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On existence of a smooth arc without bifurcations joining source-sink diffeomorphisms on 2-sphere

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Abstract. In this paper we construct a smooth arc without bifurcation points joining source-sink diffeomorphisms on the two-dimensional sphere.

1. Introduction and statement of results

One of the fifty problems of dynamical systems compiled by J. Palis and C. Pugh is the problem of the existence of an arc with a finite or countable set of bifurcations joining two Morse-Smale systems on a smooth closed manifold [16]. In [14] Sh. Newhouse and M. Peixoto proved that every Morse-Smale vector fields can be joined by *a simple arc*. Simplicity means that the arc consists of Morse-Smale systems with the exception of a finite set of points where the vector field has the least deviation from a Morse-Smale system. For discrete dynamical systems the situation is different. Two orientation preserving Morse-Smale diffeomorphisms on a circle can be joined by a simple arc if and only if they have the same rotation numbers. As follows from the works by Sh. Matsumoto [11] and by P. Blanchard [2], any orientable closed surface admits isotopic Morse-Smale diffeomorphisms that can not be joined by a simple arc. It is said that two isotopic Morse-Smale diffeomorphisms belong to the same *simple isotopic class* if they can be joined by a simple arc. It follows from the papers [11, 2] that there exist infinitely many simple isotopy classes of Morse-Smale diffeomorphisms on any orientable surface inside an isotopy class that admits Morse-Smale diffeomorphisms.

In dimension 3, the problem of the existence of a simple arc is complicated by the presence of Morse-Smale diffeomorphisms with wildly embedded separatrices. The first “wild” example was constructed by D. Pixton [17]. This diffeomorphism belongs to the class (called in [7] *Pixton class*) consisting of three-dimensional Morse-Smale diffeomorphisms for which the non-wandering set consists of exactly four points: two sinks, a source and a saddle. As follows from the work by Ch. Bonatti, V. Grines, V. Medvedev, O. Pochinka [5], any Pixton diffeomorphism is connected by a simple arc with some source-sink diffeomorphism. This effect is due to the fact that for any diffeomorphism from the Pixton class at least one one-dimensional separatrix of the saddle point is tame [4]. Using the operation of a taking the connected sum of two 3-spheres on which Pixton diffeomorphisms with wildly embedded separatrices are given, it is easy to construct a diffeomorphism for which all the separatrices of all saddles are wildly imbedded. In the paper by V. Grines and O. Pochinka [8] it was proved that such a diffeomorphism is not joined by a simple arc with a source-sink diffeomorphism.

Gradient-like diffeomorphisms on surfaces, in contrast to the three-dimensional case, do not reveal the wild behavior of separatrices, which creates the prerequisites for an exhaustive solution of the Palis-Pugh problem. The first steps in this direction have been taken in this paper. In more detail.

Consider 2-sphere $\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. We denote by $Diff(\mathbb{S}^2)$ the set of diffeomorphisms on 2-sphere, that is the smoothness of considered diffeomorphism more or equal than 1. Denote by $J(\mathbb{S}^2) \subset Diff(\mathbb{S}^2)$ set of orientation preserving *source-sink diffeomorphisms*, that is, diffeomorphisms whose non-wandering sets consist of precisely two hyperbolic points: a source, a sink. The existence of a smooth arc joining any orientation-preserving diffeomorphisms and, therefore, any source-sink diffeomorphisms of $\mathbb{S}^2$ is a classical result by S. Smale [18]. We show that this arc can be chosen in such a way that it consists of source-sink diffeomorphisms only.

Theorem 1. *For any diffeomorphisms $f, f' \in J(\mathbb{S}^2)$ there is a smooth arc $\{f_t \in J(\mathbb{S}^2)\}$ their joining.*

Thus, all orientation preserving source-sink diffeomorphisms on 2-sphere belong to the same simply connected component. According to [13] the boundary of this component consists of systems with the following properties: (1) all stable, strong stable, unstable, and strong unstable manifolds intersect transversally; (2) it has no cycles and has exactly one nonhyperbolic periodic orbit which is either a flip or a noncritical saddle-node.

