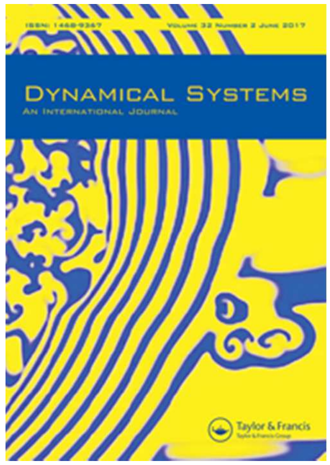

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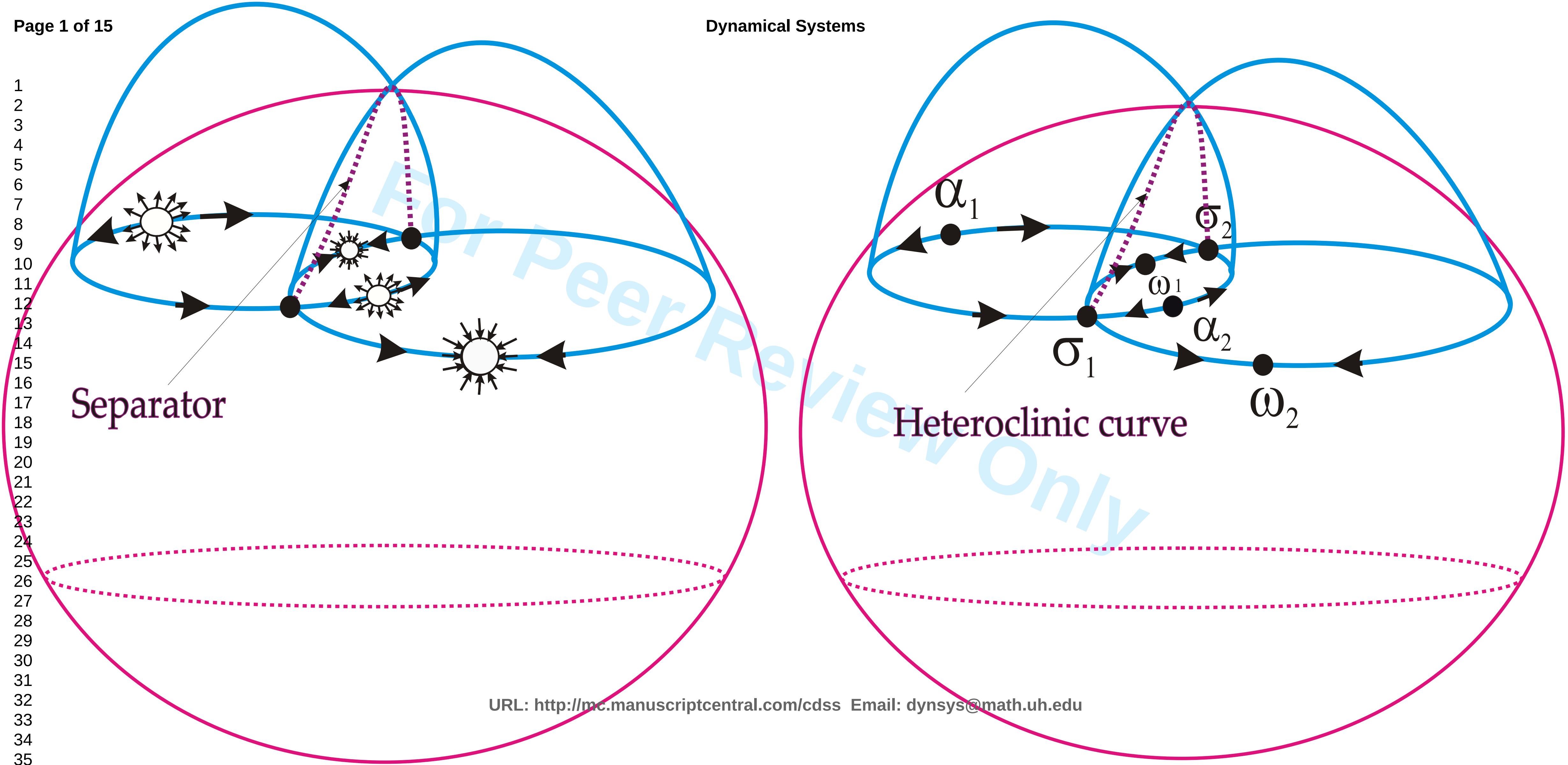
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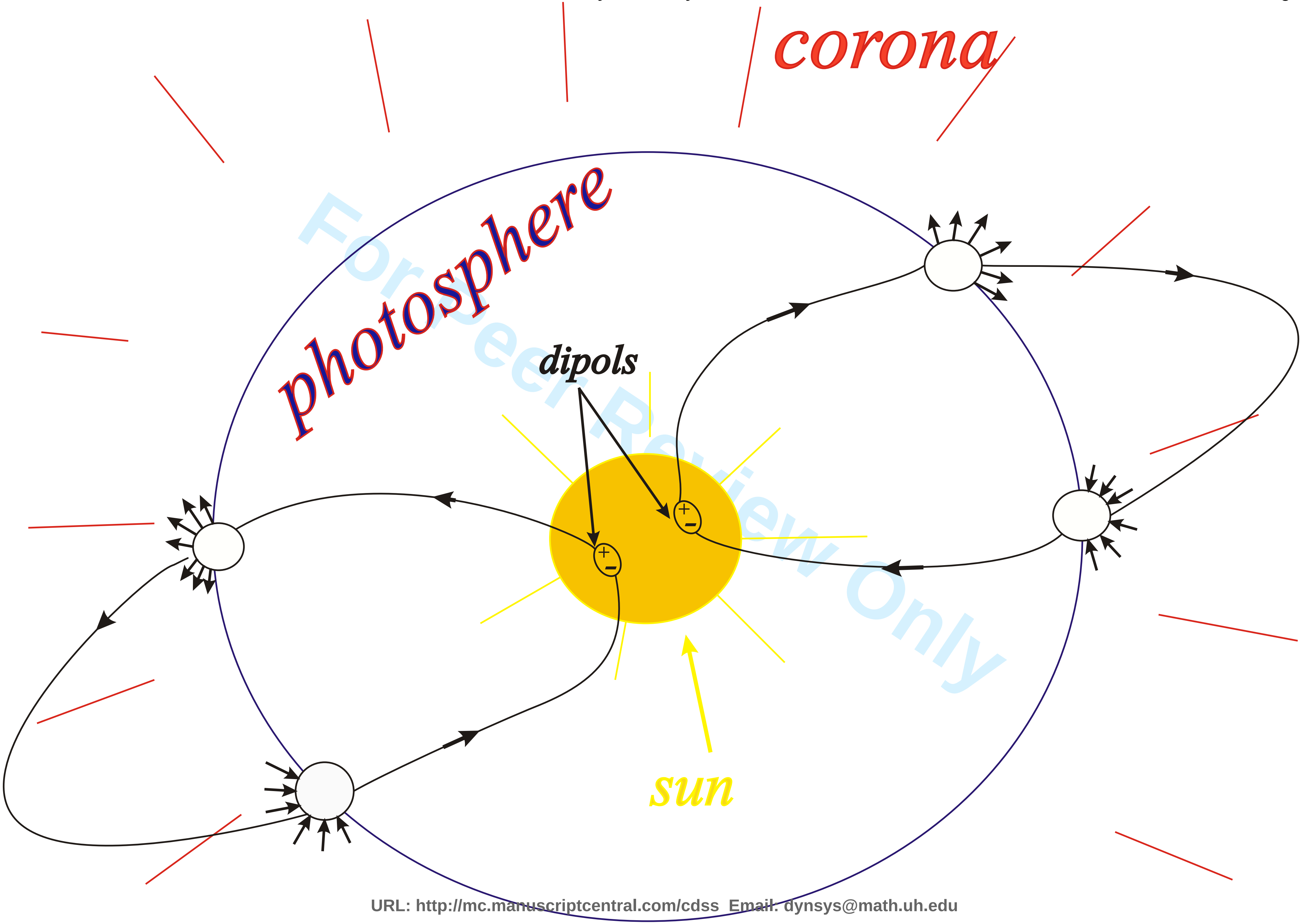
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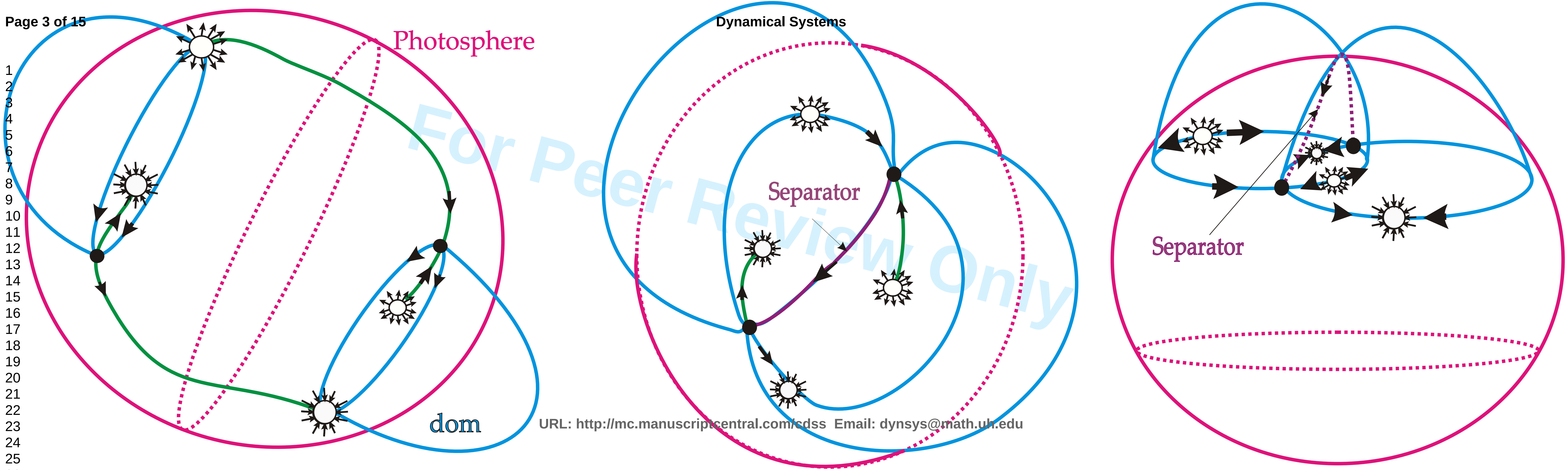
