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On LMI Approach to Robust State-Feedback $\mathcal{H}_\infty$ Control for Discrete-Time Descriptor Systems with Uncertainties in All Matrices ⋆

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Abstract: This paper deals with a state feedback $\mathcal{H}_\infty$ control problem for linear discrete-time time-invariant (LDTI) uncertain descriptor systems. Considered systems contain norm-bounded parametric uncertainties in all matrices. Bounded real lemma (BRL) for descriptor systems with all known matrices is extended on the class of uncertain systems. The control design procedure based on the conditions of BRL for uncertain descriptor systems is proposed. Numerical example is included to illustrate the effectiveness of the present result.

Keywords: Descriptor systems, suboptimal control, uncertain linear systems, $\mathcal{H}_\infty$ control, robust performance, robust control.

1. INTRODUCTION

In the last few decades interest of researchers to the area of descriptor systems (also called singular systems) has grown. Descriptor systems found their various applications in different sciences, for example, in economics, biology, electrical engineering, etc (see (Dai, 1989; Duan, 2010) and references therein). Being a general case of normal state-space systems, descriptor systems contain both algebraic and differential (or difference) constraints. Due to this fact, a specific behavior may occur while solving descriptor systems' equations. In LDTI descriptor systems non-causal behavior which is unusual for normal state-space systems may cause difficulties in control and modeling of the system's dynamics. Motivated by this fact, many efforts have been made towards generalizing methods developed for state-space control problems on descriptor systems. Among them is $\mathcal{H}_\infty$ control problem.

The $\mathcal{H}_\infty$ control problem is one of the most popular among disturbance attenuation problems. In this case, a disturbance is assumed to be a square summable sequence. The $\mathcal{H}_\infty$ norm of the closed-loop systems stands for the system's gain from the input to the controllable output and is required to be less than a given scalar $\gamma > 0$. $\mathcal{H}_\infty$ control problem was successfully generalized on a class of certain and uncertain LDTI descriptor systems.

There are a lot of different approaches for solving this problem for certain descriptor systems (Chadli & Darouach, 2012; Feng & Yagoubi, 2013; Rehm & Allgöwer, 2002; Xu & Lam, 2006). To this end, several versions of BRL are proposed in literature to solve $\mathcal{H}_\infty$ performance analysis problem.

Robust $\mathcal{H}_\infty$ control problem for discrete-time descriptor systems with parametric uncertainties is investigated in (Chadli & Darouach, 2014; Coutinho et al., 2014; Ji et al., 2007; Xu & Lam, 2006). The paper (Coutinho et al., 2014) deals with linear discrete-time descriptor systems with polytopic-type parametric uncertainties. The obtained conditions are *bilinear*, so the iterative procedure of control design with γ-value minimization is given. In (Chadli & Darouach, 2014; Ji et al., 2007; Xu & Lam, 2006) norm-bounded uncertainties are under consideration. The solution of control design problem proposed in (Chadli & Darouach, 2014) is also based on *bilinear* matrix inequalities (BMI). Results from (Ji et al., 2007) and (Xu & Lam, 2006) have several disadvantages. In both papers the $\mathcal{H}_\infty$ performance analysis for linear discrete-time descriptor systems with uncertainties in all the system's matrices is not presented. In (Xu & Lam, 2006) the proposed algorithm of $\mathcal{H}_\infty$ control problem solution, based on *nonlinear* matrix inequalities approach, is difficult to compute. The approach described in (Ji et al., 2007) requires auxiliary matrix variables and fails for high order systems. So the problem of numerically effective robust $\mathcal{H}_\infty$ control design for discrete-time systems with norm-bounded uncertainties in terms of linear matrix inequalities (LMI) is an open problem.

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This paper presents a new approach to robust $\mathcal{H}_\infty$ control problem for discrete-time descriptor systems in terms of LMI. The proposed method is based on the modified BRL, proved in (Feng & Yagoubi, 2013). The obtained conditions are linear with respect to unknown parameters and numerically effective.

