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Game equilibria and unification dynamics in networks with heterogeneous agents

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Abstract
We study game equilibria in a model of production and externalities in network with two types of agents who possess different productivities. Each agent may invest a part of her endowment (it may be, for instance, time or money) in the first of two time periods; consumption in the second period depends on her own investment and productivity as well as on the investments of her neighbors in the network. Three ways of agent's behavior are possible: passive (no investment), active (a part of endowment is invested), and hyperactive (the whole endowment is invested). For star network with different productivities of agents in the center and in the periphery, we obtain conditions for existence of inner equilibrium (with all active agents) and study comparative statics. We introduce adjustment dynamics and study consequences of junction of two complete networks with different productivities of agents. In particular, we study how the behavior of nonadopters (passive agents) changes when they connect to adopters (active or hyperactive) agents.

Keywords
Network, game equilibrium, heterogeneous agents, network formation, productivity

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Introduction

Social network analysis became an important research field, both as a subject area and as a methodological approach applicable to analysis of interrelations in various complex network structures, not only social but also political, economic, and urban. There is also a permanent exchange of ideas among researchers doing network analysis in social sciences, biology, physics, computer science, engineering, and many other fields.[1] A special place in this multidisciplinary research activity is played by methods of network economics and network games.[2–7] Economic models assume that agents/actors in network act as rational decision makers whose actions are results of solving optimization problems, and the profile of actions of all agents in the network is a game equilibrium. Decision of each agent is supposed to be influenced by behavior (or by knowledge) of her neighbors in the network. Such approach, despite being, in some sense, one sided, is found to be very productive analytically and allows finding new perspectives which can be further elaborated by use of other approaches in the multidisciplinary framework.

In majority of research on game equilibria in networks,[2,8–10] the agents are assumed to be homogeneous (except their positions in the network), and the problem is to study the relation between the agents' positions in the network and their behavior in the game equilibrium. The models demonstrate that the agents' behavior and well-being depend on their position in the network which is characterized by one or another measure of centrality. Thus, research on equilibria is in a close connection with research focusing on the network structure. For instance, Cinelli et al.,[11] Ferraro and Iovanella,[12] and Ferraro et al.[13] consider two types of nodes in a network. Different approaches to describe interrelations in networks are used,

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for example, intensiveness of interplay between nodes in the study by Ferraro et al.[13] is given exogenously, while in the study by Matveenko and Korolev,[9] it depends endogenously on the agent's behavior in network, since in the latter work agents' interplay depends on the externalities created by investments of neighbors in the network.

Diversity and heterogeneity have become an important aspect of contemporary social and economic life (international working teams is a typical example; many other examples are described by researchers of inclusiveness and social cohesion[14]). Correspondingly, along with accounting for position of agents in the network, an important task is to account for heterogeneity of agents as a factor defining differences in their behavior and well-being. This direction of research is forming in the literature. For example, in the study by Bramoullé et al.,[15] agents possess different marginal costs.

In the present article, we add heterogeneity of agents into the Romer's[16] two-period consumption-investment model (where a special case of complete network is considered) and the more general model by Matveenko and Korolev.[9] These models consider situations in which in time period 1, each agent in network, at the expense of diminishing current consumption, makes investment of some resource (such as money or time) with the goal to increase her consumption in period 2. The latter depends not only on her own investment and productivity but also on investments by her neighbors in the network. Total utility of each agent depends on her consumption in the two time periods. Such situations are typical for families, communities, international organizations, innovative industries, and so on. In the framework of the model, questions concerning interrelations between the network structure, incentives, and behavior are studied.

We use the concept of "Nash equilibrium with externalities" similar to the one used by Romer[16] and Lucas.[17] As in the common Nash equilibrium, agents maximize their payoffs (utilities), and in equilibrium, no one agent finds gainful to change her behavior if others do not change their behaviors. However, the agent's maximization problem under the present concept is such that the agent is not able to change her behavior so "free" as it is allowed by the common Nash equilibrium concept. In some degree, the agent is attached to the equilibrium of the game. Namely, it is assumed that the agent makes her decision being in a definite environment formed by herself and by her neighbors in the network. Though she participates herself in formation of the environment, the agent in the moment of decision-making considers the environment as exogenously given.