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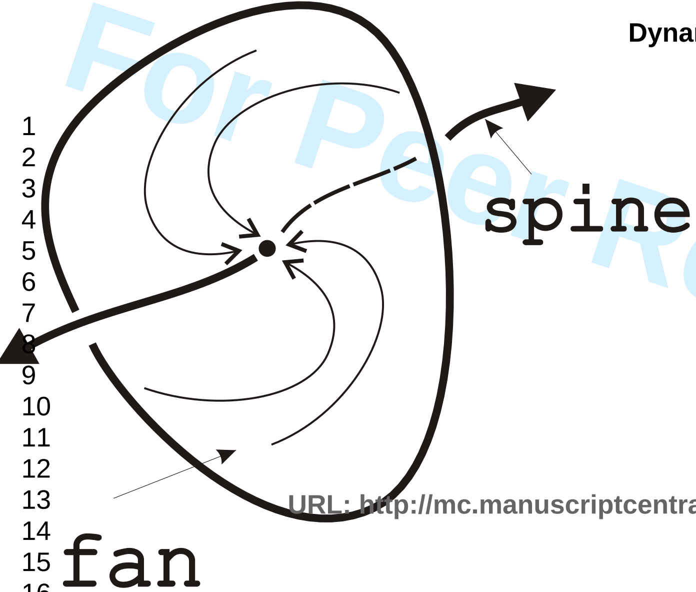
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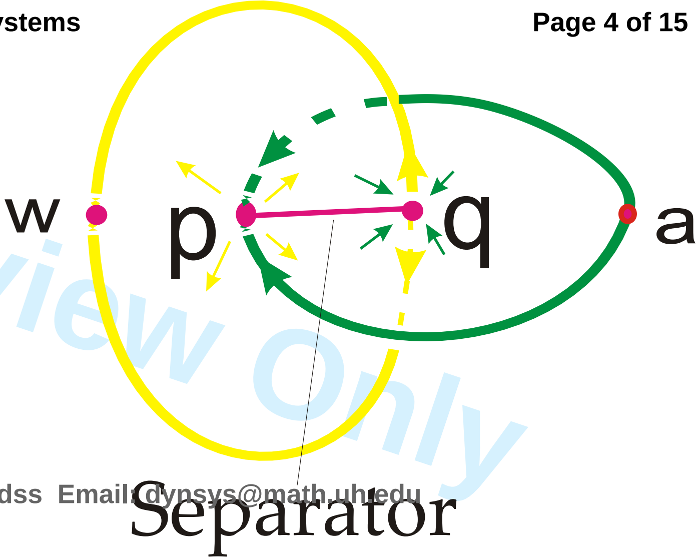
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Topological classification of global magnetic fields in the solar corona