The paper is organized as follows. In Section 2, basic definitions and background are introduced. Main results of the paper, consisting of the bounded real lemma in terms of LMI and suboptimal robust control procedure for uncertain systems, are represented in Section 3. In Section 4, numerical example is given.

2. PRELIMINARIES

In this section, main definitions, concepts, and theorems from the theory of descriptor systems are given (Dai, 1989; Xu & Lam, 2006).

A state-space representation of discrete-time descriptor systems is

$$Ex(k+1) = Ax(k) + Bf(k), \tag{1}$$

$$y(k) = Cx(k) + Df(k) \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state, $f(k) \in \mathbb{R}^m$ is the input, and $y(k) \in \mathbb{R}^p$ is the output signals. E, A, B, C and D are constant real matrices of appropriate dimensions. The matrix $E \in \mathbb{R}^{n\times n}$ is singular, i.e. $\operatorname{rank}(E) = r < n$.

The following notations are used throughout the paper:

- I_n in an identity $n \times n$ matrix;
- $\overline{\sigma}(A)$ stands for the maximal singular value of the matrix A: $\overline{\sigma}(A) = \sqrt{\rho(A^*A)}$, where A^* is the Hermitian conjugate of the matrix A;
- $\operatorname{sym}(A)$ stands for symmetrization of the matrix A: $\operatorname{sym}(A) = A + A^{\mathrm{T}}$.

Definition 1. System (1) is called admissible if it is

(1) regular ($\exists\lambda : \det(\lambda E - A) \neq 0$),
(2) causal ($\deg\det(\lambda E - A) = \operatorname{rank} E$),
(3) stable ($\rho(E, A) = \max|\lambda|_{\lambda\in z|\{\det(zE-A)=0\}} < 1$).

Regularity stands for existence and uniqueness of the solution for consistent initial conditions (Dai, 1989). Hereinafter, we suppose that considered systems are regular.

Definition 2. The transfer function of system (1)–(2) is defined by the expression

$$P(z) = C(zE - A)^{-1}B + D,\ z \in \mathbb{C}. \tag{3}$$

For the singular matrix E there exist two nonsingular matrices $\overline{W}$ and $\overline{V}$ such that $\overline{W}E\overline{V} = \operatorname{diag}(I_r, 0)$ (see (Dai, 1989)).

Consider the following change of variables:

$$\overline{V}^{-1}x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{4}$$

where $x_1(k) \in \mathbb{R}^r$ and $x_2(k) \in \mathbb{R}^{n-r}$.

By left multiplying of system (1) on the matrix $\overline{W}$ and using the change of variables (4), one can rewrite system (1)–(2) in the form

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1f(k), \tag{5}$$

$$0 = A_{21}x_1(k) + A_{22}x_2(k) + B_2f(k), \tag{6}$$

$$y(k) = C_1x_1(k) + C_2x_2(k) + Df(k) \tag{7}$$

where

$$\overline{W}A\overline{V} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \overline{W}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C\overline{V} = [\, C_1\ C_2 \,]. \tag{8}$$

Matrices $\overline{W}$ and $\overline{V}$ can be found from the singular value decomposition (SVD) (see (Belov & Andrianova, 2015, 2016)).

Consider an input signal in the following form:

$$f(k) = Kx(k) + h(k) \tag{9}$$

where $K \in \mathbb{R}^{m\times n}$ is a constant real matrix, $h(k)$ is a new input signal. The equation (1) turns into

$$Ex(k+1) = (A + BK)x(k) + Bh(k). \tag{10}$$

Definition 3. System (1) is called causally controllable (stabilizable) if there exists a state feedback control in the form (9) such that closed-loop system (10) is causal (stable).

For more information about causal controllability and stabilizability see (Dai, 1989).

The following lemmas are be used below to transform LMI.

