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A New Anisotropy-Based Control Design Approach for Descriptor Systems Using Convex Optimization Techniques ⋆

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Abstract: In this paper, suboptimal anisotropy-based control problem for linear discrete-time descriptor systems is solved. The obtained conditions are given in terms of LMIs. Numerical example is given.

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1. INTRODUCTION

While constructing mathematical models of the systems in physical variables, one can get models, which contain both differential (or difference) and algebraic equations. Such models are called descriptor (or singular), and they found wide application in different fields of science and engineering Dai (1989); Stykel (2002).

The descriptor representation is more powerful than the conventional state-space form, but analysis and design methods for descriptor systems are quite different from the classical ones, sometimes they are difficult to be implemented. It is not trivial to extend the methods of normal systems analysis and design on a class of descriptor systems because of the presence of algebraic equations. Algebraic constraints provide the system with some new properties, such as impossibility to solve the system in regard to the derivative, necessity to have sufficiently smooth input signals, and noncausal behavior in discrete-time case (impulse behavior in continuous time).

Some problems, solved for normal systems, are still actual for descriptor systems. One of such problems is a developing of computationally efficient methods of analysis and control design for descriptor systems. This paper is devoted to one of such problems — suboptimal anisotropy-based controller design using convex optimization.

Anisotropy-based control theory originates from Vladimirov (1995, 1996). Information-theoretic representation of random signals lies in the basis of this approach. Anisotropy-based control theory considers the system's reaction on the influence of "colored" noises. "Spectral color" means Kullback-Leibler information divergence from the Gaussian white noise sequence. In this case, the quality criterion is anisotropic norm of the system. This norm lies between normalized $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm of the system. Anisotropy-based analysis problem for normal systems using convex optimization was solved in Tchaikovsky (2011). This result was extended on descriptor systems in Belov (2013). Generalized Riccati inequalities approach to suboptimal anisotropy-based control design was described in Andrianova (2014). But in the listed results inequalities are not strict. As the matrices in constraints are singular, the obtained inequalities are not convex. In Feng (2013), computationally efficient algorithm of suboptimal $\mathcal{H}_\infty$-control design was proposed. This paper extends this algorithm on anisotropy-based case, a novel anisotropy-based bounded real lemma in terms of LMIs (linear matrix inequalities) is formulated and proved. It allows to develop methods of anisotropy-based analysis for descriptor systems.

The paper is organized as follows. In the section 2, basics of anisotropy-based analysis and descriptor systems theory are given. The conditions of a novel bounded real lemma in terms of LMIs for normal and descriptor systems are obtained in the section 3. Suboptimal anisotropy-based controller design problem is solved, based on the novel bounded real lemma, and numerical example is given in the section 4.

2. BACKGORUND

2.1 Descriptor systems

The state-space representation of discrete-time descriptor systems is

$$Ex(k+1) = Ax(k) + Bf(k), \qquad (1)$$
$$y(k) = Cx(k) + Df(k)$$

where $x(k) \in \mathbb{R}^n$ is the state, $f(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are the input and output signals, respectively, A, B, C and D

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are constant real matrices of appropriate dimensions. The matrix $E \in \mathbb{R}^{n\times n}$ is singular, rank $(E) = r < n$.

Definition 1. The system (1) is called regular if $\exists\lambda \neq 0$: $\det(\lambda E - A) \neq 0$.

Regularity stands for the existence and uniqueness of the solution for the consistent initial conditions Stykel (2002).

Hereinafter, we suppose that the considered systems are regular. Now we give some definitions, necessary for further presentation.

Definition 2. The transfer function of the system (1) is defined by the expression

$$P(z) = C(zE - A)^{-1}B + D,\ z \in \mathbb{C}. \qquad (2)$$

$\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of the transfer function $P(z)$ are defined as follows

$$\|P\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\left(P^*(e^{i\omega})P(e^{i\omega})\right) d\omega\right)^{\frac{1}{2}},$$

$$\|P\|_\infty = \sup_{\omega\in[0,2\pi]} \sigma_{max}\left(P(e^{i\omega})\right)$$

where $\sigma_{max}\left(P(e^{i\omega})\right)$ is the maximum singular value of the transfer function $P(z)$.

Definition 3. The system (1) is called admissible if it is regular, causal ($\deg\det(zE - A) = \mathrm{rank}\, E$), and stable ($\rho(E, A) = \max|\lambda|_{\lambda\in\{z|\det(zE-A)=0\}} < 1$). For more information, see Dai (1989); Xu (2006).

For the regular system (1) there exist two nonsingular matrices Dai (1989) $\overline{W}$ and $\overline{V}$ such that $\overline{W}E\overline{V} = \mathrm{diag}(I_r, 0)$.

Consider the following change of variables

$$\overline{V}^{-1}x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \qquad (3)$$

where $x_1(k) \in \mathbb{R}^r$ and $x_2(k) \in \mathbb{R}^{n-r}$.

