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On the Number of Heteroclinic Curves of Diffeomorphisms with Surface Dynamics

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Abstract—Separators are fundamental plasma physics objects that play an important role in many astrophysical phenomena. Looking for separators and their number is one of the first steps in studying the topology of the magnetic field in the solar corona. In the language of dynamical systems, separators are noncompact heteroclinic curves. In this paper we give an exact lower estimation of the number of noncompact heteroclinic curves for a 3-diffeomorphism with the so-called "surface dynamics". Also, we prove that ambient manifolds for such diffeomorphisms are mapping tori.

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1. INTRODUCTION

It follows from the classical papers [1] and [17] that rough (structurally stable) flows on surfaces have no trajectories which connect two different saddle equilibria (heteroclinic trajectories). In the case where the ambient manifold has dimension three, the structurally stable flows can have such trajectories, and invariant manifolds of different saddle periodic points of structurally stable diffeomorphisms can intersect with curves (they can be either compact or noncompact) that are called heteroclinic curves. Heteroclinic trajectories and curves play a principal role in studying regular processes. For example, in a series of papers by E. Priest and coauthors (see [19, 20] for information), devoted to studying the topology of the magnetic field in the solar corona, considerable attention is paid to the problem of existence of separators. From dynamical systems point of view, separators are just heteroclinic trajectories and curves.

There are a few fundamental results obtained by Ch. Bonatti, V. Grines, V. Medvedev, E. Pecou, O. Pochinka, E. Zhuzhoma on the existence of heteroclinic curves for Morse – Smale 3-diffeomorphisms. In [4] the existence of heteroclinic curves was established for every Morse – Smale diffeomorphism given on a closed 3-manifold distinct from the 3-sphere $\mathbb{S}^3$ and the connected sum of a finite number of copies of $\mathbb{S}^2 \times \mathbb{S}^1$. In [7] the existence of noncompact heteroclinic curves was proved for every polar 3-diffeomorphism (a diffeomorphism with a unique sink and source) given on an irreducible 3-manifold (a manifold where each bi-collared 2-sphere bounds a 3-ball), which was effectively applied to find heteroclinic separators of magnetic fields in electrically conducting fluids. By [9], if a polar diffeomorphism is given on a lens $L_{p,q}$ and its nonwandering set contains exactly two saddle points with trivially embedded one-dimensional manifolds, then the wandering set contains at least p heteroclinic curves. In [6], a new sufficient condition for the existence of heteroclinic curves was found for gradient-like systems with surface dynamics on three-dimensional

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manifolds (exact definitions will be given below). In this paper, an exact lower estimation of the number of heteroclinic curves for such systems is given.

Let M^n be a closed n-manifold. Recall ([25]) that a diffeomorphism $f : M^n \to M^n$ satisfies *Axiom A* (f is *A-diffeomorphism*) if the following conditions hold: 1) the nonwandering set Ω_f is hyperbolic[1)]; 2) the periodic points are dense in Ω_f. According to Smale's spectral theorem [25], the nonwandering set of an A-diffeomorphism f can be represented as a finite union of pairwise disjoint closed invariant sets, called *basic sets*, each of which contains a dense trajectory. By [12, 21, 22], Axiom A and the strong transversality condition[2)] are necessary and sufficient conditions for the structural stability of f. Due to [26], Axiom A and the absence of cycles[3)] are necessary and sufficient conditions for Ω-stablity of f.

Everywhere below we will assume that f is an orientation preserving Ω-stable diffeomorphism given on an orientable 3-manifold M^3.

Definition 1. *We say that an Ω-stable diffeomorphism $f : M^3 \to M^3$ has surface dynamics (is an SD-diffeomorphism) if its nonwandering set Ω_f consists of two disjoint families Ω_+, Ω_- of basic sets such that the sets $\mathcal{A}_f = W^u_{\Omega_+}$ and $\mathcal{R}_f = W^s_{\Omega_-}$ are disjoint and each connected component of $\mathcal{A}_f$ and $\mathcal{R}_f$ is a locally flat orientable closed surface*[4)].

SD-diffeomorphisms appeared first in [8]. The most completed results concerning SD-diffeomorphisms with nonregular dynamics were obtained in [10] and [18]. It was proved there that every 3-dimensional structural stable diffeomorphism whose nonwandering set consists of two-dimensional basic sets has surface dynamics, moreover, it is a locally direct product of a hyperbolic automorphism of the 2-torus and a structurally stable diffeomorphism of the circle.

The theorem below describes the dynamics of SD-diffeomorphisms.

Theorem 1. *Let $f : M^3 \to M^3$ be an SD-diffeomorphism. Then there exist numbers $g_f \geqslant 0$ and $k_f \geqslant 1$ such that*

1. *$\mathcal{A}_f$ is an attractor and $\mathcal{R}_f$ is a repeller*[5)] *of the diffeomorphism f that consist of the same number $k_f \geqslant 1$ of the connected components homeomorphic to a closed orientable surface $\mathbb{S}_{g_f}$ of genus g_f.*

2. *The closure cl V of each connected component V of the set $M^3 \setminus (\mathcal{R}_f \cup \mathcal{A}_f)$ is homeomorphic to the direct product $\mathbb{S}_{g_f} \times [0, 1]$.*

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1) A closed f-invariant set $\Lambda \subset M^n$ is said to be *hyperbolic* if there exists a continuous Df-invariant decomposition of the tangent subbundle $T_\Lambda M^n$ into the direct sum $E^s_\Lambda \oplus E^u_\Lambda$ of the stable and the unstable subbundles such that $\|Df^k(v)\| \leqslant C\lambda^k\|v\|$, $\|Df^{-k}(w)\| \leqslant C\lambda^k\|w\|$, $\forall v \in E^s_\Lambda, \forall w \in E^u_\Lambda, \forall k \in \mathbb{N}$ for some fixed numbers $C > 0$ and $\lambda < 1$. The hyperbolicity condition implies existence of stable and unstable manifolds denoted by W^s_x and W^u_x for each point $x \in \Lambda$ which are defined as follows $W^s_x = \{y \in M^3 : d(f^k(x), f^k(y)) \to 0, k \to +\infty\}$, $W^u_x = \{y \in M^3 : d(f^k(x), f^k(y)) \to 0, k \to -\infty\}$, where d is the metric on Λ induced by the Riemannian metric on $T_\Lambda M^n$.

2) The strong transversality condition means that all intersections of the stable and the unstable manifolds of nonwandering points are transversal.

3) A *k-cycle* ($k \geqslant 1$) is a collection of mutually disjoint basic sets $\Lambda_0, \Lambda_1, \dots, \Lambda_k$ such that $\Lambda_0 \prec \Lambda_1 \prec \cdots \prec \Lambda_k \prec \Lambda_0$, where $\Lambda_i \prec \Lambda_j$ means that $W^s_{\Lambda_i} \cap W^u_{\Lambda_j} \neq \emptyset$.

4) Let $\mathbb{S}_g$ be an orientable surface (closed 2-dimensional manifold) of genus g and $e : \mathbb{S}_g \to M^3$ be a topological embedding. A surface $S_g = e(\mathbb{S}_g)$ is called *locally flat* if for every point $p \in S_g$ there exists a neighborhood $U_p \subset M^3$ and a homeomorphism $h_p : U_p \to \mathbb{R}^3$ such that the set $h_p(S_g \cap U_p)$ is a coordinate plane in $\mathbb{R}^3$. According to [3], an orientable locally flat surface is *bi-collared*, that is, there exists a topological embedding $h : \mathbb{S}_g \times [-1; 1] \to M^3$ such that $h(\mathbb{S}_g \times \{0\}) = S_g$.

