On embedding Morse–Smale diffeomorphisms on the sphere in topological flows

V. Z. Grines, E. Ya. Gurevich, and O. V. Pochinka

One important indicator of the adequacy of a numerical solution of an autonomous system of differential equations is the topological conjugacy of the discrete model obtained to the time-one shift map of the original flow. The most significant results in this direction have been obtained for structurally stable flows. In particular, it was shown in [1] and [2] that the Runge–Kutta discretization of a Morse–Smale flow \((n \geq 2)\) without periodic trajectories on the \(n\)-disk is topologically conjugate to the time-one shift (for a sufficiently small step size). In this connection the question, going back to Palis [3], of necessary and sufficient conditions for embedding a Morse–Smale diffeomorphism in a topological flow arises naturally.

Recall that a diffeomorphism \(f\) on a closed manifold \(M^n\) is called a Morse–Smale diffeomorphism if its non-wandering set \(\Omega_f\) is finite and consists of hyperbolic periodic points, and for any two points \(p, q \in \Omega_f\) the intersection of the stable manifold \(W^s_p\) of \(p\) and the unstable manifold \(W^u_q\) of \(q\) is transversal. In [3] the following necessary conditions for embedding a Morse–Smale diffeomorphism \(f : M^n \to M^n\) in a topological flow were stated, and we call them the Palis conditions: 1) the non-wandering set \(\Omega_f\) coincides with the set of fixed points; 2) the restriction of the diffeomorphism \(f\) to each invariant manifold of each fixed point \(p \in \Omega_f\) preserves its orientation; 3) if for any distinct saddle points \(p, q \in \Omega_f\) the intersection \(W^s_p \cap W^u_q\) is non-empty, then it contains no compact connected components.

According to [3], in the case when \(n = 2\) these conditions are not only necessary but also sufficient. In [4] examples of Morse–Smale diffeomorphisms on the three-dimensional sphere were constructed that satisfy the Palis conditions but do not embed in topological flows, and also necessary and sufficient conditions were obtained for embedding a three-dimensional Morse–Smale diffeomorphism in a topological flow. An additional obstruction to embedding such diffeomorphisms in topological flows is connected with the possibility of a non-trivial embedding of the separatrices of saddle points in the ambient manifold. In the present paper we show that for the class \(G(S^n)\) of Morse–Smale diffeomorphisms without heteroclinic intersections defined on the sphere \(S^n\) of dimension \(n \geq 4\) and satisfying the Palis conditions no such obstruction exists and the following theorem holds.

**Theorem 1.** Any diffeomorphism \(f \in G(S^n), n \geq 4\), is embedded in a topological flow.

The main tool of the proof is the scheme of a diffeomorphism, defined below. Let \(f \in G(S^n)\). It follows from the connection between the dynamics of the diffeomorphism \(f\) and the homologies of the sphere \(S^n\) that for any saddle point of \(f\) either the stable or the unstable manifold has dimension 1. Denote by \(A_f\) and \(R_f\) the unions of the closures...
of the unstable and the stable one-dimensional invariant manifolds of the saddle points, respectively; if \( f \) has no saddle points with one-dimension unstable (stable) manifolds, then it has a unique sink (source) fixed point, which we also denote by \( A_f (R_f) \). Let \( V_f = M^n \setminus (A_f \cup R_f) \). According to [5], the sets \( A_f, R_f, \) and \( V_f \) are connected, and \( A_f \) is an attractor, \( R_f \) is a repeller, and \( V_f \) consists of the wandering points of \( f \) going from \( R_f \) to \( A_f \) and contains all the saddle separatrices of codimension 1.

