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Magneto-Dimensional Resonance. Pseudospin Phase and Hidden Quantum Number

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Abstract. The Schrödinger operator for a spinless charge inside a layer with parabolic confinement profile and homogeneous magnetic field is considered. The Lorentz (cyclotron) and the confinement frequencies are assumed to be equal to each other. After inclination of the layer normal from the magnetic field direction there appears a pseudospin $su(2)$-field removing the resonance degeneracy of Landau levels. Under deviations of the layer surface from the plane shape, a longitudinal geometric current is created. In circulations around surface warping, there is a nontrivial quantum phase transition generated by an element of the π_1-homotopy group and a hidden degree of freedom (spectral degeneracy) associated with a "charge" of geometric poles on the layer. The quantization rule contains an additional parity index related to the algebraic number of geometric poles and the Landau level number. The resonance pseudospin phase-shift represents an example of general Aharonov–Bohm type topologic phenomena in quantum (semiclassical or adiabatic) systems with delta-function singularities in symplectic structure.

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1. INTRODUCTION

It is well known that free charge carriers in flat films placed perpendicularly to a homogeneous magnetic field have infinite degeneracy of the Landau levels. This degeneracy provides the opportunity for perturbations to create quasiparticles (fast cyclotron vortices) whose phase space is the given film-surface. For example, an additional small longitudinal electric field applied to the film produces the Hall current of such quasiparticles and the quantum Hall effect.

Instead of inserting an external electric potential, one can slightly deform the flat geometry of the film. Then at each Landau level, the effective vortex Hamiltonian appears with the corresponding current similar to the Hall current [1–6]. The fast vortices formed by free charge carriers move across the slopes of convexities in a curved film [7]. This current can be called "geometric"[1]

Actually, this current is given by the deviation of the intrinsic area element of the film from the Euclidean one. The current density is proportional to the square of the relative variation of the film surface from the flat shape. Thus, one can speak about the second order geometric conductivity under small deformations of the film surface.

Now note that for thin enough 3D-layers where the influence of the dimensional quantization becomes essential in defining the Landau levels, one has to take into account not only the fast cyclotron rotations of the charges but also their fast transverse oscillations in the confinement potential. If the confinement potential is parabolic[2], then instead of the one-frequency Fock-Landau oscillator, the two-frequency oscillator over fast vortex variables appears. As we shall demonstrate below, in such a situation, the geometrically induced current can register a strong growth.

Namely, if the cyclotron frequency and the dimensional frequency of confinement oscillations are in basic resonance ratio 1:1, then there appears a geometric current of another type which is proportional to the first (rather than the second) degree of the variation of the layer surface from the flat shape. Thus in the resonance case, one can speak about the first order geometric

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[1]More accurately, one must use the term "magneto-geometric" since this current also depends on the magnetic field (its projection onto the surface normal). In our case, the external magnetic field is homogeneous and fixed. Therefore, in the terminology, we stress the geometric aspect only.

[2]This approximation works well at low levels for states localized near the middle surface of the layer, where the anharmonic part of the confinement potential can be taken into account by using standard perturbation theory.

conductivity of the layer. The resonance geometric current dominates the usual geometric current for small rippling of the plane layer.

This resonance conductivity is determined by the deviation of the whole normal bundle over the layer surface from the flat configuration, i.e., by the deformation taking into account the actual 3-dimensionality of the layer.

In more details, due to resonance in the fast 2D-oscillator, there appears a noncommutative symmetry algebra $su(2)$. Its irreducible representations describe the fast spectral degeneracy. Thus one obtains the first hidden magneto-dimensional degree of freedom.

A non-Abelian pseudospin $su(2)$-field couples with a specific "magneto-dimensional" vector field along the layer. This coupling is an analog of the spin term in the Pauli equation.

The spectrum of the pseudospin is discrete (half-integer or integer) and finite at any given Landau level. The corresponding hidden quantum number removes the magneto-dimensional degeneracy mentioned above.

As a result, the effective vortex Hamiltonian and the resonance geometric current are found to be proportional to the discretely varied values of the pseudospin. The zero value of the pseudospin generates zero current, the sign change of the pseudospin inverts the direction of the current. Therefore, one may say that the pseudospin of the magneto-dimensional vortex is the very object which controls the resonance geometric current.

The discreteness of the pseudospin must imply a geometric discretization of the conductance similar to the discretization due to the Landau level quantization, as in the usual quantum Hall effect.

We also show that the geometric holonomy under vortex circulation around a compact warping on the layer surface influences the vortex spectrum. In the quantization rule, a parity index appears related to the algebraic number of geometric poles and to the Landau level number. This is due to the Dirac delta-function contribution of poles to the magnetic symplectic structure on the layer.

At the same time, the eigenfunctions depend on an additional quantum number which stays as a "charge" of geometric poles. We refer to it as to the degree of vortex polarization. This is the second hidden quantum number due to the magneto-dimensional resonance. It affects the phase of wave functions but does not affect the spectrum.