Lemma 1. (Petersen, 1987)
Let matrices $M \in \mathbb{R}^{n\times p}$ and $N \in \mathbb{R}^{q\times n}$ be nonzero, and $G = G^{\mathrm{T}} \in \mathbb{R}^{n\times n}$. The inequality

$$G + M\Delta N + N^{\mathrm{T}}\Delta^{\mathrm{T}}M^{\mathrm{T}} \leqslant 0 \tag{11}$$

is true for all $\Delta \in \mathbb{R}^{p\times q}$: $\|\Delta\|_2 \leqslant 1$ if and only if there exists a scalar value $\varepsilon > 0$ such that

$$G + \varepsilon MM^{\mathrm{T}} + \frac{1}{\varepsilon}N^{\mathrm{T}}N \leqslant 0. \tag{12}$$

Lemma 2. (Schur lemma, introduced in (Boyd et al., 1994))
Let

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix}$$

where X_{11} and X_{22} are symmetric matrices.

If $X_{11} > 0$, then $X > 0$ if and only if

$$X_{22} - X_{12}^{\mathrm{T}}X_{11}^{-1}X_{12} > 0. \tag{13}$$

If $X_{22} > 0$, then $X > 0$ if and only if

$$X_{11} - X_{12}X_{22}^{-1}X_{12}^{\mathrm{T}} > 0. \tag{14}$$

The following results are based on the modified bounded real lemma for descriptor systems in SVD equivalent form.

Denote $A_d = \overline{W}A\overline{V}$, $B_d = \overline{W}B$, $C_d = C\overline{V}$, $D_d = D$ and formulate the conditions of this lemma.

Lemma 3. (Feng & Yagoubi, 2013) System (1) is admissible and

$$\|P(z)\|_\infty < \gamma$$

if there exist matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, and a sufficiently large scalar $\alpha > 0$ such that

$$\begin{bmatrix} \Phi_{11} & \Gamma A_d & \Gamma B_d & \Phi_{41}^{\mathrm{T}} & 0 \\ A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Phi_{22} & \Pi B_d & A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Phi_{52}^{\mathrm{T}} \\ B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & B_d^{\mathrm{T}}\Pi^{\mathrm{T}} & -\gamma^2 I_m & B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Phi_{53}^{\mathrm{T}} \\ \Phi_{41} & \Gamma A_d & \Gamma B_d & -Q-Q^{\mathrm{T}} & 0 \\ 0 & \Phi_{52} & \Phi_{53} & 0 & -I_p \end{bmatrix} < 0 \quad (15)$$

where
$\Phi_{11} = -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}}$, $\Phi_{22} = \Pi A_d + A_d^{\mathrm{T}}\Pi^{\mathrm{T}} - \Theta$,
$\Phi_{41} = L - Q - \frac{1}{2}Q^{\mathrm{T}}$, $\Phi_{52} = C_d + \alpha C_d \Pi A_d$,
$\Phi_{53} = D_d + \alpha C_d \Pi B_d$,
and $\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}$, $\Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$, $\Gamma = [\, Q \;\; R \,]$.

Remark 1. In (Feng & Yagoubi, 2013) the modified bounded real lemma is given in assumption that $D = 0$. Note that the scalar parameter $\alpha > 0$ and $D = 0$ in (Feng & Yagoubi, 2013) are required to make conditions necessary and sufficient and to avoid possible conservatism while solving control problem. Due to this fact the result obtained below for $\alpha = 0$ and $D \neq 0$ is only sufficient but not necessary.

3. PROBLEM STATEMENT AND MAIN RESULTS

3.1 Problem statement

Consider the following discrete-time descriptor system:

$$Ex(k+1) = A_\Delta x(k) + B_{\Delta w} w(k) + B_{\Delta u} u(k), \quad (16)$$
$$y(k) = C_\Delta x(k) + D_{\Delta w} w(k) + D_{\Delta u} u(k) \quad (17)$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^q$ is the input disturbance, $y(k) \in \mathbb{R}^p$ is the output, $u(k) \in \mathbb{R}^m$ is the control input. The matrix E is singular, $\operatorname{rank} E = r < n$.
$A_\Delta = A + M_A \Delta N_A$, $B_{\Delta w} = B_w + M_B^w \Delta N_B^w$,
$B_{\Delta u} = B_u + M_B^u \Delta N_B^u$, $C_\Delta = C + M_C \Delta N_C$,
$D_{\Delta w} = D_w + M_D^w \Delta N_D^w$, $D_{\Delta u} = D_u + M_D^u \Delta N_D^u$.

