On embedding Morse–Smale diffeomorphisms on the sphere in topological flows

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One important indicator of the adequacy of a numerical solution of an autonomous system of differential equations is the topological conjugacy of the discrete model obtained to the time-one shift map of the original flow. The most significant results in this direction have been obtained for structurally stable flows. In particular, it was shown in [1] and [2] that the Runge–Kutta discretization of a Morse–Smale flow ($n \geqslant 2$) without periodic trajectories on the n-disk is topologically conjugate to the time-one shift (for a sufficiently small step size). In this connection the question, going back to Palis [3], of necessary and sufficient conditions for embedding a Morse–Smale diffeomorphism in a topological flow arises naturally.

Recall that a diffeomorphism f on a closed manifold M^n is called a *Morse–Smale diffeomorphism* if its non-wandering set Ω_f is finite and consists of hyperbolic periodic points, and for any two points $p, q \in \Omega_f$ the intersection of the stable manifold W^{s}_p of p and the unstable manifold W^{u}_q of q is transversal. In [3] the following necessary conditions for embedding a Morse–Smale diffeomorphism $f\colon M^n \to M^n$ in a topological flow were stated, and we call them the *Palis conditions*: 1) the non-wandering set Ω_f coincides with the set of fixed points; 2) the restriction of the diffeomorphism f to each invariant manifold of each fixed point $p \in \Omega_f$ preserves its orientation; 3) if for any distinct saddle points $p, q \in \Omega_f$ the intersection $W^{\mathrm{s}}_p \cap W^{\mathrm{u}}_q$ is non-empty, then it contains no compact connected components.

According to [3], in the case when $n = 2$ these conditions are not only necessary but also sufficient. In [4] examples of Morse–Smale diffeomorphisms on the three-dimensional sphere were constructed that satisfy the Palis conditions but do not embed in topological flows, and also necessary and sufficient conditions were obtained for embedding a three-dimensional Morse–Smale diffeomorphism in a topological flow. An additional obstruction to embedding such diffeomorphisms in topological flows is connected with the possibility of a non-trivial embedding of the separatrices of saddle points in the ambient manifold. In the present paper we show that for the class $G(S^n)$ of Morse–Smale diffeomorphisms without heteroclinic intersections defined on the sphere S^n of dimension $n \geqslant 4$ and satisfying the Palis conditions no such obstruction exists and the following theorem holds.

Theorem 1. *Any diffeomorphism $f \in G(S^n)$, $n \geqslant 4$, is embedded in a topological flow.*

The main tool of the proof is the scheme of a diffeomorphism, defined below. Let $f \in G(S^n)$. It follows from the connection between the dynamics of the diffeomorphism f and the homologies of the sphere S^n that for any saddle point of f either the stable or the unstable manifold has dimension 1. Denote by A_f and R_f the unions of the closures

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This work was supported by the Programme of Basic Research at the Higher School of Economics in 2016 (project no. 98, “Topological Methods in Dynamics”), the Russian Foundation for Basic Research (grant nos. 13-01-12452-офи-м and 15-01-03689-a), and the Russian Science Foundation (grant no. 14-41-00044).

AMS 2010 *Mathematics Subject Classification.* Primary 37D15.

DOI: https://doi.org/10.1070/RM9747.

of the unstable and the stable one-dimensional invariant manifolds of the saddle points, respectively; if f has no saddle points with one-dimension unstable (stable) manifolds, then it has a unique sink (source) fixed point, which we also denote by A_f (R_f). Let $V_f = M^n \setminus (A_f \cup R_f)$. According to [5], the sets A_f, R_f, and V_f are connected, and A_f is an attractor, R_f is a repeller, and V_f consists of the wandering points of f going from R_f to A_f and contains all the saddle separatrices of codimension 1.

Denote by $\widehat{V}_f = V_f/f$ the orbit space of the f-action on V_f, by $p_f\colon V_f \to \widehat{V}_f$ the natural projection, and by $\eta_f\colon \pi_1(\widehat{V}_f) \to \mathbb{Z}$ the epimorphism induced by p_f. Let $\widehat{L}^{\mathrm{s}}_f$ and $\widehat{L}^{\mathrm{u}}_f$ denote the unions of the projections onto $\widehat{V}_f$ of all the stable and unstable separatrices of the saddle points, respectively. The set $S_f = (\widehat{V}_f, \eta_f, \widehat{L}^{\mathrm{s}}_f, \widehat{L}^{\mathrm{u}}_f)$ is called the *scheme of the diffeomorphism* $f \in G(S^n)$. The schemes S_f and $S_{f'}$ of diffeomorphisms $f, f' \in G(S^n)$ are said to be *equivalent* if there exists a homeomorphism $\widehat{\varphi}\colon \widehat{V}_f \to \widehat{V}_{f'}$ with the following properties: 1) $\eta_f = \eta_{f'}\widehat{\varphi}_*$; 2) $\widehat{\varphi}(\widehat{L}^{\mathrm{s}}_f) = \widehat{L}^{\mathrm{s}}_{f'}$ and $\widehat{\varphi}(\widehat{L}^{\mathrm{u}}_f) = \widehat{L}^{\mathrm{u}}_{f'}$.