2. Denotations

Let M be a smooth orientable manifold. Two diffeomorphisms $f, f' : M \to M$ are smoothly isotopic if there is a smooth map $F : M \times [0,1] \to M$ (smooth isotopy) such that f_t given by the formula $f_t(x) = f(x,t)$ is a diffeomorphism for each $t \in [0,1]$ and $f_0 = f, f_1 = f'$. We say that the family $\{f_t\}$ is a smooth arc joining diffeomorphisms f and f'. The support supp $\{f_t\}$ of the isotopy $\{f_t\}$ is the closure of the set $\{x \in M : f_t(x) \neq f_0(x)$ for some $t \in [0,1]\}$. Let $Diff(M)$ be the space of the diffeomorphisms of M with C^1-topology. Notice that the existence of the smooth isotopy $F : M \times [0,1] \to M$ implies the existence of the continuous path $\gamma : [0,1] \in Diff(M)$ such that $\gamma(t) = f_t$, however the opposite is not true.

For the smooth arcs $\{f_t\}$ and $\{g_t\}$ such that $f_1 = g_0$ the usually product of the respective paths is not smooth in general, but we can define the smooth arc $\{h_t\}$ joining diffeomorphisms f_0, g_1 as follows:

$$h_t = \begin{cases} f_{2\tau(t)}, 0 \leq t \leq \frac{1}{2} \\ g_{2\tau(t)-1}, \frac{1}{2} \leq t \leq 1 \end{cases},$$

where $\tau : [0,1] \to [0,1]$ is a smooth monotone function such that $\tau(t) = 0$ for $0 \leq t \leq \frac{1}{3}$ and $\tau(t) = 1$ for $\frac{2}{3} \leq t \leq 1$ (see, for example, [12], Lemma 1.8). The smooth arc $\{h_t\}$ we will called a smooth product of the smooth arcs $\{f_t\}$ and $\{g_t\}$ and denoted $\{h_t\} = \{f_t * g_t\}$.

In the present paper we will work with the class $J(\mathbb{S}^2) \subset Diff(\mathbb{S}^2)$ of orientation preserving source-sink diffeomorphisms, that is, diffeomorphisms whose non-wandering sets consist of precisely two hyperbolic points: a source, a sink. Also we introduce a subclass $NS(\mathbb{S}^2) \subset J(\mathbb{S}^2)$ for which the source is the North pole $N(0,0,1)$ and the sink is the South pole $S(0,0,-1)$.

Define the diffeomorphisms

$$\vartheta_- : \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2,$$

$$\vartheta_+ : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$$

(the stereographic projections) by the formulas:

$$\vartheta_-(x_1, x_2, x_3) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3} \right),$$

$$\vartheta_+(x_1,x_2,x_3)=\left(\frac{x_1}{1-x_3},\frac{x_2}{1-x_3}\right).$$

Then the inverse maps

$$\vartheta_-^{-1}:\mathbb{R}^2\to\mathbb{S}^2\setminus\{S\},$$
$$\vartheta_+^{-1}:\mathbb{R}^2\to\mathbb{S}^2\setminus\{N\}$$

given by the formulas:

$$\vartheta_-^{-1}(x_1,x_2)=\left(\frac{2x_1}{x_1^2+x_2^2+1},\frac{2x_2}{x_1^2+x_2^2+1},\frac{1-(x_1^2+x_2^2)}{x_1^2+x_2^2+1}\right),$$

$$\vartheta_+^{-1}(x_1,x_2)=\left(\frac{2x_1}{x_1^2+x_2^2+1},\frac{2x_2}{x_1^2+x_2^2+1},\frac{x_1^2+x_2^2-1}{x_1^2+x_2^2+1}\right).$$

