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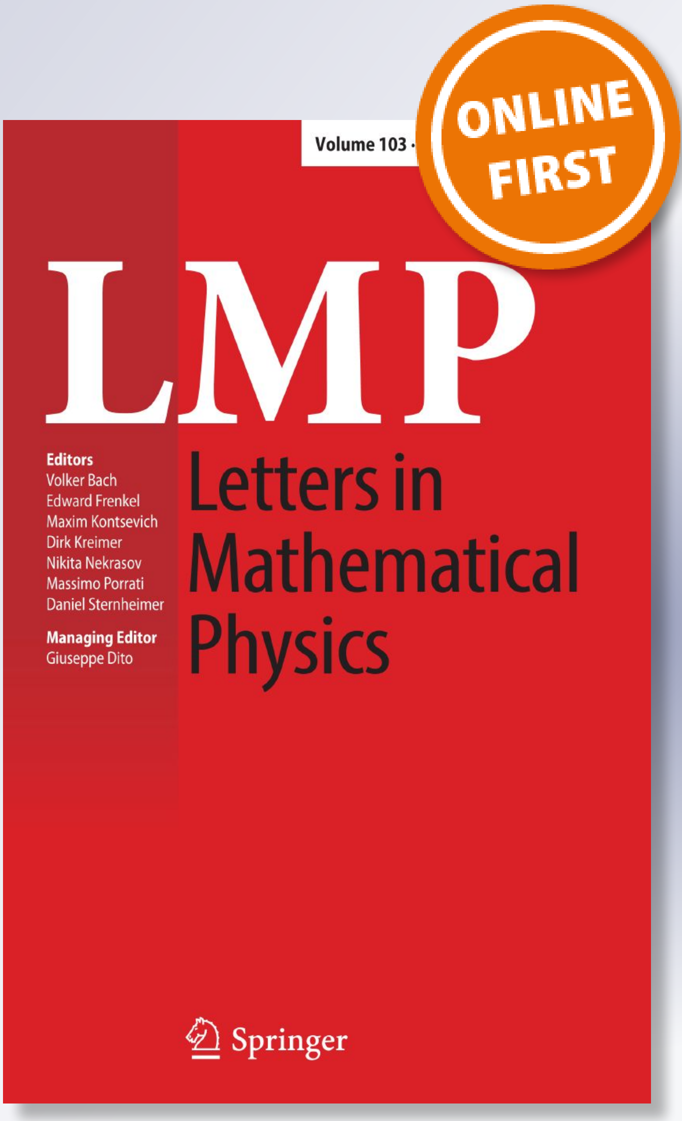
Letters in Mathematical Physics
A Journal for the Rapid Dissemination of Short Contributions in the Field of Mathematical Physics

ISSN 0377-9017

Lett Math Phys
DOI 10.1007/s11005-017-0944-0



Springer

Lett Math Phys
DOI 10.1007/s11005-017-0944-0

BKP and projective Hurwitz numbers

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Received: 7 August 2015 / Revised: 30 November 2016 / Accepted: 4 December 2016

Abstract We consider d-fold branched coverings of the projective plane $\mathbb{RP}^2$ and show that the hypergeometric tau function of the BKP hierarchy of Kac and van de Leur is the generating function for weighted sums of the related Hurwitz numbers. In particular, we get the $\mathbb{RP}^2$ analogues of the $\mathbb{CP}^1$ generating functions proposed by Okounkov and by Goulden and Jackson. Other examples are Hurwitz numbers weighted by the Hall–Littlewood and by the Macdonald polynomials. We also consider integrals of tau functions which generate Hurwitz numbers related to base surfaces with arbitrary Euler characteristics E, in particular projective Hurwitz numbers $\mathrm{E} = 1$.

Keywords Hurwitz numbers · Tau functions · BKP · Projective plane · Schur polynomials · Hall–Littlewood polynomials · Hypergeometric functions · Random partitions · Random matrices

Mathematics Subject Classification 05A15 · 14N10 · 17B80 · 35Q51 · 35Q53 · 35Q55 · 37K20 · 37K30

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Published online: 28 February 2017

1 Introduction

Okounkov [51] studied ramified coverings of the Riemann sphere, having arbitrary ramification type over 0 and ∞, and with simple ramifications elsewhere, and made the seminal observation that the generating function for the related Hurwitz numbers (numbers of non-equivalent coverings with given ramification type) is a tau function for the Toda lattice hierarchy. Later, links between the study of covers and integrable systems were further developed by Okounkov and Pandharipande [52] and by Goulden and Jackson [20]. Then a number of papers concerning the topic were written [1,2, 16,24,29,30,44,69]. A review of this topic may be found in [31] and in [35].

On the other hand, it was shown that certain matrix models also generate Hurwitz numbers [10,22,25,37,69]. This is not so surprising since tau functions used for generating Hurwitz numbers belong to a special family identified in [38] and [55,57]. These are called tau functions of *hypergeometric type*, and such tau functions were used as asymptotic expansions of matrix integrals in [27,28,60,61]. Hypergeometric tau functions are multivariable generalizations of hypergeometric series, where the Gauss hypergeometric equation plays the role of the so-called string equation [55] and matrix integrals may be viewed as analogues of the integral representation of Gauss hypergeometric series.

All the studies of Hurwitz numbers cited above are devoted to counting covers of the Riemann sphere and the links between this problem and the Toda lattice (TL) and Kadomtsev–Petviashvili (KP) hierarchies.

The covering problem of the Riemann sphere (Euler characteristic $\textsc{e} = 2$) goes back to classical results by Frobenius and Schur [18,19]. The reader is referred to the wonderful textbook [39], which considers the general case of the enumeration of covers of Riemann surfaces of higher genus. The Frobenius-type formula for the Hurwitz numbers enumerating d-fold branched coverings of connected Riemann or Klein surfaces (without boundary) of any Euler characteristic $\textsc{e}$ was obtained by A. Mednykh and G. Pozdnyakova [41,42] and also by Gareth A. Jones [33]. It contains the sum over irreducible representations λ of the symmetric group S_d [18,19,33,39,41,42]

$$H^{\textsc{e},\textsc{f}}\left(d;\Delta^{(1)}\ldots,\Delta^{(\textsc{f})}\right)=\sum_{\lambda}\left(\frac{\dim\lambda}{d!}\right)^{\textsc{e}}\prod_{i=1}^{\textsc{f}}\varphi_\lambda(\Delta^{(i)}),\tag{1}$$

where $\textsc{e}$ is the Euler characteristic of the base surface Ω, $\Delta^{(i)}$ are profiles over branch points on Ω, $\dim\lambda$ is the dimension of the irreducible representation of S_d, and

$$\varphi_\lambda(\Delta^{(i)}):=|C_{\Delta^{(i)}}|\,\frac{\chi_\lambda(\Delta^{(i)})}{\dim\lambda},\quad \dim\lambda:=\chi_\lambda\left((1^d)\right).\tag{2}$$

Here $\chi_\lambda(\Delta)$ is the character of the symmetric group S_d evaluated on a cycle type Δ, and χ_λ ranges over the irreducible complex characters of S_d, labeled by partitions $\lambda=(\lambda_1,\ldots,\lambda_\ell)$. The convenient notion of normalized character, φ_λ, comes from [1,51]. Each profile $\Delta^{(i)}$ is a partition of d, i.e., the set of nonnegative non-increasing numbers $(d_1^{(i)},d_2^{(i)},\ldots)$, which describes the ramification over point number i on the base. The weights of all partitions involved in (1) are equal: $|\lambda| := \sum_j \lambda_j = |\Delta^{(i)}| := \sum_j d_j^{(i)} = d$. The number $|C_\Delta|$ is the number of elements in the cycle class Δ in S_d. The sum (1) runs over partitions of weight d. We assume that $\varphi_\lambda(\Delta)$ vanishes whenever $|\lambda| \neq |\Delta|$.

The Hurwitz numbers form a topological field theory [15]. In string theory applications, the covering surface is the worldsheet of the string, while the base surface is the target space. Hurwitz numbers are used in mathematical physics (for instance in [15]) and in algebraic geometry [39]. A lot of interest and also a lot of developments in these studies arose from [17], which relates Hurwitz numbers to Gromov–Witten theory.

Our paper deals with the enumeration of the covers of the projective plane $\mathbb{RP}^2$, i.e., the case $\mathrm{E} = 1$ in (1). The related Hurwitz numbers will be called projective. The projective Hurwitz numbers were introduced by Mednykh and Pozdnyakova in [42] and independently in the context of topological field theory in [5].

In this case, we found that a different hierarchy of integrable equations is related to the problem: This is the BKP hierarchy, introduced by Kac and van de Leur [34].[1] In a certain sense, this hierarchy is very similar to the DKP hierarchy introduced in [32]. However, the difference between the D and B types is crucial for the counting problem we discuss here (see Remark 23 in "Appendix"). For some reason, the BKP hierarchy of Kac and van de Leur is not well known, although it has applications to the famous orthogonal and symplectic ensembles [64] and some other models of random matrices and random partitions [54,58,59,65]. We shall show that the BKP tau function of hypergeometric type introduced in [58,59] generates Hurwitz numbers for covers of $\mathbb{RP}^2$. Up to an unimportant factor, the BKP tau function of hypergeometric type may be written in the form

$$\tau^{\mathrm{B}}(N, n, \mathbf{p}) = \sum_{\substack{\lambda \in \mathrm{P} \\ \ell(\lambda) \leq N}} s_\lambda(\mathbf{p})\, c^{|\lambda|} \prod_{(i,j)\in\lambda} r(n + j - i), \tag{3}$$

where s_λ is the Schur function [40] related to a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\ell(\lambda)$ denotes the number of nonvanishing parts of λ, c is a parameter, and P denotes the set of all partitions. (In what follows, we will omit the summation range P as understood.) The product on the right-hand side ranges over all nodes of the Young diagram λ, j indicates the column and i the row of the node of λ when depicted in the English way, i.e., the diagonal spreads down and right from the origin. The two discrete parameters N and n and the set $\mathbf{p} = (p_1, p_2, \ldots)$ are called the BKP higher times [34].[2] We suppose that the tau function (3) is equal to 1 if $N = 0$ and vanishes if $N < 0$. r is an arbitrarily chosen function of one variable, which will be specified later according to our needs. The number $j - i$ is called the *content* of the node located at the ith

[1] This BKP hierarchy was called the "charged" and "fermionic" BKP hierarchy in [34]. We call it the "large" BKP hierarchy because it includes the KP hierarchy and may be related [65] to the two-component KP. The "small" KP hierarchy, introduced in [32], is a subhierarchy of the KP hierarchy.

[2] In the present paper we use the so-called power sums p_m [40] as higher time variables, rather than p_m/m as is common in soliton theory [32].

row and the jth column of the Young diagram λ, and the product over all nodes of the Young diagram on the right-hand side of (3) is called the *content product* (the generalized Pochhammer symbol). Content products play an essential role in the study of applications of the symmetric groups (see, for instance, [20,24,25] and the references therein). The special role of the content product in the study of Hurwitz numbers generated by the KP hierarchy was observed and worked out in [21].

In the present paper, we chose two different types of parameterizations of the function r which defines the content product in (3). The first is

$$\text{(I)} \qquad r(x) = \exp \sum_{m>0} \frac{1}{m} \zeta_m h^m x^m . \tag{4}$$

The second is

$$\text{(II)} \qquad r(x) = \mathfrak{t}^{x\xi_0} \exp \sum_{m\neq 1} \frac{1}{m} \xi_m \mathfrak{t}^{mx} . \tag{5}$$

The complex number $\mathfrak{t}$ and sets $\{\zeta_m,\ m > 0\}$ and $\{\xi_m,\ m \in \mathbb{Z}\}$ are free parameters. Similarly to [24,51], we introduce auxiliary parameters c and h. The powers of c count the degree of covering maps, while the powers of the parameter $1/h$, which enters (4), count the Euler characteristic of the covers. In what follows, we may put $c = 1$ and $h = 1$ in cases where we are not interested in the degree and the Euler characteristic, and hope this does not lead to confusion. Wherever there is no risk of ambiguity, we also avoid mentioning the dependence of $r(x)$ and $\tau(N, n, \mathbf{p})$ on c, h, ζ, ξ, and other parameters to make the formulae more readable.

Let us note that the use of the parametrization (I) of (4) in applications of the content product was also considered in [30], in the study of combinatorial Hurwitz numbers using Cayley graphs and Jusys–Murphy elements, as suggested by the Canadian combinatorial school [22] and developed in [24].

One of the results of our paper is explicit expressions for the content products parameterized by (4) and (5) in terms of the characters of the symmetric groups (see Propositions 2 and 3).

Let us write down the answer for case (II) [see (5)]:

$$\prod_{(i,j)\in\lambda} r(x+j-i) = \mathfrak{t}^{\xi_0 x|\lambda| + \xi_0 \varphi_\lambda(\Gamma)} \exp \sum_{m\neq 0} \frac{1}{m} \xi_m \mathfrak{t}^{mx} D_{p_1} \log s_\lambda(\mathbf{p})|_{\mathbf{p}(0,\mathfrak{t}^m)},$$
$$|\Gamma| = |\lambda| = d. \tag{6}$$

Here we first apply the Euler operator $D_{p_1} = p_1 \partial/\partial p_1$ to the Schur function $s_\lambda(\mathbf{p})$, where $\mathbf{p} = (p_1, p_2, \dots)$, and then evaluate the result at the point $\mathbf{p} = \mathbf{p}(0, \mathfrak{t}^m) = (p_1(0, \mathfrak{t}^m), p_2(0, \mathfrak{t}^m), \dots)$, where $p_k(0, \mathfrak{t}^m) = (1 - \mathfrak{t}^{mk})^{-1}$. The partition Γ is defined as follows. For $d \geq 2$, it is the partition $(1^{d-2}2)$, and it has length $\ell(\Gamma) = d - 1$. We choose the notation Γ because the Young diagram of the partition $(1^{d-2}2)$ resembles the Greek capital letter gamma. The cycle class labeled by Γ in S_d consists of all transpositions. We also keep the notation Γ for the case $d \leq 1$, when $\Gamma = (d)$.

One can see that the content product for a partition λ is expressed in terms of the Schur functions labeled by the same partition. Thanks to the characteristic map relation [40]

$$s_\lambda(\mathbf{p}) = \frac{\dim\lambda}{d!}\left(p_1^d + \sum_{\substack{\Delta\\ \Delta\neq 1^d}} \varphi_\lambda(\Delta)\mathbf{p}_\Delta\right), \tag{7}$$

formula (6) produces a series in φ_λ. Due to the summation over partitions λ in (3), this in turn allows us to consider (3) as the generating function for Hurwitz numbers (1). The content product (6) is expressed in terms of the Schur function, and formula (7) exhibits the explicit dependence of the Schur functions on $\dim\lambda$. However, in expression (6) for the content product, the dependence on $\dim\lambda$ disappears, thanks to the logarithmic derivative of the Schur function. Then one can suppose that tau function (3) generates Hurwitz numbers (1) where $\mathtt{E} = 1$ (projective Hurwitz numbers). To be precise, (3) generates weighted sums of the projective Hurwitz numbers where, as we shall see later, the weights are defined by specifications of the parameters ξ, $\mathtt{t}$).

Case (I) may be considered as the degenerate limit of case (II) when $\mathtt{t} \to 1$, whence in this case the series (3) also generates projective Hurwitz numbers.

Here and below, $\mathbf{p}_\Delta$ denotes the product $p_{d_1}p_{d_2}\cdots$, where d_i are the parts of the partition Δ: $\Delta = (d_1, d_2, \dots)$. Then the tau function (3) may be written

$$\tau^{\mathrm{B}}(N, n, \mathbf{p}) = \sum_{d\geq 0} c^d \sum_{\substack{\Delta\\ |\Delta|=d}} \mathrm{H}_r(d;\Delta)\mathbf{p}_\Delta,$$
$$\mathrm{H}_r(d;\Delta) = \sum_{\substack{\lambda\\ |\lambda|=d,\, \ell(\lambda)\leq N}} \frac{\dim\lambda}{d!}\varphi_\lambda(\Delta)\prod_{(i,j)\in\lambda} r(n+j-i), \tag{8}$$

where (for $d \leq N$) $H_r(d;\Delta)$ is a certain series of Hurwitz numbers which describe d-fold covers with ramification Δ over a point, say, 0 of $\mathbb{RP}^2$, and ramifications over additional points which are determined by the choice of r, namely the parameters in (4) or (5). We should keep in mind that, in (3), it is only the part of the sum over λ conditioned by $|\lambda| \leq N$ that generates Hurwitz numbers $H^{\mathrm{E,F}}(d = |\lambda|; \dots)$. Thus to get Hurwitz numbers for the study of d-fold coverings, one should work with the series (3) conditioned by $N \geq d$. We shall encounter the same restriction in Sect. 8, when we consider integrals over $N \times N$ matrices which generate Hurwitz numbers.

Remark 1 As we can see, the sum

$$\sum_\lambda c^{|\lambda|} \prod_{(i,j)\in\lambda} r(n+j-i) \tag{9}$$

may also be viewed as the generating function of the Hurwitz numbers when the base surface has Euler characteristic equal to zero (which corresponds to either the torus or the Klein bottle). For the specification (4), such sums may be related to the characters of the Lie algebra of differential operators on the circle, as has been studied in [11].

Both choices of the content products, (4) and (5), contain the direct analogue of the Okounkov generating series [51], but now for covers of $\mathbb{RP}^2$. It is enough to put $\zeta_m = 0$ for all m except $m = 1$ in (4), or to put all $\xi_m = 0$ for all m except $m = 0$ in (5).

Using respectively (4) and (5), we obtain two different types of generating functions for the projective Hurwitz numbers. The first, arising from (4), may be compared to the approach based on completed cycles developed in [1,52] (which studied the case $\mathbb{CP}^1$). The second, obtained from (5), is related to a "q-deformation" of the previous case (where instead of q we use the letter $\mathtt{t}$), which in turn may be compared to the approaches developed independently in [50] and [30]. We will show that, in the "$\mathtt{t}$-deformed" (or "trigonometric") case, the Hall–Littlewood and Macdonald polynomials naturally appear as weight functions in weighted sums of the projective Hurwitz numbers.