The pseudospin phase transition along ring-shape slopes in resonance layers is an analog of the known persistent current phenomenon in flat rings with boundaries [8–11].

Actually, the pseudospin phase transfer has to appear even in the plane-shape situation just under adiabatic precession of the magnetic field.

2. TOY MODEL OF RESONANCE SU(2)-PSEUDOSPIN

We assume that the transverse confining potential of the layer is parabolic and deal with the case of a magneto-dimensional resonance between the cyclotron frequency $\omega_\circ$ and the frequency $\omega_\perp$ of transverse oscillations of the charge carriers.

Let us first consider the case of flat layer. If the layer surface is perpendicular to the direction of the magnetic field, then, in the $\hbar\omega_\circ$-units, the energy of the nonrelativistic spinless charge is given by the 2D-oscillator Hamiltonian

$$\hat{H}_0 = \frac{1}{2h}\Big((\hat{k}_1^2 + \hat{k}_2^2) + \frac{\omega_\perp}{\omega_\circ}(\hat{x}^2 + \hat{p}^2)\Big) \tag{2.1}$$

over the algebra with commutation relations

$$[\hat{k}_1, \hat{k}_2] = ih, \qquad [\hat{x}, \hat{p}] = ih. \tag{2.2}$$

Here the dimensionless operators $\hat{x}$ and $\hat{p}$ in (2.1) represent the transverse coordinate and momentum, while the operators $\hat{k}_1$, $\hat{k}_2$ represent components of the longitudinal kinetic momentum of the charge in the given strong homogeneous magnetic field.

The dimensionless constant $h = \hbar/\hbar_\perp$ in (2.2) is defined via the Planck action $\hbar$ and the transverse action $\hbar_\perp = m_\perp\omega_\perp l_\perp^2$, where $m_\perp$ is the transverse mass and $l_\perp$ is the transverse scale of the layer. One can also represent h in the form

$$h = \frac{\omega_\circ}{\omega_\perp} \cdot \frac{m_\circ}{m_\perp} \cdot (\frac{l_\circ}{l_\perp})^2,$$

where $m_\circ$ is the longitudinal mass and $l_\circ$ is the magnetic length.

The constant h determines the classicality of the system. The case $h \sim 1$ corresponds to the pure quantum situation (layers with artificially made parabolic profile). In the semiclassical case $h \ll 1$, one can deal with a wider class of profiles (parabolic near the bottom and allowing anharmonic contributions). In the model parabolic situation, the spectrum of the Hamiltonian (2.1) does not at all depend on h and is given by the sequence

$$(n_\circ + \frac{1}{2}) + \frac{\omega_\perp}{\omega_\circ}(n_\perp + \frac{1}{2})$$

defined by the integers $n_\circ, n_\perp \in \mathbb{Z}_+$.

Now let us analyze what happens if one deviates the direction of the magnetic field from the layer‘s normal by a small angle ε.

Denote by x' and $q' = ({q'}^1, {q'}^2)$ the Euclidean coordinates along the magnetic field and in the perpendicular plane, so that $d{q'}^1 \wedge d{q'}^2 \wedge dx'$ represents the standard orientation of $\mathbb{R}^3$. The mid-surface of the layer is now given by the following linear equation

$$x' = f(q'), \qquad f(q') \stackrel{\text{def}}{=} \tan(\varepsilon)\, r \cdot q' \tag{2.3}$$

where r is the "slant" unit 2-covector. The dimension of coordinates is ignored in this section.

Let us introduce the normal coordinates (q, x) by the linear change

$$q' = q - rx\sin\varepsilon = q - \varepsilon r x + O(\varepsilon^2), \qquad x' = f(q) + x\cos\varepsilon = x + \varepsilon r \cdot q + O(\varepsilon^2). \tag{2.4}$$

Here x is the coordinate along the layer normals and $q = (q^1, q^2)$ are the coordinates of the feet of these normals at the mid-plane of the layer.

For any 1-form $a = \alpha' dq' + \beta' dx' = \alpha dq + \beta dx$, we have the following transformation of its coefficients: $\alpha' = \alpha - \varepsilon\beta r + O(\varepsilon^2)$, $\beta' = \beta + \varepsilon r \cdot \alpha + O(\varepsilon^2)$.

In this way, one can transform the components of the momentum and of the magnetic vector potential passing from Euclidean to normal coordinates.

The Hamiltonian of the charge carrier in the layer becomes

$$\hat{H} = \hat{H}_0 + \varepsilon \hat{H}_1 + O(\varepsilon^2), \tag{2.5}$$

where

$$\hat{H}_1 = -\frac{1}{2}\left(\frac{1}{h}\hat{x} + \sqrt{\frac{\omega_\circ}{\omega_\perp}\cdot\frac{m_\perp}{m_\circ}}\hat{p}\right)\hat{k}\cdot r^* + \sqrt{\frac{\omega_\perp}{\omega_\circ}}\left(\sqrt{\frac{m_\perp}{m_\circ}} - \sqrt{\frac{m_\circ}{m_\perp}}\right)\hat{p}\,\hat{k}\cdot r. \tag{2.6}$$

Here $\hat{k} = (\hat{k}_1, \hat{k}_2)$ are the longitudinal kinetic momenta in q-coordinates and $\hat{p}$ is the transverse momentum in x-coordinates. The commutation relations between them are canonical (2.2).