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ABSTRACT
Numerous magnetic fragments in the interior of the Sun give rise to many interesting energy processes in the solar corona, for example solar flares and prominences. Magnetic charge topology explains these phenomena by appearance and disappearance of heteroclinic trajectories (separators) — magnetic lines that belong to an intersection of stable and unstable invariant two-dimensional manifolds (fans) of different saddle singularities (nulls). Separators are the locations in the magnetic field configuration where the magnetic energy is transferred from one region (a connected component into which the fans divide the solar corona) to another. Many recent papers has gone into investigation of the configurations that arise in different concrete models. In the present paper we solve the problem of interrelation between existence of separators of any given magnetic field in the Solar corona and the type and the number of saddle singularities and charges. Following the classical definition we introduce the concept of equivalence of two magnetic fields and get a classification of such fields up to topological equivalence.

KEYWORDS
Magnetic fields, models corona, photosphere magnetic reconnection, dynamics prominences, topological classification

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1. Introduction and the formulation of the results

Understanding the energy processes in the corona of the Sun is very important to explain many a law of nature. In this paper we consider a model explaining such effects in the photosphere as flares and prominences. Their origin is connected with the restructuring of regions (domains) into which the fans and the spines of the null points of the magnetic field divide the corona of the Sun — reconnection. Therefore, the main questions for this approach are the qualitative partition of the solar corona into domains and existence of the separators (the lines of intersection of fans) — marks of an upcoming or already occurred reconnection. There are different approaches to study the topology of domains, construction of a graph that reflects the structure and the relative position of the domains [5] or footprints — traces of spines and fans on the photosphere [13] being one of them. We suggest a new approach consisting in

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distinguishing of the traces of the fans on a level surface of the potential function of the field. We prove necessary and sufficient conditions for existence of separators associated with the types of the nulls and we show the interrelation between the number of zeros and the number of charges.

More specifically, according to the topological approach the magnetic field in the corona is believed to arise from a large number of dipoles in the solar interior. The dipoles are represented on the photosphere by pairs of locations where flux tubes originating in the solar interior break through the photosphere and spread out into the atmosphere. We use the assumptions of Magnetic Charge Topology [12] where photospheric flux patches (places of the intersections of fluxes with the photosphere) are modeled as point sources (charges) on the photosphere (see Figure 1). This suggestion violates the solenoidal condition $\nabla \cdot \mathbf{B} = 0$ for a magnetic field $\mathbf{B}$ but since each source is considered to represent a flux tube passing through the solar surface and spreading out into the overlying corona this simplification is reasonable. Following [2], for a model of the magnetic field $\mathbf{B}$ with point sources the two-dimensional sphere $P = \{(x, y, z, w) \in \mathbb{S}^3 \mid w = 0\}$ in the three-dimensional sphere $\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$ is used as the photosphere and the region $\{(x, y, z, w) \in \mathbb{S}^3 \mid w > 0\}$ as the solar corona. Moreover, we suppose $\mathbf{B}$ to be symmetrically extended to the region $\{(x, y, z, w) \in \mathbb{S}^3 \mid w < 0\}$ (the mirror corona) and, hence, $\mathbf{B}$ is defined on $\mathbb{S}^3$ and is assumed to be tangent to the photosphere $\mathbb{S}^2$.

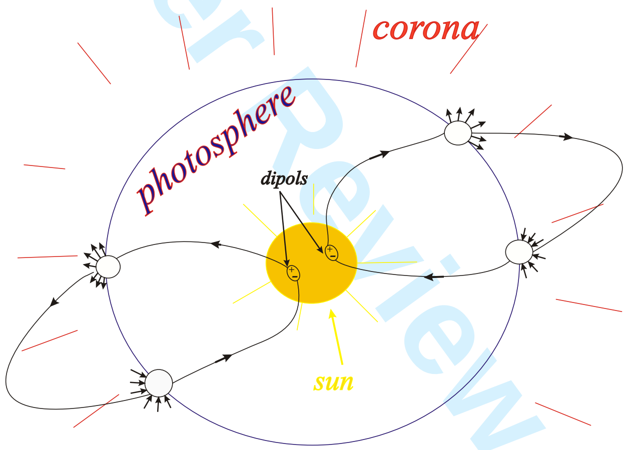

Figure 1. The dipoles in the solar interior

The coronal magnetic field is often considered to be force-free. Since we study the topology of this field for the sake of simplicity we assume that the field $\mathbf{B}$ is potential, that is $\mathbf{B} = -\nabla\Phi$, where Φ is a scalar potential. It is natural to assume that the potential Φ is a *Morse function*. Recall that a C^2-function ϕ on a n-manifold is called a *Morse function* if for each of its critical points p there exists an open neighborhood V_p with coordinate system $x = (x_1, \ldots, x_n)$ and an integer $\gamma_p \in [0, n]$ — *index of* p, such that $\phi(x)|_{V_p} = \phi(p) - \sum\limits_{i=1}^{\gamma_p} x_i^2 + \sum\limits_{i=\gamma_p+1}^{n} x_i^2$.