Through the stereographic projections the diffeomorphisms from $NS(\mathbb{S}^2)$ connected with the contractions

$$S(\mathbb{R}^2)=\{\vartheta_+f\vartheta_+^{-1},f\in NS(\mathbb{S}^2)\}$$

and extension

$$N(\mathbb{R}^2)=\{\vartheta_-f\vartheta_-^{-1},f\in NS(\mathbb{S}^2)\}.$$

Let us define the model diffeomorphisms $g:\mathbb{S}^2\to\mathbb{S}^2$, $\bar g:\mathbb{R}^2\to\mathbb{R}^2$ by the formulas

$$g(x_1,x_2,x_3)=\left(\frac{4x_1}{5-3x_3},\frac{4x_2}{3x_3-5},\frac{5x_3-3}{5-3x_3}\right),$$

$$\bar g(x_1,x_2)=\left(\frac{x_1}{2},\frac{x_2}{2}\right).$$

It is easy to check that $g\in NS(\mathbb{S}^2)$, $\bar g\in S(\mathbb{R}^2)$, $\bar g^{-1}\in N(\mathbb{R}^2)$, $\bar g^{-1}=\vartheta_-g\vartheta_-^{-1}$ and $\bar g=\vartheta_+g\vartheta_+^{-1}$.

Denote by $E_g\subset NS(\mathbb{S}^2)$ the set of diffeomorphisms such that for each diffeomorphism $h\in E_g$ there are neighborhoods $V_h(N)$, $V_h(S)$ of the points N, S, where $h|_{V_h(N)\cup V_h(S)}=g|_{V_h(N)\cup V_h(S)}$. Denote by $E_{\bar g}\subset S(\mathbb{R}^2)$ ($E_{\bar g^{-1}}\subset N(\mathbb{R}^2)$) the set of diffeomorphisms such that for each diffeomorphism $\bar h\in E_{\bar g}$ ($\bar h\in E_{\bar g^{-1}}$) there is a neighborhood $V_{\bar h}(O)$ of the point O, where $\bar h|_{V_{\bar h}(O)}=\bar g|_{V_{\bar h}(O)}$ ($\bar h|_{V_{\bar h}(O)}=\bar g^{-1}|_{V_{\bar h}(O)}$).

3. Construction of a smooth arc

It is easy to prove Theorem 1 if f and f' are smoothly conjugated by a diffeomorphism h. In this case, due to [18], there is a smooth isotopy $h_t:\mathbb{S}^2\to\mathbb{S}^2$ such that $h_0=id$, $h_1=h$. Thus $f_t=h_t^{-1}fh_t$ is a required arc. In the opposite case we show how to construct a smooth arc $\{l_t\in J(\mathbb{S}^2)\}$ joining an arbitrary diffeomorphism $f\in J(\mathbb{S}^2)$ with the model diffeomorphism g. Similarly, we construct a smooth arc $\{l'_t\in J(\mathbb{S}^2)\}$ that connects f' and g. Then the smooth product of smooth arcs $f_t=l_t*l'_{1-t}$ is the required arc. The arc l_t is a smooth product of tree smooth arcs $\beta_t,\gamma_t,\delta_t$ ($l_t=\beta_t*\gamma_t*\delta_t$), whose construction is given in the corresponding sentences below:

(i) l_t is a diffeomorphism from the class $J(\mathbb{S}^2)$ for all $t\in[0,\frac{1}{3}]$, $l_0=f$ and $l_{\frac{1}{3}}$ is a diffeomorphism from the class $NS(\mathbb{S}^2)$ (see Proposition 1);

(ii) l_t is a diffeomorphism from the class $NS(\mathbb{S}^2)$ for all $t\in[\frac{1}{3},\frac{2}{3}]$, $l_{\frac{2}{3}}\in E_g$ (see Proposition 2);

(iii) l_t is a diffeomorphism from the class $J(\mathbb{S}^2)$ for all $t\in[\frac{2}{3},1]$, $l_1=g$ (see Proposition 3).