The "cross" vector $r^* = Jr$ in (2.6) is obtained from the slant covector r by applying the matrix $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ representing the Poisson structure on the layer due to the magnetic field contribution.

At the next stage, the Hamiltonian (2.5) can be unitarily transformed to a commutative form:

$$U_\varepsilon^{-1} \cdot \hat{H} \cdot \hat{U}_\varepsilon = \hat{H}_0 + \varepsilon \underline{\hat{H}}_1 + O(\varepsilon^2), \tag{2.7}$$

$$[\hat{H}_0, \underline{\hat{H}}_1] = 0. \tag{2.8}$$

The summand of order ε in (2.7) will be nonzero if and only if 1:1 resonance takes place: $\omega_\perp = \omega_\circ$. Otherwise, the averaging of H_1 by trajectories of H_0 vanishes: $\underline{H}_1 = 0$.

In the resonance case, the main oscillator

$$\hat{H}_0 = \frac{1}{2h}(\hat{k}_1^2 + \hat{k}_2^2) + \frac{1}{2h}(\hat{x}^2 + \hat{p}^2) \tag{2.9}$$

has the following symmetries:

$$L_1 = \frac{1}{2h}(\hat{k}_1\hat{p} - \hat{k}_2\hat{x}), \qquad L_2 = \frac{1}{2h}(\hat{k}_1\hat{x} + \hat{k}_2\hat{p}), \qquad L_3 = \frac{1}{4h}(\hat{k}_1^2 + \hat{k}_2^2 - \hat{x}^2 - \hat{p}^2). \tag{2.10}$$

They obey the relations

$$
[\hat{H}_0, L_j] = 0 \qquad (j = 1, 2, 3), \tag{2.11}
$$

$$
[L_1, L_2] = -iL_3, \qquad [L_2, L_3] = -iL_1, \qquad [L_3, L_1] = -iL_2, \tag{2.12}
$$

$$
L_1^2 + L_2^2 + L_3^2 = (\hat{H}_0^2 - 1)/4. \tag{2.13}
$$

The Hamiltonian $\underline{\hat{H}}_1$ in (2.7) is given by

$$
\underline{\hat{H}}_1 = \frac{1}{2\pi} \int_0^{2\pi} e^{it\hat{H}_0} \hat{H}_1 e^{-it\hat{H}_0}\, dt.
$$

From this formula and (2.6), we derive

$$
\underline{\hat{H}}_1 = v \cdot \underset{\rightarrow}{L}, \qquad \underset{\rightarrow}{L} \stackrel{\text{def}}{=} (L_1, L_2), \tag{2.14}
$$

where the 2-vector v is defined by

$$
v \stackrel{\text{def}}{=} \bigg(-\frac{1}{2} + h\bigg(\sqrt{\frac{m_\circ}{m_\perp}} - \sqrt{\frac{m_\perp}{m_\circ}}\bigg)\bigg) r - \frac{h}{2}\sqrt{\frac{m_\perp}{m_\circ}}\, r^*. \tag{2.15}
$$

In view of (2.12), $[L_3, v \cdot \underset{\rightarrow}{L}] = iv^* \cdot \underset{\rightarrow}{L}$, and therefore

$$
e^{-i\varphi L_3}\, v \cdot \underset{\rightarrow}{L}\, e^{i\varphi L_3} = (e^{\varphi J} v) \cdot \underset{\rightarrow}{L}. \tag{2.16}
$$

Thus, if one chooses φ from the relation $v^1 + iv^2 = |v|e^{i\varphi}$, then $e^{\varphi J} v = |v|\binom{1}{0}$, and hence

$$
(v \cdot \underset{\rightarrow}{L}) e^{i\varphi L_3} = |v| e^{i\varphi L_3} L_1. \tag{2.17}
$$

Thus $\underline{\hat{H}}_1$ (2.14) is unitary equivalent to the operator $|v|L_1$.

The generator L_1 (2.10) is one-half of the “angular-momentum” over the (k_1, x)-plane; its spectrum is given by half-integer or integer numbers in the nth irreducible representation of the $su(2)$-algebra (2.12). The number n is the quantum number for the oscillator $\hat{H}_0$ (its eigenvalues are equal to $n = 1, 2, \ldots$ and each eigenvalue is of multiplicity n).