By left multiplying the system (1) on the matrix $\overline{W}$ and using the change of variables (3), one can rewrite the system (1) in the form Dai (1989)

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_1 f(k), \\ 0 &= A_{21}x_1(k) + A_{22}x_2(k) + B_2 f(k), \qquad (4) \\ y(k) &= C_1x_1(k) + C_2x_2(k) + Df(k) \end{aligned}$$

where

$$\overline{W}A\overline{V} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \overline{W}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$C\overline{V} = [\, C_1 \ C_2 \,]. \qquad (5)$$

Matrices $\overline{W}$ and $\overline{V}$ are found from the singular value decomposition (SVD)

$$E = U\,\mathrm{diag}(S, 0)H^{\mathrm{T}}.$$

Here U and H are real ortogonal matrices, S is a diagonal $r\times r$-matrix, it is formed by nonzero singular values of the matrix E

$$\overline{W} = \mathrm{diag}(S^{-1}, I_{n-r})U^{\mathrm{T}}, \qquad \overline{V} = H.$$

Representation (4) is called SVD canonical form Dai (1989). Note that the system is causal if $\det(A_{22}) \neq 0$, and stable if $\rho(A_{11} - A_{12}A_{22}^{-1}A_{21}) < 1$ Xu (2006).

While solving control problems for descriptor systems it is necessary not only to provide stability of dynamical subsystem, but also to avoid undesirable noncausal behavior. So, for descriptor systems there exist such concepts as causal controllability and stabilizability. Discuss them in detail. Consider a state feedback control in the following form:

$$f(k) = F_c x(k) + h(k) \qquad (6)$$

where $F_c \in \mathbb{R}^{m\times n}$ is a constant real matrix, $h(k)$ is a new input signal. The closed-loop system may be written in the form

$$Ex(k+1) = (A + BF_c)x(k) + Bh(k). \qquad (7)$$

Definition 4. The system (1) is called causal controllable if there exists a state feedback control in the form (6) such that the closed-loop system (7) is causal.

Causal controllability can be easily checked by the following rank condition Dai (1989).

Theorem 5. The system (1) is causal controllable if

$$\mathrm{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = \mathrm{rank}\,(E) + n.$$

Stabilizability of descriptor systems is characterized by ability to control nonstable modes of the dynamical subsystem.

Definition 6. The system (1) is called stabilizable if there exists a state feedback control in the form $f(k) = F_{st}x(k)$ such that the pair $(E, A + BF_{st})$ is stable.

2.2 Mean anisotropy of the sequence and anisotropic norm of the system

Let $W = \{w(k)\}_{k\in\mathbb{Z}}$ be a stationary sequence of square-integrable random m-dimensional vectors. The sequence W can be generated from the Gaussian white noise sequence V with zero mean and identity covariance matrix by an admissible shaping filter with a transfer function $G(z) = C_G(zE_G - A_G)^{-1}B_G + D_G$. Mean anisotropy of the signal is Kullback-Leibler information divergence from probability density function (p.d.f.) of the signal to p.d.f. of the Gaussian white noise sequence.

Mean anisotropy of the sequence may be defined by the filter's parameters, using the expression

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\frac{mS(\omega)}{\|G\|_2^2} d\omega$$

where $S(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega)$, $(-\pi \leqslant \omega \leqslant \pi)$, $\widehat{G}(\omega) = \lim_{l\to 1} G(le^{i\omega})$ is a boundary value of the transfer function $G(z)$.

Remark 1. Mean anisotropy of the random sequence W, generated by shaping filter $G(z)$, is fully defined by its parameters, so the notations $\overline{\mathbf{A}}(G)$ and $\overline{\mathbf{A}}(W)$ are equivalent.

Mean anisotropy of the signal characterizes its "spectral color", i.e. the difference between the signal and the Gaussian white noise sequence. If $\overline{\mathbf{A}}(W) = 0$, then the signal is the Gaussian white noise sequence. If $\overline{\mathbf{A}}(W) \to \infty$, the signal is a determinate sequence. For more information, see Vladimirov (2006, 1995).

Let $Y = PW$ be an output of the linear discrete-time descriptor system $P \in \mathcal{H}_\infty{}^{p\times m}$ with a transfer function

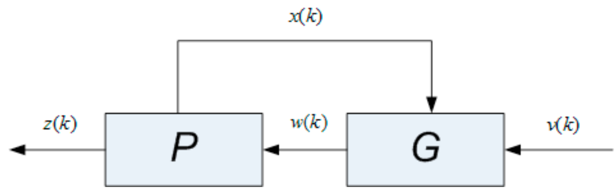

Fig. 1. To the system's a-anisotropic norm computation

$P(z)$, which is analytic in the identity circle $|z| < 1$, $P(z)$ has a finite $\mathcal{H}_\infty$-norm.

Definition 7. For a given constant value $a \geqslant 0$ a-anisotropic norm of the system P is defined as

$$|||P|||_a = \sup\{\|PG\|_2/\|G\|_2 : \ G \in \mathbf{G}_a\}, \quad (8)$$

i.e. the maximum value of the system's gain with respect to the class of shaping filters

$$\mathbf{G}_a = \left\{G \in \mathcal{H}_2{}^{m\times m} : \ \overline{\mathbf{A}}(G) \leqslant a\right\}.$$

So, a-anisotropic norm $|||P|||_a$ describes the stochastic gain of the system P with respect to the input sequence W.