The matrix $\Delta \in \mathbb{R}^{s\times s}$ is unknown norm-bounded, i.e. $\|\Delta\|_2 \leqslant 1$ (or Frobenius norm-bounded matrix as $\|\Delta\|_2 \leqslant \|\Delta\|_F$). Note that $\|\Delta\|_2 := \overline{\sigma}(\Delta) \leqslant 1$ iff $\Delta^{\mathrm{T}}\Delta \leqslant I_s$.

Assume that

(1) system (16) is causally controllable;
(2) system (16) is stabilizable;
(3) a scalar value $\gamma > 0$ is known.

The $\mathcal{H}_\infty$ robust performance analysis problem is formulated as follows.
Problem 1. Suppose that $u(k) = 0$ and the pair (E, A_Δ) is admissible for all Δ from the given set. For the given scalar $\gamma > 0$ the problem is to check the condition

$$\|P_\Delta(z)\|_\infty < \gamma$$

where

$$P_\Delta(z) = C_\Delta(zE - A_\Delta)^{-1}B_\Delta + D_\Delta.$$

The $\mathcal{H}_\infty$ control problem is formulated as follows.
Problem 2. For the given scalar $\gamma > 0$ the problem is to find a state-feedback control

$$u(k) = Fx(k),$$

for which the closed-loop system with a transfer function $P_\Delta^{cl}(z) = (C_\Delta + D_{\Delta u}F)(zE - (A_\Delta + B_{\Delta u}F))^{-1}B_{\Delta w} + D_{\Delta w}$ is admissible and

$$\|P_\Delta^{cl}(z)\|_\infty < \gamma.$$

3.2 Main Results

To solve $\mathcal{H}_\infty$ robust performance analysis problem we formulate a bounded real lemma for the uncertain system (16)–(17).

For system (16)–(17) in SVD equivalent form the following denotations are used
$A_d = \overline{W}A\overline{V}$, $B_{wd} = \overline{W}B_w$, $B_{ud} = \overline{W}B_u$, $C_d = C\overline{V}$,
$D_{wd} = D_w$, $D_{ud} = D_u$, $M_B^{wd} = \overline{W}M_B^w$, $N_B^{wd} = N_B^w$,
$M_A^d = \overline{W}M_A$, $N_A^d = N_A\overline{V}$, $M_C^d = M_C$, $N_C^d = N_C\overline{V}$,
$M_B^{ud} = \overline{W}M_B^u$, $N_B^{ud} = N_B^u$.

Theorem 1. (Bounded real lemma) System (16)–(17) for $u(k) = 0$ is admissible and $\|P_\Delta\|_\infty < \gamma$ for all Δ from the given set if there exist a scalar $\varepsilon > 0$ and matrices $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $L \in \mathbb{R}^{r\times r}$, $L > 0$ such that

$$\begin{bmatrix} \Sigma + \varepsilon N_1^{\mathrm{T}} N_1 & M_1 \\ M_1^{\mathrm{T}} & -\varepsilon I_{4s} \end{bmatrix} < 0. \quad (18)$$

Here

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Gamma A_d & \Gamma B_{wd} & \Sigma_{41}^{\mathrm{T}} & 0 \\ A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Sigma_{22} & \Pi B_{wd} & A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & C_d^{\mathrm{T}} \\ B_{wd}^{\mathrm{T}}\Gamma^{\mathrm{T}} & B_{wd}^{\mathrm{T}}\Pi^{\mathrm{T}} & -\gamma^2 I_m & B_{wd}^{\mathrm{T}}\Gamma^{\mathrm{T}} & D_{wd}^{\mathrm{T}} \\ \Sigma_{41} & \Gamma A_d & \Gamma B_{wd} & -Q-Q^{\mathrm{T}} & 0 \\ 0 & C_d & D_{wd} & 0 & -I_p \end{bmatrix},$$