Theorem 2.1. *Let the plane layer be placed into the homogeneous magnetic field deviated by the angle ε from the layer‘s normal. Then under the magneto-dimensional resonance $\omega_\circ = \omega_\perp$, the Hamiltonian of the free charge carrier, expressed in $\hbar\omega_\circ$-units, is unitary equivalent to $\hat{H}_0 + \varepsilon\, \underline{\hat{H}}_1 + O(\varepsilon^2)$, where $\hat{H}_0$ is the oscillator* (2.9) *and $\underline{\hat{H}}_1 = v^1 L_1 + v^2 L_2$. Here the vector v is given by* (2.15)*, the L_j are generators of the symmetry* su(2)*-algebra* (2.11)–(2.13)*. The spectrum of this Hamiltonian is represented by the sequence*

$$
\lambda_{n,s} = n + \varepsilon s|v| + O(\varepsilon^2), \qquad n = 1, 2, \ldots; \quad s = -\frac{n-1}{2}, -\frac{n-3}{2}, \ldots, \frac{n-3}{2}, \frac{n-1}{2}. \tag{2.18}
$$

The eigenfunctions are derived from (2.17):

$$
\psi_{n,s} = e^{i\varphi L_3} \psi_{n,s}^{(1)} + O(\varepsilon), \tag{2.19}
$$

where $\psi_{n,s}^{(1)}$ is the sth eigenfunction of L_1 in the nth irreducible representation of the symmetric su(2)*-algebra*[3].

The angle $\varphi = \arctan(v_2/v_1)$ in (2.19) *is determined by components of the vector v* (2.15) *related to the geometric slant and cross directions, r and $r^* = Jr$, in the layer.*

The case of the lowest quantum number $n = 1$ (the lowest Landau level) is not interesting for our consideration since, in this case, the multiplicity equals 1 and the number s takes the only value $s = 0$.

The first exited Landau level with $n = 2$ has multiplicity 2 and the number $s \in \{\pm 1/2\}$ takes two possible half-integer values. The corresponding irreducible representation of the algebra (2.12) is given by the Pauli matrices $L_j \sim \frac{1}{2}\sigma_j$ $(j = 1, 2, 3)$.

In the case $n = 3$, the number s takes three integer values $s \in \{-1, 0, 1\}$. And so on.

[3]Note that $\psi_{n,s}^{(1)}$ can be obtained by the standard procedure from the vacuum vector of $L_2 - iL_3$ by using powers of the creation operator $L_2 + iL_3$ in the nth irreducible representation of $su(2)$.

For these reasons, we call the operators (2.10) the *pseudospin operators* of the charge carrier under the magneto-dimensional resonance.

The perturbation term $\varepsilon\underline{H}_1 = \varepsilon v \cdot \underset{\rightarrow}{L}$ in (2.7) looks like a coupling between the in-plane magnetic field and the pseudospin. The length $\varkappa = |v|$ is given by

$$\varkappa = \frac{1}{2}\left[\left(1 - 2h\left(\sqrt{\frac{m_\circ}{m_\perp}} - \sqrt{\frac{m_\perp}{m_\circ}}\right)\right)^2 + h^2\frac{m_\perp}{m_\circ}\right]^{1/2}, \tag{2.20}$$

which contributes to the value of pseudospin "magneton" $\mu = \varkappa\frac{\hbar e}{m_\circ c}$.

Remark 2.1. Note that in the nth irreducible representation the spectrum of the operator L_3 is given by the same set of numbers as the spectrum of L_1. For instance, for $n = 2$ the spectrum of L_3 is $\{\pm 1/2\}$. Thus the eigenvalues of the unitary transformation in (2.19) are $\exp\{\pm i\varphi/2\}$. Now let us adiabatically gyrate the direction of the layer normal around the direction of the magnetic field. This means that we cyclically turn the vector v in (2.14). Then for the Landau level with $n = 2$, after passing the complete 2π-circle in the angle φ, we obtain the nontrivial geometric phase $\exp\{\pm i\pi\}$ in the gauge transformation $e^{2\pi i L_3}$ of the eigenstates (2.19).

This is the resonance pseudospin manifestation of the general Ehrenberg–Siday–Aharonov–Bohm phenomenon in gauge field theory. The gauge potential (Berry connection form) here equals $L_3\, d\varphi$. The corresponding gauge field (curvature) is zero everywhere outside the origin $v = 0$. But the phase shift (holonomy) is nontrivial due to the nonzero total flux generated by the delta-function contribution to curvature from the singularity at $v = 0$. This contribution is a multiple of the eigenvalues of L_3. See details in the next section.

3. HIDDEN DEGREE OF FREEDOM AT POLES

Assume now that the layer is not flat anymore but its deviations from the planar shape are small. Denote by l the longitudinal scale and introduce $\tilde{l}_\perp = \sqrt{m_\perp/m_\circ}\, l_\perp$. The small perturbation parameter is $\varepsilon = (\tilde{l}_\perp/l)^2$.

Let the mid-surface of the layer be given by the equation

$$x' = \sqrt{\varepsilon}\; \tilde{l}_\perp f(q'/l), \tag{3.1}$$

where f is a dimensionless smooth function.

Here, as before, we denote by $q' = ({q'}^1, {q'}^2)$ the Euclidean coordinates along the plane perpendicular to the direction of the magnetic field, and x' is the Euclidean coordinate along the magnetic field. However, the coordinates are no longer dimensionless; they are measured in units of length.