Magnetic *nulls* are the points where the magnitude of the magnetic field vector vanishes. Due to the solenoidal condition the three eigenvalues λ_1, λ_2, λ_3 of a critical point satisfy $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Since $\mathbf{B}$ is potential all eigenvalues are real. As Φ is a Morse function each eigenvalue is non zero and each null of $\mathbf{B}$ is a saddle point.

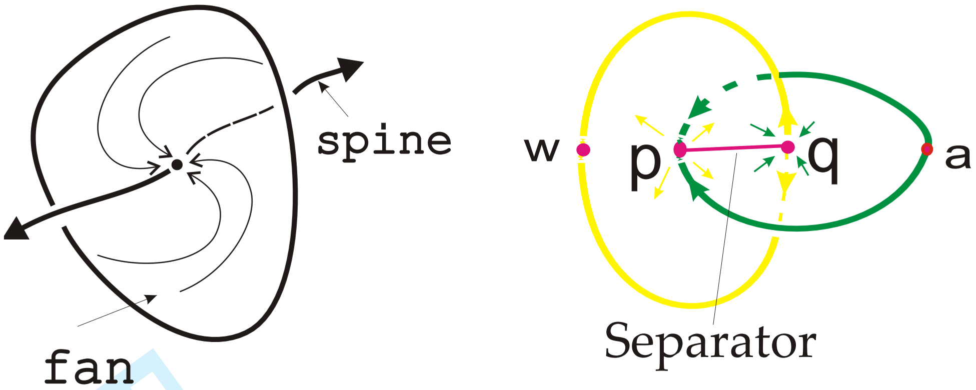

Figure 2. The separator

Two quite distinct families of field lines tend to a null point: the *spine* is a line and the *fan* is a surface (see Figure 2). A null is called *positive (negative)* if $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 > 0$ ($\lambda_1 \cdot \lambda_2 \cdot \lambda_3 < 0$). The topological structure of a magnetic field $\mathbf{B}$ is largely defined by null points, spines, fans, and separators, the union of which forms the so-called *skeleton* of the magnetic field. There are several types of nulls. A null which belongs to the photosphere is called *photospheric*. A photospheric null point whose spine lies in the photosphere is called *prone*, whereas a photospheric null with a spine directed vertically is called *upright*. *The coronal null* is a null above the photosphere [1] .

When two fans have intersection they form a *separator*, which joins two oppositely signed null points (see Figure 2). Fans divide the corona into different regions called *domains*. Appearance and disappearance of separators change the topology of domains splitting. Such situation is called a *separator reconnection*, which is one of the major reconnection mechanisms [20] . The simplest case where a separator occurs is known as the *intersecting state* [2]. Figure 3 shows the phase portrait of a magnetic field outside the photosphere, here the domes are half of the fans of photospheric nulls.

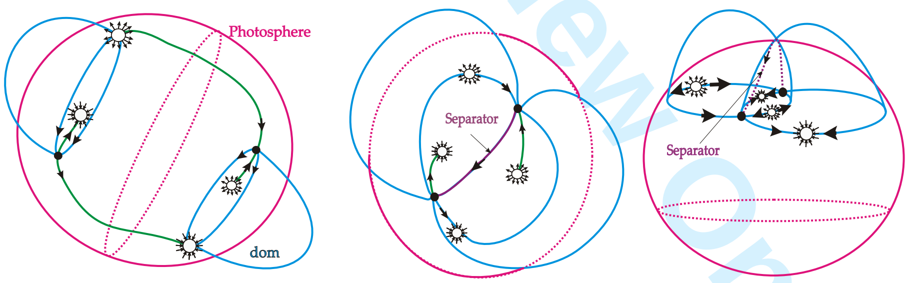

Figure 3. The intersecting state

Papers [5] , [14] , [15] , [16] studied classification of the magnetic field configurations that arise from such point-source models. It is natural to introduce the following definition which goes back to the classic paper [19], see also [22] .

Definition 1.1. *One says that two coronal magnetic fields* $\mathbf{B}, \mathbf{B}'$ *are topologically equivalent if there is a homeomorphism* $H : M \to M$ *sending magnetic lines of* $\mathbf{B}$ *to magnetic lines of* $\mathbf{B}'$ *while preserving orientation on the lines.*

It is known (see, for example, [6]) that a dense set of gradient vector fields ξ generated by a Morse function ϕ have a so called *self-indexing energy function* φ, that is a

Morse function with the following properties:

(1) the sets of the critical points of ϕ and φ coincide;
(2) for each critical point p it holds $\varphi(x) = \phi(x) + const$ for $x \in V_p$ and $\varphi(p) = \gamma_p$;
(3) $\xi(\varphi) < 0$ out of the critical points.

Denote by $\mathcal{B}$ the set of magnetic fields $\mathbf{B}$ possessing an energy function φ. In this paper we solve the problem of existence of a separator on the fan with respect to the type of the null of $\mathbf{B} \in \mathcal{B}$ and we find an interrelation between the number of zeros and the number of charges. We also get classification of the magnetic fields from $\mathcal{B}$ up to topological equivalence.

Specifically, let p be a null of $\mathbf{B}$. Denote by F_p the fan of p and by S_p the spine of p. Let $T_p = F_p \cap P$, that is T_p is the trace of the fan F_p on the photosphere P.

