons escaping the microcavity, or the detectors can be also located inside the microcavity if the corresponding change of the microcavity quality factor would be negligible. The quasi-particles forming the local normal component contribute to the local line broadening, which affects the local average deviation of the tangent of the angle between the path of...
the escaping photon and the normal to the microcavity, defined as

\[ \Delta \tan \alpha(r) = \sqrt{\frac{p^2(r)}{\rho^2(r)}}. \tag{31} \]

In Eq. (31) \( p_x(r) = \pi h/L(r) \) is the momentum component in the \( z \) direction along the axis of the cavity, and the average squared momentum component in the \((x,y)\) plane \( \overline{p^2} \) is given by

\[ \overline{p^2}(r) = \frac{1}{\rho(r)} \int p^2 \rho(p) \, dp \tag{32} \]

where \( e = c(r) P \) for \( 0 \leq r < r_1 \). In Eq. (32) it was assumed that the broadening of the photon angle distribution for a weakly interacting photon gas is formed only by the contribution of quasi-particles.

For the region of the trap with the superfluid phase \( (0 \leq r < r_1) \) after integration in Eq. (32), one obtains

\[ \overline{p^2}(r) = \frac{\Gamma(n) \zeta(4) k_B^4 T^4}{2 \pi^2 \hbar^2 \zeta(n) n(r)}, \tag{33} \]

where \( \Gamma(n) \) is the gamma function \((\Gamma(4) = 6)\) and \( \zeta(z) \) is the Riemann zeta function \( (\zeta(4) \approx 1.0823) \).

The broadening of the photon angle distribution for the region \( 0 \leq r < r_1 \) (where the superfluid component exists) is formed by the contribution of quasi-particles according to Eq. (33). Substituting Eq. (33) into Eq. (31), one obtains the following expression:

\[ \Delta \tan \alpha(r, T) = \left( \frac{3 \zeta(4)}{\pi n(r)} \right)^{1/2} \frac{k_B^2 T^2}{\hbar \rho(r)} \Delta \tan \alpha(r). \tag{34} \]

Substituting Eq. (34) into Eq. (23), we get

\[ n_s(r, T) = \frac{\zeta(3)}{4} \left( \frac{3 n(r)}{\zeta(4) e} \right)^{1/2} \frac{c k_B T}{\hbar \zeta(n) r} \Delta \tan \alpha(r, T). \tag{35} \]

Therefore, one obtains the profile of the density of the normal component \( n_s(r) \) through an experimental measurement of the profile of \( \Delta \tan \alpha(r, T) \).

The profiles for \( \Delta \tan \alpha \) in a trap for the total number of photons \( N = 10^5 \) at \( T = 300 \) K for the region, where the superfluid component exists, are presented in Fig. 4 for different parameters of strength of photon–photon coupling \( A \) and radii of the photon spot \( r_0 \). According to Fig. 4, \( \Delta \tan \alpha \) increases up to the edge of the superfluid spot \( r_1 \) and the latter one increases with the increase of \( A \) and \( r_0 \). The dependence of the profile of \( \Delta \tan \alpha \) on the total number of photons \( N \) and the radius of the photon spot \( r_0 \) at \( T = 300 \) K is presented in Fig. 5. In Fig. 5, \( N \) is the total number of photons which equals to the number of photons in BEC \( N_{BEC} \) at \( T = 0 \) K, given by Eq. (17). According to Fig. 5, \( \Delta \tan \alpha \) decreases with the increase of \( N \) at the fixed radius of the photon spot \( r_0 \). According to Figs. 4 and 5, \( \Delta \tan \alpha \) always increases with \( r \) at the fixed \( r_0 \), \( N \), \( A \), and \( T \), because the concentration of the normal component always increases with \( r \) in the region \( 0 \leq r < r_1 \) according to Fig. 2, because only the quasi-particles from the normal component contribute to \( \Delta \tan \alpha \). It follows from Figs. 4 and 5 that by measuring \( \Delta \tan \alpha \) experimentally, one can obtain the parameter of strength of photon–photon coupling \( A \), and, therefore, one can study photon–photon interaction.

