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GENERALIZED YANGIANS AND THEIR POISSON COUNTERPARTS

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By generalized Yangians, we mean Yangian-like algebras of two different classes. One class comprises the previously introduced so-called braided Yangians. Braided Yangians have properties similar to those of the reflection equation algebra. Generalized Yangians of the second class, RTT-type Yangians, are defined by the same formulas as the usual Yangians but with other quantum R-matrices. If such an R-matrix is the simplest trigonometric R-matrix, then the corresponding RTT-type Yangian is called a q-Yangian. We claim that each generalized Yangian is a deformation of the commutative algebra $\mathrm{Sym}(gl(m)[t^{-1}])$ *if the corresponding R-matrix is a deformation of the flip operator. We give the explicit form of the corresponding Poisson brackets.*

Keywords: current R-matrix, braided Yangian, quantum symmetric polynomial, quantum determinant, Poisson structure, deformation property

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Dedicated to the memory of P. P. Kulish

1. Introduction

In [1], we introduced the notion of a braided Yangian associated with a wide class of rational and trigonometric R-matrices. This notion is a new generalization of the Yangian $\mathbf{Y}(gl(m))$ introduced by Drinfeld [2]. According to one definition, $\mathbf{Y}(gl(m))$ is the algebra generated by the coefficients of the matrix-valued function

$$L(u) = \sum_{k\geq 0} L[k]u^{-k}. \tag{1.1}$$

The Laurent coefficients $L[k]$ are finite $m\times m$ matrices, and $L[0] = I$, where I is the identity matrix. The elements of the matrix $L(u)$ satisfy the system of flip relations

$$R(u, v)L_1(u)L_2(v) - L_1(v)L_2(u)R(u, v) = 0, \tag{1.2}$$

where $R(u, v) = P - aI/(u - v)$ is Yang's famous R-matrix,[1] I is the identity matrix, and P is the usual flip operator or its matrix. The matrix $L(u)$ (and similar matrices considered below) are called *generating matrices.*

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[1] We note that this R-matrix is often used in a different form $\mathcal{R}(u, v) = PR(u, v)$.

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The Yang R-matrix is the simplest example of a current R-matrix (i.e., depending on a spectral parameter), which is one solution of the quantum Yang–Baxter equation

$$R_{12}(u,v)R_{23}(u,w)R_{12}(v,w) = R_{23}(v,w)R_{12}(u,w)R_{23}(u,v). \tag{1.3}$$

If the Yang R-matrix is replaced with another current R-matrix $R(u,v)$ in relation (1.2), then we obtain a class of Yangian-like algebras. The best-known example is the so-called q-Yangian, which corresponds to the simplest trigonometric R-matrix related to the quantum group $U_q(\widehat{sl(m)})$. Below, all objects related to $U_q(\widehat{sl(m)})$ or $U_q(sl(m))$ are said to be *standard*.

In addition to standard R-matrices, there exists a large family of rational and trigonometric current R-matrices, which are constructed from constant involutive and Hecke R-matrices using the Baxterization procedure (the exact definition is given in Sec. 2). We call the corresponding algebras with relations (1.2) *RTT-type Yangians*, denoted by $\mathbf{Y}_{RTT}(R)$. Moreover, in contrast to the usual Yangian, we do not impose the condition $L[0] = I$.

Another class of algebras, braided Yangians, was introduced in [1]. The generating matrix of each such algebra satisfies the relation

$$R(u,v)L_1(u)RL_1(v) - L_1(v)RL_1(u)R(u,v) = 0, \tag{1.4}$$

where $R(u,v)$ is the abovementioned current R-matrix resulting from the Baxterization of a constant involutive or Hecke R-matrix. The generating matrix $L(u)$ is also a formal series given by formula (1.1) with the additional condition $L[0] = I$. A braided Yangian is denoted by $\mathbf{Y}(R)$.

The Yangian-like algebras defined above are called *generalized Yangians*.

We note that the properties of the generalized Yangians $\mathbf{Y}_{RTT}(R)$ and $\mathbf{Y}(R)$ differ substantially. In particular, they have different bialgebra structures. The relation between the comultiplication and multiplication operations in the tensor square is usual in the RTT-type Yangian, while it is more complicated in the braided Yangian and is similar to the situation in superalgebras, only the role of the superflip is played by a more complicated operator constructed from the defining R-matrix. Moreover, in the two classes of generalized Yangians, the evaluation maps[2] are completely different, which leads to a large difference in the theory of representations of these algebras.

Evaluation maps for braided Yangians are analogous to similar maps in the classical case. The image of this map is contained in the reflection equation (RE) algebra determined by the original R-matrix R. The modified RE algebra (which differs from the usual one by a linear shift of some generators by the unit element) can be interpreted as a twisted analogue of $U(gl(m))$. In particular, the categories of finitely generated representations of $U(gl(m))$ and the modified RE algebra are similar in many respects [3]. The evaluation map hence allows constructing a large set of representations of braided Yangians.

In the case of RTT-type Yangians, the evaluation map is less interesting because is image is in algebras whose representation theories are unknown in the general case. An example of the RTT-type q-Yangian related to the standard trigonometric R-matrix was presented in [4].

Our main purpose in this paper is to study the *deformation properties* of the generalized Yangians of both classes.[3] We say that an associative algebra A_h depending on some parameter h has the deformation property if it becomes a commutative algebra $A = A_0$ for $h = 0$ and a new product $\star_h$ can be defined in this commutative algebra, induced from the algebra A_h and depending smoothly on h. This in turn means

[2] An evaluation map is a map to a *finitely generated* algebra as one step in constructing representations of a Yangian.

[3] All the algebras we consider are defined via generators and relations. The parameters in the relations are not formal and can be specialized.

that for a generic h, there exist isomorphisms $\alpha_h\colon A \to A_h$ of linear spaces that depend smoothly on h and such that $\alpha_0 = \mathrm{Id}$. Then the induced product in the algebra A is given by the rule

$$f \star_h g = \alpha_h^{-1}(\alpha_h(f) \circ \alpha_h(g)),$$

where $\circ$ denotes the product in A_h. The map α_h is usually constructed using a special basis (sometimes called a Poincaré–Birkhoff–Witt basis) in A_h. In what follows, we show that each generalized Yangian is a deformation of the commutative algebra $\mathrm{Sym}(gl(m)[t^{-1}])$ under the additional condition that R is a deformation of the flip operator.