5) Let us recall that a set A_f is called *attractor* of a diffeomorphism $f : M^3 \to M^3$ if it has a *trapping neighborhood*, that is, a closed neighborhood $U_{A_f} \subset M^3$ such that $f(U_{A_f}) \subset int\ U_{A_f}$ and $A_f = \bigcap\limits_{i\in\mathbb{N}} f^i(U_{A_f})$. A set R_f is called *a repeller* of the diffeomorphism f if it is an attractor for f^{-1}.

Below, the numbers k_f, g_f, defined in Theorem 1, will be associated with every SD-diffeomorphism f.

To describe a topology of the ambient manifold of SD-diffeomorphisms, we recall that a mapping torus $M^3_{g,\tau}$ is a factor space $\mathbb{S}_g \times [0,1]/\sim$, where $(z,1) \sim (\tau(z),0)$ for a homeomorphism $\tau : \mathbb{S}_g \to \mathbb{S}_g$ (*gluing map*) of the closed surface $\mathbb{S}_g$ of genus g. It is easy to prove (see Section 3.1) that the mapping tori $M^3_{g,\tau}$, $M^3_{g,\tau'}$ are homeomorphic if the homeomorphisms τ, τ' are isotopic. From this fact and the Dehn – Nielsen and Baer Theorems (see Proposition 2) it follows that the set of nonhomeomorphic mapping tori is not greater than a countable set. The fact that every homeomorphism of a compact surface is isotopic to a diffeomorphism (see, for example, [13]) allows us to further assume that the gluing map τ in the definition of the mapping torus is a diffeomorphism.

Theorem 2. *Let $f : M^3 \to M^3$ be a SD-diffeomorphism. Then there exists a diffeomorphism $\tau_f : \mathbb{S}_{g_f} \to \mathbb{S}_{g_f}$ such that M^3 is diffeomorphic to the mapping torus $M^3_{g_f,\tau_f}$.*

Below we will focus on gradient-like SD-diffeomorphisms. Let us recall that the diffeomorphism $f : M^n \to M^n$ of a connected closed smooth manifold M^n of dimension n is called a *Morse – Smale diffeomorphism* if its nonwandering set Ω_f is finite and consists of hyperbolic periodic points, and for different saddle periodic points $p, q \in \Omega_f$ the invariant manifolds W^s_p, W^u_q either are disjoint or intersect transversely. Let p, q be different saddle periodic points of a Morse – Smale diffeomorphism $f : M^n \to M^n$. If $dim(W^s_p \cap W^u_q) = 0$, then each point of the set $W^s_p \cap W^u_q$ is called a *heteroclinic point*. The diffeomorphism f is called *gradient-like* if the condition $W^s_p \cap W^u_q \neq \emptyset$ leads to the fact $dim\ W^u_p < dim\ W^u_q$. So if the wandering set of f does not contain heteroclinic points, then f is gradient-like.

S. Smale showed in [24] (Theorem A) that a gradient flow generated by a Morse function given on a manifold M^n can be arbitrarily closely approximated (in C^1 topology) with a Morse – Smale flow without closed trajectories, which proves the existence of a gradient-like diffeomorphism on any manifold (for example, the time-1 map of such a Morse – Smale flow).

Theorem 3. *For any integer $g \geqslant 0$ and a diffeomorphism $\tau : \mathbb{S}_g \to \mathbb{S}_g$ there is a gradient-like SD-diffeomorphism on $M^3_{g,\tau}$.*

Remark 1. The result of Theorem 3 contrasts with the result of [10], where it was proved that the manifold admitting structurally stable SD-diffeomorphisms with only two-dimensional basic sets is mapping tori $M^3_{1,\tau}$ such that the induced homomorphism $\tau_* : \pi_1(\mathbb{S}_1) \to \pi_1(\mathbb{S}_1)$ is either hyperbolic or defined by matrix $\pm Id$.

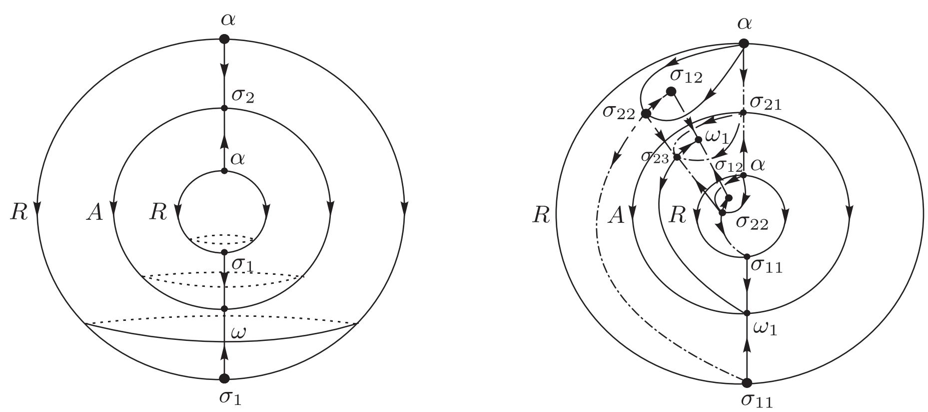
Let $f : M^3 \to M^3$ be a gradient-like diffeomorphism and let p, q be its different saddle periodic points such that $dim(W^s_p \cap W^u_q) = 1$. Then every connected component of the set $W^s_p \cap W^u_q$ is called a *heteroclinic curve.*

Theorem 4.

1. *Let $f : M^3_{g_f,\tau_f} \to M^3_{g_f,\tau_f}$ be a gradient-like diffeomorphism with surface dynamics. Then a number of noncompact heteroclinic curves is not less than $12 g_f k_f$.*

2. *The estimation is exact, namely, for all integers $k > 0, g \geqslant 0$ there is a gradient-like SD-diffeomorphism $f : \mathbb{S}_g \times \mathbb{S}^1 \to \mathbb{S}_g \times \mathbb{S}^1$ such that its wandering set contains exactly $12gk$ noncompact heteroclinic curves.*