Proposition 1. *For any diffeomorphism $f \in J(\mathbb{S}^2)$ there exists a smooth arc $\{\beta_t \in (J(\mathbb{S}^2) \cap Diff(\mathbb{S}^2))\}$ joining $\beta_0 = f$ with a diffeomorphism $\beta_1 \in NS(\mathbb{S}^2)$.*

Proof. Let $f \in J(\mathbb{S}^2)$. We denote by α the source and through the ω sink of the diffeomorphism f. Let $D_\alpha,\ D_\omega$ ($D_S,\ D_N$) be pairwise disjoint 2-disks containing $\alpha,\ \omega$ ($S,\ N$). According to [9], there exists a smooth arc $\{H_t \in Diff(\mathbb{S}^2)\}$ with the following properties $H_0 = id$, $H_1(D_N) = D_\alpha$, $H_1(D_S) = D_\omega$, $H_1(N) = \alpha$ and $H_1(S) = \omega$. Then $\beta_t = H_t^{-1} f H_t$ is the desired isotopy joining the diffeomorphism $f = \beta_0$ with the diffeomorphism $\beta_1 = H_1^{-1} f H_1 \in NS(\mathbb{S}^2)$. □

Proposition 2. *For any diffeomorphism $\beta \in NS(\mathbb{S}^2)$ there exists a smooth arc $\{\gamma_t \in NS(\mathbb{S}^2)\}$ joining $\gamma_0 = \beta$ with a diffeomorphism $\gamma_1 \in E_g$.*

Proof. Define $\bar\beta^+ = \vartheta_+ \beta \vartheta_+^{-1}$, $\bar\beta^- = \vartheta_- \beta \vartheta_-^{-1}$. Then $\bar\beta^+ \in S(\mathbb{R}^2)$, $\bar\beta^- \in N(\mathbb{R}^2)$. If $\bar\beta^+ = \bar g$ in some neighborhood of the origin O, then we define $\{\bar\gamma_t^+ = \bar\beta^+\}$. In the opposite case we show below how to construct a smooth arc $\{\bar\gamma_t^+ \in S(\mathbb{R}^2)\}$ joining $\bar\beta^+$ with a diffeomorphism $\bar\gamma_1^+ \in E_{\bar g}$ such that $\bar\gamma_t^+$ for every $t \in [0,1]$ coincides with $\bar\beta^+$ out of a neighborhood $V^+ \subset \mathbb{B}^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1\}$ of the origin O. Similarly if $\bar\beta^- = \bar g^{-1}$ in some neighborhood of the origin O, then we define $\{\bar\gamma_t^- = \bar\beta^-\}$. In the opposite case similarly to $\bar\gamma_t^+$ we construct a smooth arc $\{\bar\gamma_t^- \in N(\mathbb{R}^2)\}$ joining $\bar\beta^-$ with a diffeomorphism $\bar\gamma_1^- \in E_{\bar g^{-1}}$ such that $\bar\gamma_t^-$ for every $t \in [0,1]$ coincides with $\bar\beta^-$ out of a neighborhood $V^- \subset \mathbb{B}^2$ of the origin O. Thus the required arc γ_t given by the formula:

$$\gamma_t(w) = \begin{cases} \vartheta_+^{-1}(\gamma_t^+(\vartheta_+(w))), w \in \vartheta_+^{-1}(V^+); \\ \vartheta_-^{-1}(\gamma_t^-(\vartheta_-(w))), w \in \vartheta_-^{-1}(V^-); \\ \beta(w), w \in (\mathbb{S}^2 \setminus (\vartheta_+^{-1}(V^+) \cup \vartheta_-^{-1}(V^-)). \end{cases}$$

Let us construct a smooth arc $\{\bar\gamma_t^+ \in S(\mathbb{R}^2)\}$ joining $\bar\beta^+$ with a diffeomorphism $\bar\gamma_1^+ \in E_{\bar g}$ such that $\bar\gamma_t^+$ for every $t \in [0,1]$ coincides with $\bar\beta^+$ out of a neighborhood $V^+ \subset \mathbb{B}^2$ (the construction of the arc $\bar\gamma_t^-$ is similar). The arc $\bar\gamma_t^+$ will be a smooth product of arcs $\bar\rho_t^+$, $\bar\eta_t^+$ and $\bar\xi_t^+$, where