Romer's[16] and Matveenko and Korolev[9] consider only the case of homogeneous (in their preferences and productivities) agents. In the present article, we assume that there are two types of agents with different productivities.

We find conditions under which agent behaves in equilibrium in some definite way, being "passive" (does not invest), "active" (invests a part of the available endowment), or "hyperactive" (invests the whole endowment). We prove that the agent's utility depends monotonously on her environment and study dependence of the investment on the externality received by the agent. For complete networks, we prove the uniqueness of the inner equilibrium (in which all agents are active).

We study the influence of the nonhomogeneity on the game equilibria. For instance, we show that if in star network the central node (type 1) has a different productivity than the peripheral agents (type 2), an increase of productivity of any of these types always leads to decrease of investment by *another* type in equilibrium. However, influence of changes in productivity on *own* investment is ambiguous and depends on the counterpart's productivity. If productivity of type i is sufficiently low (less than a threshold value), then an increase in productivity of type j $(j \neq i)$ leads to decrease of type j's investment in equilibrium. But if productivity of type i is higher than the threshold, then increase in productivity of type j leads to increase of investment of j. The abovementioned threshold for the productivity of peripheral node decreases with respect to the number of the nodes. Thus, if initially the center's investment *increases* with respect to the center's productivity, but then the number of peripheral nodes rises (in different contexts, it can be because of appearance of new members of a collective, new divisions of a firm, new districts of an agglomeration, etc.), then the rise of the center's productivity may lead already to *decrease* in its investment.

Another question studied in the article is consequences of unification of networks with different types of agents. We study junction of complete networks and find conditions under which the initial equilibrium holds after unification, as well as conditions under which the equilibrium changes. In particular, we study how the behavior of non-adopters (passive agents) changes when they connect to adopters (active or hyperactive) agents.

We introduce adjustment dynamics into the model and study dynamics of transition to the new equilibrium. The dynamics pattern and the nature of the resulting equilibrium depend on the parameters characterizing the heterogeneous agents.

For instance, if complete network 1 with initially active agents of type 1 unifies with complete network 2 with initially passive agents of type 2, and the type 1 productivity is higher than the type 2 productivity by at least a certain threshold value (which we show to be inversely proportional to the number of the first type agents), then there is no transition process: The network stays in the (dynamically unstable) equilibrium in which the first type agents are active and the second type agents are passive. In the opposite case, a transition process starts in the unified network. If productivity of type 2 is higher than the abovementioned threshold but still rather low, then the transition process leads to the stable equilibrium in which the first type agents are hyperactive and the second type agents are active, that is, all agents increase their investment levels.

Under a higher productivity of type 2, the transition process leads to a stable equilibrium in which agents of both types are hyperactive.

We show that all these results are in power not only for complete networks but also for a wider class of "cognate" regular (equidegree) networks.

Such kind of results can be useful in analysis of functioning of real social, organizational, economic, and political structures.

The article is organized in the following way. The model is formulated in "The model" section. Agent's behavior in equilibrium is studied in "Indication of agent's ways of behavior" section. "Equilibria in complete network with two types of agents" section studies equilibria in complete network with heterogeneous agents. "Adjustment dynamics and dynamic stability of equilibria" section introduces and studies the adjustment dynamics which may start after a small disturbance of initial inner equilibrium or after a junction of networks. "Junction of two complete networks" section studies consequences of junction of two complete networks with different types of agents. "Equilibria in star network with heterogeneous agents" section considers equilibria in star network with heterogeneous agents. The final section is conclusion.