3. NOVEL ANISOTROPY-BASED BOUNDED REAL LEMMA AND A-ANISOTROPIC NORM COMPUTATION

3.1 Novel bounded real lemma for normal systems

Consider a normal discrete-time system, written in the following form:

$$x(k+1) = Ax(k) + Bw(k), \quad (9)$$
$$y(k) = Cx(k) + Dw(k)$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ is a random stationary sequence with known mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$, $y(k) \in \mathbb{R}^q$ is an observable output, A, B, C and D are constant real matrices of appropriate dimensions. The transfer function of the system (9) is defined by

$$T(z) = C(zI - A)^{-1}B + D.$$

We suppose that the system (9) is stable, and the constants $a \geqslant 0$ and $\gamma > 0$ are known. The problem is to satisfy the inequality

$$|||T|||_a < \gamma.$$

The following lemma gives the answer to this problem Tchaikovsky (2011).

Lemma 8. Let the system (9) with a transfer function $T(z) \in \mathcal{H}_\infty{}^{q\times m}$ be stable. For the given scalar values $a \geqslant 0$ and $\gamma > 0$ a-anisotropic norm is bounded by a given scalar value γ, i.e.

$$|||T|||_a < \gamma$$

if there exist such scalar value $\eta > \gamma^2$ and $n \times n$-matrix $\Phi = \Phi^{\mathrm{T}} > 0$ that the following inequalities hold true

$$\eta - (e^{-2a}\det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} < \gamma^2, \quad (10)$$

$$\begin{bmatrix} A^{\mathrm{T}}\Phi A - \Phi + C^{\mathrm{T}}C & A^{\mathrm{T}}\Phi B + C^{\mathrm{T}}D \\ B^{\mathrm{T}}\Phi A + D^{\mathrm{T}}C & B^{\mathrm{T}}\Phi B + D^{\mathrm{T}}D - \eta I_m \end{bmatrix} < 0. \quad (11)$$

■

To modify anisotropy-based bounded real lemma for descriptor systems we formulate and prove the following auxiliary theorem.

Theorem 9. Let the system (9) with a transfer function $T(z) \in \mathcal{H}_\infty{}^{q\times m}$ be stable. For the given scalar values $a \geqslant 0$ and $\gamma > 0$ a-anisotropic norm is bounded by γ, i.e.

$$|||T|||_a < \gamma$$

if there exist such scalar value $\eta > \gamma^2$, $n \times n$-matrix $\Phi = \Phi^{\mathrm{T}} > 0$ and a random $n \times n$-matrix Y that the following inequalities hold true:

$$\eta - (e^{-2a}\det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} < \gamma^2, \quad (12)$$

$$\begin{bmatrix} -\frac{1}{2}Y - \frac{1}{2}Y^{\mathrm{T}} & YA & YB & \Phi^{\mathrm{T}} - Y^{\mathrm{T}} - \frac{1}{2}Y & 0 \\ A^{\mathrm{T}}Y^{\mathrm{T}} & -\Phi & 0 & A^{\mathrm{T}}Y^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}}Y^{\mathrm{T}} & 0 & -\eta I_m & B^{\mathrm{T}}Y^{\mathrm{T}} & D^{\mathrm{T}} \\ \Phi - Y - \frac{1}{2}Y^{\mathrm{T}} & YA & YB & -Y - Y^{\mathrm{T}} & 0 \\ 0 & C & D & 0 & -I_q \end{bmatrix} < 0. \quad (13)$$

Proof. Suppose the inequalities (12) and (13) hold true. Rewrite the expression (13) in the form

$$\Xi + \Upsilon^{\mathrm{T}}Y^{\mathrm{T}}\Psi + \Psi^{\mathrm{T}}Y\Upsilon < 0 \quad (14)$$

where $\Psi = [\, I_n \ 0 \ 0 \ I_n \,]$, $\Upsilon = \left[-\frac{1}{2}I_n \ A \ B \ -I_n \right]$, and a symmetric matrix Ξ is given by

$$\Xi = \begin{bmatrix} 0 & 0 & 0 & \Phi \\ 0 & C^{\mathrm{T}}C - \Phi & C^{\mathrm{T}}D & 0 \\ 0 & D^{\mathrm{T}}C & D^{\mathrm{T}}D - \eta I_m & 0 \\ \Phi & 0 & 0 & 0 \end{bmatrix}.$$

Using the projection lemma Boyd (1994), we get that the inequality (14) is solvable if and only if

$$M^{\mathrm{T}}\Xi M < 0 \text{ and } N^{\mathrm{T}}\Xi N < 0$$

for $M^{\mathrm{T}} = \begin{bmatrix} 0 & I_n & 0 & 0 \\ 0 & 0 & I_m & 0 \\ -I_n & 0 & 0 & I_n \end{bmatrix}$, $N^{\mathrm{T}} = \begin{bmatrix} I_n & 0 & 0 & -\frac{1}{2}I_n \\ 0 & I_n & 0 & A^{\mathrm{T}} \\ 0 & 0 & I_m & B^{\mathrm{T}} \end{bmatrix}$.