According to [6], the scheme is a complete topological invariant for diffeomorphisms in $G(S^n)$. The key and most non-trivial point for embedding a diffeomorphism $f \in G(S^n)$, $n \geqslant 4$, in a flow is the equivalence of the scheme S_f to the following standard object. Let a_0^t be a flow on $\mathbb{R}^n \setminus \{O\}$ defined by the formula $a_0^t(x_1, \dots, x_n) = (2^{-t}x_1, \dots, 2^{-t}x_n)$, let a_0 be the time-one shift along trajectories of a_0^t, let $\widehat{V}_{a_0}$ be the orbit space of the a_0-action on $\mathbb{R}^n \setminus \{O\}$ (which is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$), and let $p_{\widehat{V}_{a_0}}\colon \mathbb{R}^n \setminus \{O\} \to \widehat{V}_{a_0}$ be the natural projection. On the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ k we choose smooth pairwise disjoint $(n-2)$-spheres $S_1^{n-2}, \dots, S_k^{n-2}$. Let $\widetilde{c}_i = \bigcup_{t\in\mathbb{R}} a_0^t(S_i^{n-2})$ and $c_i = p_{\widehat{V}_{a_0}}(\widetilde{c}_i)$. We choose an integer $m \in [0,k]$ and let $\widehat{L}^{\mathrm{s}}_{a_0} = \bigcup_{i=1}^m c_i$ and $\widehat{L}^{\mathrm{u}}_{a_0} = \bigcup_{i=m+1}^k c_i$. The set $S_{a_0} = \{\widehat{V}_{a_0}, \eta_{\widehat{V}_{a_0}}, \widehat{L}^{\mathrm{s}}_{a_0}, \widehat{L}^{\mathrm{u}}_{a_0}\}$ is called the *standard scheme.*

Lemma. *The scheme S_f of a diffeomorphism $f \in G(S^n)$, $n \geqslant 4$, is equivalent to the standard scheme for some k and m.*

This lemma allows one to use the method in [4] to construct a flow X^t whose time-one shift has a scheme equivalent to S_f. Since the scheme is a complete invariant, there exists a homeomorphism $h\colon S^n \to S^n$ such that $f = hX^1h^{-1}$. Hence f is embedded in the flow $Y^t = hX^th^{-1}$.

Bibliography

[1] B. M. Garay, *Acta Math. Univ. Comenian. (N. S.)* **62**:2 (1993), 249–275.

[2] B. M. Garay, *Numer. Math.* **72**:4 (1996), 449–479.

[3] J. Palis, *Topology* **8**:4 (1969), 385–404.

[4] В. З. Гринес, Е. Я. Гуревич, В. С. Медведев, О. В. Починка, *Матем. сб.* **203**:12 (2012), 81–104; English transl., V. Z. Grines, E. Ya. Gurevich, V. S. Medvedev, and O. V. Pochinka, *Sb. Math.* **203**:12 (2012), 1761–1784.

[5] В. З. Гринес, Е. В. Жужома, В. С. Медведев, О. В. Починка, *Дифференциальные уравнения и топология*. II, Тр. МИАН, **271**, МАИК, М. 2010, с. 111–133; English transl., V. Z. Grines, E. V. Zhuzhoma, V. S. Medvedev, and O. V. Pochinka, *Proc. Steklov Inst. Math.* **271** (2010), 103–124.

[6] В. З. Гринес, Е. Я. Гуревич, О. В. Починка, Проблемы матем. анализа, **79**, Тамара Рожковская, Новосибирск 2015, с. 73–82; English transl., V. Grines, E. Gurevich, and O. Pochinka, *J. Math. Sci.* (*N. Y.*) **208**:1 (2015), 81–90.

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Presented by Yu. S. Ilyashenko
Accepted 12/OCT/16
Translated by E. GUREVICH