The structure of the paper is as follows. In Sect. 2, we explain the notion of Hurwitz numbers for Klein surfaces. In Sect. 2.2, we present links between Hurwitz numbers for base surfaces with different Euler characteristics E. There we also explain the special role of the ramification described by the one-row Young diagram (d) (*maximally ramified profile*) in the enumeration of the d-fold covers presented in Proposition 1. This means that the BKP hypergeometric function also generates Hurwitz numbers for the d-fold covers of any Klein surface whenever at least one of the profiles is maximally ramified.

In Sect. 3, we find the content products for cases (I) and (II) in terms of the characters of the symmetric groups. The answers are given by Propositions 2 and 3, respectively.[3]

In Sect. 4, we introduce weighted sums of the projective Hurwitz numbers (which we will show in further sections to be generated by the BKP tau functions). For the weighting, we use in particular the Macdonald, Jack, and Hall–Littlewood polynomials, which naturally appear via specifications of the parametrization of r. We will then be ready to use tau functions.

In Sect. 5, we recall the notion of the BKP hierarchy and the special family of the BKP tau functions referred to as hypergeometric. We show that the BKP hypergeometric tau function may be obtained from the two-component KP hypergeometric tau function (which may be related to the semi-infinite TL equations) by an action of a special heat operator which was introduced in Sect. 2.2. This action relates hierarchies serving $\mathbb{CP}^1$ and $\mathbb{RP}^2$ Hurwitz counting problems. At the end of this section, we get hypergeometric BKP tau functions with content products (4) and (5) in terms of an action of vertex operators on a special tau function τ_1^{B}.

Section 6 discusses the examples of hypergeometric tau functions which are related to the different choices of the parameters in (4) and (5).

Section 7.2 gathers our main results. We show that the tau function (3) together with either (4) or (5) generates projective Hurwitz numbers and weighted sums of the projective Hurwitz numbers. We show that, choosing the content product as in (5),

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[3] Proposition 2 is actually a new version of the known results for completed cycles presented in [1], but we have not yet written down the correspondence in an explicit way.

sums weighted by Hall–Littlewood polynomials with the parameter $\mathtt{t}$ arise naturally. We present the BKP tau functions which generate Hurwitz numbers with an arbitrary profile over 0 and additional branch points with two types of profile: maximally ramified profiles and minimally ramified profiles (the simple branch points).

In the last section (Sect. 8), we present certain integrals over matrices, which generate projective Hurwitz numbers. Note that the well-known $\beta = 2$ ensemble (i.e., the unitary ensemble or one-matrix model) counts both $\mathbb{CP}^1$ Hurwitz numbers and ribbon graphs with a given number of faces, vertices, and edges [14], and as shown in [50, Sect. 6], the simplest way to get it is to present the one-matrix model as a hypergeometric tau function [26]. We do not succeed in doing the same in the $\mathbb{RP}^2$ case. The analogues of the unitary ensemble are the $\beta = 1, 4$ (orthogonal and symplectic) ensembles, which produce Feynman graphs. Each may be embedded either in an orientable surface (if it is a ribbon graph) or a non-orientable one (if it is a ribbon graph with cross-caps). It was shown in [64] that partition functions of these ensembles are BKP tau functions. However, the perturbation series written as series of Schur functions [58,59] are not the series we need.[4] To get the projective Hurwitz numbers, we suggest other matrix integrals. These integrals contain the simplest BKP tau function τ_1^B in the integration measure, and this is widely used in our paper [see (18), (70), (74), (77), and (106)]. Such an integral of a BKP tau function may or may not be another BKP tau function, the latter case occurring when the integral generates Hurwitz numbers with arbitrary profiles at two or more branch points. We also point out that the multiple use of the BKP tau function τ_1^{B} to deform integration measures of matrix integrals allows one to get Hurwitz numbers related to base surfaces with arbitrary Euler characteristics [see, for instance, (110)]. This approach is further developed in [62].

To end this introduction, note that if in (3) we take r as in (4) and choose $\mathbf{p} = (1, 0, 0, \dots)$, then (3) is a discrete version of the partition function of the orthogonal ensemble of random matrices:

$$\tau^{\mathrm{B}} = \frac{1}{g(n)N!} \sum_{h_1,\dots,h_N \geq 0} \prod_{i<j} |h_i - h_j| \prod_{i=1}^{N} \frac{e^{V(p^*,h_i)}}{h_i!}, \tag{10}$$

$$V(p^*, x) := \sum_{m>0} \frac{1}{m} x^m p_m^* \tag{11}$$

where as we shall see, the variables ζ and $\mathbf{p}^*$ are related via $V(\mathbf{p}^*, x-1) - V(\mathbf{p}^*, x) = V(\zeta, x)$. From [64] we know that (10) is the BKP tau function with the variables $\mathbf{p}^*$ playing the role of BKP higher times. The factor $g(n)$ is given in “Appendix B.”

In a similar way, we may obtain a discrete analogue of the circular $\beta = 1$ ensemble by choosing (5), see Remark 20 in Sect. 5 which proves that for a certain specification of $\mathbf{p}$ the series (3) is a BKP tau function with respect to the variables ξ.

[4] Recently, paper [13] has investigated the graph counting of the $\beta = 1, 2$ ensembles.

Relation (10) may be interesting because the $\beta = 1$ ensemble generates Mobius graphs related to n-gulations of non-orientable surfaces (see [12,46] and references therein).

We now study the above in detail. This paper is a development of our preprint [50].

2 Hurwitz numbers

2.1 Definitions and examples

For a partition Δ of a number $d = |\Delta|$, let $\ell(\Delta)$ be the number of nonvanishing parts. For the Young diagram corresponding to Δ, the number $|\Delta|$ is the weight of the diagram and $\ell(\Delta)$ is the number of rows. Let $(d_1, \dots, d_\ell)$ denote the Young diagram with rows of length $d_1, \dots, d_\ell$ and the corresponding partition of $d = \sum d_i$. We shall need the notion of the colength of a partition Δ, which is $\ell^*(\Delta) := |\Delta| - \ell(\Delta)$.

Let us consider a connected compact surface without boundary Ω and a branched covering $f : \Sigma \to \Omega$ by a connected or non-connected surface Σ. We will consider a covering f of degree d. This means that the preimage $f^{-1}(z)$ consists of d points $z \in \Omega$, except at some finite number of points. These points are called *critical values of f*.

Consider the preimage $f^{-1}(z) = \{p_1, \dots, p_\ell\}$ of $z \in \Omega$. Let d_i be the degree of f at p_i. This means that in the neighborhood of p_i the function f is homeomorphic to $x \mapsto x^{d_i}$. The set $(d_1 \dots, d_\ell)$ is the partition of d, called the *topological type of z*.

Now fix points $z_1, \dots, z_{\mathrm{F}}$ and partitions $\Delta^{(1)}, \dots, \Delta^{(\mathrm{F})}$ of d. Let

$$\widetilde{C}_{\Omega(z_1 \dots, z_{\mathrm{F}})}\left(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})}\right)$$

denote the set of all branched coverings $f : \Sigma \to \Omega$ with critical points $z_1, \dots, z_{\mathrm{F}}$ of topological types $\Delta^{(1)}, \dots, \Delta^{(\mathrm{F})}$.

The coverings $f_1 : \Sigma_1 \to \Omega$ and $f_2 : \Sigma_2 \to \Omega$ are said to be isomorphic if there exists a homeomorphism $\varphi : \Sigma_1 \to \Sigma_2$ such that $f_1 = f_2\varphi$. Let $\mathtt{Aut}(f)$ be the group of automorphisms of the covering f. Isomorphic coverings have isomorphic groups of automorphisms of degree $|\mathtt{Aut}(f)|$.

Consider now the set $C_{\Omega(z_1 \dots, z_{\mathrm{F}})}(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})})$ of isomorphic classes in $\widetilde{C}_{\Omega(z_1 \dots, z_{\mathrm{F}})}(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})})$. This is a finite set. The sum

$$H^{\mathrm{E,F}}\left(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})}\right) = \sum_{f \in C_{\Omega(z_1 \dots, z_{\mathrm{F}})}(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})})} \frac{1}{|\mathtt{Aut}(f)|}$$

does not depend on the location of the points $z_1 \dots, z_{\mathrm{F}}$ and is called the *Hurwitz number*. Here F denotes the number of branch points, and E is the Euler characteristic of the base surface.

Example Let $f : \Sigma \to \mathbb{RP}^2$ be a covering without critical points. If Σ is connected, then $\Sigma = \mathbb{RP}^2$, $\deg f = 1$ or $\Sigma = S^2$, $\deg f = 2$. Therefore, if $d = 3$, then $\Sigma = \mathbb{RP}^2 \coprod \mathbb{RP}^2 \coprod \mathbb{RP}^2$ or $\Sigma = \mathbb{RP}^2 \coprod S^2$. Thus $H^{1,0}(3) = 1/3! + 1/2! = 2/3$.

The Hurwitz numbers arise in different fields of mathematics: from algebraic geometry to integrable systems. They are well studied for orientable Ω. In this case, the Hurwitz number coincides with the weighted number of holomorphic branched coverings of a Riemann surface Ω by other Riemann surfaces, having critical points $z_1, \dots, z_{\mathrm{F}} \in \Omega$ of the topological types $\Delta^{(1)}, \dots, \Delta^{(\mathrm{F})}$, respectively. The well-known isomorphism between Riemann surfaces and complex algebraic curves gives the interpretation of the Hurwitz numbers as the numbers of morphisms of complex algebraic curves.

Similarly, the Hurwitz number for a non-orientable surface Ω coincides with the weighted number of dianalytic branched coverings of the Klein surface without boundary by another Klein surface and coincides with the weighted number of morphisms of real algebraic curves without real points [7,47,48]. An extension of the theory to all Klein surfaces and all real algebraic curves leads to Hurwitz numbers for surfaces with boundaries [5,49].

The Hurwitz numbers have a purely algebraic description. Any branched covering $f : \Sigma \to \Omega$ with critical points $z_1, \dots, z_{\mathrm{F}} \in \Omega$ generates a homomorphism $\phi : \pi_1(u, \Omega\backslash\{z_1, \dots z_F\}) \to S_\Gamma$, where u is a point in Ω, to the group of permutations of the set $\Gamma = f^{-1}(u)$ by the monodromy along contours of $\pi_1(u, \Omega\backslash\{z_1, \dots z_F\})$. Moreover, if $l_i \in \pi_1(u, \Omega\backslash\{z_1, \dots z_F\})$ is a contour around z_i, then the cyclic type of the permutation $\phi(l_i)$ is $\Delta^{(i)}$. Let

$$\mathtt{Hom}_\Omega(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})})$$

be the group of all homomorphisms $\phi : \pi_1(u, \Omega\backslash\{z_1, \dots z_F\}) \to S_\Gamma \cong S_d$ with this property. Isomorphic coverings generate elements of $\mathtt{Hom}_\Omega(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})})$ conjugated by S_d. Thus we construct the one-to-one correspondence between $C_{\Omega(z_1 \dots, z_{\mathrm{F}})}(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})})$ and the conjugacy classes of the homomorphism group $\mathtt{Hom}_\Omega(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})})$.

Consider the last set in more detail. Any $s \in S_d$ generates the interior automorphism $I_s(g) = sgs^{-1}$ of S_d. Therefore, S_d acts on $\mathtt{Hom}_\Omega(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})})$ by $s(h) = I_s h$. The orbit of this action of $I = \{I_s\}$ corresponds to an equivalence class of coverings. Moreover, the group $\mathtt{A} = \{s \in S_d | s(h) = h\}$ is isomorphic to the group $\mathtt{Aut}(f)$, where the covering f corresponds to the homomorphism h.

Consider the splitting $\mathtt{Hom}_\Omega(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})}) = \bigcup_{i=1}^r H_i$ of orbits by I. Then the cardinality $|H_i|$ is $d!/|\mathtt{A}(h_i)| = d!/|\mathtt{Aut}(f_i)|$, where $h_i \in H_i$. On the other hand, the orbits H_i are in one-to-one correspondence with the classes of coverings. Therefore,

$$\frac{1}{d!}\left|\mathtt{Hom}_\Omega(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})})\right| = \frac{1}{d!}\sum_{i=1}^r |H_i| = \sum_{i=1}^r \frac{1}{|\mathtt{Aut}(f_i)|}$$

is the Hurwitz number $H_\Omega(d; \Delta^{(1)} \dots, \Delta^{(\mathrm{F})})$.

We now find $|\mathtt{Hom}_\Omega(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})})|$ in terms of the characters of S_d. Recall that the cyclic type of $s \in S_d$ is specified by the cardinalities $\Delta = (d_1, \dots, d_\ell)$ of the subsets into which the permutation s splits the set $\{1, \dots, d\}$. Any partition Δ of d generates the set $C_\Delta \subset S_d$, consisting of permutations of cyclic type Δ. The cardinality of C_Δ is equal to

$$|C_\Delta| = \frac{|\Delta|!}{z_\Delta}, \qquad z_\Delta = \prod_{i=1}^{\infty} i^{m_i}\, m_i!, \tag{12}$$

where m_i denotes the number of parts equal to i in the partition Δ. The partition Δ is often then denoted by $(1^{m_1}2^{m_2}\cdots)$. Moreover, if $s_1, s_2 \in C_\Delta$, then $\chi(s_1) = \chi(s_2)$ for any complex character χ of S_d. Thus for a partition Δ, we can define $\chi(\Delta)$ by $\chi(\Delta) = \chi(s)$ for $s \in C_\Delta$.

The Mednykh–Pozdnyakova–Jones formula is [33,39,41,42]

$$\left|\mathtt{Hom}_\Omega(d; \Delta^{(1)}, \ldots, \Delta^{(\mathrm{F})})\right| = d! \sum_\lambda \left(\frac{\dim\lambda}{d!}\right)^{\mathrm{E}} \prod_{i=1}^{\mathrm{F}} |C_{\Delta^{(i)}}| \frac{\chi(\Delta^{(i)})}{\dim\lambda},$$

where $\mathrm{E} = \mathrm{E}(\Omega)$ is the Euler characteristic of Ω and χ ranges over the irreducible complex characters of S_d, associated with Young diagrams of weight d. Thus we obtain (1). In particular, for the projective plane $\mathbb{RP}^2$, we get relation (1) with $\mathrm{E} = 1$.

Example Let $\mathrm{E} = 1$, $\mathrm{F} = 0$, and $d = 3$. Then,

$$H^{1,0}(3) = \sum_{|\lambda|=3} \frac{\dim\lambda}{d!} = \frac{4}{6} = \frac{2}{3}.$$

In general, for the unbranched covering of $\mathbb{RP}^2$, we get the following generating function [compare with (70)]

$$e^{c+c^2/2} = \sum_{d\geq 0} c^d H^{1,0}(d). \tag{13}$$

The exponent reflects the fact that the connected unbranched covers of the projective plane may consist of either the projective plane (onefold cover, the term c) or the Riemann sphere (twofold cover, the term $c^2/2$, where the 2 in the denominator is the order of the automorphisms of the covering by the sphere). Finally, we write down a purely combinatorial definition of the projective Hurwitz numbers [33,42]. Let us consider the symmetric group S_d and the equation

$$R^2 X_1 \cdots X_{\mathrm{F}} = 1, \quad R, X_i \in S_d, \quad X_i \in C_{\Delta^{(i)}}, \;\; i = 1, \ldots, \mathrm{F}, \tag{14}$$

where $C_{\Delta^{(i)}}$, $i = 1, \ldots, \mathrm{F}$, are the cycle classes of a given set of partitions $\Delta^{(i)}$, $i = 1, \ldots, \mathrm{F}$, of a given weight d. Then $H^{1,\mathrm{F}}(d; \Delta^{(1)}, \ldots, \Delta^{(\mathrm{F})})$ is the number of solutions to (14) divided by $d!$. Hence, for an unbranched threefold covering, we get 4 solutions to $R^2 = 1$ in S_3: the unit element and three transpositions. Thus $H^{1,0} = 4 : 3!$ as obtained in the example above. The number of solutions to $R^2 = 1$ in S_d is given by (13).

2.2 Remarks on the Mednykh–Pozdnyakova–Jones character formula

We start from the following preliminary:

Remark 2 It follows from paper [15] by Dijkgraaf that the Hurwitz numbers for closed orientable surfaces form a 2D topological field theory. An extension of this result to the case of Klein surfaces (thus to orientable and non-orientable surfaces) was found in Theorem 5.2 of [5] (see also Corollary 3.2 in [6]). On the other hand, the Mednykh–Pozdnyakova–Jones formula describes the Hurwitz numbers in terms of characters of the symmetric groups. In this section, we interpret the axioms of the Klein topological field theory [5] for Hurwitz numbers in terms of characters of symmetric groups. This approach differs from the one in [5].

(A) We begin with the following simple statement

Lemma 1

$$\begin{aligned} &H^{\mathrm{E}+\mathrm{E}_1,\mathrm{F}+\mathrm{F}_1}\left(d;\Delta^{(1)},\ldots,\Delta^{(\mathrm{F}+\mathrm{F}_1)}\right)\\ &\quad=\sum_{\Delta}\frac{d!}{|C_\Delta|}H^{\mathrm{E}+1,\mathrm{F}+1}\left(d;\Delta^{(1)},\ldots,\Delta^{(\mathrm{F})},\Delta\right)H^{\mathrm{E}_1+1,\mathrm{F}_1+1}\\ &\qquad\times\left(d;\Delta,\Delta^{(\mathrm{F}+1)},\ldots,\Delta^{(\mathrm{F}_1)}\right). \end{aligned} \tag{15}$$

In particular,

$$H^{\mathrm{E}-1,\mathrm{F}}\left(d;\Delta^{(1)},\ldots,\Delta^{(\mathrm{F})}\right)=\sum_{\Delta}H^{\mathrm{E},\mathrm{F}+1}\left(d;\Delta^{(1)},\ldots,\Delta^{(\mathrm{F})},\Delta\right)\chi(\Delta), \tag{16}$$

where $\chi(\Delta)=d!H^{1,1}(d;\Delta)/|C_\Delta|$ *are rational numbers explicitly defined in the following way by a partition* Δ*:*

$$\chi(\Delta)=\sum_{\substack{\lambda\\|\lambda|=|\Delta|}}\chi_\lambda(\Delta)=\left[\prod_{i>0,\,\text{even}}e^{\frac{i}{2}\frac{\partial^2}{\partial p_i^2}}\cdot p_i^{m_i}\prod_{i>0,\,\text{odd}}e^{\frac{i}{2}\frac{\partial^2}{\partial p_i^2}+\frac{\partial}{\partial p_i}}\cdot p_i^{m_i}\right]_{\mathbf{p}=0}, \tag{17}$$

and $\chi_\lambda(\Delta)$ *is the character of the representation* λ *of the symmetric group* S_d*,* $d=|\lambda|$*, evaluated on the cycle class* $\Delta=(1^{m_1}2^{m_2}\cdots)$*.*

As a corollary, we get that the Hurwitz numbers of the projective plane may be obtained from the Hurwitz numbers of the Riemann sphere, while the Hurwitz numbers of the torus and the Klein bottle [see (9)] may be obtained from the Hurwitz numbers of the projective plane.