Remark 2. Theorem 4 gives an exact lower estimate of the number of noncompact heteroclinic curves. However, for any integers $g \geqslant 0; k \geqslant 1$ there is a gradient-like SD-diffeomorphism $f : \mathbb{S}_g \times \mathbb{S}^1 \to \mathbb{S}_g \times \mathbb{S}^1$ such that the set $\mathcal{A}_f \cup \mathcal{R}_f$ consists of $2k$ surfaces of genus $g \geqslant 0$ and the wandering set contains an arbitrary number (greater than $12 g_f k_f$) of noncompact heteroclinic curves. Figure 1 shows an idea how to increase the number of noncompact heteroclinic curves in the case $g = 0$, $k = 1$. The figure shows phase portraits of diffeomorphisms on unfolding of $\mathbb{S}_0 \times \mathbb{S}^1$. The first of them (Fig. 1a) is a diffeomorphism without heteroclinic curves, the second (Fig. 1b) is a diffeomorphism with heteroclinic curves (they are marked by the dash-and-dot line). Both diffeomorphisms can be defined as a direct product of a gradient-like diffeomorphism on the 2-sphere $\mathbb{S}_0$ and a structurally stable diffeomorphism on the circle $\mathbb{S}^1$, whose nonwandering set consists of exactly one sink and one source. Thus, both diffeomorphisms have two invariant spheres A, R, which are an attractor and a repeller, respectively. In the first case the gradient-like diffeomorphism on the 2-sphere $\mathbb{S}_0$ has a nonwandering set consisting of exactly one sink and one source, and the nonwandering set of the resulting diffeomorphism consists of a source α, a sink ω and saddles σ_1, σ_2. In the second case the gradient-like diffeomorphism on the 2-sphere $\mathbb{S}_0$ has a nonwandering set consisting of four points: one source, one saddle and two sinks, so the nonwandering set of the resulting diffeomorphism consists of eight points: a source α, sinks ω_1, ω_2 and saddles $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \sigma_{23}$. Heteroclinic curves in this example are generated by the intersections $W^u_{\sigma_{22}} \cap W^s_{\sigma_{12}}$, $W^u_{\sigma_{22}} \cap W^s_{\sigma_{11}}$, $W^s_{\sigma_{23}} \cap W^u_{\sigma_{21}}$, $W^u_{\sigma_{22}} \cap W^s_{\sigma_{23}}$.



(a) (b)

Fig. 1. Examples of gradient-like SD-diffeomorphisms on $\mathbb{S}_0 \times \mathbb{S}^1$: a) without heteroclinic curves; b) with heteroclinic curves.

2. DYNAMICS OF SD-DIFFEOMORPHISMS

This section is devoted to a proof of Theorem 1. Let $f \in SD(M^3)$. Without loss of generality we can assume that every connected component of the sets V_f, $\mathcal{A}_f$ and $\mathcal{R}_f$ is f-invariant (in the opposite case one can consider an appropriate degree of the diffeomorphism f that does not change the conclusion). Let us provide the proof by steps.

1. Let us prove that $\mathcal{A}_f$ is an attractor and $\mathcal{R}_f$ is a repeller of the diffeomorphism f.

According to S. Smale [25], the nonwandering set Ω_f is uniquely represented as a disjoint union $\Omega_f = \Lambda_1 \cup \dots \cup \Lambda_m$ of compact invariant and topologically transitive sets, which are called basic sets, and $M^3 = \bigcup\limits_{i=1}^{m} W^s_{\Lambda_i} = \bigcup\limits_{i=1}^{m} W^u_{\Lambda_i}$. Moreover, it is possible to define a partial order among the basic sets as follows: $\Lambda_i \prec \Lambda_j$ for different basic sets Λ_i, Λ_j if and only if $W^s_{\Lambda_i} \cap W^u_{\Lambda_j} \neq \emptyset$. Since f has no cycles, this partial order can be extended to a complete order relation (we will assume that $\Lambda_1 \prec \dots \prec \Lambda_m$). In this case the manifold M^3 admits a *filtration* (see, for example, [23]), that is, a sequence $M_1, \dots, M_{n-1}$ of n-dimensional submanifolds with a smooth boundary such that $M^3 = M_n \supset M_{n-1} \supset \dots \supset M_1 \supset M_0 = \emptyset$ such that for every $i \in \{1, \dots, n\}$ the following conditions hold:

1. $f(M_i) \subset int\ M_i$;
2. $\Lambda_i \subset int\ (M_i \setminus M_{i-1})$;

3. $\Lambda_i = \bigcap\limits_{l\in\mathbb{Z}} f^l(M_i \setminus M_{i-1})$;

4. $\bigcap\limits_{l\geqslant 0} f^l(M_i) = \bigcup\limits_{j\leqslant i} W^u_{\Lambda_j} = \bigcup\limits_{j\leqslant i} cl\ W^u_{\Lambda_j}$.

Let $\Lambda_a \subset \Omega_+$ and $\Lambda_r \subset \Omega_-$ be arbitrary basic sets connected by the relation $\prec$. It follows from the definition of the surface dynamics that $W^u_{\Lambda_a} \subset \mathcal{A}_f$ and $W^s_{\Lambda_r} \subset \mathcal{R}_f$, hence $\Lambda_a \prec \Lambda_r$. Thus, without loss of generality we can assume that $\Omega_+ = \bigcup\limits_{j=1}^{i_*} \Lambda_j$ for some $1 < i_* < n$. Then M_{i_*} is a trapping neighborhood for $\mathcal{A}_f$ and $\mathcal{A}_f$ is an attractor of the diffeomorphism f. Similar arguments for f^{-1} prove that $\mathcal{R}_f$ is a repeller.

2. Let us prove that the boundary of every connected component V of the set $M^3 \setminus (\mathcal{R}_f \cup \mathcal{A}_f)$ consists of exactly one connected component of the set $\mathcal{A}_f$ and one connected component of the set $\mathcal{R}_f$.

Let A be a connected component of the set $\mathcal{A}_f$ and let U_A be a trapping neighborhood of A, which is a closed set such that $f(U_A) \subset int\ U_A$ and $\bigcap\limits_{n\in\mathbb{Z}} f^n(U_A) = A$. Let us denote by $Q_A = \bigcup\limits_{n\in\mathbb{Z}} f^n(U_A)$ the basin of the attractor A. By construction, Q_A is a connected open set and the set $M^3 \setminus \mathcal{R}_f$ is a disjoint union of the Q_A, $A \in \mathcal{A}_f$.

Since A is bi-collared, the set $U_A \setminus A$ consists of two connected components U_+, U_-, so $Q_A \setminus A$ also consists of two connected components. Then the boundary of every connected component $V \subset V_f$ contains exactly one connected component of the set $\mathcal{A}_f$. Applying similar arguments to f^{-1}, one finds that the boundary of every connected component of $V \subset V_f$ contains exactly one connected component of the set $\mathcal{R}_f$.

3. Let us prove that there is an integer $g_f \geqslant 0$ such that $cl\ V$ is homeomorphic to the direct product $\mathbb{S}_{g_f} \times [0, 1]$.

To prove the item, we use the following important statements, which were proved in [9] (see Lemma 3.1 and Theorems 3.1 and 3.3)[6].

Let P^3 be a 3-manifold whose boundary consists of two disjoint connected components B_1, B_2. The boundary components B_1 and B_2 are said to be separated by a surface $S \subset int\ P^3$ if B_1, B_2 belongs to different connected components of the set $P^3 \setminus S$.