1) $\{\bar\rho_t^+ \in S(\mathbb{R}^2)\}$ joining $\bar\beta^+$ with a diffeomorphism $\bar\rho_1^+$ such that $\bar\rho_t^+$ for every $t \in [0,1]$ coincides with $\bar\beta^+$ out of a neighborhood $V_1^+ \subset \mathbb{B}^2$ of the origin O and $\bar\rho_1^+$ coincides with the differential $D\bar\beta_O^+$ (denote it by Q) in a neighborhood $U_1^+ \subset V_1^+$ of the origin O;

2) $\{\bar\eta_t^+ \in S(\mathbb{R}^2)\}$ joining $\bar\rho_1^+$ with a diffeomorphism $\bar\eta_1^+$ such that $\bar\varrho_t^+$ for every $t \in [0,1]$ coincides with $\bar\rho_1^+$ out of a neighborhood $V_2^+ \subset V_1^+$ of the origin O and $\bar\eta_1^+$ coincides with a diffeomorphism G given by the normal Jordan form of the diffeomorphism Q in a neighborhood $U_2^+ \subset V_2^+$ of the origin O;

3) $\{\bar\xi_t^+ \in S(\mathbb{R}^2)\}$ joining $\bar\eta_1^+$ with a diffeomorphism $\bar\xi_1^+ \in E_{\bar g}$ such that $\bar\xi_t^+$ for every $t \in [0,1]$ coincides with $\bar\eta_1^+$ out of a neighborhood $V_3^+ \subset V_2^+$.

1) As O is a hyperbolic sink for $\bar\beta_t^+$ then there is a metric $\|\cdot\|$ on $\mathbb{R}^2$ such that for some λ, $0 < \lambda < 1$, we have:

$$\|Q(v)\| \le \lambda \|v\|,$$

for each v from the tangent space $T_O\mathbb{R}^2 = \mathbb{R}^2$ (see, for example, [10]). Identify a small neighborhood $U_O \subset \bar\beta^+(\mathbb{B}^2)$ of O with a neighborhood of the zero-section of $T_O\mathbb{R}^2$ by the exponential map $e : U_O \to T_O\mathbb{R}^2$. Let $\tilde U = e(U_O)$ and define $\tilde\beta^+$ on $\tilde U$ by the formula $\tilde\beta^+ = e^{-1}\bar\beta^+ e$. For every $v \in \tilde U$ we have that $\tilde\beta^+(v) = Q(v) + \|v\| a(v)$, where $a(v)$ goes to 0 as $\|v\|$ goes to 0. As $\lambda < 1$ and $a(v)$ goes to 0 as $\|v\|$ goes to 0 then there is a number $\ell > 0$ such that in the neighborhood $\tilde V = \{(x, y) \in T_O\mathbb{R}^2 : x^2 + y^2 \le \ell^2\} \subset \tilde U$ of O for all non-zero $v \in \tilde V$ the following inequality holds $\|\tilde\beta^+(v)\| < \|v\|$. Let us define on $\tilde V$ an arc $\tilde\chi_t^+ : T_O\mathbb{R}^2 \to T_O\mathbb{R}^2$ by the formula