$\Sigma_{11} = -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}}$, $\Sigma_{41} = L - Q - \frac{1}{2}Q^{\mathrm{T}}$,
$\Sigma_{22} = \Pi A_d + A_d^{\mathrm{T}}\Pi^{\mathrm{T}} - \Theta$,

$$M_1 = \begin{bmatrix} \Gamma M_A^d & \Gamma M_B^d & 0 & 0 \\ \Pi M_A^d & \Pi M_B^d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Gamma M_A^d & \Gamma M_B^d & 0 & 0 \\ 0 & 0 & M_C^d & M_D^w \end{bmatrix}, \; N_1 = \begin{bmatrix} 0 & N_A^d & 0 & 0 & 0 \\ 0 & 0 & N_B^d & 0 & 0 \\ 0 & N_C^d & 0 & 0 & 0 \\ 0 & 0 & N_D^w & 0 & 0 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}, \; \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}, \; \Gamma = [\, Q \;\; R \,].$$

Proof.

Transform expression (15) for system (16)–(17), noting that $\alpha = 0$. We get the inequality

$$\Sigma + \text{sym}\,(M_1 \Delta N_1) < 0. \quad (19)$$

Applying Lemmas 1 and 2 to the inequality (19), we get

$$\Sigma + \frac{1}{\varepsilon}M_1 M_1^{\mathrm{T}} + \varepsilon N_1^{\mathrm{T}} N_1 < 0,$$
$$\Sigma + \varepsilon N_1^{\mathrm{T}} N_1 - M_1(-\varepsilon I_{4s})^{-1}M_1^{\mathrm{T}} < 0,$$
$$\begin{bmatrix} \Sigma + \varepsilon N_1^{\mathrm{T}} N_1 & M_1 \\ M_1^{\mathrm{T}} & -\varepsilon I_{4s} \end{bmatrix} < 0.$$

Consequently, the conditions of the Lemma 3 hold true for system (16)–(17), it means that its norm is bounded by the positive scalar value γ, i.e. $\|P_\Delta(z)\|_\infty < \gamma$. ■

Substituting control law from Problem 2 into system (16)–(17), we get the closed-loop system

$$Ex(k+1) = (A_\Delta + B_{\Delta u}F)x(k) + B_{\Delta w}w(k), \quad (20)$$
$$y(k) = (C_\Delta + D_{\Delta u}F)x(k) + D_{\Delta w}w(k). \quad (21)$$

The robust $\mathcal{H}_\infty$ control problem is solvable if the following conditions hold.

Theorem 2. For a given scalar $\gamma > 0$ Problem 2 is solvable if there exist matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $Z \in \mathbb{R}^{n\times m}$, and a scalar $\epsilon > 0$ such that

$$\begin{bmatrix} \Lambda + \varepsilon M^{\mathrm{T}} M & N \\ N^{\mathrm{T}} & -\varepsilon I_{6s} \end{bmatrix} < 0. \tag{22}$$

N is defined by (23),

$$M = \begin{bmatrix} 0 & \left(M_A^d\right)^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & \left(M_B^{ud}\right)^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & 0 & \left(M_C^d\right)^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & \left(M_D^u\right)^{\mathrm{T}} & 0 & 0 \\ 0 & \left(M_B^{wd}\right)^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & 0 & \left(M_D^w\right)^{\mathrm{T}} & 0 & 0 \end{bmatrix}, \tag{24}$$

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & \Lambda_{41}^{\mathrm{T}} & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{32}^{\mathrm{T}} & \Lambda_{21} & \Lambda_{52}^{\mathrm{T}} \\ \Lambda_{31} & \Lambda_{32} & -\gamma^2 I_q & \Lambda_{31} & \Lambda_{53}^{\mathrm{T}} \\ \Lambda_{41} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & -(Q+Q^{\mathrm{T}}) & 0 \\ 0 & \Lambda_{52} & \Lambda_{53} & 0 & -I_q \end{bmatrix}, \tag{25}$$