8. CONCLUSIONS

In conclusion, we considered the BEC of trapped two-dimensional gas of photons with the coordinate-dependent effective mass and photon–photon coupling parameter in an optical microcavity, filled by a dye solution, with the photons being confined due to the coordinate-dependent width of the optical microcavity. The coordinate dependence of the cavity photon...
effective mass and photon–photon coupling parameter describes the photons in a cavity with the mirrors of smaller radius with the higher trapping frequency, which provides BEC and superfluidity for a smaller critical number of photons at the same temperature. The photon condensate density profile was obtained in the Thomas–Fermi approximation. The condition for the chemical potential, corresponding to the trapped photonic BEC, was formulated. The local coordinate-dependent densities of the superfluid and normal components of the trapped photon system were obtained at a fixed temperature. The profiles of the superfluid and normal regions were presented at a fixed temperature. The profiles for the local mean-field phase transition temperature \( T_0^0(r) \) and local Kosterlitz–Thouless transition temperature \( T_c(r) \) for trapped microcavity photons were derived. The experiments to measure the density profiles for the normal and superfluid components were suggested.

**APPENDIX A: VALIDITY OF THE THOMAS–FERMI APPROXIMATION FOR THE PARAMETERS OF THE CALCULATIONS**

In this appendix, we discuss validity of the Thomas–Fermi approximation and the condition of negligibility of the derivative \( \frac{dL(r)}{dr} \) for the parameters of the calculations.

We assume that polariton–polariton interaction is so weak that the mean-field approximation is valid. LDA is applicable when the characteristic condensate inhomogeneity length, which is the characteristic size of the condensate \( r_0 \), is much larger than all parameters of the problem with the length dimensionality such as the healing length \( \xi \). When the mean-field approximation and the local density approximation are applicable, then the Thomas–Fermi approximation, which we are using, is valid.

Let us justify the validity of the Thomas–Fermi approximation that is used above. The condition of applicability of the Thomas–Fermi approximation implies neglecting all terms with derivatives in Eq. (5). Negligibility of these terms, except the term with \( \frac{dL(r)}{dr} \), can be achieved if the radius of the photon BEC spot \( r_0 \) is larger than the characteristic length of decrease of the condensate wave function \( \phi(r) \), which is the healing length \( \xi \). The healing length \( \xi \), which corresponds to the characteristic length of the changes of the condensate wave function in the Gross–Pitaevskii equation. The healing length \( \xi \) is defined as [1]

\[
\xi = \frac{\hbar}{\sqrt{2m\hbar^2\gamma}}, \quad (A1)
\]

where \( \bar{n} = N_{\text{BEC}}/\pi r_0^2 \) is the average 2D concentration of the photons.

In addition, negligibility of the term containing \( \frac{dL(r)}{dr} \) can be achieved at the additional condition when the change of the microcavity width \( \Delta L = L(0) - L(R) \) is sufficiently smaller than the transverse size of the microcavity \( R \). The Thomas–Fermi approximation is applicable if the size of the condensate \( r_0 \) is much larger than the healing length \( \xi \).

Using \( N_{\text{BEC}} = \pi \bar{n} r_0^2 \), the inequality

\[
\xi < r_0
\]

turns into the inequality

\[
N_{\text{BEC}} > \frac{\pi \hbar^2}{2m \hbar^2 \gamma}. \quad (A3)
\]

Substituting Eqs. (10) and (11) into Eq. (A3), and assuming \( r = 0 \) [which increases the r.h.s. of Eq. (A3)], one obtains the following estimate for \( N_{\text{BEC}} \) when the Thomas–Fermi approximation is applicable:

\[
N_{\text{BEC}} > \frac{L^2(0)}{\bar{A}}. \quad (A4)
\]

For the parameters used for our calculations from Eq. (A4), one obtains \( N_{\text{BEC}} > 10^5 \). Therefore, for the used value \( N_{\text{BEC}} = 1.7 \times 10^5 \) the Thomas–Fermi approximation can be applied.

Substituting at \( r = 0 \) Eqs. (10) and (11) into Eq. (A3), and assuming \( \bar{n} = N_{\text{BEC}}/(\pi r_0^2) \), one obtains

\[
\xi = \frac{r_0 L(0)}{\sqrt{\pi \bar{A} N_{\text{BEC}}}}, \quad (A5)
\]

which for the parameters used in our calculations results in \( \xi = 8.691 \times 10^{-6} \) m. Therefore, for the system under consideration the inequality \( \xi < r_0 \) holds.