$$\tilde\chi_t^+ = (1-t)\tilde\beta^+ + tQ.$$

Then $\|\tilde{\chi}_t^+(v)\| < \|v\|$ for all non-zero $v \in \tilde{V}$. Let $U_+^1 = e^{-1}(\tilde{V})$ and $\bar{\chi}_t^+ = e^{-1}\tilde{\chi}_t^+ e$. Notice that the origin O is a fixed point for every diffeomorphism $\bar{\chi}_t^+$ and $\bar{\chi}_t^+(U_1^+) \subset int\ U_1^+$ for every $t \in [0,1]$. Let $V_+^1 = (\bar{\beta}^+)^{-1}(U_1^+)$. Let us consider the isotopy $h_t = (\bar{\beta}^+)^{-1}\bar{\chi}_t^+$ which joints the identity map with the diffeomorphism $(\bar{\beta}^+)^{-1}Q$. By the construction $h_t(U_1^+) \subset V_1^+$ for every $t \in [0,1]$. Then, using Thom's result [12, Theorem 5.8] about the extension of an isotopy there is an isotopy $H_t : \mathbb{R}^2 \to \mathbb{R}^2$ which is h_t on U_1^+ and is identity out of V_1^+. Then the required arc is given by the formula

$$\bar{\rho}_t^+ = \bar{\beta}^+ H_t.$$

2) If $Q = \bar{g}$ then let us define $\{\bar{\eta}_t^+ = \bar{\xi}_t^+ = \bar{\rho}_1^+\}$. In the opposite case for a construction of an arc $\{\bar{\eta}_t^+\}$ let us recall that the diffeomorphism Q is smoothly conjugated to a diffeomorphism G given by the normal Jordan form of Q (see, for example, [6] [Chapter 3]). Then there is a preserving orientation diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $G = hQh^{-1}$. Due to [15][Proposition 5.4], h is isotopic to identity by means an isotopy h_t with $h_0 = id$ and $h_1 = h$. Let us choose $r_2 \in (0,1)$ such that $h_t(U_2^+) \subset U_1^+$ for the disc $U_2^+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq r_2\}$ and every $t \in [0,1]$. Using Thom's result [12] about the extension of an isotopy there is an isotopy $H_t : \mathbb{R}^2 \to \mathbb{R}^2$ which is h_t on U_2^+ and is identity out of $V_2^+ = U_1^+$. Thus, the required arc is given by the formula

$$\bar{\eta}_t^+ = H_t \bar{\rho}_1^+ H_t^{-1}.$$

3) If $G = \bar{g}$ then let us define $\{\bar{\xi}_t^+ = \bar{\eta}_1^+\}$. In the opposite case, without loss of generality we will assume that the eigenvalues of Q are different (in the opposite case we can a little bit extend the arc $\bar{\rho}_t^+$ to get $\bar{\rho}_1^+$ with the needed property). Then the diffeomorphism G is given by either a matrix A, or a matrix B of the following forms:

$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where $0 < \lambda_1 < \lambda_2 < 1$;

$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, where $0 < \alpha^2 + \beta^2 < 1$.

In the both cases we see that G contracts every disc of the form $\{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq r\}$. Let $V_3^+ = U_2^+$ and let us choose $r_3 \in (0, r_2)$ such that the disc $U_3^+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq r_3\}$ contains $G(V_3^+)$. Let us define an arc $\bar{\tau}_t^+ : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$\bar{\tau}_t^+ = (1-t)G + tg.$$

Let us consider the isotopy $\lambda_t = G^{-1}\bar{\tau}_t^+$ which joints the identity map with the diffeomorphism $G^{-1}g$. By the construction $\lambda_t(U_3^+) \subset V_3^+$ for every $t \in [0,1]$. Then, using Thom's result [12, Theorem 5.8] about the extension of an isotopy there is an isotopy $\Lambda_t : \mathbb{R}^2 \to \mathbb{R}^2$ which is λ_t on U_3^+ and is identity out of V_3^+. Then the required arc is given by the formula

$$\bar{\xi}_t^+ = \bar{\eta}_1^+ \Lambda_t.$$

□

Proposition 3. *For any diffeomorphism $\gamma \in E_g$ there is a smooth arc $\{\delta_t \in J(\mathbb{S}^2)\}$ joining $\delta_0 = \gamma$ with $g = \delta_1$.*