The model

There is a network (undirected graph) with n nodes, $i = 1, 2, \dots, n$; each node represents an agent. In period 1, each agent i possesses initial endowment of good, e, (it may be, for instance, time or money) and uses it partially for consumption in the first period, c_1^i, and partially for investment into knowledge, k_i:

$$c_1^i + k_i = e, \quad i = 1, 2, \dots, n$$

Investment immediately transforms one to one into knowledge which is used in production of good for consumption in the second period, c_2^i.

Preferences of agent i are described by quadratic utility function

$$U_i(c_1^i, c_2^i) = c_1^i(e - ac_1^i) + d_i c_2^i$$

where $d_i > 0$ and a is a satiation coefficient. It is assumed that $c_1^i \in [0, e]$, the utility increases in c_1^i, and is concave (the marginal utility decreases) with respect to c_1^i. A sufficient condition leading to the assumed property of the utility is $0 < a < 1/2$. We assume that this inequality is satisfied.

Production in node i is described by production function

$$F(k_i, K_i) = g_i k_i K_i, \quad g_i > 0$$

which depends on the state of knowledge in ith node, k_i, and on *environment*, K_i. The environment is the sum of investments by the agent himself and her neighbors

$$K_i = k_i + \tilde{K}_i, \quad \tilde{K}_i = \sum\nolimits_{j \in N(i)} k_j$$

where $N(i)$ is the set of neighboring nodes of node i. The sum of investments of neighbors, $\tilde{K}_i$, will be referred as *pure externality*.

We will denote the product $d_i g_i$ by b_i and assume that $a < b_i$. Since increase of any of parameters d_i, g_i promotes increase of the second period consumption, we will call b_i "productivity." We will assume that $b_i \neq 2a, i = 1, 2, \dots, n$. If $b_i > 2a$, we will say that ith agent is *productive*, and if $b_i < 2a$, we will say that ith agent is *unproductive.*

Three ways of behavior are possible: agent i is called *passive* if she makes zero investment, $k_i = 0$ (i.e. consumes the whole endowment in period 1); *active* if $0 < k_i < e$; *hyperactive* if she makes maximally possible investment e (i.e. consumes nothing in period 1).

Let us consider the following game. Players are the agents $i = 1, 2, \dots, n$. Possible actions (strategies) of player i are values of investment k_i from the segment [0, e]. *Nash equilibrium with externalities* (for shortness, *equilibrium*) is a profile of knowledge levels (investments; $k_1^*, k_2^*, \dots, k_n^*$), such that each k_i^* is a solution of the following problem $P(K_i)$ of maximization of ith player's utility given environment K_i:

$$U_i(c_1^i, c_2^i) \underset{c_1^i, c_2^i, k^i}{\longrightarrow} \max$$

$$\begin{cases} c_1^i = e - k^i, \\ c_2^i = F(k^i, K^i), \\ c_1^i \geq 0, c_2^i \geq 0, k^i \geq 0, \end{cases}$$

where the environment K_i is defined by the profile $(k_1^*, k_2^*, \dots, k_n^*)$

$$K_i = k_i^* + \sum\nolimits_{j \in N(i)} k_j^*.$$

Substituting the constrains equalities into the objective function, we obtain a new function (*payoff function*)

$$V_i(k_i, K_i) = U_i\Big(e - k_i, F_i(k_i, K_i)\Big) = (e - k_i)\Big(e - a(e - K_i)\Big) + b_i k_i K_i = e^2(1 - a) - k_i e(1 - 2a) - ak_i^2 + b_i k_i K_i \quad (1)$$

If all players' solutions are internal $(0 < k_i^* < e, i = 1, 2, \dots, n)$, that is, all players are active, the equilibrium will be referred to as *inner* equilibrium. Clearly, the inner equilibrium (if it exists for given values of parameters) is defined by the system

$$D_1 V_i(k_i, K_i) = 0 \quad i = 1, 2, \dots, n. \quad (2)$$

Here

$$D_1 V_i(k_i, K_i) = e(2a - 1) - 2ak_i + b_i K_i \quad (3)$$

We will use the following notation: I is the unit $n \times n$ matrix; $\tilde{b}$ is the diagonal matrix with $b_1, b_2, \dots, b_n$ on the main diagonal; M is the adjacency matrix of the network (in the adjacency matrix, $M_{ij} = M_{ji} = 1$, if in the network there is a link between nodes i and j, and $M_{ij} = M_{ji} = 0$ otherwise; $M_{ii} = 0$ for all $i = 1, 2, \ldots, n$).