First we prove the second equality in (17). It follows from the relations

$$e^{\sum_{i>0}\frac{i}{2}\frac{\partial^2}{\partial p_i^2}+\sum_{i>0,\,\mathrm{odd}}\frac{\partial}{\partial p_i}} = \sum_\lambda s_\lambda(\tilde{\partial}), \tag{18}$$

$$\left[s_\lambda(\tilde{\partial})\cdot s_\mu(\mathbf{p})\right]_{\mathbf{p}=0} = \delta_{\lambda,\mu}, \quad p_1^{m_1}p_2^{m_2}\cdots =: p_\Delta = \sum_\lambda \chi_\lambda(\Delta)s_\lambda(\mathbf{p}), \tag{19}$$

where $s_\lambda(\tilde{\partial})$ is $s_\lambda(\mathbf{p})$ and each p_i is replaced by $i\partial/\partial p_i$. The heat operator on the left-hand side of (18) plays an important role. Relations (19) may be found in [40]. Relation (18) is derived from the known relation

$$\sum_\lambda s_\lambda(\mathbf{p}(\mathbf{x})) = \prod_{i<j}\frac{1}{1-x_ix_j}\prod_i\frac{1}{1-x_i}, \quad p_m(\mathbf{x}) := \sum_i x_i^m, \tag{20}$$

which may also be found in [40].

The equality (15) follows from the orthogonality relation for characters:

$$\sum_\Delta |C_\Delta|\chi_\lambda(\Delta)\chi_\mu(\Delta) = d!\delta_{\mu,\lambda},$$

where $|\mu| = |\lambda| = |\Delta| = d$, which yields $\sum_\Delta \varphi_\lambda(\Delta)\chi(\Delta) = d!/\dim(\lambda)$. Then formula (1) gives (15).

In (72), we shall see that the heat operator which enters (17) also links solutions of 2KP (TL) and BKP hierarchies.

(B) Another remark is as follows. Let us use the so-called Frobenius notation [40] for a partition λ: $\lambda = (\alpha_1,\ldots,\alpha_\kappa|\beta_1,\ldots,\beta_\kappa)$, $\alpha_1 > \cdots > \alpha_\kappa \geq 0$, $\beta_1 > \cdots > \beta_\kappa \geq 0$. The integer $\kappa = \kappa(\lambda)$ denotes the length of the main diagonal of the Young diagram λ, while the length of λ is denoted by $\ell(\lambda)$.

Lemma 2 *The normalized character labeled by λ evaluated at the cycle (d) (as usual, $d = |\lambda|$) vanishes if $\kappa(\lambda) > 1$. Moreover,*

$$\varphi_\lambda((d)) = (-1)^{\ell(\lambda)+1}\left(\frac{d!}{\dim\lambda}\right)\frac{1}{d}\delta_{1,\kappa(\lambda)}. \tag{21}$$

For the proof, we first note that the Schur function of a one-hook partition, say, $(\alpha_i|\beta_j)$, has the form

$$s_{(\alpha_i|\beta_j)}(\mathbf{p}) = \frac{1}{d}(-1)^{\beta_j}p_{\alpha_i+\beta_j+1} + \cdots,$$

where dots denote terms which do not depend on p_a, $a \geq \alpha_i+\beta_j+1$ [this fact may be derived, say, from the Jacobi–Trudi formula $s_\lambda(\mathbf{p}) = \det s_{(\lambda_i-i+j)}(\mathbf{p})$]. Then from the Giambelli identity, viz., $s_\lambda(\mathbf{p}) = \det s_{(\alpha_i|\beta_j)}(\mathbf{p})$, it follows that s_λ does not depend on p_a, $a > \alpha_1+\beta_1+1$. Thus it does not depend on p_a, $a \geq d > \alpha_1+\beta_1+1$, when

$\kappa(\lambda) > 1$. Due to the character map relation, this means that $\varphi_\lambda((d)) = 0$ if $\kappa(\lambda) > 1$. For a one-hook partition $\lambda = (\alpha_1|\beta_1)$, we have $\alpha_1 + \beta_1 + 1 = d$ and the character map formula (7) yields

$$s_{(\alpha_1|\beta_1)}(\mathbf{p}) = \frac{\dim\lambda}{d!}(p_d\varphi_\lambda((d)) + \cdots),$$

where dots denote terms which do not depend on p_d. We compare the last two formulae and get (21).

Relation (21) allows us to equate Hurwitz numbers related to different Euler characteristics of base Klein surfaces if in both cases there are nonvanishing numbers of ramification profiles (d). Hence, the Mednykh–Pozdnyakova–Jones character formula (1) yields the following:

Proposition 1 *For any natural number g*

$$\begin{aligned} &H^{\mathrm{E}-2g,\mathrm{F}+1}\left(d;\Delta^{(1)},\ldots,\Delta^{(\mathrm{F})},(d)\right) \\ &\quad = d^{2g}H^{\mathrm{E},\mathrm{F}+2g+1}\left(d;\Delta^{(1)},\ldots,\Delta^{(\mathrm{F})},(d),\underbrace{(d),\ldots,(d)}_{2g}\right). \end{aligned} \tag{22}$$

This was first proven by Zagier (for the case of even E), see Appendix A in [39]. We get it in a different way. For d-fold covers we shall say that a branch point is *maximally ramified* if its ramification profile is (d).

Remark 3 Notice that the presence of the profile (d) means that the Hurwitz numbers of the connected and disconnected covering are equal: $H^{\mathrm{E},\mathrm{F}}_{\text{connected}}(d;(d),\ldots) = H^{\mathrm{E},\mathrm{F}}(d;(d),\ldots)$, where dots denote the same set of ramification profiles.

Remark 4 In Appendix A of [39], Zagier considered the polynomial

$$\mathrm{R}_\Delta(\mathrm{q}) := \frac{\prod_{i=1}^{\ell\Delta}(1-\mathrm{q}^{d_i})}{1-\mathrm{q}} =: \sum_r (-1)^r \mathrm{q}^r \chi_r(\Delta).$$

It was shown that χ_r $(0 \le r \le d-1)$ is the character of the irreducible representation of S_d given by $\chi_r(g) = \mathrm{tr}(g,\pi_r)$, $g \in S_d$, $\pi_r = \wedge^r(\mathrm{St}_d)$. Here St_d is the vector space $\{(x_1,\ldots,x_d) \in \mathbb{C}^d | x_1+\cdots+x_d = 0\}$ and S_d acts by permutations of the coordinates. It can be shown that χ_r coincides with χ_λ, where $\lambda = (d-r|r)$. To do this, let us consider the Schur function $s_\lambda(\mathbf{p}(\mathrm{q},0))$, where $p_m(\mathrm{q},0) := 1-\mathrm{q}^m$ and λ is not yet fixed. We get [40]

$$s_\lambda(\mathbf{p}) = (-1)^{\ell(\lambda)-1}(1-\mathrm{q})\mathrm{q}^{\ell(\lambda)-1}\delta_{\kappa(\lambda),1}.$$

In the last relation, let $k = \ell(\lambda) - 1$. Then $\lambda = (d-k|k)$. On the other hand, (7) tells us that

$$s_{(d-k|k)}(\mathbf{p}(\mathtt{q},0)) = (1-\mathtt{q})\sum_{\Delta}\frac{|C_\Delta|}{d!}\chi_{(d-k|k)}(\Delta)\mathrm{R}_\Delta(\mathtt{q})$$
$$= (1-\mathtt{q})\sum_{r=0}^{d-1}(-\mathtt{q})^r\sum_{\Delta}\frac{|C_\Delta|}{d!}\chi_{(d-k|k)}(\Delta)\chi_r(\Delta).$$

We now compare the above relations. The orthogonality of characters results in $\chi_k = \chi_{(d-k|k)}$. This means that, in the presence of a maximally ramified branch point, the summation range in (1) is restricted to one-hook partitions λ. Note also that the polynomials R_Δ are related to the ramification weights in (47).

In what follows, we shall see that tau functions generate Hurwitz numbers containing the maximally ramified branch points.

3 Content products

The content product which enters (3) may be written in the form of a generalized Pochhammer symbol

$$\prod_{(i.j)\in\lambda} r(x+j-i) = \prod_{i=1}^{\ell(\lambda)} r_{\lambda_i}(x-i+1), \tag{23}$$

where $r_n(x) := r(x)r(x+1)\cdots r(x+n-1)$, and also in the form of a sort of Boltzmann weight

$$\prod_{(i.j)\in\lambda} r(x+j-i) = e^{-U_\lambda(x)} := \prod_{i=1}^{\ell(\lambda)} e^{U_{h_i(0)+x}-U_{h_i(\lambda)+x}} = \prod_{i=1}^{\kappa(\lambda)} e^{U_{\alpha_i+x}-U_{-\beta_i+x}}, \tag{24}$$

where α_i, β_i, $i = 1, \dots, \kappa$ are the Frobenius coordinates of the partition λ, $\lambda = (\alpha_1, \dots \alpha_\kappa|\beta_1, \dots, \beta_\kappa)$, with $\kappa = \kappa(\lambda)$ the length of the main diagonal of the Young diagram of the partition λ, and

$$h_i(\lambda) := \lambda_i - i, \quad r(x) =: \exp(U_{x-1} - U_x). \tag{25}$$

The numbers U_x may be fixed by $U_{x_0} = 0$ with a chosen x_0. In the present paper, U_x is chosen as either $V(\zeta, x)$ (parametrization I) or $V(\xi, \mathtt{t}^x) + \xi_0 \log \mathtt{t}$ (parametrization II).

3.1 Parametrization I

Consider the sums of all normalized characters φ_λ evaluated on partitions Δ with a given weight $d = |\lambda| = |\Delta|$ and a given length $\ell(\Delta) = d - k$:

$$\phi_k(\lambda) := \sum_{\substack{\Delta \\ \ell(\Delta)=d-k}} \varphi_\lambda(\Delta), \quad k = 0, \ldots, d-1. \tag{26}$$

Remark 5 Let us note that $\phi_0(\lambda) = 1$. There are two other special cases when the sum of normalized characters (26) contains a single term:

(a) $\phi_1(\lambda) = \varphi_\lambda(\Gamma), \Gamma = (1^{d-2}2)$ (for $d > 1$). This is related to the *minimally ramified* profile, i.e., the one with colength equal to 1. It is the profile of the simple branch point, which is of interest in many applications [15].
(b) $\phi_{d-1}(\lambda) = \varphi_\lambda((d))$. This is related to the cyclic profile which describes the *maximally ramified* profile. It plays a specific role, as described in Proposition 1.

In what follows, we shall use the sums ϕ_k as building blocks to construct weighted sums of the Hurwitz numbers [see, for instance, (59)]. Then cases (a) and (b) produce, not the weighted sums, but the Hurwitz numbers themselves [see (97)].

Remark 6 The quantity $d - \ell(\lambda)$ used in the definition (26) is called the *colength* of a partition λ and will be denoted by $\ell^*(\lambda)$. The colength enters the so-called Riemann–Hurwitz formula, which relates the Euler characteristic E of a base surface to the Euler characteristic E' of its d-branched cover by

$$\mathrm{E}' - d\mathrm{E} + \sum_i \ell^*(\Delta^{(i)}) = 0,$$

where the sum ranges over all branch points.

Let us introduce

$$\deg \phi_k(\lambda) = k. \tag{27}$$

This degree is equal to the colength of the ramification profiles in (26), and due to Remark 6, it will be important later to define the Euler characteristic of the covering surfaces in the parametrization I cases. Next we need:

Lemma 3 *The power sums of the contents of all nodes of a Young diagram λ may be expressed in terms of the normalized characters and ratios of the Schur functions, and they are polynomials in the variables ϕ_k, $k = 1, 2, \ldots$:*

$$\Phi_m(\lambda) := \sum_{(i.j)\in\lambda} (j-i)^m \tag{28}$$

$$= \frac{m}{2\pi i}\oint a^m \prod_{k=1}^{m}\left(1+\sum_{\Delta}\left(e^{2\pi i \frac{k}{m}}a\right)^{-\ell^*(\Delta)}\varphi_\lambda(\Delta)\right)\frac{da}{a} \tag{29}$$

$$= (-1)^{m+1} m \frac{1}{2\pi i}\oint a^m \log\frac{s_\lambda(\mathbf{p}(a))}{s_\lambda(\mathbf{p}_\infty)}\frac{da}{a} \tag{30}$$

$$= (-1)^{m+1} m \frac{1}{2\pi i}\oint a^m \log\left(1+\sum_{k=1}^{d-1} a^{-k}\phi_k(\lambda)\right)\frac{da}{a} \tag{31}$$

$$= m \sum_{\substack{\mu \\ |\mu|=m,\mu_1<d}} (-1)^{\ell^*(\mu)}(\ell(\mu)-1)!\frac{\phi_\mu(\lambda)}{\mathrm{Aut}\,\mu}, \tag{32}$$

where $m \geq 0$, $|\lambda| = |\Delta|$ *and* $\ell^*(\mu) := |\mu| - \ell(\mu)$ *is the colength of the partition* μ. *Here,*

$$\phi_\mu(\lambda) := \prod_{i=k}^{d-1}(\phi_k(\lambda))^{m_k} = \prod_{i=1}^{\ell(\mu)}\phi_{\mu_i}(\lambda),$$
$$\mu = (1^{m_1}\ldots(d-1)^{m_{d-1}}) = (\mu_1,\ldots,\mu_\ell). \tag{33}$$

In (30) $\mathbf{p}(a) = (a, a, \ldots)$ *and* $\mathbf{p}_\infty = (1, 0, 0, \ldots)$, *and in* (32) $\mathrm{Aut}\,\mu = \prod_{i=1}^{\ell(\mu)} m_i!$, *where* m_i *denotes the number of times a part* i *enters the partition* $\mu = (1^{m_1}2^{m_2}\cdots)$.

As we can see from (32)–(33), each integer Φ_m is a quasi-homogeneous polynomial in the rational numbers ϕ_k, and according to (27), we assign the degree as follows:

$$\deg \Phi_m(\lambda) = m. \tag{34}$$

Let us write down the first three $\Phi_m(\lambda)$ for $|\lambda| = d \geq 4$ in terms of normalized characters φ_λ, using (26), (32), and (33). We obtain

$$\Phi_1(\lambda) = \varphi_\lambda(\Gamma), \qquad \Phi_2(\lambda) = (\varphi_\lambda(\Gamma))^2 - 2\varphi_\lambda\left(\left(1^{d-4}2^2\right)\right) - 2\varphi_\lambda\left(\left(1^{d-3}3^1\right)\right),$$
$$\Phi_3(\lambda) = (\varphi_\lambda(\Gamma))^3 - 3\varphi_\lambda(\Gamma)\left(\varphi_\lambda\left(\left(1^{d-4}2^2\right)\right) + \varphi_\lambda\left(\left(1^{d-3}3^1\right)\right)\right)$$
$$+3\varphi_\lambda\left(\left(1^{d-4}4^1\right)\right) + 3\varphi_\lambda\left(\left(1^{d-5}2^1 3^1\right)\right) + 3\varphi_\lambda\left(\left(1^{d-6}2^3\right)\right). \tag{35}$$

We assume that $d > 2$, $m > 2$. As we can see from (26), (32), and (33), each $\Phi_m(\lambda)$ has the form $(\varphi_\lambda(\Gamma))^m + \cdots$, where the dots denote the contribution of cyclic classes marked by partitions, say Δ, whose lengths $\ell(\Delta)$ belong either to the interval $[d-2, d-m]$ if $m < d$, or to the interval $[d-2, 1]$ if $m \geq d$.

The proof that the right-hand side of (28) is equal to (29) is based on the two relations

$$\lim_{n\to\infty}\prod_{k=1}^{m}\left(1-n^{-\frac{1}{m}}e^{2\pi i\frac{k}{m}}x\right)^{n}=e^{-x^m}$$

and

$$\prod_{(i.j)\in\lambda}(a+j-i)=a^{|\lambda|}\left(1+\sum_{\Delta}\varphi_\lambda(\Delta)a^{\ell(\Delta)-|\lambda|}\right)$$
$$=a^{|\lambda|}\left(1+\sum_{k=1}^{d-1}\phi_k(\lambda)a^{-k}\right), \tag{36}$$

which may be obtained from relations in [40]. This relation is important and will be further exploited to get Hurwitz numbers and weighted sums of Hurwitz numbers.

The proof that the right-hand side of (28) is equal to (31) follows from (36):

$$\prod_{i,j\in\lambda}e^{-\sum_{m>0}\frac{1}{m}(-a)^{-m}(j-i)^m}=\prod_{i,j\in\lambda}\left(1+\frac{j-i}{a}\right)=1+\sum_{k=1}^{d-1}\phi_k(\lambda)a^{-k}.$$

Remark 7 By comparing the first and last terms in (36), we conclude that

$$\phi_k(\lambda)=0,\quad \text{if } k>d-\kappa(\lambda),$$

where $\kappa(\lambda)$ in the length of the main diagonal of the Young diagram of the partition λ. Now take $k=d-1$ as in (b) of Remark 5. Then it follows that $\varphi_\lambda\left((d)\right)$ is nonvanishing only for one-hook Young diagrams $\lambda=(d-a,1^a)$, $a=0,1,\ldots,d$.