Denote by \( \tilde{V}_f = V_f / f \) the orbit space of the \( f \)-action on \( V_f \), by \( p_f : V_f \to \tilde{V}_f \) the natural projection, and by \( \eta_f : \pi_1(\tilde{V}_f) \to \mathbb{Z} \) the epimorphism induced by \( p_f \). Let \( \tilde{L}^s_i \) and \( \tilde{L}^u_i \) denote the unions of the projections onto \( \tilde{V}_f \) of all the stable and unstable separatrices of the saddle points, respectively. The set \( S_f = (\tilde{V}_f, \eta_f, \tilde{L}^s_i, \tilde{L}^u_i) \) is called the scheme of the diffeomorphism \( f \in G(S^n) \). The schemes \( S_f \) and \( S_{f'} \) of diffeomorphisms \( f, f' \in G(S^n) \) are said to be equivalent if there exists a homeomorphism \( \tilde{\varphi} : \tilde{V}_f \to \tilde{V}_{f'} \) with the following properties: 1) \( \eta_f = \eta_{f'} \tilde{\varphi} \); 2) \( \tilde{\varphi}(\tilde{L}^s_i) = \tilde{L}^s_{i'}, \) and \( \tilde{\varphi}(\tilde{L}^u_i) = \tilde{L}^u_{i'} \).

According to [6], the scheme is a complete topological invariant for diffeomorphisms in \( G(S^n) \). The key and most non-trivial point for embedding a diffeomorphism \( f \in G(S^n) \), \( n \geq 4 \), in a flow is the equivalence of the scheme \( S_f \) to the following standard object. Let \( a^0_0 \) be a flow on \( \mathbb{R}^n \setminus \{O\} \) defined by the formula \( a^0_0(x_1, \ldots, x_n) = (2^{-t}x_1, \ldots, 2^{-t}x_n) \), let \( a_0 \) be the time-one shift along trajectories of \( a^0_0 \), let \( \overline{V}_{a_0} \) be the orbit space of the \( a_0 \)-action on \( \mathbb{R}^n \setminus \{O\} \) (which is diffeomorphic to \( S^{n-1} \times S^1 \)), and let \( p_{\overline{V}_{a_0}} : \mathbb{R}^n \setminus \{O\} \to \overline{V}_{a_0} \) be the natural projection. On the unit sphere \( S^{n-1} \subset \mathbb{R}^n \) we choose smooth pairwise disjoint \((n - 2)\)-spheres \( S^{n-2}_1, \ldots, S^{n-2}_k \). Let \( \overline{c}_i = \bigcup_{t \in \mathbb{R}} a^0_0(S^{n-2}_i) \) and \( c_i = p_{\overline{V}_{a_0}}(\overline{c}_i) \). We choose an integer \( m \in [0, k] \) and let \( \overline{L}^s_{a_0} = \bigcup_{i=1}^{m} c_i \) and \( \overline{L}^u_{a_0} = \bigcup_{i=m+1}^{k} c_i \). The set \( S_{a_0} = (\overline{V}_{a_0}, \eta_{\overline{V}_{a_0}}, \overline{L}^s_{a_0}, \overline{L}^u_{a_0}) \) is called the standard scheme.

**Lemma.** The scheme \( S_f \) of a diffeomorphism \( f \in G(S^n) \), \( n \geq 4 \), is equivalent to the standard scheme for some \( k \) and \( m \).

This lemma allows one to use the method in [4] to construct a flow \( X^t \) whose time-one shift has a scheme equivalent to \( S_f \). Since the scheme is a complete invariant, there exists a homeomorphism \( h : S^n \to S^n \) such that \( f = hX^1h^{-1} \). Hence \( f \) is embedded in the flow \( Y^t = hX^t h^{-1} \).

**Bibliography**


Vyacheslav Z. Grines
National Research University
Higher School of Economics;
Lobachevski State University of Nizhni Novgorod
E-mail: vgrines@hse.ru

Elena Ya. Gurevich
National Research University
Higher School of Economics
E-mail: egurevich@hse.ru

Olga V. Pochinka
National Research University
Higher School of Economics
E-mail: opochinka@hse.ru

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