Theorem 1.2. *For the nulls $p, p_1, \dots, p_n$ of $\mathbf{B}$ the following statements hold:*

(1) if l_p is a connected component of $S_p \setminus p$ then $cl\ l_p \setminus (l_p \cup p)$[1] is exactly one charge q_i for some $i \in \{1, \dots, k\}$;
(2) the fan F_p does not contain a separator if and only if $cl\ F_p \setminus F_p$ is exactly one charge q_i for some $i \in \{1, \dots, k\}$;
(3) if there are nulls $p_1, \dots, p_n$ such that $\bigcup\limits_{i=1}^{n} cl\ S_{p_i}$ is a simple closed curve then F_{p_i} contains at least one separator for each $i \in \{1, \dots, n\}$.

Theorem 1.3. *For an arbitrary coronal magnetic field $\mathbf{B} \in \mathcal{B}$ the following statements hold:*

- *the fan F_p of each coronal or upright null p contains at least one separator;*
- *the fan F_p of a prone null p does not contain a separator if and only if $cl\ T_p \setminus T_p$ is exactly one charge q_i for some $i \in \{1, \dots, k\}$.*

Denote by m the number of the nulls for $\mathbf{B} \in \mathcal{B}$ and let $g = \frac{m-k+2}{2}$.

Theorem 1.4. *For an arbitrary coronal magnetic field $\mathbf{B} \in \mathcal{B}$ the following statements hold:*

- *g is integer and $g \geq 0$;*
- *$\mathbf{B}$ has at least $2g$ nulls whose fans contain at least one separator;*
- *the level set $\Sigma = \varphi^{-1}(\frac{3}{2})$ is an orientable surface of genus g.*

Denote by N_+ (N_-) the set of positive (negative) nulls of $\mathbf{B}$. Let $F_+ = \bigcup\limits_{p \in N_+} F_p$ ($F_- = \bigcup\limits_{p \in N_-} F_p$).

Definition 1.5. *We say two self-indexing energy functions φ, φ' for magnetic fields $\mathbf{B}, \mathbf{B}'$ to be consistently equivalent if there exists an orientation-preserving homeomorphism $H : M \to M'$ such that:*

(1) $\varphi' H = \varphi$;
(2) $H(\Sigma \cap F_+) = \Sigma' \cap F'_+$, $H(\Sigma \cap F_-) = \Sigma' \cap F'_-$.

Theorem 1.6. *Magnetic fields $\mathbf{B}, \mathbf{B}'$ are topologically equivalent if and only if their self-indexing energy functions φ, φ' are consistently equivalent.*

---

[1]Here $cl\ A$ is the closure of A.

2. Proof of the main results

To prove the results we complete the magnetic field lines in the places of point-charge by the bundle of straight lines. This idea was used in [10] to find the separators of magnetic fields in electrically conducting fluids. Now the magnetic lines of the field $\mathbf{B}$ coincide geometrically on M with trajectories of a three-dimensional flow $f^t : \mathbb{S}^3 \to \mathbb{S}^3$ with the following properties:

(1) f^t has no periodic trajectories and the non-wandering set $\Omega(f^t)$ of f^t consists of a finite number of hyperbolic equilibrium states;
(2) all the node points belong to the photosphere and the number of the sinks equals the number of the sources;
(3) all the trajectories of f^t are symmetric with respect to the photosphere P;
(4) f^t possesses a self-indexing energy function $\tilde{\varphi} : \mathbb{S}^3 \to [0,3]$.

Denote by G the set of flows with these properties. By the construction the flow f^t of a magnetic field $\mathbf{B} \in \mathcal{B}$ belongs to the class G (see Figure 4) have the following properties:

- charges coincide with sink and source equilibrium states,
- null points coincide with saddle equilibrium states,
- the fan (spine) of each null coincides with the two-dimensional (one-dimensional) invariant manifold of the corresponding saddle,
- separators coincide with *heteroclinic curves* — connected components of the intersection of two-dimensional invariant manifolds of the saddle points,
- the magnetic lines of $\mathbf{B}$ coincide with the trajectories of f^t on M,
- the function $\tilde{\varphi}$ coincides with φ on $\mathbb{S}^3 \setminus \bigcup\limits_{i=1}^{k} U_i$ where U_i is a neighborhood of q_i.

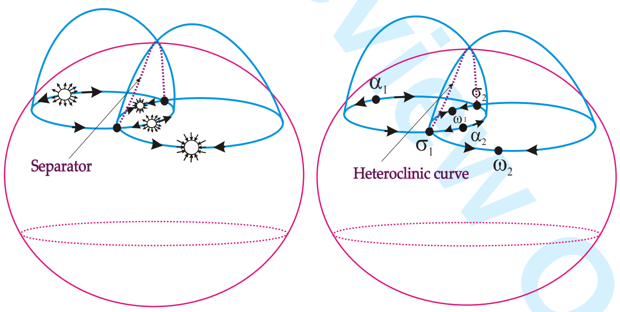

Figure 4. The completeness of the magnetic lines

Let $f^t \in G$ and let σ be a saddle point of f^t, W^u_σ and W^s_σ being its unstable and stable manifolds. Denote by Ω_1 (Ω_2) the set of the saddle points σ of f^t such that dim $W^u_\sigma = 1$ (dim $W^u_\sigma = 2$) and denote by Ω_0 (Ω_3) the set of the sinks (sources). Let

$$A = \bigcup_{\sigma \in \Omega_1} cl\ W^u_\sigma, \qquad R = \bigcup_{\sigma \in \Omega_2} cl\ W^s_\sigma.$$

The following proposition is due to [22] (see also [7] for details).

Proposition 2.1. *For every flow $f^t \in G$ the following statements hold:*

(1) $\mathbb{S}^3 = \bigcup\limits_{x\in\Omega(f^t)} W^s_x = \bigcup\limits_{x\in\Omega(f^t)} W^u_x$ *and each invariant manifold* W^s_x (W^u_x) *is a submanifold of* $\mathbb{S}^3$*;*

(2) $cl\ W^u_x \cap W^u_y \neq \emptyset$ *if and only if* $W^u_x \cap W^s_y \neq \emptyset$*;*

(3) the sets A, R *are pairwise disjoint and each of them is connected.*

Proof of Theorem 1.2

Theorem 1.2 follows from Lemma 2.2.