Remark 8 It follows from (36) that

$$a^d\left(1+\sum_{k=1}^{d-1}\phi_k(\lambda)a^{-k}\right)=0,$$

if a is integer and also if $-\lambda_1<a<\ell(\lambda)$.

Remark 9 From (36) we see that

$$s_{(1^k)}\left(\Phi(\lambda)\right)=\phi_k(\lambda), \tag{37}$$

where $\Phi(\lambda)=(\Phi_1(\lambda),\Phi_2(\lambda),\ldots)$.

Proposition 2 *Let*

$$r(\zeta,h;x)=\exp V(\zeta,hx), \tag{38}$$

where ζ is the infinite set of parameters $\zeta = (\zeta_1, \zeta_2, \dots)$ and V is defined by (11). Then the related content product may be expressed in terms of characters in the following explicit way:

$$\prod_{(i.j)\in\lambda} r(\zeta, h; j-i) = \exp \sum_{m>0} \frac{1}{m} h^m \zeta_m \Phi_m. \tag{39}$$

From (24) we get

$$\prod_{(i.j)\in\lambda} r(\zeta, j-i) = \prod_{i=1}^{\kappa(\lambda)} e^{V(\mathbf{p}^*,\alpha_i)-V(\mathbf{p}^*,-\beta_i-1)} = \prod_{i=1}^{L} e^{V(\mathbf{p}^*,h_i(\lambda))-V(\mathbf{p}^*,0)}, \tag{40}$$

where $h_i(\lambda) = \lambda_i - i$ and α_i, β_i are the Frobenius coordinates of the partition $\lambda = (\alpha|\beta)$. Then the variables $\mathbf{p}^* = (p_1^*, p_2^*, \dots)$ are related to the variables ζ by the triangle transformation given by

$$V(\zeta, x) = V(\mathbf{p}^*, x-1) - V(\mathbf{p}^*, x). \tag{41}$$

In particular, we get the discrete version of the orthogonal ensemble given by (10).

Remark 10 With the help of (40), Proposition 2 may be related to the well-known results [1,52] on Hurwitz numbers and completed cycles as follows. In [1], the generating function for Hurwitz numbers of covers of $\mathbb{CP}^1$ was studied in the form

$$\tau^{\mathrm{TL}}(\mathbf{p}^{(1)}, \mathbf{p}^{(2)}|\mathbf{p}^*) = \sum_{\lambda} e^{\sum_{m>0} \frac{1}{m} p_m^* \mathrm{C}_\lambda(m)} s_\lambda(\mathbf{p}^{(1)}) s_\lambda(\mathbf{p}^{(2)}), \tag{42}$$

and identified with a specification of the KP hypergeometric tau function [38,57]. The exponential prefactor in this KP hypergeometric tau function coincides with the right-hand side of (40). Then it follows from (41) that

$$\sum_{i=1}^{\ell(\lambda)} \left((\lambda_i - i)^m - (-i)^m\right) =: \mathrm{C}_\lambda(m) = \sum_{k=1}^{m-1} \frac{(-1)^{m-k}}{(m-k)!} \frac{(m-1)!}{(k-1)!} \Phi_k(\lambda).$$

We may collect several further remarks on (42).

Remark 11 (A) Let $\mathbf{p}^{(1)} = \mathbf{p}^{(2)} = (1, 0, 0, \dots)$ in (42). Then the variables $\mathbf{p}^*$ may be identified with the KP higher times, because expression (42) yields a discrete version of the one-matrix model (the unitary ensemble), in a similar way to (10), which describes a discrete model of the orthogonal ensemble. (B) If we choose $\mathbf{p}^{(1)} = \mathbf{p}^{(2)} = \mathbf{p}(0, \mathtt{t})$ (see introduction for the notation) and specify p_m^*, we obtain the partition functions of the $U(N)$ Chern–Simons model on S^3 with coupling constant $g_s = -\log \mathtt{t}$ [8,63]. (C) If we take $\mathbf{p}^{(2)} = \mathbf{p}(0, \mathtt{t})$ and $p_m^* = 0$, $m > 2$, then the right-hand side of (42) generates the Marino–Vafa relations for the Hodge integrals [68] (where p, $\lambda\tau$, and λ are $\mathbf{p}^{(1)}$, p_2^*, and $\sqrt{-1}\log \mathtt{t}$, respectively, in our notation). (D) It was first noticed in

[38] (see also [60]) that, for the choice $\mathbf{p}^{(1)} = (1, 0, 0, \dots)$, $p_m^{(2)} = \sum x_i^m$, the series (42) is a discrete version of the Kontsevich model:

$$\tau^{\mathrm{TL}}(\mathbf{x}, \mathbf{p}^*) = \frac{1}{N!} \sum_{h_1,\dots,h_N} \prod_{i<j} (h_i - h_j) \prod_{i=1}^{N} \frac{1}{h_i!} e^{V(\mathbf{p}^*, h_i) + \mathrm{L}_i h_i}, \quad x_i = e^{\mathrm{L}_i}.$$

3.2 Parametrization II

If $j-i$ is the content of the node of λ, the number $\mathtt{t}^{j-i}$ is called the *quantum content* of the node.

Lemma 4 *The power sum of the quantum contents $\mathtt{t}^{j-i}$ of all nodes of the Young diagram λ is expressed in terms of the parts of λ, the Schur functions, and the normalized characters φ_λ by*

$$T_\lambda(\mathtt{t}) := \sum_{(i.j)\in\lambda} \mathtt{t}^{j-i} \tag{43}$$

$$= \sum_{i=1}^{\ell(\lambda)} \mathtt{t}^{1-i} \frac{1-\mathtt{t}^{\lambda_i}}{1-\mathtt{t}} = \frac{\mathtt{t}}{\mathtt{t}-1} \sum_{i=1}^{\ell(\lambda)} \left(\mathtt{t}^{h_i(\lambda)} - \mathtt{t}^{h_i(0)} \right) \tag{44}$$

$$= p_1 \frac{\partial}{\partial p_1} \log s_\lambda(\mathbf{p})|_{\mathbf{p}=\mathbf{p}(0,\mathtt{t})} \tag{45}$$

$$= \frac{d + \sum'_\Delta m_1(\Delta) A_\Delta(\lambda, \mathtt{t})}{1 + \sum'_\Delta A_\Delta(\lambda, \mathtt{t})}, \quad A_\Delta(\lambda, \mathtt{t}) = \varphi_\lambda(\Delta) \frac{(1-\mathtt{t})^d}{\prod_{j=1}^{\ell(\Delta)} \left(1 - \mathtt{t}^{d_j}\right)}, \tag{46}$$

where $h_i(\lambda) = \lambda_i - i$, $|\lambda| = |\Delta| = d$, and $\sum'$ denotes the sum over all partitions except the partition (1^d). The partition Δ is written either as $(d_1, \dots, d_{\ell(\Delta)})$ or as $(1^{\mathrm{m}_1} 2^{\mathrm{m}_2} \cdots)$, and $\mathrm{m}_i = \mathrm{m}_i(\Delta)$ denotes the number of parts of Δ equal to i. In (45), we first take the derivative with respect to p_1, then evaluate the power sum variables $\mathbf{p}$ as $\mathbf{p} = \mathbf{p}(0, \mathtt{t}^m) = (p_1, p_2, \dots)$, where $p_k = p_k(0, \mathtt{t}^m) = (1 - \mathtt{t}^{km})^{-1}$.

The proof is similar to the previous case, but instead of (36) we use another relation:

$$\prod_{(i.j)\in\lambda} \frac{1 - \mathtt{q}\mathtt{t}^{j-i}}{1 - \tilde{\mathtt{q}}\mathtt{t}^{j-i}} = \frac{s_\lambda(\mathbf{p}(\mathtt{q}, \mathtt{t}))}{s_\lambda(\mathbf{p}(\tilde{\mathtt{q}}, \mathtt{t}))} = \left(\frac{1-\mathtt{q}}{1-\tilde{\mathtt{q}}}\right)^{|\lambda|} \frac{1 + \sum'_\Delta \varphi_\lambda(\Delta) w(\Delta, \mathtt{q}, \mathtt{t})}{1 + \sum'_\Delta \varphi_\lambda(\Delta) w(\Delta, \tilde{\mathtt{q}}, \mathtt{t})}, \tag{47}$$

where $\mathbf{p}(\mathtt{q}, \mathtt{t}) = (p_1(\mathtt{q}, \mathtt{t}), p_2(\mathtt{q}, \mathtt{t}), \dots)$, with

$$p_m(\mathtt{q}, \mathtt{t}) = \frac{1 - \mathtt{q}^m}{1 - \mathtt{t}^m}, \tag{48}$$

and

$$w(\Delta,\mathtt{q},\mathtt{t}) = \frac{(1-\mathtt{t})^d}{(1-\mathtt{q})^d}\prod_{i=1}^{\ell(\Delta)}\frac{1-\mathtt{q}^{d_i}}{1-\mathtt{t}^{d_i}} \tag{49}$$

may be called the $\mathtt{q},\mathtt{t}$-ramification weight. We have $w(\Delta,e^{ah},e^h)\to a^{\ell^*(\Delta)}$ as $h\to 0$. Equation (47) is easily obtained from known relations presented in [40]. For the proof we put $\tilde{\mathtt{q}}=0$, replace $\mathtt{q}\to\mathtt{q}/n$, and consider the n th power of (47), obtaining (46) from the right-hand side of (47), where we insert (49). Then (45) follows from (46).

Remark 12 Apart from (43)–(46), we may also write

$$T_\lambda(\mathtt{t}^m) = \frac{1}{2\pi i}\oint \mathtt{q}^{-1-m}\log\frac{s_\lambda(\mathbf{p}(\mathtt{q},\mathtt{t}))}{s_\lambda(\mathbf{p}(0,\mathtt{t}))}d\mathtt{q},\quad m>0,$$

which is the analogue of (31).

Remark 13 We get $\Phi_m(\lambda)=(\mathtt{t}\partial/\partial\mathtt{t})^m\cdot T_\lambda(\mathtt{t})|_{\mathtt{t}=1}$.

Proposition 3 *Let*

$$r(\xi,x|\mathtt{t}) = e^{V(\xi_+,\mathtt{t}^x)+\xi_0 x\log\mathtt{t}+V(\xi_-,\mathtt{t}^{-x})} = e^{\sum_{m\neq 0}\frac{1-\mathtt{t}^m}{m\mathtt{t}^m}\mathtt{p}^*_m\mathtt{t}^{mx}+\xi_0 x\log\mathtt{t}}, \tag{50}$$

where ξ is the collection of parameters ξ_0 and $\xi_\pm=(\xi_{\pm1},\xi_{\pm2},\dots)$, and where V is defined by (11)*. Then*

$$\prod_{(i.j)\in\lambda} r(\xi,x+j-i|\mathtt{t}) = e^{\xi_0(\varphi_\lambda(\Gamma)+|\lambda|x)\log\mathtt{t}+\sum_{m\neq0}\frac{1}{m}\xi_m\mathtt{t}^{mx}T_\lambda(\mathtt{t}^m)} \tag{51}$$

$$= \prod_{i=1}^{\ell(\lambda)} e^{\frac{\xi_0\log\mathtt{t}}{2}\left((x+h_i(\lambda))^2+(x+h_i(\lambda))-(x+h_i(0))^2-(x+h_i(0))\right)+\sum_{m\neq0}\frac{1}{m}\mathtt{p}^*_m\left(\mathtt{t}^{(h_i(\lambda)+x)m}-\mathtt{t}^{(h_i(0)+x)m}\right)}, \tag{52}$$

with $\mathtt{p}^*_m=\xi_m\frac{\mathtt{t}^m}{\mathtt{t}^m-1}$, $h_i(\lambda)=\lambda_i-i$, and $h_i(0)=-i$.

Remark 14 The right-hand side of (47) may be obtained by specifying the parameters: $x=0$, $\xi_m=0$, $m\le 0$, and $\xi_m=\tilde{\mathtt{q}}^m-\mathtt{q}^m$, $m>0$ in (51). Relation (47) can be used to get Hurwitz numbers in special cases. However, we need to explain how we would treat the denominator on the right-hand side. Among others, let us consider two different ways to do this.
(A) Let us fix $\tilde{\mathtt{q}}$. Then $w(\Delta,\tilde{\mathtt{q}},\mathtt{t})$ tends to zero if $\mathtt{t}\to 1$ for $\Delta\neq(1^d)$. This allows us to expand the denominator on the right-hand side of (47) and also $T_\lambda(\mathtt{t})$ in (46) as Taylor series in the normalized characters of S_d for $\mathtt{t}$ close to 1. The limit $\mathtt{t}\to 1$ returns us to the case studied in Sect. 3.1.

There is a different limiting procedure which allows us to get rid of the infinite sum arising from the character expansion of $s_\lambda(\mathbf{p}(\tilde{\mathtt{q}},\mathtt{t}))$ in the denominator of the right-hand side of (47) when the leading term in the denominator is the term with $\Delta=(d)$: (B) Here we take $\tilde{\mathtt{q}}$ to be close to 1.

Lemma 5 *Let ϵ be a small parameter. Then*

$$\varphi_\lambda((d))\prod_{(i,j)\in\lambda}\frac{1-\mathtt{q}\mathtt{t}^{j-i}}{1-e^\epsilon\,\mathtt{t}^{j-i}}=\frac{1}{\epsilon}\delta_{1,\kappa(\lambda)}\frac{1-\mathtt{t}^d}{d}\frac{s_\lambda(\mathbf{p}(\mathtt{q},\mathtt{t}))}{s_\lambda(\mathbf{p}_\infty)}+O(1), \tag{53}$$

where we use the notation of Lemma 2, and where $O(1)$ denotes terms of order ϵ^k, $k=0,1,\ldots$.

The lemma follows from (47), from $s_\lambda(\mathbf{p}_\infty)=\dim\lambda/d!$, and from (7), where the power sums are specified by (48). In particular, we have

$$s_\lambda(\mathbf{p}(e^\epsilon,\mathtt{t}))=\epsilon\frac{\dim\lambda}{d!}\varphi_\lambda((d))\frac{d}{1-\mathtt{t}^d}+o(\epsilon). \tag{54}$$

Note the similarity between the relations (48)–(49) and the scalar product of the power sum symmetric functions, where the Macdonald symmetric functions are orthogonal [40]. We have

Remark 15 For $\xi_0=\xi_-=0$, let us rewrite (51) in the form

$$\prod_{(i.j)\in\lambda} r(\xi,x+j-i|\mathtt{t})=e^{\sum_{m>0}\frac{1}{m}(1-\mathtt{t}^m)\mathtt{p}^*_m\mathtt{t}^{mx-m}T_\lambda(\mathtt{t}^m)}$$
$$=\sum_\mu \mathtt{t}^{(x-1)|\lambda|}P_\mu(\mathtt{p}^*;0,\mathtt{t})Q_\mu\left(\mathtt{T}_{\lambda,\mathtt{t}};0,\mathtt{t}\right), \tag{55}$$

where P_μ and Q_μ are Macdonald polynomials with parameters $\mathtt{q}$ and $\mathtt{t}$ evaluated at $\mathtt{q}=0$ (so these are the Hall–Littlewood polynomials). Here the notation is the same as in [40], but P_μ and Q_μ are written as functions of power sum variables. The latter are $\mathtt{p}^*=(\mathtt{p}^*_1,\mathtt{p}^*_2,\mathtt{p}^*_3,\ldots)$ for P_μ and $\mathtt{T}_{\lambda,\mathtt{t}}=\left(T_\lambda(\mathtt{t}),T_\lambda(\mathtt{t}^2),T_\lambda(\mathtt{t}^3),\ldots\right)$ for the second Hall–Littlewood polynomial Q_μ. The polynomials Q_μ may also be viewed as symmetric functions of the d variables which are the quantum contents $\mathtt{t}^{j-i}$, $(i,j)\in\lambda$. We note also that the scalar products of the power sums and of the Macdonald polynomials with the parameters $\mathtt{q}$ and $\mathtt{t}$ may be written as [40]

$$\langle p_\lambda,p_\mu\rangle=z_\mu\prod_{i=1}^{\ell(\mu)}\frac{1-\mathtt{q}^{\mu_i}}{1-\mathtt{t}^{\mu_i}}\delta_{\mu,\lambda},\quad \langle P_\lambda,Q_\mu\rangle=\delta_{\mu,\lambda}.$$

The number z_μ is defined by (12). The reason for the appearance of the Hall–Littlewood polynomials is not clear.

4 Weighted sums of Hurwitz numbers

Below we will consider combinations of normalized characters written as

$$\sum_{\substack{|\lambda|=d\\ \ell(\lambda)\le N}} (*)\varphi_\lambda(\Delta)\frac{\dim\lambda}{d!},$$

where (*) denotes a chosen (polynomial or non-polynomial) function in many variables, and the role of the variables is played by the normalized characters φ_λ evaluated at all possible different partitions of the number d. According to (1), when $d \le N$, this is a weighted sum of the projective Hurwitz numbers. However, the parameter N is an arbitrary integer and may be chosen large enough, so in this work we do not need to care about this inequality. The point is that the sums below may obtained as specifications of $\mathrm{H}_r(d, \Delta)$ in (8) resulting from the choice of r in either (4) or (5). Other examples of specifications are also presented in Sect. 6.

The weighted sums below may be compared with the weighted sums in [25,29], which investigate the statistics of the $\mathbb{CP}^1$ Hurwitz numbers compatible with the property of integrability of the related generating series. Note that, although we cannot choose functions (*) arbitrarily, there are infinitely many ways to choose them, and we are interested in those which are related to BKP tau functions in a natural way. The factor (*) appears due to the content product in the formula for the hypergeometric tau functions. The weighted sums below are labeled by a given partition $\mu = (\mu_1, \mu_2, \dots)$. Our examples are as follows.

4.1 Parametrization I

This is the case described in (4). Here we weight the Hurwitz numbers with symmetric functions of the contents viewed as functions of the power sum variables, the role of the power sums being played by $(\Phi_1(\lambda), \Phi_2(\lambda), \dots)$ as defined in (32) and $(\phi_1(\lambda), \phi_2(\lambda), \dots)$ as defined in (26).