Note that

$$N^{\mathrm{T}}\Xi N = \begin{bmatrix} -\Phi & \Phi A & \Phi B \\ A^{\mathrm{T}}\Phi & C^{\mathrm{T}}C & C^{\mathrm{T}}D \\ B^{\mathrm{T}}\Phi & D^{\mathrm{T}}C & D^{\mathrm{T}}D \end{bmatrix} < 0. \quad (15)$$

As $\Phi = \Phi^{\mathrm{T}} > 0$, using Schur complement, we may transform the inequality (15) into

$$\begin{bmatrix} C^{\mathrm{T}}C & C^{\mathrm{T}}D \\ D^{\mathrm{T}}C & D^{\mathrm{T}}D \end{bmatrix} - \begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} \Phi(-\Phi)^{-1}\Phi\,[\, A \ B \,] < 0.$$

Hence,

$$\begin{bmatrix} A^{\mathrm{T}}\Phi A - \Phi + C^{\mathrm{T}}C & A^{\mathrm{T}}\Phi B + C^{\mathrm{T}}D \\ B^{\mathrm{T}}\Phi A + D^{\mathrm{T}}C & B^{\mathrm{T}}\Phi B + D^{\mathrm{T}}D - \eta I_m \end{bmatrix} < 0.$$

Consequently, the conditions of this theorem are equivalent to the conditions of Lemma 8, proved in Tchaikovsky (2011).

■

3.2 Novel bounded real lemma for descriptor systems

Consider a discrete-time descriptor system in the state-space representation

$$Ex(k+1) = Ax(k) + Bw(k), \qquad (16)$$
$$y(k) = Cx(k) + Dw(k)$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ is a random sequence with a given mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$, $y(k) \in \mathbb{R}^q$ is an observable output, E, A, B, C and D are given matrices of appropriate dimensions, $\operatorname{rank} E = r < n$. Its transfer function $P(z)$ is given by the expression (2). Suppose that the system (16) is admissible, the scalar values $a \geqslant 0$ and $\gamma > 0$ are known. The problem is to find sufficient conditions to satisfy the inequality

$$|\!|\!| P |\!|\!|_a < \gamma.$$

As the system is regular, there exist two transformation matrices $\overline{W}$ and $\overline{V}$, and the system (16) may be rewritten in the equivalent form (4). We use the following denotations: $E_d = \overline{W}E\overline{V}$, $A_d = \overline{W}A\overline{V}$, $B_d = \overline{W}B$, $C_d = C\overline{V}$, $D_d = D$.

Now we formulate the conditions of anisotropic norm boundedness for the system (16).

Theorem 10. Let the system (16) with a transfer function $P(z) \in \mathcal{H}_\infty{}^{q\times m}$ be admissible. Suppose that

$$\operatorname{rank} E = \operatorname{rank}\,[E \;\; B].$$

For given scalar values $a \geqslant 0$ and $\gamma > 0$ a-anisotropic norm of the system is bounded by the value γ, i.e.

$$|\!|\!| P |\!|\!|_a < \gamma$$

if there exist such matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, scalar values $\eta > \gamma^2$ and $\alpha > 0$ that the following inequalities hold true

$$\eta - (e^{-2a}\det(\eta I_m - B_d^{\mathrm{T}}\Pi B_d - D_d^{\mathrm{T}} D_d))^{1/m} < \gamma^2 \qquad (17)$$

and (18)

where $\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}$, $\Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$, $\Gamma = [\, Q \;\; R \,]$.

Proof.

Suppose the inequality (18) holds. For the equivalent form (4) of the system (16) it is not difficult to get

$$Z = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & 0 \\ z_{12}^{\mathrm{T}} & -L & z_{23} & 0 & z_{25} & z_{26} \\ z_{13}^{\mathrm{T}} & z_{23}^{\mathrm{T}} & z_{33} & z_{34} & z_{35} & z_{36} \\ z_{14}^{\mathrm{T}} & 0 & z_{34}^{\mathrm{T}} & -\eta I_m & z_{45} & z_{46} \\ z_{15}^{\mathrm{T}} & z_{25}^{\mathrm{T}} & z_{35}^{\mathrm{T}} & z_{45}^{\mathrm{T}} & z_{55} & 0 \\ 0 & z_{26}^{\mathrm{T}} & z_{36}^{\mathrm{T}} & z_{46}^{\mathrm{T}} & 0 & -I_q \end{bmatrix} < 0$$