Proof. As $\gamma \in E_g$ then there are neighborhoods $V_\gamma(N)$, $V_\gamma(S)$ of the points N, S, where $\gamma|_{V_\gamma(N)\cup V_\gamma(S)} = g|_{V_\gamma(N)\cup V_\gamma(S)}$. Let us define a diffeomorphism $\psi_\gamma : \mathbb{S}^2 \setminus \{S\} \to \mathbb{S}^2 \setminus \{S\}$ by the formula $\psi_\gamma(w) = g^k(\gamma^{-k}(w))$, where $k \in \mathbb{Z}$ such that $\gamma^{-k}(w) \in V_\gamma(N)$ for $w \in \mathbb{S}^2 \setminus \{S\}$. Then $\gamma = \psi_\gamma^{-1} g \psi_\gamma$. If ψ_γ can be smoothly extended to S by $\psi_\gamma(S) = S$ then, due to [18] there is a smooth isotopy $\rho_t : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\rho_0 = \psi$, $\rho_1 = id$. Thus $\delta_t = \rho_t^{-1} g \rho_t$ is a required arc.

In the opposite case we show that there is a smooth arc $\{\zeta_t \in E_g\}$ joining the diffeomorphism $\zeta_0 = \gamma$ with some diffeomorphism ζ_1 such that ψ_{ζ_1} can be smoothly extended to S by $\psi_{\zeta_1}(S) = S$.

Define $\mathbb{B}_r(O) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ and $K_r = cl(\mathbb{B}_r \setminus \mathbb{B}_{\frac{r}{2}})$ for $r > 0$. Let $\bar{\gamma} = \vartheta_+ \gamma \vartheta_+^{-1}$ and $\bar{\psi}_\gamma = \vartheta_+ \psi_\gamma \vartheta_+^{-1}$. Then $\bar{\gamma} \in E_{\bar{g}}$ and, hence, there is $r_0 > 0$ such that $\bar{\gamma} = \bar{g}$ on $\mathbb{B}_{r_0}$ and the annulus K_{r_0} is a fundamental domain of the diffeomorphism $\bar{g}$ (also $\bar{\gamma}$) on $\mathbb{R}^2 \setminus \{O\}$. Let us represent $\mathbb{T}^2$ as the orbit space $(\mathbb{R}^2 \setminus \{O\})/\bar{g}$. Denote by $p : \mathbb{B}_{r_0} \setminus \{O\} \to \mathbb{T}^2$ the natural projection. Then curves $a = p(Ox_1), b = p(\partial \mathbb{B}_{r_0})$ are generators of the fundamental group $\pi_1(\mathbb{T}^2)$. As $\bar{\psi}_\gamma$ sends the orbits of $\bar{g}$ to the orbits $\bar{\gamma}$ and K_{r_0} is the common fundamental domain for $\bar{g}, \bar{\gamma}$ on $\mathbb{R}^2 \setminus \{O\}$ then $\bar{\psi}_\gamma$ can be projected to $\mathbb{T}^2$ as a diffeomorphism given by the formula $\hat{\psi}_\gamma = p\bar{\psi}_\gamma p^{-1}$. Then, the induced isomorphism $\hat{\psi}_{\gamma*} : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$ preserves the homotopic class of the generator a and, hence, given by the matrix

$$\begin{pmatrix} 1 & n_0 \\ 0 & 1 \end{pmatrix}$$

for some integer n_0. Let us introduce the polar coordinates ρ, φ on $\mathbb{R}^2$.

The arc ζ_t will be the smooth product of arcs ν_t and μ_t, where

1) $\{\nu_t \in J(\mathbb{S}^2)\}$ is a smooth arc joining the diffeomorphism $\nu_0 = \gamma$ with some diffeomorphism $\nu_1 \in E_g$ such that $\hat{\psi}_{\nu_1}$ induces the identity action in $\pi_1(\mathbb{T}^2)$;

2) $\{\mu_t \in E_g\}$ is a smooth arc joining the diffeomorphism $\mu_0 = \nu_1$ with a diffeomorphism μ_1 such that $\hat{\psi}_{\mu_1} = id$, it means that $\psi_{\mu_1} = g^k$ for some $k \in \mathbb{Z} \setminus \{0\}$.