(a) Hurwitz numbers weighted by power sum monomials built from $(\Phi_1(\lambda), \Phi_2(\lambda), \dots)$, where $\Phi_\mu(\lambda) := \prod_{i=1}^{\ell(\mu)} \Phi_{\mu_i}(\lambda)$:

$$\mathrm{c}_\mu(d;\Delta) := \sum_{\substack{\lambda\\ |\lambda|=d}} \Phi_\mu(\lambda)\,\varphi_\lambda(\Delta)\frac{\dim\lambda}{d!}. \tag{56}$$

This is a linear combination of Hurwitz numbers of (both connected and disconnected) d-fold covers with the profile Δ at ∞ and $\ell(\mu)$ different branch points. The Euler characteristic of the covers is $\mathrm{E}' = \ell(\Delta) - d - |\mu|$. This follows from the Hurwitz formula $\mathrm{E}' - Ed = \sum_i (\ell(\Delta_i) - d)$ for a d-fold covering, where the sum ranges over all branch points, and E' and E are the Euler characteristics of the cover and the base, respectively.

If we choose $\mu = (1^b)$, the integer $\mathrm{c}_\mu(d;\Delta)$ counts the number of non-equivalent branched coverings of the projective plane with a given ramification profile at some point and b simple branch points:

$$\mathrm{C}_{(1^b)}(\Delta) = H^{1,b+1}\left(d; \underbrace{\Gamma, \ldots, \Gamma}_{b}, \Delta\right), \quad |\Gamma| = |\Delta| = d. \tag{57}$$

For $\mu = (1^b 2)$, by (35), we obtain

$$\mathrm{C}_{(1^b 2)}(\Delta) = H^{1,b+3}\left(d; \underbrace{\Gamma, \ldots, \Gamma}_{b+2}, \Delta\right) - 2H^{1,b+2}\left(d; \underbrace{\Gamma, \ldots, \Gamma}_{b}, \left(1^{d-4}2^2\right), \Delta\right)$$
$$-2H^{1,b+2}\left(d; \underbrace{\Gamma, \ldots, \Gamma}_{b}, \left(1^{d-3}3^1\right), \Delta\right).$$

(b) Hurwitz numbers weighted by Jack polynomials. In our case, the Jack polynomials are homogeneous symmetric polynomials in d variables which are integers, namely the contents of all nodes of λ. At the same time, the Jack polynomials may be rewritten as (quasi-homogeneous) polynomials in the power sum variables, that is, in the integers $(\Phi_1(\lambda), \Phi_2(\lambda), \ldots)$, which in turn are also quasi-homogeneous in the variables $\phi_k(\lambda)$ according to (32). The last fact allows us to use the content product to define the weighted Hurwitz numbers as follows:

$$\mathrm{J}_\mu^{(\alpha)}(d; \Delta) := \sum_{\substack{\lambda \\ |\lambda|=d}} Q_\mu^{(\alpha)}(\Phi(\lambda))\, \varphi_\lambda(\Delta) \frac{\dim\lambda}{d!}, \tag{58}$$

where $Q_\mu^{(\alpha)}$ is the (dual) Jack polynomial in the notation of [40, Sect. 10, Chap. VI]. The Euler characteristic of the cover is $\mathrm{E}' = \ell(\Delta) - d - |\mu|$, similarly to the previous example.

(c) Perhaps the most important example is the sum of Hurwitz numbers which may be called projective Goulden–Jackson Hurwitz numbers [20]:

$$\begin{aligned} \mathrm{S}_\mu(d; \Delta) &:= \sum_\lambda \frac{\dim\lambda}{d!} \varphi_\lambda(\Delta) \prod_{s=1}^{k} \phi_{\mu_s}(\lambda) \\ &= \sum_{\substack{\Delta^{(1)}, \ldots, \Delta^{(s)} \\ \ell^*(\Delta^{(s)}) = \mu_s, \quad s=1,\ldots,k}} H^{1,k+1}(d; \Delta^{(1)}, \ldots, \Delta^{(k)}, \Delta), \end{aligned} \tag{59}$$

recalling that ϕ_i were introduced in (26). This is the sum of the Hurwitz numbers of all d-branched covers of $\mathbb{RP}^2$ with $k+1$ ramification profiles, given by an arbitrary partition Δ and partitions $\Delta^{(s)}$, $s = 1, \ldots, k$, whose lengths are given numbers: $\ell(\Delta^{(s)}) = d - \mu_s$. The Euler characteristic of the cover is $\mathrm{E}' = \ell(\Delta) - d - |\mu|$. Each weighted sum of Hurwitz numbers obtained from BKP is actually a linear combination of (59).

Remark 16 Let us consider the case where the sum S_μ reduces to a single term, whence it is not a sum of Hurwitz numbers, but a Hurwitz number itself. This occurs if we choose the partition μ to be $\mu(b,m) := (1^b(d-1)^m)$. We get

$$\mathtt{S}_{\mu(b,m)}(d,\Delta) = H^{1,b+m+1}\left(d;\Delta,\underbrace{\Gamma,\ldots,\Gamma}_{b},\underbrace{(d),\ldots,(d)}_{m}\right), \tag{60}$$

which counts d-fold covers of $\mathbb{RP}^2$ with the following set of ramification profiles: an arbitrary profile Δ, say, over 0, then b simple branch points and m maximally ramified profiles. When $m > 0$, this Hurwitz number coincides with the Hurwitz numbers of connected covers. This follows from Remarks 3 and 5.

4.2 Parametrisation II

This is the case described in (4). Here we weight the Hurwitz numbers with symmetric functions viewed as functions of the power sum variables, the role of the set of power sums being played by the set $(T_\lambda(\mathtt{t}), T_\lambda(\mathtt{t}^2),\ldots)$ as defined in (43). We denote this set by $\mathtt{T}_{\lambda,\mathtt{t}}$. In this case, the weighted sums contain Hurwitz numbers for covers with different Euler characteristics, so there is no sense in introducing the analogue of the constant h. In the examples below, the prefactor (*) is not a polynomial function of φ_λ. For a given partition μ, we introduce $\mathtt{t}$-dependent sums.
(d) Hurwitz numbers weighted by the power sum monomials built from $(T_\lambda(\mathtt{t}), T_\lambda(\mathtt{t}^2),\ldots)$:

$$\mathtt{K}_\mu(d;\Delta|\mathtt{t}) := \sum_{\substack{\lambda\\ |\lambda|=d}} T_\lambda(\mu|\mathtt{t})\,\varphi_\lambda(\Delta)\,\frac{\dim\lambda}{d!}, \quad |\Delta| = d, \tag{61}$$

where $T_\lambda(\mu|\mathtt{t}) = \prod_{i=1}^{\ell(\mu)} T_\lambda(\mathtt{t}^{\mu_i})$ and $T_\lambda(\mathtt{t}^{\mu_i})$ are defined by (43).
(e) $\mathtt{t}$-dependent sums weighted by Jack polynomials:

$$\mathtt{J}_\mu^{(\alpha)}(d;\Delta|\mathtt{t}) := \sum_{\substack{\lambda\\ |\lambda|=d}} Q_\mu^{(\alpha)}(\mathtt{T}_{\lambda,\mathtt{t}})\,\varphi_\lambda(\Delta)\,\frac{\dim\lambda}{d!}, \quad |\Delta| = d, \tag{62}$$

where $Q_\mu^{(\alpha)}$ is the Jack polynomial. It may be viewed either as the homogeneous symmetric function of the d variables which are the quantum contents of the diagram λ, or alternatively as quasi-homogeneous functions of power sum variables $\mathtt{T}_{\lambda,\mathtt{t}} = \big(T_\lambda(\mathtt{t}), T_\lambda(\mathtt{t}^2), T_\lambda(\mathtt{t}^3),\ldots\big)$, expressed in terms of S_d characters via Lemma 4 (see also Remark 15).
(f) Sums weighted by Macdonald polynomials:

$$\mathtt{M}_\mu^{\mathtt{q},\mathtt{t}}(d;\Delta) := \sum_{\substack{\lambda\\ |\lambda|=d}} Q_\mu^{\mathtt{q},\mathtt{t}}(\mathtt{T}_{\lambda,\mathtt{t}})\,\varphi_\lambda(\Delta)\,\frac{\dim\lambda}{d!}, \quad |\Delta| = d, \tag{63}$$

where $Q_\mu^{\mathtt{q},\mathtt{t}}(\mathtt{T}_{\lambda,\mathtt{t}})$ are Macdonald polynomials viewed as functions of the power sum variables $\mathtt{T}_{\lambda,\mathtt{t}} = \big(T_\lambda(\mathtt{t}), T_\lambda(\mathtt{t}^2), T_\lambda(\mathtt{t}^3), \dots\big)$ (see Remark 15). Here the polynomials $Q_\mu^{\mathtt{q},\mathtt{t}}$ may also be written as symmetric functions in the d variables which are the quantum contents of the diagram λ.

Note that the idea of weighting (the $\mathbb{CP}^1$) Hurwitz numbers by symmetric functions was first worked out in [25], where $\{h_\mu, m_\mu\}$, $\{e_\mu, f_\mu\}$, and also $\{s_\mu\}$, $\{p_\mu\}$ were used as basis sets [40]. In our approach, the notion of q-deformed Hurwitz numbers introduced in [25] is based on q-dependent specifications of the parameters ζ in the parametrization I (4), while the parametrization II (5) was not considered in [25].

(g) Remark 14 suggests considering the following weighted sums of Hurwitz numbers:

$$\mathtt{F}(d, \Delta, (d), \{\mathtt{q}_s, \mathtt{t}_s\}) := \sum_{\substack{\lambda \\ |\lambda|=d}} \varphi_\lambda((d))\, \varphi_\lambda(\Delta) \frac{\dim\lambda}{d!} \prod_{s=1}^{k} \frac{s_\lambda(\mathbf{p}(\mathtt{q}_s, \mathtt{t}_s))}{s_\lambda(\mathbf{p}_\infty)}$$

$$= \sum_{\substack{\lambda \\ |\lambda|=d}} \varphi_\lambda((d))\, \varphi_\lambda(\Delta) \frac{\dim\lambda}{d!} \prod_{s=1}^{k} \left(1 + \sum_{\mu \neq 1^d} \varphi_\lambda(\mu) w(\Delta^{(s)}, \mathtt{q}_s, \mathtt{t}_s) \right). \quad (64)$$

As we can see this sum describes covers with the following set of profiles on $\mathbb{RP}^2$: an arbitrary profile Δ over 0, the maximally ramified profile (d) over another point, special weighted sums of profiles $\Delta^{(s)}$, $s = 1, \dots, k$, over each of k additional branch points with the ramification weights $w(\Delta^{(s)}, \mathtt{q}_s, \mathtt{t}_s)$ of (49). (Here we skip the details because this will be published in a more detailed way in another paper. Such sums allow us to count the d-fold covers whose profiles $\Delta^{(s)}$ over the additional branch points contain given numbers of parts which are multiples of other given numbers playing the role of a chosen set of degrees of roots of unity. This is achieved by studying limits where the parameters $\mathtt{q}_s$ and $\mathtt{t}_s$ are chosen to be close to the roots of unity.)

We shall show below that the numbers $\mathtt{C}_\mu(d;\Delta)$, $\mathtt{J}_\mu(d;\Delta)$, $\mathtt{S}(d;\Delta)$, and $\mathtt{K}_\mu(d;\Delta|\mathtt{t})$, $\mathtt{M}_\mu^{\mathtt{q},\mathtt{t}}(d;\Delta)$, and $\mathtt{F}(d, \Delta, (d), \{\mathtt{q}_s, \mathtt{t}_s\})$ are generated by the special BKP tau functions considered in Sects. 6 and 7. For instance, the number (59) is generated by (82) and the number (64) is generated by (94).

5 BKP tau functions

5.1 BKP hierarchy of Kac and van de Leur

There are two different BKP hierarchies of integrable equations. One was introduced by the Kyoto group in [32] and the other by Kac and van de Leur [34]. We shall need the last one here. This hierarchy includes the celebrated KP hierarchy as a particular reduction. In a certain sense (see [65]), the BKP hierarchy may be related to the three-component KP hierarchy introduced in [32] (described earlier in [66,67] using L–A pairs of differential operators with matrix-valued coefficients). For a detailed description of the BKP hierarchies, we refer readers to the original work [34]. Here we write down the first non-trivial equations (Hirota equations) for the BKP tau function:

$$\frac{1}{2}\frac{\partial\tau(N,n,\mathbf{p})}{\partial p_2}\tau(N+1,n+1,\mathbf{p})-\frac{1}{2}\tau(N,n,\mathbf{p})\frac{\partial\tau(N+1,n+1,\mathbf{p})}{\partial p_2}$$
$$+\frac{1}{2}\frac{\partial^2\tau(N,n,\mathbf{p})}{\partial^2 p_1}\tau(N+1,n+1,\mathbf{p})+\frac{1}{2}\tau(N,n,\mathbf{p})\frac{\partial^2\tau(N+1,n+1,\mathbf{p})}{\partial^2 p_1}$$
$$-\frac{\partial\tau(N,n,\mathbf{p})}{\partial p_1}\frac{\partial\tau(N+1,n+1,\mathbf{p})}{\partial p_1}$$
$$=\tau(N+2,n+2,\mathbf{p})\tau(N-1,n-1,\mathbf{p}). \tag{65}$$

The BKP tau functions depend on the set of higher times $t_m = p_m/m$, $m > 0$, and the discrete parameter N. In [58,59], a second discrete parameter n was included, and the simplest Hirota equation relating the BKP tau functions for neighboring values of n is

$$\frac{1}{2}\tau(N,n+1,\mathbf{p})\frac{\partial^2\tau(N+1,n+1,\mathbf{p})}{\partial^2 p_1}-\frac{1}{2}\frac{\tau(N,n+1,\mathbf{p})}{\partial^2 p_1}\tau(N+1,n+1,\mathbf{p})$$
$$=\frac{\partial\tau(N+2,n+2,\mathbf{p})}{\partial p_1}\tau(N-1,n,\mathbf{p})-\frac{\partial\tau(N+1,n+2,\mathbf{p})}{\partial p_1}\tau(N,n,\mathbf{p}). \tag{66}$$

The complete set of Hirota equations with two discrete parameters is written down in "Appendix."

The general solution to the BKP Hirota equations may be written as

$$\tau\,(N,n,\mathbf{p})=\sum_\lambda A_\lambda(N,n)s_\lambda(\mathbf{p}), \tag{67}$$

where A_λ satisfies the Plucker relations for an isotropic Grassmannian and may be written in Pfaffian form (as one can show using the Wick formula).

5.2 BKP tau function of hypergeometric type

We are interested in a certain subclass of the BKP tau functions (67) introduced in [58,59] and called BKP hypergeometric tau functions. These may be compared with a similar class of TL and KP tau functions found in [38,55].

Similarly to [55], we proceed as follows. Suppose that λ is a Young diagram. Given an arbitrary function r of one variable, we construct the product

$$r_\lambda(x):=\prod_{(i.j)\in\lambda} r(x+j-i), \tag{68}$$

which is called the content product (or, sometimes, the generalized Pochhammer symbol attached to a Young diagram λ). Examples were considered above.

Remark 17 (1) If $r=fg$, then $r_\lambda(x)=f_\lambda(x)g_\lambda(x)$. (2) If $\tilde{r}(x)=(r(x))^n$, $n\in\mathbb{C}$, then $\tilde{r}_\lambda(x)=(r_\lambda(x))^n$.

We consider sums over partitions of the form

$$\sum_{\substack{\lambda \\ \ell(\lambda)\le N}} r_\lambda(n) c^{|\lambda|} s_\lambda(\mathbf{p}) =: \tau_r^{\mathrm{B}}(N, n, \mathbf{p}), \tag{69}$$

where s_λ are the Schur functions [40] and $\mathbf{p}$ denotes the semi-infinite set $(p_1, p_2, \dots)$. It was shown in [58,59] that, up to a factor, (69) defines the BKP tau function:

Proposition 4 *For any given r, the tau function $g(n)\tau_r^{\mathrm{B}}(N, n, \mathbf{p})$ solves the BKP Hirota equations. Here $g(n)$ is a function of the parameter n defined by* (118) *in "Appendix B."*

Let us make two points. Although discrete parameters enter the Hirota equations, for our purposes (a) the factor $g(n)$ is unimportant, and (b) the cutoff N should be chosen large enough, and we can take $N = +\infty$.

We call such tau functions hypergeometric because both the so-called generalized hypergeometric functions and the basic hypergeometric functions of one variable may be obtained as special cases of (69). For instance, one can choose $p_m = x^m$. Then a rational function r in (69) yields the generalized hypergeometric function, while a trigonometric r results in the basic hypergeometric function. However, the key tau function is the simplest one.

Example Consider $r(x) = 1$ for any x. The resulting tau function does not depend on n and will be denoted by $\tau_1(N, \mathbf{p})$. Other hypergeometric tau functions may be obtained by action of a specially chosen vertex operator on $\tau_1(N, \mathbf{p})$, e.g., see (74). If we take $N = +\infty$, we obtain

$$\tau_1^{\mathrm{B}}(\infty, \mathbf{p}) = \sum_\lambda c^{|\lambda|} s_\lambda(\mathbf{p}) = e^{\sum_{m>0}\left(\frac{c^2}{2m} p_m^2 + c\frac{p_{2m-1}}{2m-1}\right)}. \tag{70}$$

Remark 18 Each tau function τ_r^{B} may be expressed as a Pfaffian [58,59].