Lemma 2.2. *Let* $f^t \in G$ *and* $\sigma, \sigma_1, \ldots, \sigma_n \in \Omega_1$*. Then the following statements hold:*

(1) if l_σ *is a connected component of* $W^u_\sigma \setminus \sigma$ *then* $cl\ l_\sigma = \sigma \cup l_\sigma \cup \omega$ *for some sink* $\omega \in \Omega_0$*;*

(2) W^s_σ *contains no heteroclinic curve if and only if* $cl\ W^s_\sigma \setminus W^s_\sigma = \alpha$ *for some source* $\alpha \in \Omega_3$*;*

(3) if there are saddle points $\sigma_1, \ldots, \sigma_n$ *such that* $\bigcup\limits_{i=1}^{n} cl\ W^u_{\sigma_i}$ *is a simple closed curve then* $W^s_{\sigma_i}$ *contains at least one heteroclinic curve for each* $i \in \{1, \ldots, n\}$*.*[2]

Proof:

(1) Let l_σ be a connected component of $W^u_\sigma \setminus \sigma$. It follows from the properties of the self-indexing energy function $\tilde{\varphi}$ for f^t that it decreases along l_σ. As $\tilde{\varphi}(\sigma) = 1$ from the fact that $W^u_\sigma \cap W^s_\omega \neq \emptyset$ for some point $\omega \in \Omega(f^t)$ (see item 1) of Proposition 2.1) it follows that ω is a sink point. Since l_σ is connected and since stable manifolds of different sinks are pairwise disjoint it follows from item 2 of Proposition 2.1 that there is a unique sink $\omega \in \Omega_0$ such that $cl\ l_\sigma = \sigma \cup l_\sigma \cup \omega$.

(2) Suppose that W^s_σ does not contain a heteroclinic curve. Similar to the proof above we get that $cl\ (W^s_\sigma) \setminus (W^s_\sigma) = \alpha$ for some source $\alpha \in \Omega_3$. Inversely, if $cl\ (W^s_\sigma) \setminus (W^s_\sigma) = \alpha$ for some source $\alpha \in \Omega_3$ then by item 2 of Proposition 2.1 the stable manifold W^s_σ intersects only one of the unstable manifolds W^u_α. Hence, $W^s_\sigma \cap W^u_\delta = \emptyset$ for any saddle point $\delta \in \Omega_2$ which means that W^s_σ does not contain a heteroclinic curve.

(3) Let $\sigma_1, \ldots, \sigma_n$ be saddle points such that $\bigcup\limits_{i=1}^{n} cl\ W^u_{\sigma_i}$ is a simple closed curve C. Suppose that $W^s_{\sigma_i}$ does not contain heteroclinic curves. By the item 2) $cl\ (W^s_{\sigma_i}) = W^s_{\sigma_i} \cup \alpha_i$ for some source $\alpha_i \in \Omega_3$. Then $cl\ (W^s_{\sigma_i})$ is the 2-sphere (see [7] for details). By Morse theory each level set of $\tilde{\varphi}$ near α_i is the 2-sphere. As $\tilde{\varphi}$ is the self-indexing energy function for f^t such level set intersects $cl\ (W^s_{\sigma_i})$ along a unique circle. Due to item 1 of Proposition 2.1 and the criteria in [11] the sphere $cl\ (W^s_{\sigma_i})$ is cylindrically embedded[3] to $\mathbb{S}^3$, hence, by Generalized Schoenflies theorem [4] $cl\ (W^s_\sigma)$ bounds a 3-ball B_σ in $\mathbb{S}^3$. Then the simple closed curve C intersects ∂B_σ at least two points, that is a contradiction with the equality $\partial B_\sigma \cap C = \sigma$.

$\diamond$

Proof of Theorem 1.3

Proof:

- By item 1 of Theorem 1.2 $cl\ S_p \setminus S_p$ consists of charges. If p is a coronal or upright

[2] For saddle points from the set Ω_2 we can prove similar statements if we consider the flow f^{-t}.

[3] A 2-sphere S^2 in a 3-manifold X is said to be *cylindrical* or *cylindrically embedded* into X if there is a homeomorphism to its image $h : \mathbb{S}^2 \times [-1, 1] \to X$ such that $h(\mathbb{S}^2 \times \{0\}) = S^2$.

null then using the symmetry of $\mathbf{B}$ with respect to P we get that S_p is a subset of a closed curve composed of the closures of spines. By item 3 of Theorem 1.2 the fan F_p contains a separator.

- By item 2 of Theorem 1.2 the fan F_p does not contain a separator if and only if $cl\ F_p \setminus F_p$ is exactly one charge q_i for some $i \in \{1, \dots, k\}$. If p is a prone null then T_p is not empty. Thus the fan F_p contains no separators if and only if $cl\ T_p \setminus T_p$ is exactly one charge q_i for some $i \in \{1, \dots, k\}$.

$\diamond$

Proof of Theorem 1.4

Notice that $k = |\Omega_0 + \Omega_3|$ and $m = |\Omega_1 + \Omega_2|$ where $|.|$ stands for the cardinality. Recall that $g = \frac{m-k+2}{2}$. Then Theorem 1.4 follows from the following lemma.