where

$$z_{11} = -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}},\; z_{12} = QA_{11} + RA_{21},$$
$$z_{13} = QA_{12} + RA_{22},\; z_{14} = QB_1 + RB_2,$$
$$z_{15} = L^{\mathrm{T}} - Q^{\mathrm{T}} - \frac{1}{2}Q,\; z_{23} = A_{21}^{\mathrm{T}}S^{\mathrm{T}},$$
$$z_{25} = A_{11}^{\mathrm{T}}Q^{\mathrm{T}} + A_{21}^{\mathrm{T}}R^{\mathrm{T}},\; z_{26} = C_1^{\mathrm{T}} + \alpha A_{21}^{\mathrm{T}}S^{\mathrm{T}}C_2^{\mathrm{T}}$$
$$z_{33} = SA_{22} + A_{22}^{\mathrm{T}}S^{\mathrm{T}},\; z_{34} = SB_2,$$
$$z_{35} = A_{12}^{\mathrm{T}}Q^{\mathrm{T}} + A_{22}^{\mathrm{T}}R^{\mathrm{T}},\; z_{36} = C_2^{\mathrm{T}} + \alpha A_{22}^{\mathrm{T}}S^{\mathrm{T}}C_2^{\mathrm{T}}$$
$$z_{45} = B_1^{\mathrm{T}}Q^{\mathrm{T}} + B_2^{\mathrm{T}}R^{\mathrm{T}},\; z_{46} = D^{\mathrm{T}} + \alpha B_2^{\mathrm{T}}S^{\mathrm{T}}C_2^{\mathrm{T}},$$
$$z_{55} = -Q - Q^{\mathrm{T}}.$$

Using matrix properties, we get $KZK^{\mathrm{T}} < 0$ for a nonsingular matrix K. Choose $K = \begin{bmatrix} I_r & 0 & 0 & 0 & 0 & 0 \\ 0 & I_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & I_q \\ 0 & 0 & I_{n-r} & 0 & 0 & 0 \end{bmatrix}$.

Hence, $KZK^{\mathrm{T}} = \begin{bmatrix} z_{11} & z_{12} & z_{14} & z_{15} & 0 & z_{13} \\ z_{12}^{\mathrm{T}} & -L & 0 & z_{25} & z_{26} & z_{23} \\ z_{14}^{\mathrm{T}} & 0 & -\eta I_m & z_{45} & z_{46} & z_{34}^{\mathrm{T}} \\ z_{15}^{\mathrm{T}} & z_{25}^{\mathrm{T}} & z_{45}^{\mathrm{T}} & z_{55} & 0 & z_{35}^{\mathrm{T}} \\ 0 & z_{26}^{\mathrm{T}} & z_{46}^{\mathrm{T}} & 0 & -I_q & z_{36}^{\mathrm{T}} \\ z_{13}^{\mathrm{T}} & z_{23}^{\mathrm{T}} & z_{34} & z_{35} & z_{36} & z_{33} \end{bmatrix} < 0.$

Consider the expression $KZK^{\mathrm{T}} = W + W^{\mathrm{T}}$ where

$$W = \begin{bmatrix} w_{11} & 0 & 0 & 0 & 0 & 0 \\ w_{21} & w_{22} & 0 & w_{24} & w_{25} & w_{26} \\ w_{31} & 0 & w_{33} & w_{34} & w_{35} & w_{36} \\ w_{41} & 0 & 0 & w_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & w_{55} & 0 \\ w_{61} & 0 & 0 & w_{64} & w_{65} & w_{66} \end{bmatrix},$$

$$w_{11} = -\frac{1}{2}Q,\; w_{21} = w_{24} = A_{11}^{\mathrm{T}}Q^{\mathrm{T}} + A_{21}^{\mathrm{T}}R^{\mathrm{T}},$$
$$w_{22} = -\frac{1}{2}L,\; w_{25} = C_1^{\mathrm{T}} + \alpha A_{21}^{\mathrm{T}}S^{\mathrm{T}}C_2^{\mathrm{T}},$$
$$w_{26} = A_{21}^{\mathrm{T}}S^{\mathrm{T}},\; w_{31} = w_{34} = B_1^{\mathrm{T}}Q^{\mathrm{T}} + B_2^{\mathrm{T}}R^{\mathrm{T}},$$
$$w_{33} = -\frac{\eta}{2}I_m,\; w_{35} = D^{\mathrm{T}} + \alpha B_2^{\mathrm{T}}S^{\mathrm{T}}C_2^{\mathrm{T}},$$
$$w_{36} = B_2^{\mathrm{T}}S^{\mathrm{T}},\; w_{41} = L - Q - \frac{1}{2}Q^{\mathrm{T}},$$
$$w_{44} = -Q,\; w_{55} = -\frac{1}{2}I_q,\; w_{65} = C_2^{\mathrm{T}} + \alpha A_{22}^{\mathrm{T}}S^{\mathrm{T}}C_2^{\mathrm{T}},$$
$$w_{61} = w_{64} = A_{12}^{\mathrm{T}}Q^{\mathrm{T}} + A_{22}^{\mathrm{T}}R^{\mathrm{T}},\; w_{66} = A_{22}^{\mathrm{T}}S^{\mathrm{T}}.$$