2KP and BKP hypergeometric tau functions The role of the hypergeometric functions of matrix argument in the form of KP tau functions presented in [57] was discussed in [20] in the context of combinatorial problems. The hypergeometric tau function of the two-component KP (2KP) may be written as

$$\sum_{\substack{\lambda \\ \ell(\lambda)\le N}} r_\lambda(n)\, c^{|\lambda|}\, s_\lambda(\mathbf{p})\, s_\lambda(\bar{\mathbf{p}}) =: \tau_r^{\mathrm{2KP}}(N, n, \mathbf{p}, \bar{\mathbf{p}}), \tag{71}$$

where $r_\lambda(n)$ is the same as in (69). Here two independent sets $\mathbf{p} = (p_1, p_2, \dots)$ and $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2,)$ and two discrete parameters N and n play the role of 2KP higher times. (We do not indicate the dependence of the right-hand side on the constant c, since it is trivial.) Then the hypergeometric tau functions of the 2KP and BKP hierarchies are related:

$$\left[e^{\sum_{i>0}\frac{i}{2}\frac{\partial^2}{\partial \bar{p}_i^2}+\sum_{i>0,\,\mathrm{odd}}\frac{\partial}{\partial \bar{p}_i}} \cdot \tau_r^{2\mathrm{KP}}(N,n,\mathbf{p},\bar{\mathbf{p}}) \right]_{\bar{\mathbf{p}}=0} = \tau_r^{\mathrm{B}}(N,n,\mathbf{p}), \tag{72}$$

which follows from (18) and (19):

$$\left[e^{\sum_{i>0}\frac{i}{2}\frac{\partial^2}{\partial p_i^2}+\sum_{i>0,\,\mathrm{odd}}\frac{\partial}{\partial p_i}} \cdot s_\lambda(\mathbf{p}) \right]_{\mathbf{p}=0} = 1. \tag{73}$$

Hypergeometric tau functions via the vertex operators From the bosonization formulae given in [32], the tau functions (3) were presented in [58,59] in terms of an action of the vertex operators. For r given by (5) [or indeed by (50)], the tau function (3) may be written as

$$\tau_r^B(N,n,\mathbf{p}) = \frac{1}{g(n)}\, e^{\xi_0 \hat{h}_2(n)\log \mathtt{t}+\sum_{m\neq 0}\mathtt{p}_m^*\hat{h}(n,\mathtt{t}^m)} \cdot \sum_{\substack{\lambda\\ \ell(\lambda)\le N}} c^{|\lambda|} s_\lambda(\mathbf{p}), \tag{74}$$

where $\hat{h}(n,\mathtt{t}^m)$, $m\in\mathbb{Z}$, are commuting operators defined as vertex operators:

$$\hat{h}(n,\mathtt{t}) := \mathtt{t}^n \operatorname*{res}_z \frac{dz}{z} e^{\sum_{i>0}(\mathtt{t}^i-1)\frac{z^i p_i}{i}}\, e^{-\sum_{i>0}(\mathtt{t}^{-i}-1)z^{-i}\frac{\partial}{\partial p_i}}, \tag{75}$$

and where $\hat{h}_2(n)$ is determined by the generating series $\hat{h}(n,e^\epsilon) =: 1+\sum_{i\ge 0}\frac{\epsilon^{i+1}}{(i+1)!}\hat{h}_i(n)$. The operators $\hat{h}_i(n)$ were written down in [1,45] in the most explicit way. From (75), we get

$$\hat{h}_0(n) = n, \qquad \hat{h}_1(n) = n^2 + \sum_{i>0} i p_i \frac{\partial}{\partial p_i},$$
$$\hat{h}_2(n) = n^3 + \sum_{i,j}\left((i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j}\frac{\partial^2}{\partial p_i \partial p_j}\right). \tag{76}$$

In particular, the operator $\hat{h}_2(0)$ is known as the cut-and-join operator, first introduced in [21].

For r given by (4) [or indeed by (38)], the tau function (3) may be written as

$$\tau_r^B(N,n,\mathbf{p}) = \frac{1}{g(n)}\, e^{\sum_{m>0}p_m^*\hat{h}_m(n)} \cdot \sum_{\lambda,\,\ell(\lambda)\le N} c^{|\lambda|} s_\lambda(\mathbf{p}).$$

Example For $N=+\infty$, $n=0$, and $r(x)=e^{\zeta_1 x}$, i.e., $p_m^*=0$, $m>2$ [see (41)], we get

$$\tau_r^{\mathrm{B}}(\mathbf{p}) = \sum_\lambda e^{\zeta_1\varphi_\lambda(\Gamma)} c^{|\lambda|} s_\lambda(\mathbf{p}) = e^{\zeta_1\hat{h}_2(0)} \cdot e^{\sum_{m>0}\frac{c^2}{2m}p_m^2+\frac{c}{2m-1}p_{2m-1}}. \tag{77}$$

6 Examples of the BKP hypergeometric tau functions

When we use parameters to describe r, say, the parameters ζ in (4), we shall write $\tau^{\mathrm{B}}(N, n, \mathbf{p}|\zeta)$ instead of $\tau_r^B(N, n, \mathbf{p})$. Let us use Propositions 2 and 3 and the relations (36), (47) to construct examples of BKP tau functions. In view of (8), each example may be considered as the generating function for certain sums of Hurwitz numbers. More specific examples will be discussed in Sect. 7, where the tau functions generate the sums introduced in Sect. 4.

Example 0 The simplest hypergeometric tau function $\sum_{\ell(\lambda)\le N} s_\lambda(\mathbf{p})$ is related to $\zeta = 0$.

Example I First we choose (38) for the content product. Using (17), we write down the following example:

$$\tau^{\mathrm{B}}(N, 0, \mathbf{p}|h, \zeta) = \sum_{d\ge 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\, \ell(\lambda)\le N}} s_\lambda(\mathbf{p}) \exp \sum_{m>0} \frac{1}{m} h^m \zeta_m \Phi_m(\lambda). \tag{78}$$

If $h = 1$ it may be suitable to introduce the dependence on the variable n after performing the triangular change in variables $\zeta \to \mathbf{p}^*$ given by $V(x-1,\zeta) - V(x,\zeta) = V(x,\mathbf{p}^*)$. Then

$$\tau^{\mathrm{B}}(N, n, \mathbf{p}|\mathbf{p}^*) = \sum_{d\ge 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\, \ell(\lambda)\le N}} s_\lambda(\mathbf{p}) \prod_{i=1}^{N} e^{V\left(h_{i+n}(\lambda),\mathbf{p}^*\right)}, \tag{79}$$

where $h_i(\lambda) = \lambda_i - i$.

Remark 19 The specialization $p_m = \mathrm{tr} R^m = \sum_{a=1}^N x_a^m$, where $x_i = e^{y_i}$, allows (78) to be rewritten as

$$\tau^{\mathrm{B}}(N, 0, \mathbf{p}|\zeta) = \frac{1}{\Delta_N(\mathbf{x})} \sum_{h_1,\dots,h_N=1}^{M} e^{V(h,\mathbf{p}^*)} \det\left(e^{y_j h_i}\right) \mathrm{sgn}\Delta_N(h), \tag{80}$$

which is a discrete analogue of the two-matrix integral

$$\int dU \int dR \exp\left(\mathrm{Tr}\left(UYU^\dagger R + \sum_{m\neq 0} \frac{1}{m} p_m^* R^m\right)\right), \tag{81}$$

where the first integral represents integration over unitary matrices and the second is the integral over real symmetric ones, dU and dR denoting the corresponding Haar measures. Y is any diagonal matrix (a source). The matrices are $N \times N$. This integral may be viewed as an analogue of the Kontsevich integral.

Example Ia In (78), one can specify the variables ζ as

$$\zeta_m = \sum_{s=1}^{k} \mathtt{n}_s(-\mathtt{a}_s)^{-m}, \quad \zeta_0 = -\mathtt{n}_s \log \mathtt{a}_s,$$

where $\mathtt{a}_s \in \mathbb{C}$. If we restore the dependence of the tau function on n, we obtain

$$\tau^{\mathrm{B}}(N, n, \mathbf{p}|h, \{\mathtt{a}_s, \mathtt{n}_s\}) = \sum_{d\geq 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\, \ell(\lambda)\leq N}} s_\lambda(\mathbf{p}) \prod_{s=1}^{k} \prod_{(i.j)\in\lambda} \left(1 + h\frac{n+j-i}{a_s}\right)^{-\mathtt{n}_s}, \tag{82}$$

where $\mathtt{a}$ and $\mathtt{n}$ are respectively the collections of complex parameters $\mathtt{a}_1, \ldots, \mathtt{a}_k$ and $\mathtt{n}_1, \ldots, \mathtt{n}_k$. For $\mathtt{n}_s = \pm$, we obtain the Pfaffian version of the hypergeometric function of a matrix argument [57].

Example Ib Let us take all the $\mathtt{n}_s$ equal to $\mathtt{n}(\alpha) = 1/\alpha$ in the previous example. We then obtain

$$\tau^{\mathrm{B}}(N, n, \mathbf{p}|h, \mathtt{a}, \mathtt{n}(\alpha)) = \sum_{d\geq 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\, \ell(\lambda)\leq N}} s_\lambda(\mathbf{p}) \sum_{\mu} h^{|\mu|} P_\mu^{(\alpha)}(-\mathtt{a}(n))\, Q_\mu^{(\alpha)}(\Phi(\lambda)), \tag{83}$$

where P_μ^α and Q_μ^α is the pair of dual Jack polynomials written in the notation of [40, Chap. IV]. Here the first Jack polynomial P_μ^α is a symmetric function of the variables $-\mathtt{a}(n) = (-a_1 - n, \ldots, -a_k - n)$, while the second Jack polynomial $Q_\mu^\alpha(\Phi(\lambda))$ may be viewed either as a quasi-homogeneous polynomial in the power sum variables $\Phi = (\Phi_1(\lambda), \Phi_2(\lambda), \ldots)$, or alternatively as a symmetric function in d variables, viz. the contents of the diagram λ.

Example II Next we use (50) and (17) to obtain

$$\tau^{\mathrm{B}}\left(N, n, \mathbf{p}\middle|\left\{\mathtt{p}^{*(s)}, \mathtt{t}_s\right\}\right) = \sum_{d\geq 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\, \ell(\lambda)\leq N}} s_\lambda(\mathbf{p}) \prod_{s=1}^{k} e^{\xi_0(\varphi_\lambda(\Gamma)+nd)\log \mathtt{t}_s + \sum_{m\neq 0} \xi_m^{(s)} \mathtt{t}_s^{mn} T_\lambda(\mathtt{t}_s^m)} \tag{84}$$

$$= \sum_{d\geq 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\, \ell(\lambda)\leq N}} s_\lambda(\mathbf{p}) \prod_{s=1}^{k} e^{\xi_0(\varphi_\lambda(\Gamma)+nd)\log \mathtt{t}_s - \sum_{m\neq 0} \frac{1}{m}(1-\mathtt{t}_s^m)\mathtt{p}_m^{*(s)} \mathtt{t}_s^{mx-m} T_\lambda(\mathtt{t}_s^m)}. \tag{85}$$

The variables $\mathtt{p}^{*(s)}$ are related to the variables $\xi^{(s)}$ by $\mathtt{p}_m^* = \xi_m \frac{\mathtt{t}^m}{\mathtt{t}^m - 1}$.

For $k = 1$ (here we will write $\mathtt{p}^{*(1)} \to -\mathtt{p}^*$) and $\mathtt{p}^*_m = 0,\ m < 0$, we have

$$\tau^{\mathrm{B}}(N, n, \mathbf{p}|\mathtt{p}^*, \mathbf{t}) = \sum_{d\geq 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\ \ell(\lambda)\leq N}} s_\lambda(\mathbf{p}) \prod_{s=1}^{k} \mathtt{t}^{\xi_0 \varphi_\lambda(\Gamma)+d\xi_0} \times \sum_{\mu} P_\mu^{0,\mathtt{t}}(\mathtt{p}^*) Q_\mu^{0,\mathtt{t}}(T_{\lambda,\mathtt{t}}), \tag{86}$$

where $P_\lambda^{0,\mathtt{t}}$ and $Q_\lambda^{0,\mathtt{t}}$ are the Macdonald polynomials specified by $\mathtt{q} = 0$ (Hall–Littlewood polynomials), which may be written either as quasi-homogeneous polynomials of the power sum variables $T_{\lambda,\mathtt{t}} = (T_\lambda(\mathtt{t}), T_\lambda(\mathtt{t}^2), \dots)$, or as symmetric polynomials in d variables, viz. the quantum contents of λ (see Remark 15).

Remark 20 Given s, let us specify $\mathbf{p} = \mathbf{p}(\mathtt{q}, \mathtt{t})$ according to (48). Then the series (84) solves the BKP Hirota equations with respect to the variables $\mathtt{p}^*$. When $|\mathtt{t}| = 1$ and is not a root of 1, τ^{B} in (84) is basically a discrete version of the circular $\beta = 1$ ensemble, viz.,

$$\frac{1}{N!} \sum_{h_1,\dots,h_N} \prod_{i<j} |\mathtt{t}^{h_i} - \mathtt{t}^{h_j}| \prod_{i=1}^{N} e^{V\left(\mathtt{p}^*, \mathtt{t}^{h_i}\right)} \mu(h_1; \mathtt{q}, \mathtt{t}),$$

with a certain weight function μ independent of $\mathtt{p}^*$ [58,59]. Compare with Remark 11 and the discrete version of the orthogonal ensemble (10).

Consider three specifications of the variables ξ in (84).

Example IIa First, we put each $\xi_m^{(s)} = 0,\ s = 1, \dots, k$. Then the content product depends only on the parameter ξ_0. We obtain the BKP analogue of Okounkov's TL tau function presented in [51]:

$$\tau^{\mathrm{B}}(N, n, \mathbf{p}|\xi_0) = \sum_{d\geq 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\ \ell(\lambda)\leq N}} s_\lambda(\mathbf{p}) \prod_{(i.j)\in\lambda} e^{(n+j-i)\xi_0}. \tag{87}$$

Example IIb Now taking $\xi_0 = 0$ and

$$\xi_m^{(s)} = \frac{\mathtt{t}^m - 1}{\mathtt{t}^m} \mathtt{p}_m^{*(s)} = \mathtt{n}_s \mathtt{q}_s^m, \quad m > 0, \tag{88}$$

we obtain

$$\tau^{\mathrm{B}}(N, n, \mathbf{p}|\{\mathtt{t}_s, \mathtt{q}_s, \mathtt{n}_s\}) = \sum_{d\geq 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\ \ell(\lambda)\leq N}} s_\lambda(\mathbf{p}) \prod_{s=1}^{k} \prod_{(i.j)\in\lambda} \left(1 - \mathtt{q}_s \mathtt{t}_s^{n+j-i}\right)^{-\mathtt{n}_s}, \tag{89}$$

where $\mathtt{t}, \mathtt{q}, \mathtt{n}$ are sets of complex numbers $\mathtt{t}_s, \mathtt{q}_s, \mathtt{n}_s, s = 1, \dots, k$. When $\mathtt{n}_s = \pm 1$, $s = 1, \dots, k$, the tau function (89) is the Pfaffian version of Milne's hypergeometric function [43,56].

Example IIc Next, taking $\xi_0 = 0$ and

$$\xi^{(s)}_{\pm m} = \frac{\mathtt{t}^{\pm m} - 1}{\mathtt{t}^{\pm m}} \mathtt{p}^{*(s)}_{\pm m} = (-1)^m \mathtt{n}_s \frac{\mathtt{q}_s^{\frac{m}{2}} \mathtt{t}_s^{\pm \mathtt{a}_s m}}{1 - \mathtt{q}_s^m}, \quad s = 1, \dots, k, \quad m > 0,$$

and putting $\mathtt{q}_s = e^{2\pi i \tau_s}$, $\mathtt{t}_s = e^{2c_s \pi i}$, the relation (84) takes the form

$$\begin{aligned} &\tau^{\mathrm{B}}(N, n, \mathbf{p}|\{\mathtt{c}_s, \tau_s, \mathtt{a}_s, \mathtt{n}_s\}) \\ &= \sum_{d \geq 0} c^d \sum_{\substack{\lambda \\ |\lambda| = d,\, \ell(\lambda) \leq N}} s_\lambda(\mathbf{p}) \prod_{s=1}^{k} \left(\theta_\lambda(\mathtt{c}_s(n + \mathtt{a}_s), \tau_s)\right)^{-\mathtt{n}_s}, \end{aligned} \tag{90}$$

where $\{\mathtt{c}, \tau, \mathtt{a}, \mathtt{n}\}$ are sets of complex numbers $\{\mathtt{c}_s, \tau_s, \mathtt{a}_s, \mathtt{n}_s, s = 1, \dots, k\}$, and where

$$\theta_\lambda(c_s(n + \mathtt{a}_s), \tau_s) := \prod_{(i.j) \in \lambda} \theta(c_s(n + \mathtt{a}_s + j - i), \tau_s)$$

is the elliptic version of the Pochhammer symbol, and θ is the Jacobi theta function

$$\begin{aligned} \theta(c_s x, \tau_s) &:= \sum_{k \in \mathbb{Z}} \exp(\pi i k^2 \tau_s + 2 c_s \pi i k x) \\ &= (\mathtt{q}_s; \mathtt{q}_s)_\infty \prod_{k=1}^{\infty} \left(1 + \mathtt{q}_s^{k - \frac{1}{2}} \mathtt{t}_s^x\right) \left(1 + \mathtt{q}_s^{k - \frac{1}{2}} \mathtt{t}_s^{-x}\right), \end{aligned}$$

with $(\mathtt{q}_s; \mathtt{q}_s)_\infty$ the Dedekind function. For this example, we chose $c = (\mathtt{q}_s; \mathtt{q}_s)_\infty$ in (84). For $\mathtt{n}_s = \pm 1$, we obtain the Pfaffian version of the elliptic hypergeometric function considered in [55].