The another method to check the validity of the Thomas–Fermi approximation is to substitute the condensate wave function in the form \( \psi(r) = \psi_0(r) + \delta(r) \) into Eq. (5), where \( \psi_0(r) \) is the condensate wave function in the Thomas–Fermi approximation, satisfying Eq. (7), and \( \delta(r) \) is the small perturbation to the condensate wave function, caused to the deviation from the Thomas–Fermi approximation.

Assuming that the derivatives of \( \delta(r) \) vanish, one obtains

\[
F(r) = -\frac{\hbar}{2\pi \sqrt{e}} \left( L(r) \frac{d^2}{dr^2} + \frac{dL(r)}{dr} \frac{d}{dr} + \frac{L(r)}{r} \frac{d}{dr} \right) \psi_0(r)
\]

\[
= -\frac{\hbar c D(r)}{2\pi \sqrt{e}}, \quad (A6)
\]

where \( \psi_0(r) = \sqrt{n(r)} \), \( n(r) \) is given by Eq. (12), and

\[
D(r) = D_1(r) + D_2(r) + D_3(r), \quad (A7)
\]

where

\[
D_1(r) = \frac{L(r)}{2\psi_0(r)} \left[ \frac{d^2n(r)}{dr^2} - \frac{1}{2n(r)} \left( \frac{dn(r)}{dr} \right)^2 \right], \quad (A8)
\]

where

\[
\frac{dn(r)}{dr} = -\frac{2\mu\gamma r}{A(V_0 + \frac{r^2}{2})}, \quad (A9)
\]

\[
\frac{d^2n(r)}{dr^2} = \frac{2\mu\gamma(\frac{3}{2}r^2 - V_0)}{A(V_0 + \frac{r^2}{2})^3}, \quad (A10)
\]

\[
D_3(r) = \frac{\mu L(r) \gamma^2 r^2}{A \psi_0(r) (V_0 + \frac{r^2}{2})^3}, \quad (A11)
\]
Substituting the expansion \( \varphi(r) = \varphi_0(r) + \delta(r) \) and Eq. (A6) into Eq. (5), applying Eq. (7) for \( \varphi_0(r) \), and keeping only linear terms with respect to \( \delta(r) \), one obtains \( \delta(r) \) in the following form:

\[
\delta(r) = - \frac{\pi \hbar}{\sqrt{2} r L(r) \mu} \left( F(r) - g(r) \mu r \right). 
\]  

(A13)

Substituting the parameters used for our calculations into Eq. (A13), one obtains \( |\delta(0)/\varphi_0(0)| = 0.071 \ll 1 \). Since it follows from Eq. (A13) that \( |\delta(0)/\varphi_0(0)| = 0.071 \ll 1 \), we conclude that the Thomas–Fermi approximation is valid for the parameters used for our calculations. At the distance from the center of the trap \( r = 7 \mu m \), this ratio becomes \( |\delta(r)/\varphi_0(r)| = 0.1 \). Let us mention that we present the ratio \( \delta/\varphi_0 \) in the center of the trap, since closer to the edges of the spot it becomes close to one due to the well-known fact that closer to the edges of the spot the Thomas–Fermi approximation is not valid, and the Gross–Pitaevskii equation has to be solved [25]. While in the framework of the Thomas–Fermi approximation the condensate density vanishes at the edges of the spot, the solution of the Gross–Pitaevskii equation demonstrates the asymptotic decrease of the condensate profile [25] due to the essential role of the spatial derivatives of the condensate wave function.

Our assumption about the slowly varying width of the cavity corresponds to the inequality \( \Delta L \ll R \), where \( \Delta L \) is the change of the width of the cavity, defined as

\[
\Delta L = L(0) - L(R) = \frac{\pi \hbar}{\sqrt{\varepsilon V_0}} - \frac{\pi \hbar}{\sqrt{\varepsilon (V_0 + \gamma R^2/2)}}. 
\]  

(A14)

In Eq. (A14) \( R \) is the radius of the microcavity, which is \( R = 0.5 \text{ mm} \) [4]. For the parameters used for our calculations, Eq. (A14) results in \( \Delta L = 1.204 \times 10^{-6} \text{ m} \).

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