Lemma 2.3. *For an arbitrary flow $f^t \in G$ the following statements hold:*

- *g is integer and $g \geq 0$;*
- *the flow f^t has at least g saddles in Ω_1 whose stable manifold contain a separator;*
- *the level set $\Sigma = \tilde{\varphi}^{-1}(\frac{3}{2})$ is an orientable surface of genus g.*

Proof: A flow $f^t \in G$ has a self-indexing energy function $\tilde{\varphi}$ which is a Morse function with Ω_i points of index i, $i = 0, 1, 2, 3$. By Morse theory [18] the alternating sum $|\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3|$ equals to the Euler characteristic of $\mathbb{S}^3$, which is 0 since $\mathbb{S}^3$ is an odd-dimensional, closed oriented manifold. Hence, $|\Omega_0| - |\Omega_1| = |\Omega_3| - |\Omega_2|$. Since $f^t \in G$ we have $|\Omega_0| = |\Omega_3|$, therefore, $|\Omega_1| = |\Omega_2|$. Thus

$$|\Omega_1| - |\Omega_0| + 1 = |\Omega_2| - |\Omega_3| + 1 = g.$$

Let $Q = \tilde{\varphi}^{-1}([0, \frac{3}{2}])$. According to the Morse theory Q is obtained by gluing $|\Omega_1|$ 1-handles[4] to the union of $|\Omega_0|$ 3-balls. Moreover, it is connected since each point in Q can be connected by a trajectory of the flow f^t with the set A which is connected due to Proposition 2.1. Therefore Q is a handlebody of some genus $\tilde{g} \geq 0$. As Q has the homotopic type of the bouquet with $\tilde{g}$ loops the Euler characteristic of Q is $1 - \tilde{g}$. On the other hand Q is homotopic to A and, hence, the Euler characteristic of Q is $|\Omega_0| - |\Omega_1|$. Then $\tilde{g} = |\Omega_1| - |\Omega_0| + 1$ and so $\tilde{g} = g$. From $\Sigma = \partial Q$ it follows that Σ is an orientable surface of genus g.

From the properties of A it follows that A has exactly g closed different curves consisting of closures of unstable manifolds of saddle points from Ω_1. By Lemma 2.2 at least g stable manifolds of saddles from Ω_1 contain heteroclinic curves. $\diamond$

Proof of Theorem 1.6

In papers [17], [9] the self-indexing energy function was used for the topological classification of different classes of *Morse-Smale flows* — structural stable flows with finite number of non-wandering trajectories. In the class G we deal with flows which are not structurally stable in general but Lemma 2.5 shows that the consistently equivalent class of self-indexing functions (see definition below) is the complete topological invariant for flows from G.

Definition 2.4. *We say that two self-indexing energy function $\tilde{\varphi}, \tilde{\varphi}'$ for flows f^t, f'^t are consistently equivalent if there exists an orientation-preserving homeomorphism $\tilde{H} : \mathbb{S}^3 \to \mathbb{S}^3$ such that:*

[4] The 3-dimensional 1-handle is the product of the interval and the 2-disc. The gluing is made along the top and bottom disks (see Section 3 in [18]).

(1) $\tilde{\varphi}'\tilde{H} = \tilde{\varphi}$;
(2) $\tilde{H}(W^s_{\Omega_1} \cap \Sigma) = W^s_{\Omega'_1} \cap \Sigma'$, $\tilde{H}(W^u_{\Omega_2} \cap \Sigma) = W^u_{\Omega'_2} \cap \Sigma'$.

Then Theorem 1.6 follows from the following lemma.

Lemma 2.5. *Two flows f^t, f'^t are topologically equivalent if and only if their self-indexing energy functions $\tilde{\varphi}, \tilde{\varphi}'$ are consistently equivalent.*

Proof: Let $x \in \mathbb{S}^3$ be an arbitrary point. Denote by l_x (l'_x) the trajectory of f^t (of f'^t) passing through x. According to Proposition 2.1 there is a unique pair of the equilibrium states $\alpha(l_x)$, $\omega(l_x)$ ($\alpha(l'_x)$, $\omega(l'_x)$) such that $l_x \subset (W^u_{\alpha(l_x)} \cap W^s_{\omega(l_x)})$ ($l'_x \subset (W^u_{\alpha(l'_x)} \cap W^s_{\omega(l'_x)})$). For each $c \in [0,3]$ let $\Sigma_c = \tilde{\varphi}^{-1}(c)\,(\Sigma'_c = (\tilde{\varphi}')^{-1}(c))$. Let $C^s = W^s_{\Omega_1} \cap \Sigma$, $(C'^s = W^s_{\Omega'_1} \cap \Sigma')$, $C^u = W^u_{\Omega_2} \cap \Sigma$, $(C'^u = W^u_{\Omega'_2 1} \cap \Sigma')$. Note that each connected component of the sets C^s, (C'^s), C^u, (C'^u) is a closed curve.

Necessity.

Let $\tilde{\varphi}$ and $\tilde{\varphi}'$ be self-indexing energy functions of topologically equivalent flows f^t and f'^t from G and let $h : \mathbb{S}^3 \to \mathbb{S}^3$ be a homeomorphism sending the trajectories of f^t to the trajectories of f'^t. It follows from the definition of a self-indexing function and the properties of the homeomorphism h that for any equilibrium state p of the flow f^t it holds $\tilde{\varphi}(p) = \tilde{\varphi}'(h(p))$. We construct a homeomorphism $\tilde{H}$ realizing the consistent equivalence of $\tilde{\varphi}$ and $\tilde{\varphi}'$ as follows.

For $x' = h(x)$ we have $l_{x'} = h(l_x)$. It follows from the properties of h that

$$\alpha(l_{x'}) = h(\alpha(l_x)), \quad \omega(l_{x'}) = h(\omega(l_x)).$$

Moreover,

$$\tilde{\varphi}(\alpha(l_x)) = \tilde{\varphi}'(\alpha(l_{x'})), \quad \tilde{\varphi}(\omega(l_x)) = \tilde{\varphi}'(\omega(l_{x'})).$$

Let $c \in [\tilde{\varphi}(\omega(l_x)), \tilde{\varphi}(\alpha(l_x))]$. Note that by the definition of a self-indexing energy function the intersection $\Sigma_c \cap l_x$ ($\Sigma'_c \cap l'_{x'}$) consists of exactly one point. Thus, the map $\tilde{H}$ sending each point $y = \Sigma_c \cap l_x$ to the point $y' = \Sigma'_c \cap l'_{x'}$ is well a defined homeomorphism of $\mathbb{S}^3$. By construction $\tilde{H}$ maps the trajectories of f^t and the level sets of the function $\tilde{\varphi}$ into the trajectories of f'^t and the level sets of $\tilde{\varphi}'$. Hence,

$$\tilde{H}(C^s) = C'^s, \; \tilde{H}(C^u) = C'^u.$$

Sufficiency.