If an algebra A_h has the deformation property, then the $\star_h$ product can be expanded in a series in the parameter:

$$f \star_h g = f \cdot g + hc_1(f,g) + h^2 c_2(f,g) + \ldots,$$

where $\cdot$ denotes the commutative product in A. In this case, there exists a Poisson bracket on A defined by the antisymmetrization of the term c_1:

$$A^{\otimes 2} \ni f \otimes g \mapsto \{f,g\} = \frac{1}{2}(c_1(f,g) - c_1(g,f)) \in A.$$

Our second purpose is to calculate the explicit Poisson structures corresponding to the Yangians of both classes. We write the quadratic Poisson brackets and their linearization for both classes.

This paper is organized as follows. In Sec. 2, we recall the definitions of RTT and RE algebras and Yangians of both classes and describe some of their properties. First, we consider the so-called Baxterization, which allows constructing current R-matrices from involutive and Hecke R-matrices. We then describe the target algebras for the evaluation maps of the corresponding braided Yangians and construct quantum analogues of some symmetric polynomials. Finally, we present a form of the Cayley–Hamilton–Newton matrix identity for the generating matrix of braided Yangians.

Section 3 is devoted to the question of the deformation properties of generalized Yangians. We present arguments supporting the assertion that generalized Yangians are deformations of the algebra $\mathrm{Sym}(gl(m)[t^{-1}])$ if the initial matrix R is a deformation of the flip operator. We then calculate the quadratic Poisson brackets corresponding to this deformation.

Section 4 is devoted to linearization of the quadratic Poisson brackets. We present two explicit examples of such brackets in the case of low dimensions.

The ground number field $\mathbb{K}$ is assumed to be the field of complex ($\mathbb{C}$) or real ($\mathbb{R}$) numbers.

In conclusion, we once more stress that the properties of braided Yangians (and, in particular, their Poisson structure) are quite similar to these of the usual Yangians. Using this similarity, we can generalize the method for quantizing the rational Gaudin model, proposed by Talalaev [5], to the trigonometric case. We plan to elucidate these questions in our subsequent publications.

2. Quantum matrix algebras and generalized Yangians

We begin with consideration of quantum matrix (QM) algebras similar to the Yangians of both classes but associated with *constant* R-matrices. We recall that an operator $R\colon V^{\otimes 2} \to V^{\otimes 2}$, where V is a finite-dimensional vector space ($\dim V = m$) is called a *twist* if it satisfies quantum Yang-Baxter equation (1.3) with the parameter dependence omitted. A twist R is called a *Hecke* or *involutive* symmetry if it additionally satisfies the relation

$$(qI - R)(q^{-1}I + R) = 0, \tag{2.1}$$

where the numerical nonzero parameter q satisfies the respective condition $q^2 \neq 1$ or $q^2 = 1$. Below, we assume that if $q \neq 1$, then the value of q is is chosen to be *generic*, i.e., $q^n \neq 1$ for any integer n.

An associative algebra generated by elements of a matrix $L = \|l_i^j\|_{1\le i,\, j\le m}$ satisfying the matrix relation

$$RL_1RL_1 - L_1RL_1R = h(RL_1 - L_1R) \tag{2.2}$$

is called a *modified RE algebra* if $h \neq 0$. If $h = 0$, then we omit "modified." Algebra (2.2) is denoted by $\mathcal{L}(R,h)$ for nonzero values of the parameter and by $\mathcal{L}(R)$ if $h = 0$. As usual, subscripts on matrices and operators (e.g., L_i, $R_{i\,j}$, etc.) indicate the position of the factor (or factors) in $V^{\otimes n}$ where the operator acts. Moreover, for the operator $R_{i\,i+1}$, we use the brief notation R_i.

We also consider the RTT algebra associated with a twist R and defined by the system of relations for the elements of the generating matrix,

$$RT_1T_2 - T_1T_2R = 0, \qquad T = \|t_i^j\|_{1\le i,\, j\le m}. \tag{2.3}$$

This algebra is denoted by $\mathcal{T}(R)$.

The algebras $\mathcal{L}(R)$ and $\mathcal{T}(R)$ are examples of QM algebras defined in the general case by a pair of compatible twists [6].

A twist R is said to be *skew-invertible* if there exits an operator $\Psi\colon V^{\otimes 2} \to V^{\otimes 2}$ such that

$$\mathrm{Tr}_2\, R_{12}\Psi_{23} = P_{13} \quad \Leftrightarrow \quad R_{ij}^{kl}\Psi_{lp}^{jq} = \delta_i^q\delta_p^k. \tag{2.4}$$

All the twists R we consider in this paper are assumed to be skew-invertible involutive or Hecke symmetries.

For any skew-invertible twist R, we can define the so-called R-trace $\mathrm{Tr}_R\, A$, where A is an arbitrary $m{\times}m$ matrix. This operation has many useful properties and applications. For example, the R-trace appears in the construction of *quantum symmetric polynomials* in the algebras $\mathcal{T}(R)$ and $\mathcal{L}(R)$ and also in the generalized Yangians of both classes. The explicit form of this operation is written below for the generalized Yangians. We note that these quantum symmetric polynomials generate the so-called characteristic subalgebras $Ch(\mathcal{T}(R))$ and $Ch(\mathcal{L}(R))$ of $\mathcal{T}(R)$ and $\mathcal{L}(R)$. The properties of $Ch(\mathcal{T}(R))$ and $Ch(\mathcal{L}(R))$ strongly differ from each other: the latter subalgebra is central in the RE algebra, while $Ch(\mathcal{T}(R))$ is commutative but not central. A more detailed exposition can be found in [6], [7].

We now turn to the generalized Yangians. For this, we must first indicate the explicit form of the current R-matrices in formulas (1.2) and (1.4). The following statement was proved in [1].