We introduce the normal coordinates (q, x) by formulas similar to (2.4),

$$q' = q - \varepsilon\frac{\nabla f(q/l)x}{\sqrt{1 + \varepsilon^2|\nabla f(q/l)|^2}} = q - \varepsilon\nabla f(q/l)x + O(\varepsilon^2),$$
$$x' = \frac{x}{\sqrt{1 + \varepsilon^2|\nabla f(q/l)|^2}} + \sqrt{\varepsilon}\; \tilde{l}_\perp f(q/l) = x + \sqrt{\varepsilon}\; \tilde{l}_\perp f(q/l) + O(\varepsilon^2).$$

The Hamiltonian of a charge carrier in (q, x)-coordinates has the form (2.5) with the covector $r = \nabla f(q/l)$.

Now let us introduce the (dimensionless) guiding center coordinates

$$\hat{Q} \overset{\text{def}}{=} q/l + \sqrt{\varepsilon} J\hat{k}. \tag{3.2}$$

Then $[\hat{k}_j, \hat{Q}^l] = [\hat{x}, \hat{Q}^l] = [\hat{p}, \hat{Q}^l] = 0$ $(j, l = 1, 2)$, and

$$[\hat{Q}^1, \hat{Q}^2] = -ih\varepsilon. \tag{3.3}$$

Therefore, one can replace the coordinates q/l by the new coordinates $\hat{Q}$ with accuracy $O(\sqrt{\varepsilon})$ at least[4]. These new coordinates commute with all "fast" operators $\hat{k}$, $\hat{x}$, $\hat{p}$, and with the operators (2.10).

[4]Actually, the accuracy is $O(\varepsilon)$ since the $\sqrt{\varepsilon}$-order corrections contain odd powers of the fast operators $\hat{k}$, $\hat{x}$, $\hat{p}$ and vanish after the averaging by H_0-trajectories in (3.4).

So our Hamiltonian takes the form (2.5) with the covector $r = \nabla f(\hat{Q})$ depending on the "slow" operators whose mutual commutator (3.3) is small. Thus, following the usual adiabatic approximation algorithm, we have to compute the eigenvalues and eigenfunctions of the Hamiltonian under frozen slow coordinates Q.

Applying the results of the previous section, under the resonance condition $\omega_\circ = \omega_\perp$, we first transform the Hamiltonian to a commutative form similar to (2.7), (2.14):

$$\hat{H}_0 + \varepsilon \underline{\hat{H}}_1 + O(\varepsilon^2), \quad \underline{\hat{H}}_1 = v(\hat{Q}) \cdot \underset{\to}{\hat{L}} \tag{3.4}$$

with zero commutation relations $[\hat{H}_0, \hat{Q}^l] = 0, \quad [\hat{Q}^l, \hat{L}_j] = 0$.

In (3.4) the "magneto-dimensional" vector field v is defined similarly to (2.15):

$$v \stackrel{\text{def}}{=} \left(-\frac{1}{2} + h\left(\sqrt{\frac{m_\circ}{m_\perp}} - \sqrt{\frac{m_\perp}{m_\circ}}\right)\right) I\nabla f - \frac{h}{2}\sqrt{\frac{m_\perp}{m_\circ}} J\nabla f, \tag{3.5}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the Poisson tensor generated by the inverse magnetic strength tensor on the layer surface and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is up to $O(\varepsilon^2)$ the inverse metric tensor on the surface written in Euclidean coordinates.

The pseudospin $\underset{\to}{L}$ in (3.4) can be regarded as the operator-valued differential 1-form on the layer surface $\underset{\to}{L} = L_1\, dQ^1 + L_2\, dQ^2$, where the generators L_j (2.10) represent the $su(2)$-algebra (2.12) with the Casimir element (2.13) given by the oscillator $\hat{H}_0$ (2.9). This pseudospin 1-form is coupled in (3.4) with the magneto-dimensional vector field v along the layer.

The adiabatic terms are obtained by formula (2.18)

$$\lambda_{n,s}(\hat{Q}) = n + \varepsilon s |v|(\hat{Q}) + O(\varepsilon^2). \tag{3.6}$$

Now if one takes into account the noncommutativity of the slow coordinates $\hat{Q}$ and the presence of a small parameter in (3.3), then we conclude that in the "classical limit" $\varepsilon \to 0$ the terms (3.6) generate the classical Hamiltonian flow[5] in the layer

$$\frac{dQ}{dt} = J\nabla |v(Q)|. \tag{3.7}$$

In view of (3.5), since the vectors $I\nabla f$ and $J\nabla f$ are mutually orthogonal, we have $|v| = \varkappa |\nabla f|$ with the constant $\varkappa$ given by (2.20). So, the geometric flow equation (3.7) reads

$$\frac{dQ}{dt} = \varkappa J D^2 f(Q) \frac{\nabla f(Q)}{|\nabla f(Q)|}. \tag{3.7a}$$

Actually the pseudospin value s has to remain as a multiplier at the right of (3.7),(3.7a).

