ON ARITHMETIC OF PLANE TREES

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ABSTRACT. In [3] L. Zapponi studied the arithmetic of plane bipartite trees with prime number of edges. He obtained a lower bound on the degree of a tree's definition field. Here we obtain a similar lower bound in the following case. There exists a prime p such, that: a) the number of edges is divisible by p, but not by p^2; b) for any proper subset of white (or black) vertices the sum of their degrees is not divisible by this p.

1. INTRODUCTION

Let T be a plane (i.e. embedded into plane) bipartite tree with N edges, n white vertices $v_1,\ldots,v_n$ and m black ones $u_1,\ldots,u_m$: $n+m=N+1$. Let $k_1,\ldots,k_n$ be degrees of vertices $v_1,\ldots,v_n$, respectively, and $l_1,\ldots,l_m$ be degrees of $u_1,\ldots,u_m$. Thus, $P=\langle k_1,\ldots,k_n\,|\,l_1,\ldots,l_m\rangle$ is the passport of the tree T.

We will assume that $N=p\,r$, where p is a prime and r is coprime with p.

If there exists a proper subset

$$\{k_{i_1},\ldots,k_{i_t}\}\subset\{k_1,\ldots,k_n\}$$

such, that the sum $k_{i_1}+\ldots+k_{i_t}$ is divisible by p, then the passport P will be called *white-decomposable*. Analogously can be defined a *black-decomposable* passport.

A passport cannot be simultaneously white- and black-indecomposable. This statement is a consequence of the following proposition.

Proposition 1.1. *Let Π is a partition of a number x, l — the number of elements in Π and $l>\frac{x}{2}$. Then for each y, $0<y<x$, there exists a subset of Π such, that the sum of its elements is y.*

Proof. Let us assume that for some y such subset doesn't exist. For each subset $\Xi\in\Pi$ such, that the sum S_Ξ of its elements is $<y$, let d_Ξ be the difference $d_\Xi=y-S_\Xi$. And let for a subset Ξ_0 this difference be minimal.

The partition Π has units as its elements (otherwise, the sum of elements of Π will be greater, than x). All this units belong to the subset Ξ_0 (otherwise, the move of one unit from $\Pi\setminus\Xi_0$ to Ξ_0 diminishes d_{Ξ_0}).

Let a be the minimal element in $\Pi\setminus\Xi_0$ and let Π has b units (and all of them are in Ξ_0). We have,

$$a+b+2(l-b-1)=a-2-b+2l\leqslant x\Rightarrow a-2-b<0\Rightarrow b>a-2.$$

Thus, Ξ_0 contains not less, than $a-1$ units, and by moving a from $\Pi\setminus\Xi_0$ to Ξ_0 and $a-1$ units from Ξ_0 to $\Pi\setminus\Xi_0$ we diminish d_{Ξ_0}. Contradiction. $\square$

Let T be a plane bipartite tree with the passport P and let b be a Shabat polynomial for T such, that white vertex v_1 be at the origin and black vertex u_1 — at the point 1. This choice defines positions of all other vertices. So, let $x_1 = 0, x_2, \dots, x_n$ be coordinates of white vertices $v_1, v_2, \dots, v_n$, respectively, and $y_1 = 1, y_2, \dots, y_m$ be coordinates of black vertices $u_1, \dots, u_m$. Then

$$b(z) = \prod_{i=1}^{n} (z - x_i)^{k_i}.$$

As $c = b(1)$ is the value of the Shabat polynomial b in black vertices, then

$$b(z) - c = \prod_{i=1}^{m} (z - y_i)^{l_i}.$$

Let K be the *big definition field* that contains coordinates of all white and black vertices (and the number c also). Let ρ be a prime divisor (see [1]) of K that divides prime p and v_ρ be the corresponding valuation, i.e. $v_\rho(x) = a$, if $x = \rho^a y$, where y and ρ are coprime. Let $O = \{x \in K \,|\, v_\rho(x) \geqslant 0\}$ be the set of ρ-integral numbers and $I = \{x \in K \,|\, v_\rho(x) > 0\}$ be the maximal ideal in O. Then O/I is a finite field of characteristic p. And let, at last, $e_K = v_\rho(p)$ be the ramification index.

In Section 2 we prove the existence of *normalized model*, i.e. such Shabat polynomial of our tree T, that one white vertex is at the origin, one black vertex is at 1 and coordinates of all vertices are ρ-integral.

The main result of this work is Theorem 3.1: if the passport is white-indecomposable, then in the scope of normalized model $v_\rho(x_i) = e_K/(n-1)$ for all $x_i \neq 0$.

2. A normalized model

In this section we will construct a special Shabat polynomial for our tree T — a *normalized model* [3], and will study its properties.