Proposition 1. *We consider the sum*

$$R(u,v) = R + g(u,v)I, \tag{2.5}$$

where R is a twist, $g(u,v) = f(u-v)$, and $f(z)$ is a nonconstant meromorphic function. If R is an involutive symmetry, then $R(u,v)$ is a current R-matrix iff

$$g(u,v) = \frac{a}{u-v}. \tag{2.6}$$

If $R = R_q$ is a Hecke symmetry, then $R(u,v)$ is a current R-matrix iff

$$g(u,v) = \frac{q-q^{-1}}{b^{u-v}-1}. \tag{2.7}$$

Here, a and $b \neq 1$ are arbitrary nonzero complex numbers.

In particular, setting $b = q^{-2/a}$, we obtain

$$R(u,v) = R_q - \frac{q^{(u-v)/a}}{((u-v)/a)_q} I. \tag{2.8}$$

In the limit $q \to 1$, this R-matrix tends to

$$R_1 - \frac{a}{u-v} I \tag{2.9}$$

under the condition that the Hecke symmetry R_q tends to an involutive matrix R_1.

Changing the variables $b^{-u} \to u$ and $b^{-v} \to v$ in (2.7)), we obtain the trigonometric current R-matrix in the form

$$R(u,v) = R - \frac{u(q-q^{-1})}{u-v} I. \tag{2.10}$$

Obviously, it depends only on the ratio $x = v/u$.

In what follows, we consider RTT-type Yangians $\mathbf{Y}_{RTT}(R)$ and braided Yangians $\mathbf{Y}(R)$ respectively defined by formulas (1.2) and (1.4), where $R(u,v)$ are current R-matrices (2.9) or (2.10) and the middle terms R in (1.4) are the initial symmetries.

As noted above, each of the Yangians of both classes has a bialgebra structure, but this structure is braided in $\mathbf{Y}(R)$. The details can be found in [1].

We now consider the evaluation map for a braided Yangian. For a given braided Yangian $\mathbf{Y}(R)$, the evaluation map is defined by the formula

$$L(u) \mapsto I + \frac{M}{u}, \tag{2.11}$$

where M is the generating matrix of the *target algebra.* The concrete form of this algebra depends on the initial symmetry R. Namely, we have the following proposition.

Proposition 2 [1]**.** 1. *If the twist R is an involutive symmetry, then map* (2.11) *defines a surjective morphism* $\mathbf{Y}(R) \to \mathcal{L}(R,1)$*. Moreover, the map* $M \mapsto L[1]$ *defines an injective morphism* $\mathcal{L}(R,1) \to \mathbf{Y}(R)$.

2. *If the twist R is a Hecke symmetry, then map* (2.11) *defines a morphism* $\mathbf{Y}(R) \to \mathcal{L}(R)$.

Therefore, the target algebra is $\mathcal{L}(R,1)$ *in the first case and* $\mathcal{L}(R)$ *in the second case.*

Proposition 2 allows constructing a category of finite-dimensional representations of a braided Yangian using the results in [3], where such a category was constructed for the algebra $\mathcal{L}(R,1)$. Hence, if R is an involutive symmetry, then the evaluation map converts any $\mathcal{L}(R,1)$-module into a $\mathbf{Y}(R)$-module. If R is a Hecke symmetry, then we first convert any $\mathcal{L}(R,1)$-module into $\mathcal{L}(R)$-module and then realize a representation of the Yangian $\mathbf{Y}(R)$ in it. This is always possible because the algebras $\mathcal{L}(R)$ and $\mathcal{L}(R,1)$ are isomorphic. Their isomorphism is given by the map

$$\mathcal{L}(R) \to \mathcal{L}(R,1), \qquad L \mapsto L - \frac{1}{q-q^{-1}} I, \quad q^2 \neq 1.$$

The evaluation map for RTT-type Yangians has the form

$$L(u) \mapsto T_0 + \frac{T_1}{u}. \tag{2.12}$$

If R is a Hecke symmetry, then map (2.12) leads to a Yangian in a target algebra similar to the quantum algebra $U_q(gl(m))$. The explicit form of this algebra is given, for instance, in [4], [8]. If R is an involutive symmetry, then the target algebra is new, and its properties have not been studied. We note that if the condition $T_0 = I$ is imposed, then a contradiction appears in the permutation relations of the target algebra. This is why the condition $T[0] = I$ is not imposed on the Laurent coefficients of the generating matrix in RTT-type Yangians.

We now turn to the construction of *quantum symmetric polynomials* in braided Yangians. Moreover, we write the family of Cayley–Hamilton–Newton matrix identities analogous to the identities found in [6] for the QM algebras. With this goal, we first define quantum analogues of matrix powers of the generating matrix. We work with trigonometric R-matrices (2.8) where we set $a = 1$. The formulas for rational R-matrices (2.9) can be obtained by the limit transition $q \to 1$ to form (2.8) of the R-matrix.

We define the *quantum skew-powers* of the matrix $L(u)$ as

$$L^{\wedge k}(x) = \mathrm{Tr}_{R(2\dots k)}(\mathcal{P}^{(k)}_{12\dots k} L_{\overline{1}}(x) L_{\overline{2}}(x-1) \cdots L_{\overline{k}}(x-k+1)), \quad k \geq 2, \tag{2.13}$$

where

$$\mathcal{P}^{(k+1)}_{12\dots k+1} = \frac{(-1)^k}{(k+1)_q} R_1(1) R_2(2) \cdots R_k(k) \mathcal{P}^{(k)}_{12\dots k} \tag{2.14}$$

is the q-antisymmetrizer in the space $V^{\otimes(k+1)}$ constructed in accordance with the Hecke symmetry R. We assume that $L^{\wedge 1}(x) = L(x)$ by definition. We note that if the Hecke symmetry R is a deformation of the usual flip P, then $L^{\wedge k}(x) \equiv 0$ for all $k > m$.