So,

$$W + W^{\mathrm{T}} < 0. \qquad (19)$$

As $A_{22}^{\mathrm{T}}S^{\mathrm{T}} + SA_{22} < 0$, both matrices A_{22} and S are nonsingular. The system (16) is causal, it may be transformed into a normal system $\widehat{T}$ of reduced dimension

$$\widehat{x}(k+1) = \widehat{A}\widehat{x}(k) + \widehat{B}w(k), \qquad (20)$$
$$\widehat{y}(k) = \widehat{C}\widehat{x}(k) + \widehat{D}w(k)$$

where $x(k) \in \mathbb{R}^r$,

$$\widehat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}\,, \quad \widehat{B} = B_1 - A_{12}A_{22}^{-1}B_2\,,$$
$$\widehat{C} = C_1 - C_2A_{22}^{-1}A_{21}\,, \quad \widehat{D} = D - C_2A_{22}^{-1}B_2\,.$$

According to the rank condition, $B_2 = 0$. Hence, the inequality (17) coincides with (12) for the equivalent system (20).

Now show that the matrix $\widehat{A}$ is stable, and $|\!|\!| \widehat{T} |\!|\!|_a < \gamma$. As SA_{22} and $A_{22}^{\mathrm{T}}S^{\mathrm{T}}$ are invertible, $A_{22}^{\mathrm{T}}S^{\mathrm{T}} < 0$ and $SA_{22} < 0$, applying Schur complement to (19), we obtain

$$\begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}} & Q\widehat{A} & Q\widehat{B} & L^{\mathrm{T}} - Q^{\mathrm{T}} - \frac{1}{2}Q & 0 \\ \widehat{A}^{\mathrm{T}}Q^{\mathrm{T}} & -L & 0 & \widehat{A}^{\mathrm{T}}Q^{\mathrm{T}} & \widehat{C}^{\mathrm{T}} \\ \widehat{B}^{\mathrm{T}}Q^{\mathrm{T}} & 0 & -\eta I_m & \widehat{B}^{\mathrm{T}}Q^{\mathrm{T}} & \widehat{D}^{\mathrm{T}} \\ L - Q - \frac{1}{2}Q^{\mathrm{T}} & Q\widehat{A} & Q\widehat{B} & -Q - Q^{\mathrm{T}} & 0 \\ 0 & \widehat{C} & \widehat{D} & 0 & -I_q \end{bmatrix} < 0. \qquad (21)$$

$$\begin{bmatrix} -\frac{1}{2}Q-\frac{1}{2}Q^{\mathrm T} & \Gamma A_d & \Gamma B_d & L^{\mathrm T}-Q^{\mathrm T}-\frac{1}{2}Q & 0 \\ A_d^{\mathrm T}\Gamma^{\mathrm T} & \Pi A_d - A_d^{\mathrm T}\Pi^{\mathrm T}-\Theta & \Pi B_d & A_d^{\mathrm T}\Gamma^{\mathrm T} & C_d^{\mathrm T}+\alpha A_d^{\mathrm T}\Pi^{\mathrm T}C_d^{\mathrm T} \\ B_d^{\mathrm T}\Gamma^{\mathrm T} & B_d^{\mathrm T}\Pi^{\mathrm T} & -\eta I_m & B_d^{\mathrm T}\Gamma^{\mathrm T} & D_d^{\mathrm T}+\alpha B_d^{\mathrm T}\Pi^{\mathrm T}C_d^{\mathrm T} \\ L-Q-\frac{1}{2}Q^{\mathrm T} & \Gamma A_d & \Gamma B_d & -Q-Q^{\mathrm T} & 0 \\ 0 & C_d+\alpha C_d\Pi A_d & D_d+\alpha C_d \Pi B_d & 0 & -I_q \end{bmatrix} < 0 \tag{18}$$

According to Theorem 9, we have $\rho(\widehat{A}) < 1$ and $|||\widehat{T}|||_a < \gamma$. So, $|||P|||_a < \gamma$.
■

Remark 2. In order to avoid the product $D_d^{\mathrm T}D_d$ in the inequality (17) we introduce a new variable Ψ

$$\Psi < \eta I_m - B_d^{\mathrm T}\Pi B_d - D_d^{\mathrm T}D_d. \tag{22}$$

Using Schur complement, transform the inequality (22):

$$\Psi - \eta I_m + B_d^{\mathrm T}\Pi B_d - D_d^{\mathrm T}(-I_q)D_d < 0,$$

$$\begin{bmatrix} \Psi - \eta I_m + B_d^{\mathrm T}\Pi B_d & D_d^{\mathrm T} \\ D_d & -I_q \end{bmatrix} < 0.$$

So, the inequality (17) is equivalent to the system of inequalities

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/m} < \gamma^2 \tag{23}$$

and

$$\begin{bmatrix} \Psi - \eta I_m + B_d^{\mathrm T}\Pi B_d & D_d^{\mathrm T} \\ D_d & -I_q \end{bmatrix} < 0. \tag{24}$$

Remark 3. While computing a-anisotropic norm for descriptor systems one should solve the following optimization problem: to find $\gamma_* = \inf \gamma$ on the set $\{L, Q, R, S, \Psi, \eta, \gamma\}$, which satisfies the inequalities (18), (23) and (24). If the minimum value γ_* is found, a-anisotropic norm of the system P may be approximately found from the expression

$$|||P|||_a \approx \gamma_*. \tag{25}$$

Here the scalar value $\alpha > 0$ is set.