Statement 1. *The boundary components B_1 and B_2 are not separated by a connected orientable surface $S \subset int\ P^3$ if and only if S bounds a domain $D \subset int\ P^3$.*

Statement 2. *Let $P^3_{g'}$, $g' > 0$ be a manifold homeomorphic to the direct product $S_{g'} \times [0, 1]$ and let $S_g \subset P^3_{g'}$ be a locally flat embedded orientable surface of genus $g < g'$. Then the connected components of $\partial P^3_{g'}$ are not separated by S_g in $P^3_{g'}$.*

Statement 3. *Let P^3_g be a manifold homeomorphic to the direct product $S_g \times [0, 1]$ and let $S_g \subset P^3_g$ be a locally flat embedded orientable surface of genus g that does not bound any domain in P^3_g. Then S_g divides P^3_g into two connected components homeomorphic to $S_g \times [0, 1]$.*

Let V be a connected component of the set V_f and $\partial V = A \cup R$. Denote by g_a, g_r the genus of the surfaces A, R. Since A, R are bi-collared in M^3, there are topological embeddings $h_a : S_{g_a} \times [-1, 1] \to M^3$, $h_r : S_{g_r} \times [-1, 1] \to M^3$ such that $h_a(S_{g_a} \times \{0\}) = A$, $h_r(S_{g_r} \times \{0\}) = R$ and $h_a(S_{g_a} \times [-1, 1]) \cap h_r(S_{g_r} \times [-1, 1]) = \emptyset$. Let

$$N_a = h_a(S_{g_a} \times [-1, 1]) \cap cl\ V,\ N_r = h_r(S_{g_r} \times [-1, 1]) \cap cl\ V.$$

Let us assume for definiteness that the sets $B_a = h_a(S_{g_a} \times \{1\})$, $B_r = h_r(S_{g_r} \times \{1\})$ belong to V.

[6] Notice that in [9] all embeddings are supposed to be smooth, but the proofs of the statements below are based only on the locally flatness of the embedding.

Let us show that $g_a = g_r$. Notice that there exists a natural number n_* such that $f^{n^*}(B_r)$ belongs to $int\ N_a$. Indeed, every point $p \in B_r$ belongs to the basin Q_A, hence there exist a closed neighborhood $U(p) \subset B_r$ of the point p and a natural number $n(p)$ such that $f^n(U(p)) \subset int\ N_a$ for each $n > n(p)$. Since B_r is compact, there exists a finite subcovering of the covering $\{U(p)\}_{p \in B_r}$. Thus there exists a natural number n^* such that $f^n(B_r) \subset int\ N_a$ for any $n \geqslant n^*$.

Let $B_r^* = f^{n^*}(B_r)$. Show that A and B_a are separated by B_r^* in N_a. Assume the contrary. Then, according to Statement 1, B_r^* bounds a domain $D \subset int\ N_a$. Applying the above arguments, prove that there is a number $m^* > 0$ such that $f^{-m^*}(D \cup B_r^*) \subset int\ N_r$. Then $f^{n^*-m^*}(B_r) = f^{-m^*}(B_r^*)$ bounds the disk $f^{-m^*}(D)$ in N_r and, according to Statement 1, does not separate B_r and R in N_r. On the other hand, the surfaces $f^{n^*-m^*}(B_r)$ and R bound the set $f^{n^*-m^*}(N_r)$, which is homeomorphic to the direct product $\mathbb{S}_{g_r} \times [0;1]$. Then B_r and R are separated by $f^{n^*-m^*}(B_r)$ in N_r. This contradiction proves that A and B_a are separated by B_r^* in N_a. Then, according to Statement 2, $g_a \geqslant g_r$. Applying the above arguments to B_a, prove that $g_r \geqslant g_r$, so $g_a = g_r = g$.

Since B_r^* does not bound a domain in N_a, according to Statement 3, it divides N_a in two parts homeomorphic to $\mathbb{S}_g \times [0,1]$. Let P_1 be the part that is bounded by A and B_r^*. Then $cl\ V = P_1 \cup f^{n_*}(N_r)$ and it is homeomorphic to $\mathbb{S}_g \times [0,1]$.

As the component V was chosen arbitrary and the manifold M^3 is connected, all connected components of the set $\mathcal{A} \cup \mathcal{R}$ have the same genus g_f.

4. The fact that the sets $\mathcal{A}_f$ and $\mathcal{R}_f$ consist of the same number $k_f \geqslant 1$ of the connected components each of which is a surface of genus g_f immediately follows from items 2 and 3.

3. CLASSIFICATION OF AMBIENT MANIFOLDS OF DIFFEOMORPHISMS WITH SURFACE DYNAMICS

Let K be a connected component of $\mathcal{A}_f \cup \mathcal{R}_f$. It follows from item 3 of Theorem 1 that there exists a continuous map $H : \mathbb{S}_{g_f} \times [0,1] \to M^3$ such that the restrictions $H|_{\mathbb{S}_{g_f} \times (0,1)} : \mathbb{S}_{g_f} \times (0,1) \to M^3 \setminus K$, $H_0 = H|_{\mathbb{S}_{g_f} \times \{0\}} : \mathbb{S}_{g_f} \times \{0\} \to K$, $H_1 = H|_{\mathbb{S}_{g_f} \times \{1\}} : \mathbb{S}_{g_f} \times \{1\} \to K$ are homeomorphisms. Let $H_0(z,0) = (h_0(z), 0)$, $H_1(z,1) = (h_1(z), 1)$ and $\tau_f = h_0^{-1} h_1$. Then by construction the manifold $M^3_{g_f,\tau_f}$ is homeomorphic to the manifold M^3 by a homeomorphism $\check{H}$ which maps the equivalence class $[(z,t)]$ of $(z,t) \in \mathbb{S}_{g_f} \times [0,1]$ to the point $H(z,t) \in M^3$.

Thus M^3 is homeomorphic to a mapping torus.

3.1. On Mapping Tori

Proposition 1. *Mapping tori $M^3_{g,\tau}$, $M^3_{g,\tau'}$ are homeomorphic if the homeomorphisms τ, τ' are isotopic.*

Proof. Let $H : \mathbb{S}_g \times [0;1] \to \mathbb{S}_g$ be an isotopy connecting a map $\tau'\tau^{-1} = H(z,0)$ with the identity map $H(z,1)$. Let us denote a homeomorphism $h : \mathbb{S}_g \times [0;1] \to \mathbb{S}_g \times [0;1]$ by $h(z,t) = (H(z,t), t)$. Then a map $\hat{h} : M^3_{g,\tau} \to M^3_{g,\tau'}$ that sends an equivalence class $[(z,t)]$ to the equivalence class $[h(z,t)]$ is a homeomorphism. □

There are exactly two isotopy classes of homeomorphisms of the sphere $\mathbb{S}_0$ that consist of homeomorphisms preserving and reversing an orientation, respectively. For $g \geqslant 1$ the criterion of the existence of an isotopy between two homeomorphisms τ, τ' of the surface $\mathbb{S}_g$ follows from the Dehn–Nielsen and Baer theorems ([2, 14], see also [27], Theorem 5.15.3) and is connected with the induced homomorphisms of the fundamental groups.

More precisely, let $\tau_* : \pi_1(\mathbb{S}_g, x_0) \to \pi_1(\mathbb{S}_g, \tau(x_0))$ be an isomorphism induced by homeomorphism τ, and let $Q_\xi, Q_\eta : \pi_1(\mathbb{S}_g, \tau(x_0)) \to \pi_1(\mathbb{S}_g, x_0)$ be isomorphisms induced by paths ξ, η, respectively. Then there is an inner automorphism[7] $\varphi(x) = \xi\eta^{-1}x\eta\xi^{-1}$ such that isomorphisms $Q_\xi h_*$,

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[7] An isomorphism $\varphi_\gamma : G \to G$ of a group G is called *inner* if there exists an element $\gamma \in G$ such that $\varphi_\gamma(x) = \gamma^{-1}x\gamma$ for any $x \in G$.