We also define the *quantum matrix powers* of the generating matrix:

$$L^k(x) = \mathrm{Tr}_{R(2\dots k)}(R_1 R_2 \cdots R_{k-1} L_{\overline{1}}(x) L_{\overline{2}}(x-1) \cdots L_{\overline{k}}(x-k+1)), \quad k \geq 1. \tag{2.15}$$

The *quantum elementary symmetric polynomials* and *quantum power sums* are now respectively defined by

$$e_k(x) = \mathrm{Tr}_R(L^{\wedge k}(x)), \qquad s_k(x) = \mathrm{Tr}_R(L^k(x)). \tag{2.16}$$

Proposition 3. *The quantum skew-symmetric powers and matrix powers of $L(u)$ are related by the Cayley–Hamilton–Newton identities*

$$(-1)^{k+1} k_q L^{\wedge k}(x) = \sum_{p=1}^{k} (-q)^{k-p} L^p(x) e_{k-p}(x-p), \quad k \geq 1. \tag{2.17}$$

If R is a deformation of a flip P, then the last nontrivial Cayley–Hamilton–Newton identity becomes a quantum analogue of the classical Cayley–Hamilton identity. In this case, the highest nontrivial symmetric polynomial $e_m(x)$ is called the *quantum determinant.*

Calculating the R-trace of relation (2.17), we obtain a relation between quantum power sums and quantum elementary symmetric polynomials.

All these objects (quantum matrix powers, skew-powers, power sums, and elementary symmetric polynomials) are also well defined in RTT-type Yangians. We note that the quantum elementary symmetric polynomials and power sums in Yangians of both classes generate commutative subalgebras. Nevertheless, in contrast to the RE algebras, these commutative subalgebras are not central. The only nontrivial quantum symmetric polynomial that is central in the braided Yangian $\mathbf{Y}(R)$ is the quantum determinant. More precisely, all coefficients of the expansion of the quantum determinant in a series in the spectral parameter are central. In the Yangian $\mathbf{Y}_{RTT}(R)$, the quantum determinant is central iff it is central in the corresponding RTT algebra. Hence, the centrality of the quantum determinant is completely determined by the properties of the initial twist R.

In Sec. 4, we give an example of an RTT algebra and consequently a Yangian $\mathbf{Y}_{RTT}(R)$ with a noncentral quantum determinant.

3. Yangian deformation properties and corresponding Poisson structures

All Yangians that we consider in this paper are introduced in terms of generators and relations between them. We must therefore verify the deformation properties of the obtained algebras assuming that the involutive or Hecke twist R is a deformation of the flip operator. Below, we present arguments supporting good deformation properties of generalized Yangians. Moreover, we write explicit formulas for their Poisson structure.

But we first compare the deformation properties of the QM algebras $\mathcal{T}(R)$ and $\mathcal{L}(R)$. For this, we introduce the notation

$$L_{\overline{1}} = L_1, \qquad L_{\overline{k}} = R_{k-1} L_{\overline{k-1}} R_{k-1}^{-1} \quad \forall k \geq 2.$$

For $h = 0$, relations (2.2) can be written in a form similar to the relations in an RTT algebra:

$$R L_{\overline{1}} L_{\overline{2}} = L_{\overline{1}} L_{\overline{2}} R.$$

This notation is also useful for establishing some isomorphisms between the linear spaces $\mathcal{T}(R)$ and $\mathcal{L}(R)$. We let

$$\mathbf{T} = \mathrm{span}_{\mathbb{K}}(t_i^j), \qquad \mathbf{L} = \mathrm{span}_{\mathbb{K}}(l_i^j)$$

denote the vector spaces spanned by the respective generators t_i^j of the RTT algebra $\mathcal{T}(R)$ and l_i^j of the RE algebra $\mathcal{L}(R)$ associated with the twist R. For any positive integer k, we consider the linear map $\pi_k \colon \mathbf{T}^{\otimes k} \to \mathbf{L}^{\otimes k}$ of the tensor powers of the spaces, defined on the basis elements by the rule

$$\pi_k(T_1 \otimes T_2 \otimes \ldots \otimes T_k) = L_{\overline{1}} \otimes L_{\overline{2}} \otimes \ldots \otimes L_{\overline{k}}, \quad k \geq 1.$$

Below, we omit the tensor product sign $\otimes$ to simplify the formulas.

Proposition 4. *The relations*

$$\pi_k(T_1 \ldots T_{i-1}(R_i T_i T_{i+1} - T_i T_{i+1} R_i) T_{i+1} \ldots T_k) = \\ = L_{\overline{1}} \cdots L_{\overline{i-1}} (R_i L_{\overline{i}} L_{\overline{i+1}} - L_{\overline{i}} L_{\overline{i+1}} R_i) L_{\overline{i+1}} \cdots L_{\overline{k}} \tag{3.1}$$

hold for all $k \geq 2$, $1 \leq i \leq k-1$. The size of all matrices in these relations is $m^k \times m^k$.

The homogeneous component $\mathcal{L}(R)^{(k)} \subset \mathcal{L}(R)$ of degree $k \geq 2$ is the quotient of the space $\mathbf{L}^{\otimes k}$ over the ideal generated by the right-hand side of (3.1) *for all* $1 \leq i \leq k-1$.

Because the maps π_k are invertible, this proposition allows us to state that any basis of the homogeneous component $\mathcal{T}(R)^{(k)} \subset \mathcal{T}(R)$ translates into a basis of the component $\mathcal{L}(R)^{(k)} \subset \mathcal{L}(R)$. This in turn allows deducing the deformation properties of the algebra $\mathcal{L}(R)$ from the properties of $\mathcal{T}(R)$.

This construction is close to the transmutation map introduced in [9] and also to the formulas in [6] used to define QM algebras associated with pairs of compatible twists.