4. SUBOPTIMAL ANISOTROPY-BASED CONTROL DESIGN FOR DISCRETE-TIME DESCRIPTOR SYSTEMS

Consider the following discrete-time descriptor system:

$$\begin{aligned} Ex(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k), \\ z(k) &= Cx(k) + D_1 w(k) \end{aligned} \tag{26}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^{m_1}$ is a random stationary sequence with mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$ $(a \geqslant 0)$, $z(k) \in \mathbb{R}^q$ is an observable output, $u(k) \in \mathbb{R}^{m_2}$ is a control sequence, E, A, B_1, B_2, C, D_1 are constant real matrices of appropriate dimensions.

Assume that

(1) the whole state vector is observable;
(2) the system (26) is causal controllable;
(3) the system (26) is stabilizable;
(4) a scalar value $\gamma > 0$ is known.

The problem is to find a state-feedback control $u(k) = Fx(k)$, for which the closed-loop system P_{cl} is causal and stable, and $|||P_{cl}|||_a < \gamma$.

The system (26) is regular, so there exist two matrices $\overline{W}$ and $\overline{V}$, which transform the system (26) into the equivalent form (4). Now we use the denotations $E_d = \overline{W}E\overline{V}$, $A_d = \overline{W}A\overline{V}$, $B_{1d} = \overline{W}B_1$, $B_{2d} = \overline{W}B_2$, $C_d = C\overline{V}$, $D_{1d} = D_1$.

The following theorem contains sufficient conditions of anisotropic norm boundedness for the closed-loop system, it also gives us the feedback gain, which makes the closed-loop system causal and stable.

Theorem 11. Let the following rank condition hold true:

$$\operatorname{rank} E = \operatorname{rank}[E \;\; B_1] \text{ and } \operatorname{rank} E^{\mathrm T} = \operatorname{rank}[E^{\mathrm T} \;\; C^{\mathrm T}].$$

For the given scalar value $\gamma > 0$ and for the known mean anisotropy level of the input disturbance $a \geqslant 0$ the closed-loop system P_{cl} is causal and stable, and $|||P_{cl}|||_a < \gamma$ if there exist matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $Z \in \mathbb{R}^{n\times m_1}$, $\Psi \in \mathbb{R}^{m_1\times m_1}$, a scalar value $\eta > \gamma^2$ and a sufficiently large scalar $\alpha > 0$, such that

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{21}^{\mathrm T} & \Lambda_{31}^{\mathrm T} & \Lambda_{41}^{\mathrm T} & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{32}^{\mathrm T} & \Lambda_{21} & \Lambda_{52}^{\mathrm T} \\ \Lambda_{31} & \Lambda_{32} & -\eta I_q & \Lambda_{31} & \Lambda_{53}^{\mathrm T} \\ \Lambda_{41} & \Lambda_{21}^{\mathrm T} & \Lambda_{31}^{\mathrm T} & -(Q+Q^{\mathrm T}) & 0 \\ 0 & \Lambda_{52} & \Lambda_{53} & 0 & -I_{m_1} \end{bmatrix} < 0, \tag{27}$$

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/m_1} < \gamma^2, \tag{28}$$

$$\begin{bmatrix} \Psi - \eta I_{m_1} + B_{1d}^{\mathrm T}\Theta B_{1d} & (D_{1d}+\alpha C_d \Pi B_{1d})^{\mathrm T} \\ (D_{1d}+\alpha C_d\Pi B_{1d}) & -I_q \end{bmatrix} < 0 \tag{29}$$

where

$$\begin{aligned} \Lambda_{11} &= -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm T},\; \Lambda_{21} = A_d\Gamma^{\mathrm T} + B_{1d}Z^{\mathrm T}\Omega^{\mathrm T}, \\ \Lambda_{31} &= C_d\Gamma^{\mathrm T},\; \Lambda_{41} = L - Q - \frac{1}{2}Q^{\mathrm T}, \\ \Lambda_{22} &= LA_d^{\mathrm T} + A_dL^{\mathrm T} + \Phi Z B_{1d} + B_{1d}Z^{\mathrm T}\Phi^{\mathrm T} - \Theta, \\ \Lambda_{52} &= B_{1d}^{\mathrm T} + \alpha B_{1d}^{\mathrm T}\Pi A_d^{\mathrm T} + \alpha B_{1d}^{\mathrm T}\Phi^{\mathrm T} Z B_{2d}^{\mathrm T}, \\ \Lambda_{53} &= D_{1d} + \alpha B_{1d}^{\mathrm T}\Pi C_d^{\mathrm T} \; \Lambda_{32} = C_d\Pi^{\mathrm T}. \end{aligned}$$