where
$\Lambda_{11} = -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}}$, $\Lambda_{21} = A_d\Gamma^{\mathrm{T}} + B_{ud}Z^{\mathrm{T}}\Omega^{\mathrm{T}}$,
$\Lambda_{31} = C_d\Gamma^{\mathrm{T}} + D_{ud}Z^{\mathrm{T}}\Omega^{\mathrm{T}}$, $\Lambda_{41} = L - Q - \frac{1}{2}Q^{\mathrm{T}}$,
$\Lambda_{22} = \Pi A_d^{\mathrm{T}} + A_d\Pi^{\mathrm{T}} + \Phi Z B_{ud}^{\mathrm{T}} + B_{ud}Z^{\mathrm{T}}\Phi^{\mathrm{T}} - \Theta$,
$\Lambda_{32} = C_d\Pi^{\mathrm{T}} + D_{ud}Z^{\mathrm{T}}\Phi^{\mathrm{T}}$, $\Lambda_{52} = B_{wd}^{\mathrm{T}}$, $\Lambda_{53} = D_{wd}^{\mathrm{T}}$.
$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}$, $\Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$, $\Phi = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}$,
$\Omega = [\, I_r \;\; 0 \,]$, $\Gamma = [\, Q \;\; R \,]$.

The gain matrix can be obtained as

$$F = Z^{\mathrm{T}} \begin{bmatrix} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}}R^{\mathrm{T}}Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{bmatrix} \overline{V}^{-1}. \tag{26}$$

Proof. The dual closed-loop system is

$$E^{\mathrm{T}}x(k+1) = (A_\Delta^{\mathrm{T}} + F^{\mathrm{T}}B_{\Delta u}^{\mathrm{T}})x(k) + + (C_\Delta^{\mathrm{T}} + F^{\mathrm{T}}D_{\Delta u}^{\mathrm{T}})w'(k), \tag{27}$$

$$y(k) = B_{\Delta w}^{\mathrm{T}}x(k) + D_{\Delta w}^{\mathrm{T}}w'(k) \tag{28}$$

If system (27)–(28) is admissible and its $\mathcal{H}_\infty$ norm is less than γ, then Problem 2 is solved for system (16)–(17). Applying Lemma 1 to system (27)–(28) we obtain (22) where

$$\hat{\Sigma}_{21} = A_d\Gamma^{\mathrm{T}} + B_{ud}F_d\Gamma^{\mathrm{T}}, \tag{29}$$

$$\hat{\Sigma}_{22} = \Pi A_d^{\mathrm{T}} + A_d\Pi^{\mathrm{T}} + \Pi F_d^{\mathrm{T}}B_{ud}^{\mathrm{T}} + B_{ud}F_d\Pi^{\mathrm{T}} - \Theta, \tag{30}$$

$$\hat{\Sigma}_{31} = C_d\Gamma^{\mathrm{T}} + D_{ud}F_d\Gamma^{\mathrm{T}}, \tag{31}$$

$$\hat{\Sigma}_{32} = C_d\Pi^{\mathrm{T}} + D_{ud}F_d\Pi^{\mathrm{T}}. \tag{32}$$

Introduce the following linear change of variables

$$\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}} = Z.$$

It implies that

$$[\, Q \;\; R \,]\, F_d^{\mathrm{T}} = [\, I_r \;\; 0 \,]\, Z$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} F_d^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Z.$$

Substituting it into (29) and (30) we get Λ_{21} and Λ_{22} entries from (25). Also, after substitution (31) and (32) coincide with Λ_{31} and Λ_{32} entries from (25).

As inequality (22) holds, then the (1,1) entry implies the matrix Q is invertible. We also suppose, that the matrix S is invertible. If it does not hold, there exists a scalar $\epsilon \in (0,1)$, such that inequality (22) holds true for the scalar matrix $\bar{S} = S + \epsilon I_{n-r}$. So, we can use $\bar{S}$ instead of S.