Remark 1. If the initial symmetry R is involutive, then this property is also preserved for twists acting in the spaces $\mathbf{T}^{\otimes 2}$ and $\mathbf{L}^{\otimes 2}$:

$$T_1 \otimes T_2 \mapsto R^{-1} T_1 \otimes T_2 R, \qquad L_{\overline{1}} \otimes L_{\overline{2}} \mapsto R^{-1} L_{\overline{1}} \otimes L_{\overline{2}} R. \tag{3.2}$$

Based on this fact, we can easily construct some symmetrizers (projectors of symmetrization) in the spaces $\mathbf{T}^{(k)}$ and $\mathbf{L}^{(k)}$ for any $k \geq 2$. We note that the maps π_k are involved in this construction. It hence follows that the dimensions of the homogenous components $\mathcal{T}(R)^{(k)}$ and $\mathcal{L}(R)^{(k)}$ for $k \geq 2$ are equal to the dimensions of the corresponding components $\mathrm{Sym}(gl(m))^{(k)}$ of the commutative algebra $\mathrm{Sym}(gl(m))$. Correspondingly, the algebras $\mathcal{T}(R)$ and $\mathcal{L}(R)$ have the deformation property.

If the twist R is a Hecke symmetry, then braidings (3.2) acting in the spaces $\mathbf{T}^{\otimes 2}$ and $\mathbf{L}^{\otimes 2}$ have three eigenvalues and are not symmetries at all. Consequently, the method for proving the deformation property must be modified. In [3], symmetrizers in the components $\mathbf{T}^{(k)}$ and $\mathbf{L}^{(k)}$ were constructed for $k = 2, 3$. Using these symmetrizers, we can show that the dimensions of the homogeneous components $\mathcal{T}(R)^{(k)}$ and $\mathcal{L}(R)^{(k)}$ for $k = 2, 3$ are equal to the dimensions of the corresponding components of the algebra $\mathrm{Sym}(gl(m))$ if $q - 1$ is sufficiently small. Although the analogous symmetrizers in the higher components of these QM algebras are still not constructed, using the results in [10] (also see [11]) allows concluding that for a generic q, the dimensions of the higher components $\mathcal{T}(R)^{(k)}$ and $\mathcal{L}(R)^{(k)}$, $k \geq 4$, are equal to these in the algebra $\mathrm{Sym}(gl(m))$. This ensures the deformation property of the QM algebras related to Hecke symmetries.

We now turn to the Poisson structures corresponding to QM algebras. We use the notation $\{L_1, L_2\}$ for the $m^2 \times m^2$ matrix $L_1 L_2$, where each element $l_i^j \otimes l_k^l$ is replaced with $\{l_i^j, l_k^l\}$.

Let the twist R be a deformation of the flip P. Then the matrix $\mathcal{R} = PR$ is a deformation of the identity matrix:

$$\mathcal{R} = I - hr + O(h^2), \tag{3.3}$$

where h is the deformation parameter and r is the corresponding *classical r-matrix*, a solution of the classical Yang–Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

If R is a Hecke symmetry, then we additionally set $q = e^{-h}$.

If R is an involutive or a Hecke symmetry, then r satisfies the respective relation

$$r_{21} = -r_{12} \qquad \text{or} \qquad r_{12} + r_{21} = 2P. \tag{3.4}$$

In the latter case, we represent r as a sum $r = r_- + r_+$, where

$$r_- = \frac{r_{12} - r_{21}}{2}, \qquad r_+ = \frac{r_{12} + r_{21}}{2}$$

are the respective skew-symmetric and symmetric components of r. Then the second relation in (3.4) means that $r_+ = P$. This relation is well known for the standard Hecke symmetries, but it also holds for all other Hecke twists.

The Poisson structure corresponding to the RTT algebra is given by the Poisson bracket of the generators

$$\{T_1, T_2\} = rT_1T_2 - T_2T_1r = rT_1T_2 - T_1T_2r. \tag{3.5}$$

We note that we assume $T_1T_2 = T_2T_1$ in this relation (and similarly for L) because the Poisson bracket is defined in the commutative algebra $\mathrm{Sym}(gl(m))$.

If r arises from an involutive symmetry, then the right-hand side of (3.5) is obviously skew-symmetric. If r arises from a Hecke symmetry, then we have the same property taking $r_+ = P$ into account. Moreover, we can replace r in bracket (3.5) with r_-.

Representing a classical r-matrix r as an element of $gl(m)^{\otimes 2}$, we can write bracket (3.5) as

$$\{f, g\} = \cdot\,(\rho_l(r)^{\otimes 2}(f \otimes g) - \rho_r(r)^{\otimes 2}(g \otimes f)). \tag{3.6}$$

Here, ρ_l and ρ_r are the respective representations of $\mathfrak{g}$ by the left and right vector fields acting on the algebra $\mathrm{Sym}(gl(m))$. The symbol $\cdot$ denotes the commutative product in this algebra.

We now consider the Poisson bracket arising in the semiclassical limit of the RE algebra $\mathcal{L}(R)$ associated with an involutive or a Hecke symmetry R.

We first rewrite the defining relations of the RE algebra $\mathcal{L}(R)$ in a somewhat different form (we recall that $\mathcal{R} = PR$):

$$\mathcal{R}_{12}L_1\mathcal{R}_{21}L_2 - L_2\mathcal{R}_{12}L_1\mathcal{R}_{21} = 0.$$

Applying expansion (3.3) to the left-hand side of this relation, we obtain the Poisson bracket

$$\{L_1, L_2\} = r_{12}L_1L_2 - L_1L_2r_{21} + L_1r_{21}L_2 - L_2r_{12}L_1. \tag{3.7}$$

If r arises from an involutive symmetry, then this bracket can be written as

$$\{f, g\} = \cdot\, \rho_{\mathrm{ad}}(r)^{\otimes 2}(f \otimes g), \quad f, g \in \mathrm{Sym}(gl(m)). \tag{3.8}$$

Here, ρ_{ad} is the adjoint action of $gl(m)$ extended to the entire symmetric algebra $\mathrm{Sym}(gl(m))$ using the Leibniz rule. Hence, $\rho_{\mathrm{ad}}^{\otimes 2}(r)$ is a skew-symmetric bivector field.

If r arises from a Hecke symmetry, then the analogous formula for the bracket requires additional terms:

$$\{f, g\} = \cdot\, (\rho_{\mathrm{ad}}^{\otimes 2}(r_-) + (\rho_r \otimes \rho_l - \rho_l \otimes \rho_r)(r_+))(f \otimes g). \tag{3.9}$$

The right-hand side of bracket (3.9) is obviously skew-symmetric. But a direct verification of the Jacobi identity for it is rather tedious. Nevertheless, the Jacobi identity follows from the deformation property of the RE algebra.