$Q_\eta h_*:\pi_1(\mathbb{S}_g,x_0)\to\pi_1(\mathbb{S}_g,x_0)$ satisfy the following relation: $Q_\xi\tau_*=\varphi Q_\eta\tau_*$. Thus τ_* represents a well-defined element of the quotient group $Out(\pi_1(\mathbb{S}_g,x_0))=Aut(\pi_1(\mathbb{S}_g,x_0)/Inn(\pi_1(\mathbb{S}_g,x_0))$, where $Aut(\pi_1(\mathbb{S}_g,x_0)$, $Inn(\pi_1(\mathbb{S}_g,x_0))$ are a group of all automorphisms and a group of inner automorphisms of the group $\pi_1(\mathbb{S}_g,x_0)$, respectively. This element will be denoted by $[\tau_*]$.

Proposition 2. *Homeomorphisms $\tau,\tau':\mathbb{S}_g\to\mathbb{S}_g$, $g\geqslant 1$, are isotopic if and only if $[\tau'_*]=[\tau_*]$.*

In the case $g=1$ the group $\pi_1(\mathbb{S}_1)$ is isomorphic to $\mathbb{Z}\times\mathbb{Z}$ and the automorphism $\tau_*:\pi_1(\mathbb{S}_1)\to\pi_1(\mathbb{S}_1)$ is determined by an integer unimodular matrix $J_{\tau_*}\in GL_2(\mathbb{Z})$. The following criteria for the mapping tori $M^3_{1,\tau}$, $M^3_{1,\tau'}$ to be homeomorphic follows from [11].

Proposition 3. *Manifolds $M^3_{1,\tau}$, $M^3_{1,\tau'}$ are diffeomorphic iff there exists a matrix $A\in GL_2(\mathbb{Z})$ such that either $J_{\tau'_*}=AJ_{\tau'_*}A^{-1}$ or $J_{\tau'_*}=AJ^{-1}_{\tau'_*}A^{-1}$.*

3.2. Construction of a Diffeomorphism with Surface Dynamics on a Mapping Torus

In this section we prove the second item of Theorem 4 and Theorem 3. First for all integers $k>0, g\geqslant 0$ we construct a gradient-like SD-diffeomorphism $f:\mathbb{S}_g\times\mathbb{S}^1\to\mathbb{S}_g\times\mathbb{S}^1$ such that its wandering set contains exactly $12gk$ noncompact heteroclinic curves.

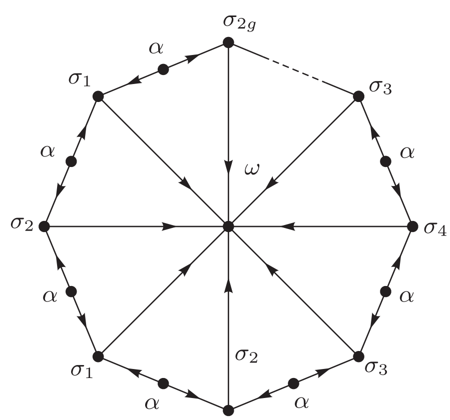

Fig. 2. Morse – Smale diffeomorphism on a surface of genus g.

Let $\psi:[0;1]\to[0,1]$ be the time-1 map of the flow $\dot r=\sin 2\pi kr$, and let $\varphi^t_g:\mathbb{S}_g\to\mathbb{S}_g$ be a gradient-like flow whose nonwandering set consists of exactly one sink ω, one source α and $2g$ saddle equilibria $\sigma_1,\ldots,\sigma_{2g}$. Figure 2 shows an unfolding of the surface $\mathbb{S}_g$ as a $2g$-gon and a phase portrait of the flow φ^t_g on it. Let $\Gamma^u=\bigcup\limits_{i=1}^{2g}W^u_{\sigma_i}$, $\Gamma^s=\bigcup\limits_{i=1}^{2g}W^s_{\sigma_i}$. Denote by f_0 the time-1 map of the flow φ^t_g and define a diffeomorphism $F:\mathbb{S}_g\times[0;1]\to\mathbb{S}_g\times[0;1]$ by the formula $F(z,r)=(f_0(z),\psi(r))$. To get $\mathbb{S}_g\times\mathbb{S}^1$, take an identical gluing map. Then the diffeomorphism F induces a gradient-like diffeomorphism $f:\mathbb{S}_g\times\mathbb{S}^1\to\mathbb{S}_g\times\mathbb{S}^1$ that maps an equivalence class $[(z,r)]$ of point $(z,r)\in\mathbb{S}_g\times[0;1]$ to equivalence class $[F(z,r)]$. To prove Theorem 3, for arbitrary integers $g\geqslant 0; k\geqslant 1$ and an orientation preserving diffeomorphism $\tau:\mathbb{S}_g\to\mathbb{S}_g$, we construct a gradient-like SD-diffeomorphism on $M^3_{g,\tau}\to M^3_{g,\tau}$ whose nonwandering set belongs to exactly $2k$ disjoint closed surfaces, each has genus g. As a small perturbation of τ does not chain the isotopy class, without loss of generality we assume that

$(*)$ Γ^u is transversal to $\tau(\Gamma^s)$;

$(**)$ $\tau(\alpha)\notin(\Gamma^u\cup\omega)$ and $\omega\notin\tau(\Gamma^s\cup\alpha)$.

Let $f_1 = \tau^{-1} f_0 \tau$ and denote by $\varphi_g^{[t]}$ a time-t map along the trajectories of φ_g^t.

Choose $r_0 \in (1 - \frac{1}{2k}, 1)$, put $r_1 = \psi^{-1}(r_0)$, $r_2 = \psi^{-1}(r_1)$ $(r_0 < r_1 < r_2)$ and define a diffeomorphism $F : \mathbb{S}_g \times [0; 1] \to \mathbb{S}_g \times [0; 1]$ by the formula

$$F(z, r) = \begin{cases} (f_0(z), \psi(r)), r \in [0; r_0]; \\ (\varphi_g^{[\frac{r_1 - r}{r_1 - r_0}]}(z), \psi(r)), r \in [r_0; r_1]; \\ (\tau^{-1} \varphi_g^{[\frac{r - r_1}{r_2 - r_1}]} \tau(z), \psi(r)), r \in [r_1; r_2]; \\ (f_1(z), \psi(r)), r \in [r_2; 1]. \end{cases}$$

By construction, the nonwandering set of diffeomorphism F is finite, hyperbolic and belongs to surfaces $\mathbb{S}_g \times \{\frac{i}{2k}\}$, $i \in \{0, \ldots, 2k\}$. The diffeomorphism F can be projected as a SD-diffeomorphism $\widetilde{F}$ on $M^3_{g,\tau}$. To show that diffeomorphism $\widetilde{F}$ is gradient-like, it is enough to show that two-dimensional manifolds of saddle points of F have a transversal intersection and one-dimensional saddle separatrices do not intersect any other saddle separatrices in $\mathbb{S}_g \times (1 - \frac{1}{2k}, 1)$.