If $\min_i v_\rho(y_i) = a < 0$ and $v_\rho(y_j) = a$, then we will perform a coordinate change: now x_i/y_j, $i = 1, \dots, n$ are coordinates of white vertices and y_i/y_j are coordinates of black ones. Let us note that the vertex v_1 remains at origin and at point 1 now is the vertex u_j. We will continue to use notations x_i, y_i and c for coordinates of white vertices, black vertices and the value of b at 1.

As all black coordinates now are ρ-integral, then the polynomial

$$\prod_{i=1}^{m} (z - y_i)^{l_i}$$

has ρ-integral coefficients. Thus, the polynomial

$$b(z) = \prod_{i=1}^{m} (z - y_i)^{l_i} + c$$

has ρ-integral coefficients (because $x_1 = 0$). Hence, all white coordinates are ρ-integral and the number c is also ρ-integral.

Definition 2.1. A Shabat polynomial b of a tree T is called its *normalized model* if

- the leading coefficient of b is 1;
- some prime number p is fixed, which divides the number of edges;

- in the big definition field some prime divisor ρ is fixed, that divides p;
- some white vertex is in origin and some black vertex is in the point 1;
- coordinates of all (black and white) vertices are ρ-integral.

The existence of normalized model was proved above. Let us consider in more details its arithmetic properties.

Let

$$b(z) = \prod_{i=1}^{n}(z-x_i)^{k_i} = z^N + a_{N-1}z^{N-1} + \ldots + a_1 z$$

be a normalized model of our tree T. On one hand

$$b'(z) = Nz^{N-1} + (N-1)a_{N-1}z^{N-2} + \ldots + a_1.$$

On the other hand

$$b'(z) = N\prod_{i=1}^{n}(z-x_i)^{k_i-1}\prod_{i=1}^{m}(z-y_i)^{l_i-1}.$$

It means, that all coefficients a_i (except, maybe, $a_p, a_{2p}, \ldots, a_{rp}$) belong to the ideal I. Thus,

$$b \bmod I = z^{rp} + b_{r-1}z^{(r-1)p} + \ldots + b_1 z^p,$$

where $b_i = a_{p\,i} \bmod I \in O/I$. The polynomial $t^r + b_{r-1}t^{r-1} + \ldots + b_1 t$ has r roots in O/I and each of them generates a root of the polynomial $b \bmod \rho$ of multiplicity p (because $x \mapsto x^p$ is the Frobenius automorphism in the field O/I). Thus, N roots of the polynomial b are partitioned into r subsets of cardinality p each, and roots in each subset are congruent modulo ρ.

Let, for example, roots $x_1, \ldots, x_t$ are pairwise congruent modulo ρ, but other roots $x_{t+1}, \ldots, x_n$ are not congruent to them. Then the number $k_1 + \ldots + k_t$ is divisible by p, i.e. the passport P is white-decomposable. Thus, we have the theorem.

Theorem 2.1. *If there exists a pair of vertices v_i and v_j such, that their coordinates are not congruent modulo ρ, then the passport P is white-decomposable. If the passport is white-indecomposable, then coordinates of all white vertices belong to I.*

Analogous statement is valid for black vertices.

3. Arithmetic of trees with a white-indecomposable passport

Let b be a normalized model of a tree T with a white-indecomposable passport P. As

$$b(z) \bmod \rho = \prod_{i=1}^{n}(z-x_i)^{k_i} \bmod \rho = z^N,$$

then $b(1) \equiv 1 \bmod \rho$. Hence, coordinates of all black vertices are not in I.

As

$$b'(z) = b(z)\sum_{i=1}^{n}\frac{k_i}{z-x_i} = N\prod_{i=1}^{n}(z-x_i)^{k_i-1}\prod_{i=1}^{m}(z-y_i)^{l_i-1}, \tag{1}$$

then the substitution $z = x_1 = 0$ gives us the relation

$$(-1)^{n-1}k_1\prod_{i=2}^{n}x_i = (-1)^{N-m}N\prod_{i=1}^{m}y_i^{l_i-1}.$$

Thus,

$$\sum_{i=2}^{n} v_\rho(x_i) = e_K \Rightarrow (n-1)v \leqslant e_K, \tag{2}$$

where $v = \min_{i>1} v_\rho(x_i) > 0$. Let $v_\rho(x_j) = v$ and let us consider new Shabat polynomial for our tree:

$$b_1(z) = x_j^{-N} b(x_j z) = \prod_{i=1}^{n} (z - \widetilde{x}_i)^{k_i} = z^N + \widetilde{a}_{N-1} z^{N-1} + \ldots + \widetilde{a}_1 z,$$

where $\widetilde{x}_i = x_i/x_j$. On one hand,

$$b_1'(z) = x_j^{-N} x_j\, b'(x_j z) = N x_j^{1-n} \prod_{i=1}^{n} (z - \widetilde{x}_i)^{k_i - 1} \prod_{i=1}^{m} (x_j z - y_i)^{l_i - 1}. \tag{3}$$

On the other hand,

$$b_1'(z) = N z^{N-1} + (N-1)\widetilde{a}_{N-1} z^{N-2} + \ldots + \widetilde{a}_1.$$

Some of the numbers $\widetilde{x}_i$ do not belong to the ideal I. Thus, there exists a coefficient $\widetilde{a}_t$ that also does not belong to I. As the passport is white-indecomposable, then $t \bmod p \neq 0$. Hence, $v_\rho(t\,\widetilde{a}_t) = 0$. But then from (3) we have that $e_K \leqslant (n-1)v$. As $(n-1)v \leqslant e_K$ (relation (2)), then we have the following theorem.