Thus one obtains the effective electric field $\mathcal{E} = s\varkappa \cdot D^2 f \frac{\nabla f}{|\nabla f|}$ which is the covector field along the layer given in the magnetic Darboux coordinates. It is proportional to the *extrinsic torsion* of the layer mid-surface. The flow generated by the cross-field $\mathcal{E}^* = J\mathcal{E}$ is an analog of the Hall current. It is controlled by the inverse magnetic tensor J which changes the direction of the torsion by 90°. It is directed across the slopes on the layer‘s mid-surface. One obtains ring circuits of free charge carriers around compact hills and dimples, or even saddles on the layer. We refer to this current as to the *resonance geometric current*.

In order to derive the rule for the quantum phase transition along trajectories of such a pseudospin flow, one needs to deal with the complete Hamiltonian $v(\hat{Q}) \cdot \underset{\to}{L}$ taking the coordinates $\hat{Q} = (\hat{Q}^1, \hat{Q}^2)$ as semiclassical operators obeying relation (3.3).

The semiclassical theory is very sensitive to the global topology of the phase space, namely, to the nontriviality of its homotopy groups π_1 or π_2 [12]. Of course, under small deformations of the planar shape of the layer, a nontrivial group π_2 cannot arise. But the π_1-group appears if the vector field v degenerates somewhere, i.e., if $v = 0$ or $\nabla f = 0$.

[5] Ignoring the value and the sign of the pseudospin.

At degeneracy points, the magnetic field is perpendicular to the layer surface. For instance, these are tops of hills (bottoms of dimples) or saddle points. Such isolated points of degeneracy we call *geometric poles*, of positive or negative sign respectively.

But let us first consider the situation with a simply connected domain on the layer where there are no degeneracies, $\nabla f \neq 0$. In such a domain, one can determine a smooth angle function $\varphi = \varphi(Q)$ such that $v_1 + iv_2 = |v| \exp(i\varphi)$.

We denote by $\hat{g} = g(\hat{Q})$ the Weyl-symmetrized functions in the set of operators $\hat{Q} = (\hat{Q}^1, \hat{Q}^2)$ with commutation relation (3.3). As in (2.17), we derive

$$\hat{v} \cdot \underset{\rightarrow}{L}\, e^{i\hat{\varphi} L_3} = e^{i\hat{\varphi} L_3} L_1 \widehat{|v|} + O(h^2\varepsilon^2). \tag{3.8}$$

Thus, in the nth irreducible representation of the algebra $su(2)$ (2.12) and at the sth eigenlevel of L_1, we reduce the Hamiltonian to

$$\hat{v} \cdot \underset{\rightarrow}{L} \sim s\widehat{|v|} + O(h^2\varepsilon^2), \tag{3.9}$$

i.e., to a Hamiltonian in slow quantum coordinates $\hat{Q}$ only.

If the energy levels of $|v| = \varkappa |\nabla f|$ are closed curves, then the semiclassical quantization rule for the eigenvalues of the operator (3.9) is standard:

$$\frac{1}{2\pi h\varepsilon} \oint_{|v|=\nu} Q^1\, dQ^2 = k + \frac{1}{2}. \tag{3.10}$$

Here the quantum numbers k are integers and big enough, $k \sim 1/h\varepsilon$. Recalling the definition of h, ε and Q, we rewrite this rule in physical dimensional units as

$$\frac{1}{2\pi l_\circ^2} \int_{\Sigma_\nu} dq^1 \wedge dq^2 = k + \frac{1}{2}. \tag{3.10a}$$

The left-hand side of (3.10a) is the magnetic flux in London's flux units through the layer area Σ_ν enclosed by the curve $\{|v| = \nu\}$.

Lemma 3.1. *In the absence of poles, the spectrum of $\hat{v} \cdot \underset{\rightarrow}{L}$ is given by the sequence $s\nu_k + O(h^2\varepsilon^2)$, where $\nu = \nu_k$ are obtained from* (3.10) *or* (3.10a).

But these formulas become incorrect if the layer area contains poles, since the angle φ is not globally defined in such a domain and one cannot perform the transformation (3.8). At poles where $|v| = \varkappa |\nabla f|$ is zero, the eigenvalues of the operator-valued symbol $v(Q) \cdot \underset{\rightarrow}{L}$ change their own multiplicity from 1 to n (at the nth Landau level).

Assume that the domain under consideration on the layer contains a finite number of simple poles (at which the curvature $D^2 f$ does not vanish). In this case, one can apply the quantum adiabatic scheme [13–17] and reduce the pseudospin Hamiltonian to the term $s\widehat{|v|}$ plus a correction given by the adiabatic "gauge potential" A as follows:

$$\hat{v} \cdot \underset{\rightarrow}{L} \sim s\widehat{|v|} + h\varepsilon\Big(s\hat{V} \cdot \hat{A} + \frac{i}{2}\langle \hat{A} J[\hat{v} \cdot \underset{\rightarrow}{L}\,, \hat{A}]\rangle_s\Big) + O(h^2\varepsilon^2). \tag{3.11}$$

Here the field $V = -J\nabla|v|$ corresponds to the leading effective Hamiltonian $|v|$, the angle brackets $\langle \dots \rangle_s$ denote the average with respect to the action operator $L^v = \frac{v}{|v|} \cdot \underset{\rightarrow}{L}$ at its eigenlevel s. The $su(2)$-valued covector field $A = (A_1, A_2)$ in (3.11) is determined on the layer's surface by the Lax-type equation

$$\nabla L^v = i[A, L^v] \tag{3.12}$$

together with the zero average condition $\langle A \rangle = 0$. See the details, for instance, in [18] (and formulas (23) and (24) therein).