For this purpose notice that a region $D = \mathbb{S}_g \times [r_1; r_2]$ is a fundamental domain of the restriction $F|_{\mathbb{S}_g \times (1 - \frac{1}{2k}, 1)}$. It follows from the construction of the diffeomorphism F that the two-dimensional stable separatrices intersect D along $\Gamma^s \times [r_1; r_2]$, two-dimensional unstable separatrices intersect D along $\tau^{-1}(\Gamma^u) \times [r_1; r_2]$, one-dimensional stable separatrices intersect D along $\alpha \times [r_1; r_2]$ and one-dimensional unstable separatrices intersect D along $\tau^{-1}(\omega) \times [r_1; r_2]$. Due to (*) two-dimensional manifolds of saddle points of F have a transversal intersection in D and hence in $\mathbb{S}_g \times (1 - \frac{1}{2k}, 1)$. Due to (**) one-dimensional saddle separatrices do not intersect any other saddle separatrices in D and hence in $\mathbb{S}_g \times \left(1 - \frac{1}{2k}, 1\right)$.

4. ON HETEROCLINIC CURVES OF GRADIENT-LIKE SD-DIFFEOMORPHISMS

Theorem 4 directly follows from Lemmas 1–3 and Corollaries 1–3 below. To prove these lemmas, we use the following two important statements. The first is proved in [25] (see Theorem 2.3), the second is a strong form of the λ-lemma proved in [16, Remarks, p. 85].

Statement 4. *If $f : M^n \to M^n$ is a Morse – Smale diffeomorphism, then for every point $p \in \Omega_f$ and a connected component l^u of the set $W^u_p \setminus p$ the equality*

$$cl\ l^u_p \setminus (l^u_p \cup p) = \bigcup_{q \in \Omega_f : W^s_q \cap l^u_p \neq \emptyset} W^u_q$$

holds.

Statement 5 (λ-lemma). *Let $f : M^n \to M^n$ be a diffeomorphism of an n-manifold, and let p be a fixed point of f, $\dim\ W^u_p = m$, $0 < m < n$. Let B^s be a compact subset of W^s_p (containing p or not) and let $F : B^s \to C^1(\mathbb{D}^m, X)$ be a continuous family of embedded closed m-disks of class C^1 transverse to W^s_p; set $F(x) := D^u_x$. Let $D^u \subset W^u_p$ be a compact m-disk and let $V \subset X$ be a compact n-ball such that D^u is a connected component of $W^u_p \cap V$. Then, as k goes to $+\infty$, the sequence $f^k(D^u_x) \cap V$ converges to D^u in the C^1 topology uniformly for $x \in B^s$.*

Let $f : M^3 \to M^3$ be a gradient-like SD-diffeomorphism, $g_f \geqslant 1$. Without loss of generality we can assume that the set Ω_f consists only of fixed points (in the opposite case one can consider an appropriate degree of the diffeomorphism f that does not change the number of heteroclinic curves). For a connected component A of the attractor $\mathcal{A}_f$ let

$$\Omega_A = \Omega_f \cap A \ \ and \ \ \Omega^i_A = \Omega^i_f \cap A, \quad i \in \{0, 1, 2\}.$$

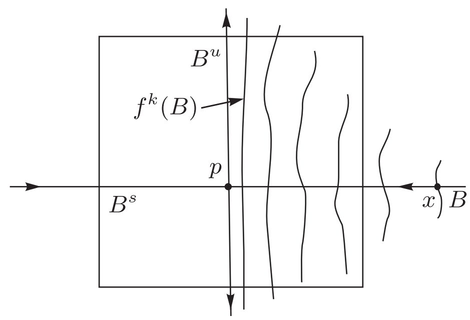

Fig. 3. Illustration to the λ-lemma.

It follows from the definition of the surface A that

$$A = \bigcup_{p \in \Omega_A} W^u_p.$$

Since the diffeomorphism f is gradient-like and A is a closed surface, for any point $q \in \Omega^1_A$ the set $(cl\ W^u_q \setminus W^u_q)$ belongs to Ω^0_A. Therefore the set $cl\ W^u_{\Omega^1_A}$ consists of a finite number of compact curves. Moreover, each connected component of the set $A \setminus cl\ W^u_{\Omega^1_A}$ is W^u_p for some point $p \in \Omega^2_A$. Thus the decomposition

$$A = \Omega^0_A \cup W^u_{\Omega^1_A} \cup W^u_{\Omega^2_A}$$

is a cellular decomposition of the surface A with genus g_f. Then, according to the Euler formula,

$$|\Omega^2_A| - |\Omega^1_A| + |\Omega^0_A| = 2 - 2g_f,$$

hence

$$|\Omega^1_A| \geqslant 2g_f. \tag{4.1}$$

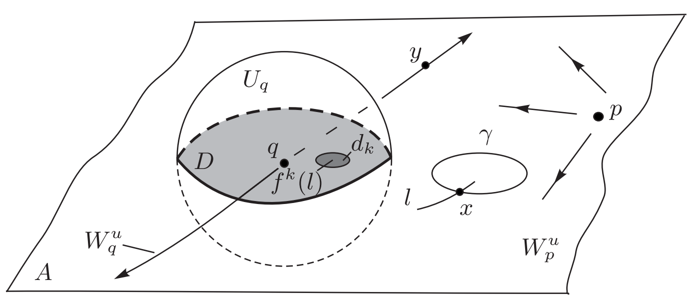

Fig. 4. Illustration to a proof of the absence of compact heteroclinic curves.

Lemma 1. *The surface A does not contain compact heteroclinic curves.*

Proof. Suppose the contrary: there are saddle points $p \in \Omega^2_A, q \in \Omega^1_f$ such that the intersection $W^u_p \cap W^s_q$ contains a compact connected component γ. Then, by Statement 4, $W^u_q \subset cl\ W^u_p$ and hence $q \in \Omega^1_A$. Since W^u_p is homeomorphic to $\mathbb{R}^2$, the curve γ bounds a unique disk d in W^u_p. As the surface A is bi-collared in M^3, there exists a neighborhood U_q of the point q in M^3 such that the intersection $D = U_q \cap A$ is homeomorphic to a 2-disk. Since γ is a subset of W^s_q, there exists a number $n_0 \in \mathbb{N}$ such that $f^n(\gamma) \subset U_q$ for all $n > n_0$. Thus $f^n(\gamma)$ belongs to D, and hence $f^n(\gamma)$ bounds a unique disk d_n, therefore $d_n = f^n(d)$.

Let us denote by B_q a compact arc in W^u_q such that $q \in int\ B_q$ and $B_q \setminus U_q \neq \emptyset$, and by V_q a neighborhood of the arc B_q, similar to the neighborhood V in the statement of the λ-lemma. Let us choose a compact arc $l \subset W^u_p$ which transversally intersects the arc γ (see Fig. 4) at a unique point x and such that one of the connected components of the set $l \setminus x$ belongs to the disk d. By construction, the arc l is transversal to the manifold W^s_q. It follows from the λ-lemma that for any $\varepsilon > 0$ there exists a number $k_0 \in \mathbb{N}$ such that for all $k > k_0$ a connected component l_k of the set $f^k(l) \cap V_q$, containing the point $x_k = f^k(x)$, is ε-close to the arc B_q.