As pointed out before, Q and S are invertible. So the feedback gain F_d for the closed-loop system (27) is $F_d = Z^{\mathrm{T}} \begin{bmatrix} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}}R^{\mathrm{T}}Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{bmatrix}$. Note that $F_d = F\overline{V}$. By the inverse change of variables we get F from (26). This completes the proof. ■

4. NUMERICAL EXAMPLE

Consider a numerical example taken from (Chadli & Darouach, 2014). The system's parameters are

$E = \begin{bmatrix} 8 & 3.2 & 1.6 \\ 4 & 1.6 & 0.8 \\ 12 & 3.2 & 5.6 \end{bmatrix}$, $A = \begin{bmatrix} 0.6 & 0.24 & 0.36 \\ 0.3 & 0.12 & 0.3 \\ 0.72 & 0.24 & 1.2 \end{bmatrix}$,

$B_u = \begin{bmatrix} 0.12 & 0.06 \\ 0.12 & 0 \\ 0.06 & 0.12 \end{bmatrix}$, $B_w = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.2 \\ 0 & 0.2 \end{bmatrix}$, $C = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}$,

$D_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\Delta \in [-1;\ 1]$,

and
$M_A = M_B^u = M_B^w = [\,0.2\ 0.3\ 0.3\,]^{\mathrm{T}}$, $M_C = M_D^w = [\,0.1\ 0.1\,]^{\mathrm{T}}$, $N_A = N_C = [\,0.2\ 0.2\ 0.2\,]$, $N_B^u = [\,0.1\ 0.1\,]^{\mathrm{T}}$, $N_B^w = N_D^w = [\,0\ 0\,]^{\mathrm{T}}$. D_u, M_D^u, and N_D^u are assumed to be zero.

Note that the system is admissible for all $\Delta \in [-1;\ 1]$ and its upper bound of $\mathcal{H}_\infty$ norm is $\overline{\gamma} = 1.245$. The objective in (Chadli & Darouach, 2014) is to design a controller which provides $\mathcal{H}_\infty$ robust performance of the closed-loop system less than $\gamma_{obj} = 0.18$. It is also mentioned in (Chadli & Darouach, 2014) that the method from (Ji et al., 2007) fails to find a feasible solution. The obtained controller in (Chadli & Darouach, 2014) is

$$F_1 = \begin{bmatrix} -1.5807 & -0.1634 & -0.2967 \\ -0.3101 & -1.0900 & -0.1515 \end{bmatrix}.$$

For the actual boundary of $\mathcal{H}_\infty$ norm of the closed-loop system with controller F_1 the following inequality is true: $\overline{\gamma_1} = 0.1975$. Hence, the method proposed in (Chadli & Darouach, 2014) does not guarantee that the norm of the closed-loop uncertain system is less than a prescribed value.

The control design procedure proposed in Theorem 2 with the same objective gives a controller

$$F_2 = \begin{bmatrix} -4.2150 & 1.8499 & 0.8112 \\ -2.8755 & -4.8516 & -10.43575 \end{bmatrix}.$$

$\mathcal{H}_\infty$ norm of the system, closed by the controller F_2, is bounded as $\overline{\gamma_2} = 0.0402$. Hence, the design objective is satisfied.

$$N = \begin{bmatrix} \Gamma\left(N_A^d\right)^{\mathrm{T}} & \Omega Z\left(N_B^{ud}\right)^{\mathrm{T}} & \Gamma\left(N_C^d\right)^{\mathrm{T}} & \Omega Z\left(N_D^u\right)^{\mathrm{T}} & 0 & 0 \\ \Pi\left(N_A^d\right)^{\mathrm{T}} & \Phi Z\left(N_B^{ud}\right)^{\mathrm{T}} & \Pi\left(N_C^d\right)^{\mathrm{T}} & \Phi Z\left(N_D^u\right)^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \Gamma\left(N_A^d\right)^{\mathrm{T}} & \Omega Z\left(N_B^{ud}\right)^{\mathrm{T}} & \Gamma\left(N_C^d\right)^{\mathrm{T}} & \Omega Z\left(N_D^u\right)^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(N_B^{wd}\right)^{\mathrm{T}} & \left(N_D^w\right)^{\mathrm{T}} \end{bmatrix} \quad (23)$$

5. CONCLUSION

The paper presents a new method of robust state-space $\mathcal{H}_\infty$ control design procedure for uncertain descriptor systems with norm-bounded uncertainties in all matrices in the right hand side of the system. It should be noted that the problem statement allows to take into account a wider set of uncertain matrices because they can be chosen independently. A numerical example has shown that the proposed method is effective when other developed methods fail.

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