1) Define a diffeomorphism $\bar{\theta}_t$ by the formula $\bar{\theta}_t(\rho e^{i\varphi}) = \begin{cases} \rho e^{i\varphi}, \rho > r_0, \\ \rho e^{i\left(\varphi + 4n_0\pi t\left(1 - \frac{\rho}{r_0}\right)\right)}, \frac{r_0}{2} \leq \rho \leq r_0; \\ \rho e^{i(\varphi + 2n_0\pi t)}, \rho < \frac{r_0}{2}. \end{cases}$

Let $\theta_t = \vartheta_+^{-1}\bar{\theta}_t\vartheta_+ : \mathbb{S}^2 \setminus \{S\} \to \mathbb{S}^2 \setminus \{S\}$, then θ_t can be smoothly extended to $\mathbb{S}^2$ by $\theta_t(S) = S$. Moreover, by the construction $\hat{\psi}_{\theta_1\gamma}$ induces the identity action in $\pi_1(\mathbb{T}^2)$. Thus $\nu_t = \theta_t\gamma : \mathbb{S}^2 \to \mathbb{S}^2$ the required arc.

2) Here we have the diffeomorphism $\nu_1 : \mathbb{S}^2 \to \mathbb{S}^2$ such that the diffeomorphism $\hat{\psi}_{\nu_1}$ induces the identity action in $\pi_1(\mathbb{T}^2)$. Thus, by [19, 3], the diffeomorphism $\hat{\psi}_{\nu_1}$ is smoothly isotopic to the identity map. Pick an open cover $U = \{U_1, \ldots, U_q\}$ of the manifold $\mathbb{T}^2$ consisting of connected sets such that each connected component of the set $p^{-1}(U_i)$ is a subset of K_{r_i} for some $r_i < \frac{r_{i-1}}{2}$. By [1, Lemma de fragmentation] there are diffeomorphisms $\hat{w}_1, \ldots, \hat{w}_q : \mathbb{T}^2 \to \mathbb{T}^2$ such that they are smoothly isotopic to the identity and

i) for each $i = \overline{1, q}$ there is $U_{j(i)} \in U$ such that for every $t \in [0, 1]$ the map $\hat{w}_{i,t}$ is the identity out of $U_{j(i)}$ where $\{\hat{w}_{i,t}\}$ is a smooth isotopy between the identity map and $\hat{w}_i$;

ii) $\hat{\psi}_{\nu_1} = \hat{w}_1 \ldots \hat{w}_q$.

Let $\bar{w}_{i,t} : \mathbb{R}^2 \to \mathbb{R}^2$ be the diffeomorphism which coincides with $(p|_{K_{r_i}})^{-1}\hat{w}_{i,t}p$ on K_{r_i} and coincides with the identity map outside K_{r_i}. Let $\bar{\mu}_t = \bar{\nu}_1\bar{w}_{1,t} \ldots \bar{w}_{q,t} : \mathbb{R}^2 \setminus \{O\} \to \mathbb{R}^2 \setminus \{O\}$. By the construction $\bar{\mu}_t \in E_{\bar{g}}$ for every $t \in [0, 1]$ and $\bar{\mu}_0 = \bar{\nu}_1$. Moreover, for the diffeomorphism $\bar{\mu}_1$ the following equality holds: $\hat{\psi}_{\mu_1} = \hat{w}_q^{-1} \ldots \hat{w}_1^{-1}\hat{\psi}_{\nu_1} = \hat{w}_q^{-1} \ldots \hat{w}_1^{-1}\hat{w}_1 \ldots \hat{w}_q = id$. Thus $\mu_t = \vartheta_+^{-1}\bar{\mu}_t\vartheta_+ : \mathbb{S}^2 \setminus \{S\} \to \mathbb{S}^2 \setminus \{S\}$ can be smoothly extended to $\mathbb{S}^2$ by $\mu_t(S) = S$ up to the desired arc. □

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