Let the self-indexing energy functions $\tilde{\varphi}, \tilde{\varphi}'$ be consistently equivalent, i.e. there exists an orientation-preserving homeomorphism $\tilde{H} : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\tilde{\varphi}'\tilde{H} = \tilde{\varphi}$; $\tilde{H}(C^s) = C'^s$, $\tilde{H}(C^u) = C'^u$. Now we construct a homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ realizing the equivalence of the flows f^t and f'^t.

For any point $x \in \Sigma$ we set $x' = \tilde{H}(x)$. Let $V = \mathbb{S}^3 \setminus (A \cup R)$, $V' = \mathbb{S}^3 \setminus (A' \cup R')$. By Proposition 2.1 we have the following possibilities for a point $x \in \Sigma$:

- $\alpha(l_x) \in \Omega_3$, $\omega(l_x) \in \Omega_0$ for $x \in \Sigma \setminus (C^u \cup C^s)$;
- $\alpha(l_x) \in \Omega_2$, $\omega(l_x) \in \Omega_0$ for $x \in (C^u \setminus C^s)$;
- $\alpha(l_x) \in \Omega_3$, $\omega(l_x) \in \Omega_1$ for $x \in (C^s \setminus C^u)$;
- $\alpha(l_x) \in \Omega_2$, $\omega(l_x) \in \Omega_1$ for $x \in (C^u \cap C^s)$.

The same holds for points of Σ'. As $\tilde{H}(C^s) = C'^s$, $\tilde{H}(C^u) = C'^u$ the homeomorphism $H_V : V \to V'$ on V is well-defined by the formula $H_V(y) = l'_{x'} \cap \Sigma'_c$ for $y = l_x \cap \Sigma_c$, $c \in (\omega(l_x), \alpha(l_x))$. Moreover, H_V sends the two-dimensional invariant manifolds of a saddle point σ of f^t to the two-dimensional invariant manifolds of a saddle point σ' of f'^t.

Now we show that $H_V(\omega(l_x)) = \omega(l'_{x'})$ for each $x \in \Sigma$.

Let $Q = \tilde{\varphi}^{-1}([0, \frac{3}{2}])$. As $\tilde{\varphi}$ decreases along trajectories of f^t the set $Q \subset W^s_{\Omega_0 \cup \Omega_1}$ and the set $D_\sigma = W^s_\sigma \cap Q$ is a 2-disk for each $\sigma \in \Omega_1$. Denote by Y a connected component of the set $Q \setminus W^s_{\Omega_1}$. There is a unique sink $\omega \in \Omega_0$ such that $\omega \in Y \subset W^s_\omega$ and there is a unique connected component K_Y of the set $\Sigma \setminus C^s$ belonging Y and such that $Y \setminus A = \bigcup\limits_{x \in K_Y} (l_x \cap Y) \cup \omega$. The same holds for the flow f'^t. Since $\tilde{H}(\Sigma \setminus C^s) = \Sigma' \setminus C'^s$ the component $\tilde{H}(K_Y)$ is a connected component of $\Sigma' \setminus C'^s$ belonging to a connected component Y' of the set $Q' \setminus W^s_{\Omega'_1}$ containing a sink $\omega' \in \Omega'_0$. By construction $H_V(Y \setminus A) = Y' \setminus A'$, hence, $H_V(\omega(l_x)) = \omega(l'_{x'})$ for each $x \in (\Sigma \setminus C^s)$. By continuity $H_V(\omega(l_x)) = \omega(l'_{x'})$ for each $x \in C^s$.

Thus H_V can be uniquely extended to the sets Ω_0, Ω_1. We keep the notation H_V for the homeomorphism thus obtained and extend this homeomorphism to the one-dimensional separatrices of the saddle fixed points. For any $c \in (0, 1)$ we have $H_V(\Sigma_c \setminus W^u_{\Omega_1}) = \tilde{H}(\Sigma_c) \setminus W^u_{\Omega'_1}$ and the sets $W^u_{\Omega_1} \cap \Sigma_c$, $W^u_{\Omega'_1} \cap \tilde{H}(\Sigma_c)$ are finite unions of finitely many points. It follows that the homeomorphism H_V extends by continuity to a homeomorphism $H_1 : W^u_{\Omega_1} \to W^u_{\Omega'_1}$. The homeomorphism $H_2 : W^s_{\Omega_2} \to W^s_{\Omega'_2}$ is constructed in the similar way. The desired homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ is defined by

$$h(x) = \begin{cases} H_V(x), & x \in \mathbb{S}^3 \setminus (W^u_{\Omega_1} \cup W^s_{\Omega_2}), \\ H_1(x), & x \in W^u_{\Omega_1}, \\ H_2(x), & x \in W^s_{\Omega_2}. \end{cases}$$

$\diamond$

3. Conclusions

Theorems 1.2, 1.3, 1.4 answer one of the main question of magnetic fields theory in corona of the Sun – the existence of separators in dependence on the type and the number of saddle singularities and charges. Using these results we get a tool for understanding of scenarios of bifurcations connected with appearance and disappearance of separators and, therefore, of flares and prominences in corona and also for describing the transitions from one state of domains to a qualitatively different state. The result concerning the topological equivalence of different types of solar activity (Theorem 1.6) can be used directly to model the energy processes in the solar corona. Moreover, it creates a possibility for description of scenarios of transitions from one state of domains to a qualitatively different state by, for example, the bifurcations from papers [3], [8].

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