Theorem 3.1. *If the passport is white-indecomposable, then*

$$v_\rho(x_i) = e_K/(n-1), \quad \forall\, i > 1.$$

The substitution $z = x_i$ in (1) and analogous reasoning give us relation

$$v_\rho(x_i - x_j) = e_K/(n-1), \quad \forall\, 1 \leqslant i < j \leqslant n.$$

Remark 3.1. We want to obtain some estimation on the degree of the field of definition (see [2]) L of a tree T. As L is a subfield of the field K, then let τ be a prime divisor in L that divides p and is divisible by ρ. If $x, y \in L$, then

$$\frac{v_\rho(x)}{v_\rho(y)} = \frac{v_\tau(x)}{v_\tau(y)}.$$

In particular,

$$\frac{e_K}{v_\rho(x)} = \frac{e_L}{v_\tau(x)}, \tag{4}$$

where $e_L = v_\tau(p)$ is the ramification index of the field L.

Remark 3.2. If $N = p^s r,\ s > 1$, then Theorem 2.1 is valid and the statement of Theorem 3.1 is as follows:

$$v_\rho(x_i) = s e_K/(n-1), \quad \forall\, i > 1.$$

4. Examples

Example 4.1. Let T be a tree of diameter 4 with the central black vertex of degree 4 and four white vertices of degrees $a < b < c < d$ (here $N = a + b + c + d$). There are 6 trees with this passport. We assume that there exists a prime p such, that $N \equiv 0 \bmod p$, $N \not\equiv 0 \bmod p^2$ and the passport is white-indecomposable.

The white vertex of degree d we put at origin and the central black vertex — at 1. Then coordinates of all other vertices are uniquely defined.

Shabat polynomial for T is a normalized model. Indeed, otherwise we have to divide all coordinates by some number with negative valuation. After that the coordinate of central black vertex will be at I and will have positive valuation. But the passport is white-indecomposable, so coordinates of all black vertices have zero valuation. We have a contradiction.

Our normalized model is defined over the field L — the definition field of the tree T. Moreover, x_c — the coordinate of the white vertex of degree c belongs to L. Thus,

$$\frac{e_K}{v_\rho(x_c)} = \frac{e_L}{v_\tau(x_c)} = n - 1 = 3.$$

It means that the degree of L is not less, than 3. But a conjugate to any tree belongs to the same Galois orbit, hence, the cardinality of each orbit is even. Hence, there is one orbit of cardinality 6.

Remark 4.1. The white-indecomposability can be obtained, if $d > p\,(r-1)$. Thus, for example, all six trees with the passport

$$\langle 15, 3, 2, 1 \,|\, 4, \underbrace{1, \ldots, 1}_{17} \rangle$$

belong to one orbit.

Remark 4.2. If $a = 1, b = 11, c = 80, d = 84$, then there are two Galois orbits: one of cardinality 4 and one of cardinality 2. But here the passport is white-decomposable, because $1 + 11 + 80 + 84 = 11 \cdot 16$.

Remark 4.3. The above results hold if $N = p^s r$ and s is coprime with 3 (see Remark 3.2).

Remark 4.4. The same reasoning can be applied: a) in the case of the passport $\langle a, b, c, c \,|\, 4, 1, \ldots, 1 \rangle$, where a, b, c are pairwise different (there are 3 trees with such passport); b) in the case of the passport $\langle a, b, c, c, c \,|\, 5, 1, \ldots, 1 \rangle$, where a, b, c are pairwise different (there are 4 trees with such passport).

Example 4.2. Let us consider a tree T with the passport

$$\langle p+1, 2, \underbrace{1, \ldots, 1}_{p-3} \,|\, p-1, \underbrace{1, \ldots, 1}_{p+1} \rangle,$$

where p is a prime. There are $p-2$ such trees. Let the white vertex of degree $p+1$ be at origin and the black vertex of degree $p-1$ be at point 1. The corresponding Shabat polynomial is a normalized model and its definition field coincides with the definition field of T. Number a — the coordinate of the white vertex of degree 2 belongs to L. Thus,

$$\frac{e_K}{v_\rho(x)} = \frac{e_L}{v_\tau(x)} = p - 2.$$

Hence, all trees with this passport are in one Galois orbit.

References

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