Bracket (3.7) is compatible with the linear $gl(m)$ Poisson bracket. In other words, these two brackets form a Poisson pencil. This can be easily verified by the procedure for linearizing bracket (3.9) (see Sec. 4). We emphasize that the modified RE algebra $\mathcal{L}(R, h)$ is a quantum counterpart of this pencil.

Remark 2. Let $\mathfrak{g}$ be a simple Lie algebra belonging to one of the classical series B_m, C_m, or D_m, and let R be a twist associated with the corresponding quantum universal enveloping algebra $U_q(\mathfrak{g})$. In this case, the corresponding RTT and RE algebras are not deformations of the commutative algebra $\mathrm{Sym}(\mathfrak{g})$. Both (3.5) and (3.7) are Poisson brackets on the corresponding Lie group G and are respectively called Sklyanin and Semenov-Tian-Shansky brackets. In addition, they are also Poisson on certain varieties in $\mathfrak{g}^*$. Details and descriptions of these varieties can be found in [12], [13].

We now turn to the case of Yangians. Similarly to the construction considered above, we introduce the vector spaces

$$\mathbf{T} = \mathrm{span}_{\mathbb{K}}(t_i^j[k]) \qquad \text{and} \qquad \mathbf{L} = \mathrm{span}_{\mathbb{K}}(l_i^j[k])$$

containing all finite linear combinations of the generators $t_i^j[k]$ of an RTT-type Yangian or $l_i^j[k]$ of a braided Yangian respectively associated with a given R-matrix (2.9) or (2.10). We let $\mathbf{T}^{\otimes k}$ and $\mathbf{L}^{\otimes k}$ denote the subspaces containing all degree-k homogenous tensor polynomials in the corresponding generators.

We also define maps similar to maps (3.1):

$$\pi_k(T_1(u_1) \otimes T_2(u_2) \otimes \ldots \otimes T_k(u_k)) = L_{\overline{1}}(u_1) \otimes L_{\overline{2}}(u_2) \otimes \ldots \otimes L_{\overline{k}}(u_k), \tag{3.10}$$

where, just as above, we set

$$L_{\overline{1}}(u) = L_1(u), \qquad L_{\overline{k}}(u) = R_{k-1}L_{\overline{k-1}}(u)R_{k-1}^{-1}, \quad k \geq 2.$$

Here, the maps π_k are written in terms of the current matrices $T(u)$ and $L(u)$. Their expression in terms of the coefficients $T[s]$ and $L[s]$ has the same form.

A statement similar to Proposition 4 holds, i.e., maps (3.10) establish vector space isomorphisms between homogeneous components of the algebras $\mathbf{Y}_{RTT}(R)$ and $\mathbf{Y}(R)$. As a consequence, we have the following proposition.

Proposition 5. *If an RTT-type Yangian has the deformation property, then the corresponding braided Yangian has the same property.*

The method in [3] mentioned in Remark 1 allows proving the following proposition.

Proposition 6. *Any generalized Yangian is a deformation of the algebra* $\mathrm{Sym}(gl(m)[t^{-1}])$ *under the condition that the initial twist R is a deformation of the flip operator.*

We will present a detailed proof of this proposition in subsequent publications. Here, we restrict ourself to some supporting arguments. By virtue of Proposition 5, we can consider only RTT-type Yangians.

We first note that the map

$$\rho\colon T_1(u)\otimes T_2(v)\mapsto R(u,v)^{-1}T_1(v)\otimes T_2(u)R(u,v) \tag{3.11}$$

is involutive. Indeed, $R(u,v)R(v,u)=\varphi(u,v)I$, where $\varphi(u,v)$ is a rational function depending on the initial involutive or Hecke symmetry R.

Applying the map ρ twice, we obtain the identity map. Therefore, the map ρ has two eigenvalues: 1 or -1. The two-sided ideal in the definition of the Yangian $\mathbf{Y}_{RTT}(R)$ is generated by the subspace $I_-=\mathrm{Im}(I-\rho)$. We consider the complementary subspace $I_+=\mathrm{Im}(I+\rho)$. We can write the defining relations of this subspace in the form

$$R(u,v)T_1(u)\otimes T_2(v)=-T_1(u)\otimes T_2(v)R(u,v).$$

Expanding the generating matrix $T(u)$ in a series in inverse powers of u and using the relation

$$\frac{1}{u-v}=\sum_{k\geq 0}v^k u^{-k-1},$$

we obtain an explicit expression for the space I_+ in terms of the elements of the matrices $T[k]$. We note that this subspace is infinite-dimensional and is generated by all possible finite quadratic polynomials in a countable set of variables $t_i^j[r]$.

We consider the subspaces $I_\pm(h)$ as a function of the deformation parameter h. At $h=0$, the subspaces $I_\pm(0)\subset\mathbf{T}^{\otimes 2}$ are generated by the elements

$$t_j^c[r]t_i^d[s]\pm t_i^d[s]t_j^c[r]. \tag{3.12}$$

The generating function of these elements has the matrix form

$$T_1(u)T_2(v)\pm T_2(v)T_1(u).$$

The subspace $I_+(h)$ is generated by the elements $E_{ij}^{cd}[k,l]$, which are coefficients of $u^{-k}v^{-l}$ in the generating series

$$E_{ij}^{cd}(u,v)=R_{ij}^{ab}(u,v)T_a^c(u)T_b^d(v)+T_i^a(v)T_j^b(u)R_{ab}^{cd}(u,v). \tag{3.13}$$

We note that at $h = 0$, the elements $E_{ij}^{cd}[k, l]$ coincide with symmetric combinations of generators (3.12). The subset of the elements $E_{ij}^{cd}[k, l]$ such that the triple (k, i, c) precedes (l, j, d) in lexicographic order forms a basis of the space $I_+(0)$. Moreover, the linear span of the elements $E_{ij}^{cd}[k, l]$ such that $k + l \leq p$ is finite-dimensional.