Lemma 3.2. (i) *The solution of* (3.12) *is given by $A = i[L^v, \nabla L^v] = \nabla\varphi\, L_3$.*

(ii) *The "gauge field" $dA - \frac{i}{2}[A \overset{\wedge}{,} A]$ (the curvature of Berry's connection) corresponding to the adiabatic gauge potential A is equal to*

$$2\pi L_3 \delta(v)\, dv^1 \wedge dv^2 = 2\pi L_3 \Big(\Sigma_\star (-1)^{\sigma_\star} \delta_\star(Q)\Big)\, dQ^1 \wedge dQ^2.$$

Here δ is the Dirac delta-function at zero, $\delta_\star$ is the delta-function at the pole $\star$, the sum $\Sigma_\star$ is taken over all poles, and their signs are denoted by $(-1)^{\sigma_\star}$. Thus this gauge field equals zero everywhere on the layer surface outside the poles.

Note that the second summand in the circle brackets at $h\varepsilon$ in (3.11) vanishes since we have $\nabla\varphi J\nabla\varphi = 0$. Thus, by this lemma, we can rewrite the pseudospin Hamiltonian as

$$\hat{v}\cdot \underset{\to}{L} \sim s\Big(\widehat{|v|} - h\varepsilon L_3 \widehat{\nabla\varphi\cdot J\nabla}|v|\Big) + O(h^2\varepsilon^2). \tag{3.13}$$

The integral of the $h\varepsilon$-correction in (3.13) along trajectories $Q = Q(t)$ (3.7) of the leading Hamiltonian $|v|$ can be written as

$$L_3\int_0^t \nabla\varphi(Q)\cdot J\nabla|v|(Q)\,dt = L_3\int_0^t \nabla\varphi(Q)\frac{dQ}{dt}\,dt = L_3\int_{Q_0}^{Q}\nabla\varphi.$$

The last integral might be not be equal to $\varphi(Q)-\varphi(Q_0)$, since the value φ is not globally smoothly defined (although the covector field $\nabla\varphi = (v^1\nabla v^2 - v^2\nabla v^1)|v|^{-2}$ is globally defined outside the poles). Therefore, in the semiclassical approximation, the contribution to the phase of the wave function from the $h\varepsilon$-correction in (3.13) has the form $\exp\{im\int_{Q_0}^{Q}\nabla\varphi\}$ at the mth eigenlevel of the operator L_3.

Thus, in view of Lemma 3.2 (ii), after passing around the closed curve $\{|v| = \nu\}$ enclosing an area Σ_ν on the layer, the phase shift will be given by a topological invariant:

$$\exp\left\{im\oint\nabla\varphi\right\} = \exp\left\{im\int_{\Sigma_\nu} d(\nabla\varphi)\right\} = \exp\left\{2\pi im\int_{\Sigma_\nu}\delta(v)\,dv^1\wedge dv^2\right\} = \exp\{2\pi imN\}. \tag{3.14}$$

Here N is the algebraic number of poles (counted with their signs) inside the area. This number is equal to the topological degree of the map $Q\to v(Q)$.

Note that m is an eigenvalue of the operator L_3 in the nth irreducible representation of su(2), i.e.,

$$m\in\big\{-(n-1)/2, -(n-3)/2,\dots,(n-3)/2,(n-1)/2\big\}. \tag{3.15}$$

In particular, for the second Landau level with $n = 2$, the number m takes two possible values $m = \pm 1/2$.

Consequently the parity $\sigma(n) \overset{\text{def}}{=} (1-(-1)^n)/2$ determines whether the number m in the phase factor (3.14) is either half-integer or integer. This topological factor comes to stationary states and changes the Planck–Bohr–Sommerfeld quantization rule.

Theorem 3.1. (a). *If in the considered domain, there are no singular points where $v = 0$ and so $\nabla\varphi$ is an exact 1-form, then the summand of order $h\varepsilon$ can be removed from* (3.13) *by the transformation* (3.8)*, the quantization rule has the standard form* (3.10), (3.10a) *and the eigenstates $\psi^{(1)}_{n,s}$ of $\hat{v}\cdot\underset{\to}{L}$* (3.9) *are not related to any particular eigenvalue m of the operator L_3; they are obtained by the transformation $e^{i\hat{\varphi}L_3}$* (3.8) *from the eigenstates $\psi^{(1)}_{n,s}$ of $L_1\widehat{|v|}$, similarly to* (2.19).