Besides, $\Theta = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}$, $\Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$, $\Phi = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}$, $\Omega = [\, I_r \;\; 0\,]$, $\Gamma = [\, Q \;\; R\,]$. The feedback gain is given as

$$F = Z^{\mathrm T}\begin{bmatrix} Q^{-\mathrm T} & 0 \\ -S^{-\mathrm T}R^{\mathrm T}Q^{-\mathrm T} & S^{-\mathrm T} \end{bmatrix}\overline{V}^{-1}. \tag{30}$$

Proof. Show, that the controller, which solves the design control problem for the canonical form of the system, solves it also for the initial system. The transfer function of the closed-loop system may be written in the form $P_{cl}(z) = C\overline{V}\overline{V}^{-1}(zE - A - B_2F)^{-1}\overline{W}^{-1}\overline{W}B_1 + D_1 =$

$C\overline{V}(z\overline{W}E\overline{V} - \overline{W}A\overline{V} - \overline{W}B_2F\overline{V})^{-1}\overline{W}B_1 + D_1 = C_d(zE_d - A_d - B_{2d}F_d)^{-1}B_{1d} + D_{1d}$.

Suppose, that the inequalities (27)–(29) hold. Then the (1,1) entry implies the matrix Q is invertible. We also suppose, that the matrix S is invertible. If it does not hold, there exists a scalar $\epsilon \in (0, 1)$, such that the inequality (27) holds true for the scalar $\tilde{S} = S + \epsilon I_{n-r}$. So, we can use $\tilde{S}$ instead of S. Replacing Z with $\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}}$ in (27), we get the conditions of the bounded real for the system, dual to the system (26). So, according to the bounded real lemma, the closed-loop system (26) is admissible, and a-anisotropic norm of its transfer function is bounded by the given scalar γ.

If the design control problem is solvable, the conditions of the Theorem 10 hold true for the system (26).These conditions also hold for the dual system. By the linear change of variables $\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}} = Z$, which implies that $[\,Q\;R\,]\,F_d^{\mathrm{T}} = [\,I_r\;0\,]\,Z$ and $\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Z$, we get the inequality (27). Moreover, as pointed out before, Q and S are invertible. So the feedback gain F_d for the closed-loop system (26) is $F_d = Z^{\mathrm{T}} \begin{bmatrix} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}}R^{\mathrm{T}}Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{bmatrix}$. By the inverse change of variables we get F from (30).

■

Remark 4. Denote $\xi = \gamma^2$, then to solve the optimal control problem it is necessary to find $\xi_* = \inf \xi$ on the set $\{L, Q, R, S, Z, \Psi, \eta, \xi\}$, which satisfies the inequalities (27)–(29).

Example. Consider the following system:

$$A = \begin{bmatrix} 0.3 & 0.5001 & 0.1002 & 0.0005 & 0.5006 \\ 0.7 & 0.7941 & 3.2909 & 0.0006 & 0.6002 \\ 0.6 & 0.8 & 0.2999 & 0.0008 & 0.8004 \\ 0.7 & 0.4989 & 0.8978 & 0.001 & 1.0003 \\ 0.6 & 0.7 & 0.2998 & 0.0004 & 0.4013 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.3 & 0.5 & 0.1 & 0 & 0.5 \\ 0.7 & 0.8 & 3.3 & 0 & 0.6 \\ 0.6 & 0.8 & 0.3 & 0 & 0.8 \\ 0.7 & 0.5 & 0.9 & 0 & 1 \\ 0.6 & 0.7 & 0.3 & 0 & 0.4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.0003 & -0.0002 \\ -0.0058 & 0.0019 \\ 0.0002 & -0.0013 \\ -0.0013 & -0.0015 \\ 0.0001 & 0.0017 \end{bmatrix},$$

$$B_2 = 10^{-3} \begin{bmatrix} 0.1 & -0.125 \\ 0.2333 & 0.2 \\ 0.2 & 0.2 \\ 0.2333 & 0.125 \\ 0.2 & 0.175 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}.$$

The system is not causal ($\deg\det(zE - A) = 3$, $\operatorname{rank} E = 4$), and it is not stable ($\rho(E, A) = 1.000$).

Design control results for $\alpha = 100$ are given in Table 1.

5. CONCLUSION

This paper is devoted to the state-feedback anisotropy-based control problem for linear discrete-time descriptor systems. New sufficient conditions of anisotropic norm boundedness in terms of LMIs for normal systems are derived, then they are generalized on the class of descriptor systems. These results allow to develop a computationally efficient algorithm for anisotropy-based analysis. Relied on these conditions, suboptimal control problem for descriptor systems is solved.

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Table 1. Results of control design for different mean anisotropy levels

a	0	0.1	0.2	0.5	1	2
$\lVert P_{cl} \rVert_a$	0.2739	0.3266	0.3430	0.3662	0.3796	0.3855
$\rho(E, A)$	0.9999	0.9998	0.9998	0.9998	0.9998	0.9998