Consequently, the elements $E_{ij}^{cd}[k, l]$ such that $k + l \leq p$ are independent in the Yangian $\mathbf{Y}_{RTT}(R)$ if h is sufficiently small. Therefore, mapping the elements $E_{ij}^{cd}[k, l] \in I_+(0)$ to their analogues in $I_+(h)$, we construct the map α_h (see Sec. 1) at the quadratic level. By an analogous method, we can construct the maps α_h for higher homogenous components of the Yangian $\mathbf{Y}_{RTT}(R)$ and thus show that this Yangian has the deformation property in the case where the twist R is a deformation of the flip P.

We turn to the Poisson structures corresponding to the generalized Yangians. We begin with the expansion $\mathcal{R}(u, v) = PR(u, v) = I - hr(u, v) + O(h^2)$ in a series in h, where $h = -a$ in formula (2.9) and $h = -\log q$ in formula (2.10). This yields the expressions for the classical current r-matrices

$$r(u, v) = r - \frac{1}{u - v}P, \qquad r(u, v) = r - \frac{2u}{u - v}P. \tag{3.14}$$

Here, the constant matrix r in the first formula corresponds to an involutive symmetry R and is skew-symmetric: $r_+ = 0$. The matrix r in the second formula arises from a Hecke symmetry R and consequently $r_+ = P$.

We emphasize that both current matrices $r(u, v)$ are skew-symmetric. We verify this for the second matrix in (3.14):

$$r_{12}(u, v) + r_{21}(v, u) = r_{12} + r_{21} - \frac{2u}{u - v}P - \frac{2v}{v - u}P = 2r_+ - 2P = 0.$$

Hereafter, the symbol $r_{21}(u, v)$ means that we interchange the elements from $gl(m)^{\otimes 2}$ without interchanging u and v.

The brackets corresponding to the Yangians $\mathbf{Y}(R)$ and $\mathbf{Y}_{RTT}(R)$ are respectively similar to (3.5) and (3.7):

$$\{T_1(u), T_2(v)\} = r(u, v)T_1(u)T_2(v) - T_1(u)T_2(v)r(u, v), \tag{3.15}$$

$$\begin{aligned}\{L_1(u), L_2(v)\} = r_{12}(u, v)L_1(u)L_2(v) + L_1(u)r_{21}L_2(v) - \\ - L_2(v)r_{12}L_1(u) - L_1(u)L_2(v)r_{21}(u, v).\end{aligned} \tag{3.16}$$

We note that the r-matrices between the generating matrices $L_1(u)$ and $L_2(v)$ in (3.16) are constant, while the external matrices are defined by one of formulas (3.14). And, finally, it is taken into account in formulas (3.15) and (3.16) that the Poisson structures are defined in the commutative algebra where the relations

$$T_1(u)T_2(v) = T_2(v)T_1(u), \qquad L_1(u)L_2(v) = L_2(v)L_1(u)$$

are satisfied.

Remark 3. We emphasize that the analogous brackets associated with Lie algebras of the series B_m, C_m, and D_m are Poisson only on the corresponding groups.

All the brackets we consider in this paper are local, i.e., they have no singularity in the limit as $u - v \to 0$. In other words, if we set $u - v = \nu$ and expand $L(u) = L(v + \nu)$ in a series in ν, then we can replace the Poisson brackets presented above with a countable set of brackets involving the generating matrices and their derivatives taken with one value of the parameter (i.e., at the point $u = v$). The quantum variant of this procedure was presented in [1].

4. Linear Poisson brackets: Examples

The linear Poisson brackets arising from the brackets presented in the preceding section by the linearization procedure are of interest. Linearizing, we obtain the linear counterpart of bracket (3.15) in the trigonometric case:

$$\{T_1(u), T_2(v)\} = r(u,v)(T_1(u) + T_2(v)) - (T_2(v) + T_1(u))r(u,v) = \\ = \left[r - \frac{2u}{u-v}P,\ T_1(u) + T_2(v)\right]. \tag{4.1}$$

In the rational case, we have

$$\{T_1(u), T_2(v)\} = \left[r - \frac{1}{u-v}P,\ T_1(u) + T_2(v)\right]. \tag{4.2}$$

Linearizing bracket (3.16) in the trigonometric case leads to the bracket

$$\{L_1(u), L_2(v)\} = -\left[2P, \frac{uL_1(u) - vL_1(v)}{u-v}\right]. \tag{4.3}$$

And, finally, in the rational case, we obtain the expression

$$\{L_1(u), L_2(v)\} = -\left[P, \frac{L_1(u) - L_1(v)}{u-v}\right]. \tag{4.4}$$

Bracket (4.3) does not contain the component r_- because the first term in bracket (3.9) vanishes under the linearization procedure. For the same reason, the constant matrix r does not appear in expression (4.4). Hence, these formulas are independent of r in contrast to brackets (4.1) and (4.2).

We note that brackets (4.2) and (4.4) coincide if we set $r = 0$ in (4.2), while brackets (4.1) and (4.3) differ substantially.

The method for constructing the above linear brackets directly implies that they are compatible with the corresponding quadratic brackets, i.e., this pair (linear and quadratic brackets) generates a Poisson pencil. The quantum algebra corresponding to this pencil can be obtained by linearizing the corresponding Yangian.

We now consider two examples. We first introduce the symmetries:

$$\widehat{R} = \begin{pmatrix} 1 & a & -a & ab \\ 0 & 0 & 1 & -b \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{4.5}$$

The first symmetry is involutive for any values of the numerical parameters $a, b \in \mathbb{K}$. We note that it was constructed in [14].

The second symmetry is a Hecke symmetry and arises from the quantum group $U_q(sl(2))$. Higher-dimensional analogues of this Hecke symmetry arise from the quantum groups $U_q(sl(m))$, $m > 2$. For all these Hecke symmetries, the quantum determinants in the corresponding RTT algebras are central.

For the RTT algebra constructed in accordance with the first symmetry in (4.5), the quantum determinant turns out to be central iff $a = b$. Consequently, the necessary and sufficient condition for the quantum determinant in the corresponding RTT-type Yangian to be central is also given by the equality $a = b$.