Let us choose a number $k > max\,\{k_0, n_0\}$. The set $B_q \setminus q$ $(l_k \setminus x_k)$ consists of two connected components B^+_q, B^-_q (l^+_k, l^-_k), then the ε-closeness of the arcs l_k and B_q leads to the fact that for any point $y \in B^\delta_q \setminus U_q$, $\delta \in \{+,-\}$ there exists a neighborhood $u_y \subset V_q$ which has a nonempty intersection with l^δ_k. This contradicts the fact that one of the components l^+, l^- belongs to the disk $d_k \subset U_q$. $\square$

Lemma 2. *For each point $q \in \Omega^1_A$ the set $(W^s_q \setminus q) \cap A$ consists of exactly two noncompact heteroclinic curves.*

Proof. Let $q \in \Omega^1_A$. Since A is a surface, it is possible to choose a neighborhood V_q of the point q in A such that the set $V_q \setminus W^u_q$ is a union of two connected components. Then there are points $p_1, p_2 \in \Omega^2_A$ (it is possible that $p_1 = p_2$) such that $W^u_q \subset cl\ W^u_{p_i}$ for all $i \in \{1, 2\}$. According to Statement 4, the intersection $W^u_{p_i} \cap W^s_q$, $i \in \{1, 2\}$ is not empty, and hence the intersection $W^s_q \cap A$ contains heteroclinic curves. Denote by Γ_q the union of these curves. According to Lemma 1, the set Γ_p does not contain compact curves. Then, due to a result in [5], Γ_q consists of a finite number of connected components.

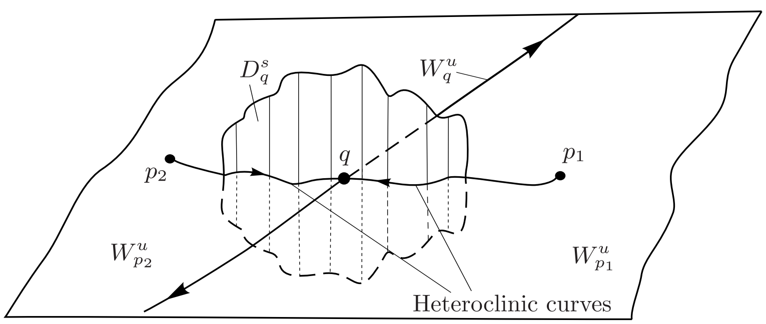

Fig. 5. Heteroclinic curves in W^s_q.

As the surface A is a bi-collared in M^3, there exists a neighborhood U_q of the point q in M^3 such that the intersection $D = U_q \cap A$ is homeomorphic to a 2-disk. Let us choose a disk $D^s_q \subset (W^s_q \cap U_q)$, $q \in int\ D^s_q$, such that each curve from Γ_q intersects the boundary of the disk D^s_q at a unique point (see Fig. 5). As the intersection $D^s_q \cap W^u_{p_i}$ is transversal, the set $D^s_q \setminus (\Gamma_q \cup q)$ intersects both connected components of the set $U_q \setminus D$. Thus the set Γ_q contains at least two noncompact heteroclinic curves.

Let us show that Γ_q consists of exactly two curves. Suppose the contrary: the number k of curves in Γ_q is greater than two. Without loss of generality we can assume that every curve in Γ_q is f-invariant (in the opposite case one can consider an appropriate degree of the diffeomorphism f that does not change the number of heteroclinic curves). Moreover, we assume that every curve from Γ_q intersects the boundary of the disk V_q at a unique point (in the opposite case we can choose on V_q a 2-disk with required properties). Then the set $V_q \setminus (\Gamma_q \cup q)$ contains a connected component avoiding W^u_q. Suppose that this component belongs to $W^u_{p_1}$ and denote it by d_1. Let $\gamma_1, \gamma_2 \subset \Gamma_q$ be heteroclinic curves which bound d_1 and let $D_1 \subset W^u_{p_1}$ be a 2-disk which is bounded by $\gamma_1 \cup p_1 \cup \gamma_2$ and contains d_1 (see Fig. 6).

Let us choose a curve $l \subset W^u_{p_1}$ which transversely intersects the arc γ_1 (see Fig. 6) at a unique point x and such that one of the connected components of the set $l \setminus x$ belongs to the disk D_1.

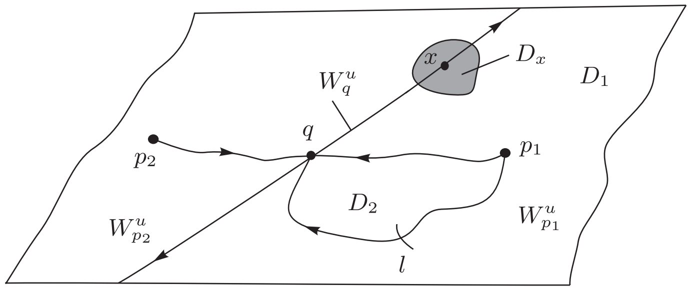

Fig. 6. Heteroclinic curves in $W_{p_1}^u \cup W_{p_2}^u$

Applying arguments similar to those used in the proof of Lemma 1, we get a contradiction with the λ-lemma. □

Applying inequality (1) together with Lemma 2, we get the following result.

Corollary 1. *The number of noncompact heteroclinic curves on the surface A is not less than $4g_f$.*

Applying Lemma 2 to diffeomorphism f^{-1} brings an estimation similar to Corollary 1 for every connected component of the attractor $\mathcal{R}_f$. Bearing in mind that the set $\mathcal{A}_f \cup \mathcal{R}_f$ contains $2k_f$ connected components, we get the following result.

Corollary 2. *The number of noncompact heteroclinic curves in the set $\mathcal{A}_f \cup \mathcal{R}_f$ is not less than $8g_f k_f$.*

Let V be a connected component of the set $M^3 \setminus (\mathcal{A}_f \cup \mathcal{R}_f)$, $A \subset \mathcal{A}_f$ and let $R \subset \mathcal{R}_f$ be connected components of the set $cl\ V \setminus V$.

Lemma 3. *The number of noncompact heteroclinic curves in the set V is not less than $2g_f$.*

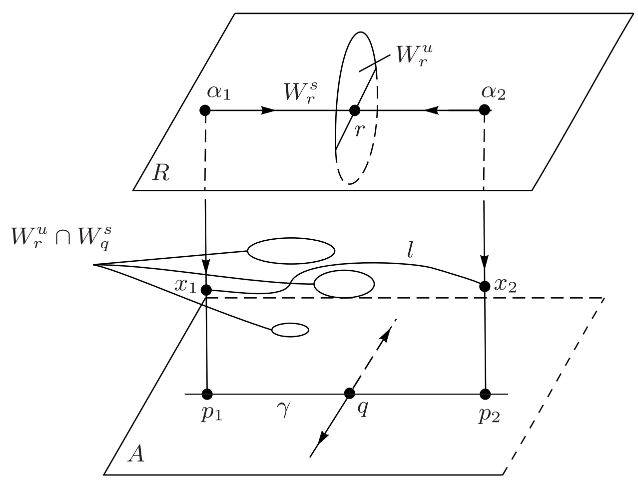

Fig. 7. Illustration to Lemma 3

Proof. It follows from Lemma 2 that it is possible to define a cellular decomposition of the surface A which is dual to the decomposition by the unstable manifolds of the points from Ω_A and such that the union of the heteroclinic curves and the saddle points from the set Ω_A^1 form one-dimensional cells, and the saddle points from the set Ω_A^2 form null-dimensional cells. Hence, on the surface A there exists a connected 1-dimensional complex containing $2g_f$ closed curves with a unique common point z_0, which are nonhomotopic to each other and to zero, and consisting of closures of traces of stable manifolds of points from Ω^1_A. Let γ be an arbitrary curve with such a property. Put $\Gamma = (\bigcup\limits_{\sigma\in\gamma} W^s_\sigma) \cap V$.