Example IId In (86) we choose $k = 1, \mathtt{n} = 1$. Taking

$$\xi_m = \frac{1 - \mathtt{t}^m}{1 - \mathtt{q}^m} \sum_{i=1}^{k} y_i^m, \quad m > 0,$$

all other variables vanish. This may be viewed as a limiting case of Example Ib, where we send $k \to \infty$. Then

$$r(x) = \prod_{m > 0} \prod_{i=1}^{k} \frac{1 - y_i q^m \mathtt{t}^{x+1}}{1 - y_i q^m \mathtt{t}^x}.$$

The content product is equal to

$$\prod_{(i.j)\in\lambda}\prod_{m>0}\prod_{i=s}^{k}\frac{1-y_i q^m \mathtt{t}^{x+1+j-i}}{1-y_s q^m \mathtt{t}^{x+j-i}} = e^{\sum_{m>0}\frac{1}{m}\frac{1-\mathtt{t}^m}{1-\mathtt{q}^m}\mathtt{t}^{mx}T_\lambda(\mathtt{t}^m)\sum_{i=1}^k y_i^m}$$
$$= \sum_\mu \mathtt{t}^{x|\mu|}P_\mu^{\mathtt{q},\mathtt{t}}(Y)Q_\mu^{\mathtt{q},\mathtt{t}}(T_{\lambda,\mathtt{t}}), \qquad (91)$$

where the Macdonald function $P_\mu^{\mathtt{q},\mathtt{t}}$ is the symmetric polynomial in $Y=(y_1,\ldots,y_k)$, and the Macdonald function $Q_\mu^{\mathtt{q},\mathtt{t}}$ may be written either as the quasi-homogeneous polynomial of the power sum variables $T_{\lambda,\mathtt{t}}=(T_\lambda(\mathtt{t}),T_\lambda(\mathtt{t}^2),\ldots)$, or as the symmetric polynomial in the quantum contents (see Remark 15). The tau function (86) takes the form

$$\tau^{\mathrm{B}}(N,n,\mathbf{p}|\mathtt{q},\mathtt{t},\xi_0,Y) = \sum_{d\geq 0}c^d \sum_{\substack{\lambda\\ |\lambda|=d,\,\ell(\lambda)\leq N}} s_\lambda(\mathbf{p})\,e^{\xi_0\varphi_\lambda(\Gamma)}$$
$$\times\sum_\mu \mathtt{t}^{n|\mu|}P_\mu^{\mathtt{q},\mathtt{t}}(Y)Q_\mu^{\mathtt{q},\mathtt{t}}(T_{\lambda,\mathtt{t}}) \qquad (92)$$
$$=\sum_{d\geq 0}c^d \sum_{\substack{\lambda\\ |\lambda|=d,\,\ell(\lambda)\leq N}} \prod_{j=1}^N e^{\xi_0(\lambda_j-j+n)^2}s_\lambda(\mathbf{p})$$
$$\times\prod_{j=1}^N\prod_{i=1}^k\prod_{m>0} e^{\frac{y_i^m}{1-q^m}\mathtt{t}^{m(\lambda_j-j+n-1)}}, \qquad (93)$$

where $P_\mu^{\mathtt{q},\mathtt{t}}$ and $Q_\mu^{\mathtt{q},\mathtt{t}}$ are Macdonald polynomials (see Remark 15). The last equality follows from (52).

Example III We choose

$$r(x)=(a+x)\prod_{s=1}^k\frac{1-\mathtt{q}_s\mathtt{t}_s^x}{1-e^{\epsilon_s}\mathtt{t}_s}(a_s+x),$$

where we used both parameterizations (see Remark 17). We obtain the tau function

$$\tau^{\mathrm{B}}(N,n,\mathbf{p}|a,\{a_s,\mathtt{t}_s,\epsilon_s\}) = \sum_{d\geq 0}c^d\sum_{\substack{\lambda\\ |\lambda|=d}} s_\lambda(\mathbf{p})\frac{s_\lambda(\mathbf{p}(a))}{s_\lambda(\mathbf{p}_\infty)}$$
$$\times\prod_s\frac{s_\lambda(\mathbf{p}(\mathtt{q}_s,\mathtt{t}_s))}{s_\lambda(\mathbf{p}(e^{\epsilon_s},\mathtt{t}_s))}\frac{s_\lambda(\mathbf{p}(a_s))}{s_\lambda(\mathbf{p}_\infty)}. \qquad (94)$$

In particular, these tau functions generate the sums $\mathtt{F}(d,\Delta,(d),\{\mathtt{q}_s,\mathtt{t}_s\})$ in (64).

Remark 21 Equation (82) may be obtained as a limiting case of (89) if we set $\mathtt{q}_s = \mathtt{t}_s^{\mathtt{a}_s}$ and send $\mathtt{t} \to 1$, taking into account the fact that, for the hypergeometric tau functions (69), we have the obvious transformation $r_\lambda \to a^{-|\lambda|} r_\lambda,\ p_m \to a p_m,\ m > 0$, which leaves them unchanged. In this limiting case, the polynomials $P^{\mathtt{q},\mathtt{t}}$ and $Q^{\mathtt{q},\mathtt{t}}$ tend to the Jack polynomials [40] [compare with (83)].

Remark 22 Similarly to (72), we may prove the relation

$$e^{\frac{1}{2}\sum_{m>0} m \frac{\partial^2}{\partial p_m^2} + \sum_{m>0,\,\mathrm{odd}} \frac{\partial}{\partial p_m}} \cdot \tau_r^{\mathrm{B}}(N, n, \mathbf{p})|_{\mathbf{p}=0}$$
$$= \sum_{d\geq 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\, \ell(\lambda)\leq N}} \prod_{(i.j)\in\lambda} r(n+j-i), \tag{95}$$

where the right-hand side generates weighted Hurwitz numbers for the torus and the Klein bottle.

7 BKP tau functions generating Hurwitz numbers

7.1 Getting the Hurwitz numbers themselves

As we shall see, the hypergeometric tau functions generate weighted sums of Hurwitz numbers. However, there are special cases when one gets the Hurwitz numbers themselves. This is based on Remark 5. We will distinguish between the parameterizations I and II.

First, let us write down the simplest case of a single branch point related to all $r = 1$ and $N = \infty$. This case is generated by τ_1^{B}, where it is reasonable to produce the change $p_m \to h^{-1} c^m p_m$. We get

$$e^{\frac{1}{h^2}\sum_{m>0} \frac{1}{2m} p_m^2 c^{2m} + \frac{1}{h}\sum_{m\,\mathrm{odd}} \frac{1}{m} p_m c^m} = \sum_{d>0} c^d \sum_{\substack{\Delta \\ |\Delta|=d}} h^{-\ell(\Delta)} \mathbf{p}_\Delta H^{1,a}(d;\Delta), \tag{96}$$

where $a = 0$ if $\Delta = (1^d)$, and $a = 1$ otherwise. Then $H^{1,1}(d;\Delta)$ is the Hurwitz number describing a d-fold covering of $\mathbb{RP}^2$ with a single branch point of type $\Delta = (d_1, \ldots, d_l)$, $|\Delta| = d$ by a (not necessarily connected) Klein surface of Euler characteristic $\mathrm{E}' = \ell(\Delta)$. For instance, for $d = 3$, $\mathrm{E}' = 1$, we get $H^{1,1}(3;\Delta) = \delta_{\Delta,(3)}/3$. For unbranched coverings (that is for $a = 0$, $\mathrm{E}' = d$), we get formula (13).

Next note that the exponent on the left-hand side may be rewritten as the generating series of the connected Hurwitz numbers

$$\frac{1}{h^2}\sum_{d=2m} c^{2m} p_m^2 H_{\mathrm{con}}^{1,1}(d;(m,m)) + \frac{1}{h}\sum_{d=2m-1} c^{2m-1} p_{2m-1} H_{\mathrm{con}}^{1,1}(d;(2m-1)),$$

where $H^{1,1}_{\rm con}$ describes a d-fold covering either by the Riemann sphere ($d = 2m$) or by the projective plane ($d = 2m - 1$). These are the only ways to cover $\mathbb{RP}^2$ by a connected surface for the case of a single branch point. The geometrical meaning of the exponent in (96) may be explained as follows. The projective plain may be viewed as the unit disk with the identification of the opposite points z and $-z$ on the boundary $|z| = 1$. If we cover the Riemann sphere by the Riemann sphere $z \to z^m$, we get two critical points with the same profiles. However, if we cover $\mathbb{RP}^2$ by the Riemann sphere, then we have the composition of the mapping $z \to z^m$ on the Riemann sphere and the factorization by antipodal involution $z \to -1/\bar{z}$. Thus we have the ramification profile (m, m) at the single critical point 0 of $\mathbb{RP}^2$. The automorphism group is the dihedral group of order $2m$, which consists of rotations by $2\pi/m$ and antipodal involution $z \to -1/\bar{z}$. Thus we get that $H^{1,1}_{\rm con}(d; (m, m)) = 1/2m$, which is the factor in the first sum in the exponent in (96). Now let us cover $\mathbb{RP}^2$ by $\mathbb{RP}^2$ via $z \to z^d$. For even d, we have the critical point 0, and in addition, each point of the unit circle $|z| = 1$ is critical (a folding), while from the beginning we restrict our consideration to isolated critical points. For odd $d = 2m - 1$, there is a single critical point 0; the automorphism group consists of rotations through the angle $2\pi/(2m - 1)$. Thus in this case $H^{1,1}(d; (2m - 1)) = 1/(2m - 1)$, which is the factor in the second sum in the exponent in (96).

Next, consider the BKP hypergeometric function in the parametrization I, setting $\zeta_k = \beta\delta_{k,1} - \sum_{i=1}^{m}(-a_i)^k$:

$$H^{1,b+m+1}\left(d; \underbrace{\Gamma, \ldots, \Gamma}_{b}, \underbrace{(d), \ldots, (d)}_{m}, \Delta\right) = h^{\rm E'}\left[\tau(N > d, 0, \mathbf{p}|\zeta)\right]_{d,b,m,\Delta}, \quad (97)$$

where the bracket $[*]_{d,b,m,\Delta}$ is the coefficient of $c^d\beta^b\mathbf{p}_\Delta\prod_{i=1}^{m} a_i^{1-d}$ which counts d-fold covers of $\mathbb{RP}^2$ with the following ramification type: There are b simple branch points, m maximally ramified branch points, and one branch point of type $\Delta = (d_1, \ldots, d_l)$. Each cover is a connected Klein surface when $m > 0$ and a not necessarily connected Klein surface when $m = 0$. The Euler characteristic of the cover is $\rm E' = \ell(\Delta) - b - m(d - 1)$.

From Proposition 1, we conclude that the projective Hurwitz number of (97) may be equated with Hurwitz numbers related to different base surfaces.

7.2 BKP Tau Function as Generating Function for Weighted Sums of Hurwitz Numbers

In this section, the power of $1/h$ counts the Euler characteristic of the covering surface denoted by $\rm E'$. For this purpose, in Propositions 5 and 6, we use $\tilde{\mathbf{p}}$ defined by

$$p_m = h^{-1}\tilde{p}_m. \quad (98)$$

First we present the simplest weighted sum of Hurwitz numbers, which is just the sum of Hurwitz numbers related to a single branch point with a fixed Euler characteristic E', viz.,

$$\sum_{d\geq 0} c^d \sum_{\ell} h^{d-l} \sum_{\substack{\Delta \\ \ell(\Delta)=l}} H^{1,1}(d;\Delta) = (1-c^2)^{-1/2h^2}\left(\frac{1+c}{1-c}\right)^{1/2h}, \tag{99}$$

where each Δ has the same weight d and length ℓ. This follows directly from (96), where all $p_m = 1$.

From the previous sections, we derive:

Proposition 5 *The tau function (78) generates the numbers $C_\mu(\Delta)$ (56) through*

$$\tau^{\mathrm{B}}(N,0,\mathbf{p}|h,\zeta) = \sum_{d\geq 0} c^d \sum_{\substack{\mu,\Delta \\ |\Delta|=d}} h^{|\mu|-\ell(\Delta)} \frac{1}{z_\mu} C_\mu(\Delta)\,\zeta_\mu\,\tilde{\mathbf{p}}_\Delta, \tag{100}$$

where z_μ is defined by (12). For $d = |\Delta| \leq N$, the numbers $C_\mu(\Delta)$ are weighted Hurwitz numbers.

Corollary 1 *In particular, let us put $\zeta_m = 0$ if $m > 1$. Then (100) reads*

$$\sum_{d\geq 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\,\ell(\lambda)\leq N}} e^{h\zeta_1\varphi_\lambda(\Gamma)} s_\lambda(\mathbf{p}) = \sum_{d,b\geq 0} c^d \sum_{\substack{\Delta \\ |\Delta|=d}} h^{b-\ell(\Delta)}\tilde{\mathbf{p}}_\Delta \frac{\zeta_1^b}{b!} H\left(d;\underbrace{\Gamma,\ldots,\Gamma}_{b},\Delta\right), \tag{101}$$

which is the $\mathbb{RP}^2$ analogue of the Okounkov generating function [51].

The representation of this series in the form of a matrix integral is given in (107).

Weighted sums of Hurwitz numbers generated by the BKP tau functions (89) and (82) were given in our previous paper [50]. The simplest example resulting from (82) is similar to the one considered in [29] and may be presented as follows. The tau function (82), with $\mathrm{n}_s = 1$ for $s = 1,\ldots,k$, generates sums S defined by (59):

Proposition 6 *It may be interesting to compare (6) with its $\mathbb{CP}^1$ analogue discussed in [36, Example 2.22]*

$$\begin{aligned}\tau^{\mathrm{B}}(N,n,\mathbf{p}|h,\{a_s\}) &= \sum_{d\geq 0} c^d \sum_{\substack{\lambda \\ |\lambda|=d,\,\ell(\lambda)\leq N}} s_\lambda(\mathbf{p}) \prod_{s=1}^{k} \prod_{(i.j)\in\lambda} \left(a_s h^{-1} + n + j - i\right) \\ &= \sum_{d\geq 0} c^d \sum_{\substack{\Delta \\ |\Delta|=d}} \sum_{\mu} (a_s h^{-1}+n)^{d-\mu_s} h^{-\ell(\Delta)} \tilde{\mathbf{p}}_\Delta\, S_\mu(d,\Delta).\end{aligned} \tag{102}$$

Proposition 7 *The tau function* (84) *generates the numbers* $K_{\mu^{(s)}}(\Delta|t_s)$ (61) *through*

$$\tau^{\mathrm{B}}(N,n,\mathbf{p}|\xi,\{t_s\}) = \sum_{d\geq 0} c^d \sum_{\substack{\mu,\Delta \\ |\Delta|=d}} \prod_{s=1}^{k} \frac{1}{z_\mu} \mathbf{p}_\Delta\, \xi_{\mu^{(s)}}\, K_{\mu^{(s)}}(\Delta|t_s), \tag{103}$$

where z_μ *is defined by* (12). *For* $d=|\Delta|\leq N$, *the numbers* $K_{\mu^{(s)}}(\Delta|t_s)$ *are weighted Hurwitz numbers.*

Proposition 8 *The tau function* (92) *generates the Hurwitz numbers* $M_\mu^{q,t}$, *weighted by Macdonald polynomials [see* (63)*]:*

$$\tau^{\mathrm{B}}(N,n,\mathbf{p}|q,t,0,Y) = \sum_{d\geq 0} c^d \sum_{\substack{\Delta \\ |\Delta|=d}} \mathbf{p}_\Delta \sum_{\mu} t^{n|\mu|} P_\mu^{q,t}(Y) M_\mu^{q,t}(d;\Delta). \tag{104}$$

Proposition 9 *The numbers* $F(d,\Delta,(d),\{q_s,t_s\})$ *given by* (64) *may be obtained as the following term in the tau function* (64)*:*

$$\begin{aligned}
&\tau^{\mathrm{B}}(N,n,\mathbf{p}|a,\{a_s,t_s,\epsilon_s\}) \\
&\quad = \sum_{d\geq 0} c^d \sum_{\lambda} s_\lambda(\mathbf{p}) \frac{s_\lambda(\mathbf{p}(a))}{s_\lambda(\mathbf{p}_\infty)} \prod_{s=1}^{k} \frac{s_\lambda(\mathbf{p}(q_s,t_s))}{s_\lambda(\mathbf{p}(e^{\epsilon_s},t_s))} \frac{s_\lambda(\mathbf{p}(a_s))}{s_\lambda(\mathbf{p}_\infty)} \\
&\quad = \sum_{d\geq 0} c^d \sum_{\substack{\Delta \\ |\Delta|=d}} \mathbf{p}_\Delta \left(F(d,\Delta,(d),\{q_s,t_s\})\, a \prod_{s=1}^{k} \frac{a_s}{\epsilon_s} + \dots \right),
\end{aligned} \tag{105}$$

where dots indicate terms of different order in any of ϵ_s, a_s $(s=1,\dots,k)$, *and* a.

8 Matrix integrals as generating functions of Hurwitz numbers

If the base surface is $\mathbb{CP}^1$, the set of examples of matrix integrals generating Hurwitz numbers was studied in [2,4,9,10,14,37,39,69]. One can show that the perturbation series in coupling constants of these integrals (Feynman graphs) may be related to TL (KP and two-component KP) hypergeometric tau functions. This actually means that these series generate Hurwitz numbers with at most two arbitrary profiles, while others are subject to certain conditions, since the origin of additional profiles is the content product factors in hypergeometric tau functions (71).

Here, very briefly, we write down a few generating series for the $\mathbb{RP}^2$ Hurwitz numbers. These series may not be tau functions themselves, but may be represented as integrals of tau functions with matrix argument. [The matrix argument, which we denote by a capital letter, say X, means that the power sum variables $\mathbf{p}$ are specified as $p_i = \mathrm{tr} X^i$, $i>0$. Then instead of $s_\lambda(\mathbf{p})$, $\tau(\mathbf{p})$, we write $s_\lambda(X)$ and $\tau(X)$.] If a matrix integral in the examples below is a BKP tau function, then it generates Hurwitz numbers with a single arbitrary profile, and all others are subject to restrictions identical to those in the $\mathbb{CP}^1$ case mentioned above. In all the examples, V is given by (11). We also recall that the limiting values of $\mathbf{p}(\mathtt{q},\mathtt{t})$ given by (48) may be $\mathbf{p}(a)=(a,a,\dots)$ and $\mathbf{p}_\infty=(1,0,0,\dots)$. Further, the numbers $H^{\mathrm{E,F}}(d;\dots)$ are Hurwitz numbers only when $d\le N$, where N is the size of the matrices.