(b). *Otherwise, in the presence of poles, the summand of order $h\varepsilon$ cannot be removed from* (3.13)*. It contributes to the quantization rule via a parity index so that* (3.10), (3.10a) *is changed as follows*

$$\frac{1}{2\pi l_\circ^2}\int_{\Sigma_\nu} dq^1\wedge dq^2 = k - \frac{\sigma((n+1)N)}{2} + \frac{1}{2}. \tag{3.16}$$

Here n is the Landau level number, N is the algebraic number of poles in the area Σ_ν enclosed by the curve $\{|v| = \nu\}$.

The asymptotics of the eigenvalues of $\hat{v}\cdot\underset{\to}{L}$, where v is the magneto-dimensional vector field (3.5) *along the layer, is given by the sequence $s\nu_{k,n} + O(h^2\varepsilon^2)$, where $\nu = \nu_{k,n}$ are obtained from* (3.16).

If n is even and N is odd, then the set of values $\nu_{k,n}$ does not coincide with the set determined by (3.10a) (*since the Maslov index is cancelled by the parity index*).

The eigenvalues of L_3, i.e., the numbers m in the set (3.15)*, do not affect the spectral sequence $s\nu_{k,n}$, but generate different phase-factors $\exp\{im\int\nabla\varphi\}$ in the quantum states and determine nontrivial phase transition around poles.*

The explicit asymptotics $\chi_{m,k,n}$ for the eigenfunctions of the operator $\hat{v} \cdot \underset{\to}{L}$ by the semiclassical parameter $h\varepsilon$ can be obtained via the coherent states technique, as in [18] (*see formula* (*11*) *therein*) *or* [19]:

$$\chi_{m,k,n} = \frac{1}{\sqrt{2\pi h\varepsilon}} \oint_{|v|=\nu_{k,n}} \exp\left\{ i \int_0^t \left(\frac{1}{h\varepsilon} Q^1 \, \nabla Q^2 + m \nabla\varphi(Q) \right) \right\} \sqrt{\dot{\mathcal{Z}}} \; |\mathcal{Z}\rangle \otimes e_{m,n} \, dt. \tag{3.17}$$

Here $e_{m,n}$ is the mth eigenvector of L_3 in nth irreducible representation of the su(2)*-algebra* (2.12), $Q = Q(t)$ *is the trajectory of* (3.7) *on the level $|v| = \nu_{k,n}$, the complex structure is introduced by $\mathcal{Z} = Q^2 + iQ^1$, the upper dot denotes the derivative in t, and $|\cdot\rangle$ denotes the standard coherent states family over the complex plane for the algebra* (3.3).

We can call the new quantum number m the *degree of vortex polarization*. Formula (2.10) for L_3 demonstrates that $m = \frac{1}{2}(n_\circ - n_\perp)$ measures the difference between longitudinal and transverse contributions to the vortex energy. Positive values of m mean that the charge carrier mostly rotates around the magnetic field direction and slightly oscillates in the transverse confinement. The negative values of m, on the contrary, mean that the charge carrier mostly oscillates in the confinement and weakly rotates around the field direction.

The appearance of the number m in (3.17) detects a degeneracy (of multiplicity n) which can be removed only in higher adiabatic approximation orders.

Thus we observe a hidden quantum number, and actually a whole hidden degree of freedom whose phase space is the coadjoint orbit in $su(2)^*$, corresponding to the nth irreducible representation.

Theorem 3.2. *The energy levels of charge carriers in a layer with the magneto-dimensional resonance regime in the presence of compact distortion of the layer's surface are given in the $\hbar\omega_\circ$-units by*

$$n + \varepsilon s \, \nu_{k,n} + O(\varepsilon^2), \tag{3.18}$$

where $\nu_{k,n}$ are obtained from (3.16). *The corresponding stationary states are parametrized by four quantum numbers rather than three: n* (*Landau level*), *s* (*pseudospin*), *k* (*magnetic flux*), *and an additional m* (*polarization*) *taking values in the set* (3.15). *The asymptotics of these states is given by two transformations* (2.7) *and* (3.11)(*controlled by the quantum number s*) *which are applied to the functions $\chi_{m,k,n}$* (3.17).

The polarization quantum number m remains in (3.17) *as a charge* (*winding number*) *at the scalar gauge potential $\nabla\varphi$ of the Aharonov–Bohm type; the corresponding scalar gauge field is given by the δ-function at geometric poles on the layer. It deforms the usual magnetic symplectic structure $\frac{1}{2l_\circ^2} J^{-1} dq \wedge dq$* (*the flux* 2*-form in London' units*) *on the layer up to the following one:*

$$\frac{1}{2} J^{-1} dq \wedge dq \left(l_\circ^{-2} + \, m \, \Sigma_\star (-1)^{\sigma_\star} \delta_\star(q) \right).$$

In the first adiabatic order, the energy levels (3.18) *do not depend on the quantum number m.*

The described two-dimensional pseudospin topologic mechanism works due to the magneto-dimensional resonance in the layer. Similar systems are well known in the framework of the generalized Dirac monopole and the Aharonov–Bohm string (solenoid) phenomena [20–26].

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