According to Theorem 1 there is an integer $g_f \geqslant 1$ such that the set $cl\ V$ is homeomorphic to the direct product $\mathbb{S}_{g_f} \times [0,1]$. Then the set $cl\ V$ can be represented as an orbit space of the set $\mathbb{U} \times [0;1]$, where $\mathbb{U}$ is the universal cover for $\mathbb{S}_{g_f}$, with respect to a motion group G (for $g_f \geqslant 2$, non-Euclidean) that acts freely on the set $\mathbb{U} \times [0;1]$ and is isomorphic to the fundamental group of the surface $\mathbb{S}_{g_f}$. Let $\Theta : \mathbb{U} \times [0;1] \to cl\ V$ be a natural projection such that $\Theta(\mathbb{U} \times \{0\}) = A, \Theta(\mathbb{U} \times \{1\}) = R$ and let $F : \mathbb{U} \times [0;1] \to \mathbb{U} \times [0;1]$ be a lift of $f|_{cl\ V}$ with respect to the cover Θ.

For a set $X \in cl\ V$ let $\Theta^{-1}(X)$ be the complete preimage of X. Let $p \in \Omega_f \cap cl\ V$ and let $\tilde{p}$ be a point in $\Theta^{-1}(p)$. Denote by $W^s_{\tilde{p}}$ $(W^u_{\tilde{p}})$ a connected component of $\Theta^{-1}(W^s_p \cap cl\, V)$ $(\Theta^{-1}(W^u_p \cap cl\, V))$ passing through the point $\tilde{p}$. We will say that $\tilde{p}$ is a sink, source or saddle point for F if p is a such point for f.

Let us show that a set $cl\ \Gamma$ contains at least one one-dimensional manifold $W^u_r \subset R$ of some saddle point $r \in R$.

Since the curve γ is nonhomologous to zero, for any two preimages $\tilde{p}_i, \tilde{p}_j$ of a point $p \in \Omega^2_A \cap \gamma$ the sets $cl\ W^s_{\tilde{p}_i}, cl\ W^s_{\tilde{p}_j}$ contain different source points $\tilde{\alpha}_i$, $\tilde{\alpha}_j$, respectively. It is clear that points $\tilde{\alpha}_i$, $\tilde{\alpha}_j$ belong to the set $cl\ \Gamma$. If the set $\Theta^{-1}(cl\ \Gamma \setminus \Gamma)$ consists only of source points $\tilde{\alpha}_1, \tilde{\alpha}_2, \dots$ of the homeomorphism F, then the set $\Theta^{-1}(\Gamma)$ is a disjoint union of subsets, each of which belongs to the unstable manifold of a point from the set $\tilde{\alpha}_1, \tilde{\alpha}_2, \dots$, which contradicts the connectivity of the set Γ. Then, as homeomorphism F has no heteroclinic points, there exists a point $r \in \Omega^2_R$ such that $W^s_r \subset cl\ \Gamma \setminus \Gamma$, $\alpha \subset cl\ W^s_r$, so $W^s_r \subset cl\ W^s_q$ for some point $q \in \Omega^1_A$. Then, according to Statement 4, $W^u_r \cap W^s_q \neq \emptyset$.

Let us prove that the set $W^u_r \cap W^s_q \cap V$ contains at least one noncompact curve. Suppose the contrary. Then the set $W^u_r \cap W^s_q \cap V$ consists of a countable set of smooth closed curves. Since A and R are an attractor and a repeller, respectively, there exist disjoint neighborhoods N_A, N_R in $cl\ V$ of the sets A, R, respectively, and natural numbers n^*, m^* such that $f^n(c) \subset N_a$, $f^{-m}(c) \subset N_r$ for any $n > n^*, m > m^*$ and for any connected component $c \in W^s_r \cap W^s_q \cap V$. Therefore the set $W^u_r \cap W^s_q \cap (V \setminus (N_a \cup N_r))$ consists of a finite number of compact connected components. Let $p_1, p_2 \in \gamma$ be points such that $W^u_{p_i} \cap W^s_q \neq \emptyset$ for $i \in \{1, 2\}$. Show that it is possible to choose a simple compact arc $l \subset (W^s_q \cap V) \setminus (N_a \cup N_r)$ with end points on $W^s_{p_1}, W^s_{p_2}$ such that $l \cap W^u_r = \emptyset$ (see Fig. 7).

As the diffeomorphism f does not contain heteroclinic points, $W^u_r \cap W^s_{p_i} = \emptyset$ for $i \in \{1, 2\}$, and hence there is a 2-disk $D_i \subset int\ V \setminus (N_a \cup N_r \cup W^u_r)$ that transversally intersects $W^s_{p_i}$ at a unique point x_i. Also, there is a closed strip (homeomorphic image of the product $[0;1] \times [0;1]$) $K_i \subset (W^s_q \setminus W^u_r) \cap cl(V)$ with the boundary consisting of arcs $e_{i1}, e_{i2}, e_{i3}, e_{i4}$ such that $e_{i1} \subset W^u_{p_i} \cap W^s_q$, e_{i2}, e_{i3} are transversal to $W^u_{p_i} \cap W^s_q$ in W^s_q and $e_{i3} = f(e_{i2})$. It follows from the λ-lemma that there exists a number $k^*_i > 0$ such that for any $k \geqslant k^*_i$ the intersection of D_i and a connected component of $f^{-k}(K_i)$, containing the set $f^{-k}(e_{i1})$, is a closed arc b_{ik}. Put $l_i = \bigcup\limits_{k=k^*_i}^{\infty} b_{ik} \cup x_i$. The set l_i is a closed arc with end points x_i and y_i. As the set $W^u_r \cap W^s_q \cap (V \setminus (N_a \cup N_r))$ consists of a finite number of compact connected components, there is a simple compact arc $l_0 \subset W^s_q \setminus W^u_r$ joining the points y_1, y_2 and such that the curve $l = l_0 \cup l_1 \cup l_2$ is the required curve.

Since W^s_q, W^u_r are invariant, $f^{-\nu}(l) \cap W^u_r = \emptyset$ for any $\nu > 0$. On the other hand, for any neighborhood N_r of W^s_r in $cl\ V$ there exists ν^* such that $f^{-\nu}(l) \subset N_r$ for any $\nu > \nu^*$. Since the intersection $W^s_r \cap W^u_r$ is transversal, the intersection $f^\nu(l) \cap W^u_r$ must be nonempty for some sufficiently large ν, which contradicts the definition of curve l. Hence the set $W^s_r \cap W^s_q \cap V$ contains at least one noncompact curve. □

Bearing in mind that the set $M^3 \setminus (\mathcal{A}_f \cup \mathcal{R}_f)$ contains $2k_f$ connected components, we get the following result.

Corollary 3. *The number of noncompact heteroclinic curves in the set* $M^3 \setminus (\mathcal{A}_f \cup \mathcal{R}_f)$ *is not less than* $4g_f k_f$.

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