For more details on the $\mathbb{RP}^2$ case, the reader is referred to [50]. A new development in [50] as compared with [61] is the use of products of matrices. Here we consider a few examples. All examples include the simplest BKP tau function with matrix argument X [58,59] defined by [compare with (20)]

$$\tau_1^{\mathrm{B}}(X):=\sum_\lambda s_\lambda(X)=e^{\frac12\sum_{m>0}\frac1m(\mathrm{tr}X^m)^2+\sum_{m>0,\mathrm{odd}}\frac1m\mathrm{tr}X^m}$$
$$=\frac{\det^{1/2}\dfrac{1+X}{1-X}}{\det^{1/2}(I_N\otimes I_N-X\otimes X)} \tag{106}$$

as part of the integration measure. Other integrands are the simplest KP tau functions $\tau_1^{\mathrm{KP}}(X,\mathbf{p}):=e^{\mathrm{tr}V(X,\mathbf{p})}$, where V is defined by (11) and the parameters $\mathbf{p}$ may be called coupling constants. The perturbation series in the coupling constants are expressed as sums of products of the Schur functions over partitions and are similar to the series we considered in the previous sections.

Example 1 (The $\mathbb{RP}^2$ Okounkov–Hurwitz series as a model of normal matrices) From the equality

$$\left(2\pi\zeta_1^{-1}\right)^{\frac12}e^{\frac{(n\zeta_0)^2}{2\zeta_1}}e^{\zeta_0nc+\frac12\zeta_1c^2}=\int_{\mathbb{R}}e^{x_in\zeta_0+(cx_i-\frac12x_i^2)\zeta_1}dx_i,$$

and in a similar way to what was done in [60] using $\varphi_\lambda(\Gamma)=\sum_{(i.j)\in\lambda}(j-i)$, we can derive

$$e^{n|\lambda|\zeta_0}e^{\zeta_1\varphi_\lambda(\Gamma)}\delta_{\lambda,\mu}=\mathrm{K}\int s_\lambda(M)s_\mu(M^\dagger)\det\left(MM^\dagger\right)^{n\zeta_0}e^{-\frac12\zeta_1\mathrm{tr}\left(\log\left(MM^\dagger\right)\right)^2}dM,$$

where K is an unimportant multiplier, M is a normal matrix with eigenvalues $z_1,\dots,z_N$ and $\log|z_i|=x_i$, and $dM=d_*U\prod_{i<j}|z_i-z_j|^2\prod_{i=1}^N d^2z_i$. Then the $\mathbb{RP}^2$ analogue of the Okounkov series (101) may be written

$$\sum_{\substack{\lambda\\ \ell(\lambda)\le N}}e^{n|\lambda|\zeta_0+\zeta_1\varphi_\lambda(\Gamma)}s_\lambda(\mathbf{p})$$
$$=\mathrm{K}\int e^{V(M,\mathbf{p})}e^{\zeta_0n\mathrm{tr}\log\left(MM^\dagger\right)-\frac12\zeta_1\left(\mathrm{tr}\log\left(MM^\dagger\right)\right)^2}\tau_1^{\mathrm{B}}(M^\dagger)dM. \tag{107}$$

A similar representation of the Okounkov $\mathbb{CP}^1$ series was presented earlier in [3].

Below we use the following notation:

- d_*U is the normalized Haar measure on $\mathbb{U}(N)$, i.e., $\int_{\mathbb{U}(N)} d_*U = 1$.
- Z is a complex matrix

$$d\Omega(Z, Z^\dagger) = \pi^{-n^2} e^{-\mathrm{tr}(ZZ^\dagger)} \prod_{i,j=1}^{N} d\Re\, Z_{ij}\, d\Im\, Z_{ij}.$$

- If M is a Hermitian matrix, the measure is defined by

$$dM = \prod_{i\le j} d\Re M_{ij} \prod_{i<j} d\Im M.$$

It is known that [40]

$$\int s_\lambda(Z) s_\mu(Z^\dagger)\, d\Omega(Z, Z^\dagger) = (N)_\lambda \delta_{\lambda,\mu}, \tag{108}$$

where $(N)_\lambda := \prod_{(i.j)\in\lambda}(N + j - i)$ is the Pochhammer symbol related to λ. A similar relation was used in [2,28,53,60,61] for models of Hermitian, complex, and normal matrices. I_N is the $N \times N$ unit matrix. We recall that

$$s_\lambda(I_N) = (N)_\lambda s_\lambda(\mathbf{p}_\infty), \quad s_\lambda(\mathbf{p}_\infty) = \frac{\dim\lambda}{d!}, \quad d = |\lambda|.$$

Example 2 (Three branch points) The generating function for $\mathbb{RP}^2$ Hurwitz numbers having three ramification points with three arbitrary profiles is

$$\sum_{\lambda,\, \ell(\lambda)\le N} \frac{s_\lambda(\mathbf{p}^{(1)}) s_\lambda(\Lambda) s_\lambda(\mathbf{p}^{(2)})}{(s_\lambda(\mathbf{p}_\infty))^2} = \int \tau_1^{\mathrm{B}}(Z_1 \Lambda Z_2) \prod_{i=1,2} e^{V\left(\mathrm{tr} Z_i^\dagger,\, \mathbf{p}^{(i)}\right)} d\Omega\left(Z_i, Z_i^\dagger\right). \tag{109}$$

If $\mathbf{p}^{(2)} = \mathbf{p}(\mathtt{q}, \mathtt{t})$ with any given parameters $\mathtt{q}$, $\mathtt{t}$, and $\Lambda = I_N$, then (109) is the hypergeometric BKP tau function.

Example 3 (Hermitian two-matrix model) The following 'projective analogue' of the well-known two-matrix model is the BKP tau function

$$\int \tau_1^{\mathrm{B}}(cM_2) e^{\mathrm{tr} V(M_1, \mathbf{p}) + \mathrm{tr}(M_1 M_2)} dM_1 dM_2 = \sum_\lambda c^{|\lambda|} (N)_\lambda s_\lambda(\mathbf{p}),$$

where M_1, M_2 are Hermitian matrices. Using the results of [21], we can show that it is a projective analogue of the generating function of the so-called strictly monotonic Hurwitz numbers introduced by Goulden and Jackson. In the projective case, these numbers count paths on the Cayley graph of the symmetric group whose initial point is a given partition, while the end point is not fixed: we take the weighted sum over all possible end points, say, Δ, and the weight is given by $\chi(\Delta)$ of Lemma 1.

Example 4 (Unitary matrices) Generating series for projective Hurwitz numbers with arbitrary profiles at n branch points and restricted profiles at other points:

$$\int e^{\mathrm{tr}(cU_1^\dagger\dots U_{n+m}^\dagger)}\left(\prod_{i=n+1}^{n+m}\tau_1^{\mathrm{B}}(U_i)d_*U_i\right)\left(\prod_{i=1}^{n}\tau_1^{\mathrm{KP}}\left(U_i,\mathbf{p}^{(i)}\right)d_*U_i\right)$$
$$=\sum_{d\geq 0}c^d\,(d!)^{1-m}\sum_{\substack{\lambda,\,|\lambda|=d\\ \ell(\lambda)\leq N}}\left(\frac{\dim\lambda}{d!}\right)^{2-m}\left(\frac{s_\lambda(I_N)}{\dim\lambda}\right)^{1-m-n}\prod_{i=1}^{n}\frac{s_\lambda(\mathbf{p}^{(i)})}{\dim\lambda}. \tag{110}$$

Here $\mathbf{p}^{(i)}$ are parameters. This series generates certain linear combinations of Hurwitz numbers for base surfaces with Euler characteristic $2-m$, $m\geq 0$. The integral (110) is a BKP tau function when the parameters are specialized as $\mathbf{p}^{(i)}=\mathbf{p}(\mathtt{q}_i,\mathtt{t}_i)$, $i=2,\dots,n$, with any values of $\mathtt{q}_i$, $\mathtt{t}_i$, and if in addition $m=1$. When $n=1$, this BKP tau function may be viewed as an analogue of the generating function of the so-called non-connected Bousquet–Melou–Schaeffer numbers [36, Example 2.16]. When $n=m=1$, we obtain the BKP tau function

$$\int \tau_1^{\mathrm{B}}(U_2)e^{\mathrm{tr}V(U_1,\mathbf{p})+\mathrm{tr}\left(cU_1^\dagger U_2^\dagger\right)}d_*U_1d_*U_2=\sum_{\substack{\lambda\\ \ell(\lambda)\leq N}}c^{|\lambda|}\frac{s_\lambda(\mathbf{p})}{(N)_\lambda}.$$

If we compare this series with those used in [23,24], we can see that it is a projective analogue of the generating function of the so-called weakly monotonic Hurwitz numbers. In the projective case, it counts paths on the Cayley graph whose initial point is a given partition, while instead of a fixed end point, we consider the sum over all possible end points Δ, with a weight given by $\chi(\Delta)$ as in Lemma 1.

Example 5 (Integrals over complex matrices) We give two examples. An analogue of the Belyi curve generating functions [10,69] is as follows [compare also with (59)]:

$$\sum_{l=1}^{N}N^l\sum_{\substack{\Delta^{(1)},\dots,\Delta^{(n+1)}\\ \ell(\Delta^{n+1})=l}}c^dH^{\mathrm{E},n+1}\left(d;\Delta^{(1)},\dots,\Delta^{(n+1)}\right)\prod_{i=1}^{n}\mathbf{p}^{(i)}_{\Delta^{(i)}}$$
$$=\sum_{\lambda}c^{|\lambda|}\frac{(d!)^{m-2}(N)_\lambda}{(\dim\lambda)^{m-2}}\prod_{i=1}^{n}\frac{s_\lambda(\mathbf{p}^{(i)})}{s_\lambda(\mathbf{p}_\infty)} \tag{111}$$
$$=\int e^{\mathrm{tr}\left(cZ_1^\dagger\dots Z_{n+m}^\dagger\right)}\left(\prod_{i=n+1}^{n+m}\tau_1^{\mathrm{B}}(Z_i)d\Omega\left(Z_i,Z_i^\dagger\right)\right)$$
$$\times\left(\prod_{i=1}^{n}\tau_1^{\mathrm{KP}}\left(Z_i,\mathbf{p}^{(i)}\right)d\Omega\left(Z_i,Z_i^\dagger\right)\right), \tag{112}$$

where $\mathrm{E} = 2 - m$ is the Euler characteristic of the base surface.

The series in the following example generates the projective Hurwitz numbers themselves. To get rid of the factor $(N)_\lambda$ in the sum over partitions, we use mixed integration over $\mathbb{U}(N)$ and over complex matrices:

$$\sum_{\Delta^{(1)},\dots,\Delta^{(n)}} c^d\, H^{1,n}\left(d;\Delta^{(1)},\dots,\Delta^{(n)}\right)\prod_{i=1}^n \mathbf{p}^{(i)}_{\Delta^{(i)}}$$

$$= \sum_{\lambda,\,\ell(\lambda)\le N} c^{|\lambda|}\frac{\mathrm{dim}\lambda}{d!}\prod_{i=1}^n \frac{s_\lambda(\mathbf{p}^{(i)})}{s_\lambda(\mathbf{p}_\infty)} \tag{113}$$

$$= \int \tau_1^{\mathrm{KP}}\left(cU^\dagger Z_1^\dagger\cdots Z_k^\dagger,\mathbf{p}^{(n)}\right)\tau_1^{\mathrm{B}}(U)d_*U\prod_{i=1}^{n-1}\tau_1^{\mathrm{KP}}\left(Z_i,\mathbf{p}^{(i)}\right)d\Omega\left(Z_i,Z_i^\dagger\right). \tag{114}$$

Here $Z, Z_i, i = 1,\dots,n-1$, are complex $N\times N$ matrices and $U \in \mathbb{U}(N)$. As in the previous examples, one can specify all sets $\mathbf{p}^{(i)} = \mathbf{p}(\mathrm{q}_i,\mathrm{t}_i)$, $i = 1,\dots,n$, except a single one which in this case has the meaning of the BKP higher times.

Acknowledgements A.O. was partially supported by RFBR grant 14-01-00860, by V.E. Zakharov's scientific school (Program for Support of Leading Scientific Schools, Grant NS-3753.2014.2), and also by the Russian Academic Excellence Project "5-100." The work of S.N. was partially supported by RFBR Grant 16-01-00409. We would first like to thank J. Harnad, A. Mironov, and J. van de Leur for important remarks and also A. Zabrodin, S. Loktev, and I. Marshall for useful discussions. Special thanks also to L. Chekhov for organizing the workshop on Hurwitz numbers (Moscow, May 2014) which inspired us to do this work.

Appendix A: Hirota Equations for the BKP Tau Function with Two Discrete Time Variables

The BKP hierarchy we are interested in was introduced in [34]. In this paper, the BKP tau function $\tau^{\mathrm{B}}(N,\mathbf{p})$ does not contain the discrete variable n. We need a slightly more general version of the BKP hierarchy which includes n as the higher time parameter [58,59,65]. The Hirota equations for the tau functions $\tau^{\mathrm{B}}(N,n,\mathbf{p})$ of this modified BKP hierarchy read

$$\oint\frac{dz}{2\pi i}z^{N'-N-1}e^{V(\mathbf{p}'-\mathbf{p},z)}\tau\left(N'-1,n,\mathbf{p}'-[z^{-1}]\right)\tau\left(N+1,n+1,\mathbf{p}+[z^{-1}]\right)$$
$$+\oint\frac{dz}{2\pi i}z^{N-N'-3}e^{V(\mathbf{p}-\mathbf{p}',z)}\tau\left(N'+1,n+2,\mathbf{p}'+[z^{-1}]\right)$$
$$\times\tau\left(N-1,n-1,\mathbf{p}-[z^{-1}]\right)$$
$$=\tau(N'+1,n+1,\mathbf{p}')\tau(N-1,n,\mathbf{p})-\frac{1}{2}\left(1-(-1)^{N'+N}\right)$$
$$\times\tau\left(N',n+1,\mathbf{p}'|g\right)\tau(N,n,\mathbf{p}) \tag{115}$$

and

$$\oint \frac{dz}{2\pi i} z^{N'-N-2} e^{V(\mathbf{p}'-\mathbf{p},z)} \tau\left(N'-1, n-1, \mathbf{p}'-[z^{-1}]\right)$$
$$\times \tau\left(N+1, n+1, \mathbf{p}+[z^{-1}]\right)$$
$$+\oint \frac{dz}{2\pi i} z^{N-N'-2} e^{V(\mathbf{p}-\mathbf{p}',z)} \tau\left(N'+1, n+1, \mathbf{p}'+[z^{-1}]\right)$$
$$\times \tau\left(N-1, n-1, \mathbf{p}'-[z^{-1}]\right)$$
$$= \frac{1}{2}\left(1-(-1)^{N'+N}\right) \tau\left(N', n, \mathbf{p}'\right) \tau\left(N, n, \mathbf{p}\right). \tag{116}$$

Here $\mathbf{p} = (p_1, p_2, \ldots)$, $\mathbf{p}' = (p'_1, p'_2, \ldots)$. The notation $\mathbf{p}+[z^{-1}]$ denotes the set

$$\left(p_1 + z^{-1}, p_2 + z^{-2}, p_3 + z^{-3}, \ldots\right)$$

and V is defined by (11). Equations (116) are the same as in [34], while (115) relate tau functions with different discrete time n and were given in [58,59,65]. Taking $N' = N+1$ and all $p_i = p'_i$, $i \neq 2$ in (116), then picking up the terms linear in $p'_2 - p_2$, we obtain (65). Taking $N' = N+1$ and all $p_i = p'_i$, $i \neq 1$ in (115), then picking up the terms linear in $p'_1 - p_1$, we obtain (66). The relation between the BKP hierarchy and the two- and three-component KP hierarchy was established in [65].

Appendix B: Hypergeometric BKP tau function—Fermionic formulae

Details may be found in [55,58,59]. Let $\{\psi_i, \psi_i^\dagger, i \in \mathbb{Z}\}$ be Fermi creation and annihilation operators that satisfy the usual anticommutation relations and vacuum annihilation conditions

$$[\psi_i, \psi_j]_+ = \delta_{i,j}, \quad \psi_i|n\rangle = \psi_{-i-1}|n\rangle = 0, \quad i < n.$$

In contrast to the DKP hierarchy introduced in [32], for the BKP hierarchy introduced in [34], we need an additional Fermi mode ϕ which anticommutes with all the other Fermi operators except itself, for which $\phi^2 = 1/2$, and $\phi|0\rangle = |0\rangle/\sqrt{2}$ [34]. Then the hypergeometric BKP tau function introduced in [58,59] may be written as

$$g(n)\tau_r^{\mathrm{B}}(N, n, \mathbf{p})$$
$$= \left\langle n \left| e^{\sum_{m>0} \frac{1}{m} J_m p_m} e^{\sum_{i<0} U_i \psi_i^\dagger \psi_i - \sum_{i\geq 0} U_i \psi_i \psi_i^\dagger} e^{\sum_{i>j} \psi_i \psi_j - \sqrt{2}\phi \sum_{i\in\mathbb{Z}} \psi_i} \right| n - N \right\rangle$$
$$= \sum_{\substack{\lambda \\ \ell(\lambda)\leq N}} e^{-U_\lambda(n)} s_\lambda(\mathbf{p}) = g(n) \sum_{\substack{\lambda \\ \ell(\lambda)\leq N}} r_\lambda(n) s_\lambda(\mathbf{p}), \tag{117}$$

where $J_m = \sum_{i\in\mathbb{Z}} \psi_i \psi^{\dagger}_{i+m}$, $m > 0$, $U_\lambda(n) = \sum_i U_{h_i+n}$, $r(i) = e^{U_{i-1}-U_i}$, and

$$g(n) := \left\langle n \middle| e^{\sum_{i<0} U_i \psi_i^{\dagger}\psi_i - \sum_{i\geq 0} U_i \psi_i \psi_i^{\dagger}} \middle| n \right\rangle = \begin{cases} e^{-U_0+\cdots-U_{n-1}} & \text{if } n > 0, \\ 1 & \text{if } n = 0, \\ e^{U_{-1}+\cdots U_n} & \text{if } n < 0. \end{cases} \tag{118}$$

In (117) the summation runs over all partitions whose lengths do not exceed N.

Remark 23 Note that, without the additional Fermi mode ϕ, the summation range in (117) does include partitions with odd partition lengths. One can avoid this restriction by introducing a pair of DKP tau functions, which seems unnatural.

Apart from (117), the same series without the restriction $\ell(\lambda) \leq N$ gives the BKP tau function. However, it is related to the single value $n = 0$. The n-dependence destroys the simple form of this tau function [58,59].

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