The current R-matrices corresponding to constant matrices (4.5) have the forms

$$\widehat{R}(u,v) = \widehat{R} - \frac{1}{u-v}I = \begin{pmatrix} 1-\dfrac{1}{u-v} & a & -a & ab \\ 0 & -\dfrac{1}{u-v} & 1 & -b \\ 0 & 1 & -\dfrac{1}{u-v} & b \\ 0 & 0 & 0 & 1-\dfrac{1}{u-v} \end{pmatrix}, \tag{4.6}$$

$$R(u,v) = R - \frac{(q-q^{-1})u}{u-v}I =$$

$$= \begin{pmatrix} \dfrac{-qv+q^{-1}u}{u-v} & 0 & 0 & 0 \\ 0 & \dfrac{(-q+q^{-1})v}{u-v} & 1 & 0 \\ 0 & 1 & \dfrac{(-q+q^{-1})u}{u-v} & 0 \\ 0 & 0 & 0 & \dfrac{-qv+q^{-1}u}{u-v} \end{pmatrix}. \tag{4.7}$$

The RTT-type Yangian associated with symmetry (4.7) is an example of the abovementioned q-Yangians. The structure of a basis of the Poincaré–Birkhoff–Witt type can be extracted from [8]. For this, we must consider "halves" of the quantum algebra $U_q(\widehat{gl(m)})$ (see [8]).

We now obtain the corresponding classical r-matrices. For this, we set $a = -h\alpha$ and $b = -h\beta$ in the matrix $\widehat{R}$ in (4.5) and correspondingly $q = e^{-h}$ in the matrix R. Expanding the R-matrices $\mathcal{R} = PR$ in a series in h, we obtain the classical r-matrices in the first-order expansion:

$$\begin{aligned} \hat{r} &= \alpha(e_{11}\otimes e_{12} - e_{12}\otimes e_{11}) + \beta(e_{12}\otimes e_{22} - e_{22}\otimes e_{12}), \\ r &= e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + 2\, e_{21}\otimes e_{12}, \end{aligned} \tag{4.8}$$

where e_{ij} are the elements of the standard basis of the Lie algebra $gl(2)$.

The r-matrix $\hat{r}$ in (4.8) is skew-symmetric. If $\alpha = \beta$ (i.e., $a = b$ in the matrix $\widehat{R}$ in (4.5)), then it is in the exterior power $sl(2)^{\wedge 2}$. Otherwise, $\hat{r}$ is an element of $gl(2)^{\wedge 2}$. The second r-matrix is not skew-symmetric; its skew-symmetric component has the form $r_- = e_{21}\otimes e_{12} - e_{12}\otimes e_{21}$.

The classical r-matrices corresponding to symmetries (4.6) and (4.7) have the respective forms

$$\hat{r}(u,v) = \hat{r} - \frac{1}{u-v}P, \qquad r(u,v) = r - \frac{2u}{u-v}P. \tag{4.9}$$

Finally, we write explicit formulas for the linear Poisson brackets corresponding to RTT-type Yangians associated with the involutive symmetry $\widehat{R}$ and to braided Yangians associated with the Hecke symmetry R. We let

$$T(u) = \begin{pmatrix} a(u) & b(u) \\ c(u) & d(u) \end{pmatrix}$$

denote the elements of the matrix $T(u)$. From (4.2), we then obtain

$$\{a(u),a(v)\}=\alpha(c(v)-c(u)),\qquad \{a(u),b(v)\}=\alpha(d(v)-a(v))+\frac{b(u)-b(v)}{u-v},$$

$$\{a(u),c(v)\}=\frac{c(v)-c(u)}{u-v},\qquad \{a(u),d(v)\}=\beta c(u)-\alpha c(v),$$

$$\{b(u),b(v)\}=(\alpha+\beta)(b(u)-b(v)),$$

$$\{b(u),c(v)\}=(\alpha+\beta)c(v)+\frac{a(u)-a(v)+d(v)-d(u)}{u-v},$$

$$\{b(u),d(v)\}=\beta(d(u)-a(u))+\frac{b(u)-b(v)}{u-v},$$

$$\{c(u),c(v)\}=0,\qquad \{c(u),d(v)\}=\frac{c(v)-c(u)}{u-v},$$

$$\{d(u),d(v)\}=\beta(c(v)-c(u)).$$

Using the same notation for the elements of the generating matrix

$$L(u)=\begin{pmatrix}a(u) & b(u)\\ c(u) & d(u)\end{pmatrix}$$

of the braided Yangian $\mathbf{Y}(R)$, from (4.3), we obtain the expressions for the linear brackets

$$\{a(u),a(v)\}=0,\qquad \{a(u),b(v)\}=\frac{2}{u-v}(ub(u)-vb(v)),$$

$$\{a(u),c(v)\}=\frac{2}{u-v}(vc(v)-uc(u)),\qquad \{a(u),d(v)\}=0,$$

$$\{b(u),b(v)\}=0,\qquad \{b(u),c(v)\}=\frac{2}{u-v}(ua(u)-va(v)+vd(v)-ud(u)),$$

$$\{b(u),d(v)\}=\frac{2}{u-v}(ub(u)-vb(v)),$$

$$\{c(u),c(v)\}=0,\qquad \{c(u),d(v)\}=\frac{2}{u-v}(vc(v)-uc(u)),$$

$$\{d(u),d(v)\}=0.$$

Poisson structure (4.4) in the case of the rational current R-matrix $\widehat{R}(u,v)$ is easily obtained from the above formulas for the brackets between the elements of $T(u)$ by simply fixing the parameter values $\alpha=0$ and $\beta=0$.

Remark 4. The construction of the affine quantum algebra $U_q(\widehat{gl(m)})$ presented in [8] is easily generalized to other current R-matrices. This construction is based on a choice of some permutation relations between two Yangian-type algebras. A similar method can be used to construct a "braided version" of this algebra taking two braided Yangians and writing the corresponding permutation relations for them. The question of the structure of the center of such an algebra remains open.

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