Remark 1.1. System of equation (2) takes the form

$$Sk = \bar{e} \tag{4}$$

where

$$S = \tilde{b} - 2aI + \tilde{b}M$$

$$k = (k_1, k_2, \ldots, k_n)^T$$

$$\bar{e} = e(1 - 2a)\mathbf{1}$$

Here, I is the identity matrix such that $\mathbf{1} = (1, 1, \ldots, 1)^T$.

Remark 1.2. Since the matrix $\tilde{b} - 2aI$ is nonsingular, we can multiply both parts of equation (4) by diagonal matrix $(\tilde{b} - 2aI)^{-1}$.

$$[I + (\tilde{b} - 2aI)^{-1}\tilde{b}M]k = (1 - 2a)e(\tilde{b} - 2aI)^{-1}\mathbf{1}$$

Let us introduce a diagonal matrix

$$\alpha = (\tilde{b} - 2aI)^{-1}\tilde{b}$$

and a vector

$$\tilde{e} = (1 - 2a)e(\tilde{b} - 2aI)^{-1}\mathbf{1}$$

If matrix S is nonsingular, then the unique solution of the system (4) takes the form

$$k^* = (I + \alpha M)^{-1}\tilde{e}$$

Thus, the equilibrium investments by the agents are defined by their generalized α-centralities in the network. Instead of one α-parameter (as in the common definition of α-centrality), we have here a diagonal matrix α, that is, the heterogeneous agents are characterized by parameters α_i depending on their productivities, b_i. Notice that two components of centrality (the agent's position in the network and her exogenous "importance") influence the equilibrium investment level in the opposite directions.

Theorem 1.1. For complete network, the inner equilibrium exists and unique.

Proof. We shall proof that complete network system of equation (4) has a unique solution. It is sufficient to check the nonsingularity of the matrix

$$\tilde{S} = \tilde{b}^{-1}S = I - 2a\tilde{b}^{-1} + M = \begin{pmatrix} \beta_1 & 1 & \ldots & 1 \\ 1 & \beta_2 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & \beta_n \end{pmatrix}$$

Here, $\beta_i = 1 - 2ab_i^{-1} = b_i^{-1}(b_i - 2a)$. Since $b_i > a, i = 1, 2, \ldots, n$, the diagonal elements satisfy condition $0 < |\beta_i| < 1, i = 1, 2, \ldots, n$.

We will prove by induction that for complete network of order n, the determinant of matrix $\tilde{S}$ is negative when n is even and positive when n is odd. For $n = 2$, the determinant of matrix $\tilde{S}$ is negative

$$\Delta_2 = \begin{vmatrix} \beta_1 & 1 \\ 1 & \beta_2 \end{vmatrix} = \beta_1\beta_2 - 1 < 0$$

For $n = 3$, the determinant is positive

$$\Delta_3 = \begin{vmatrix} \beta_1 & 1 & 1 \\ 1 & \beta_2 & 1 \\ 1 & 1 & \beta_3 \end{vmatrix} = (1 - \beta_1)(1 - \beta_2) + (\beta_3 - 1)\Delta_2 > 0$$

Suppose that we have proven the statement for all complete networks of order not higher than n. Let us prove it for the complete network of order $n + 1$. We have $\Delta_n < 0$ when n is even and $\Delta_n > 0$ when n is odd. The determinant of order $n + 1$ is

$$\Delta_{n+1} = \begin{vmatrix} \beta_1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & \beta_2 & 1 & 1 & \ldots & 1 \\ 1 & 1 & \beta_3 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \beta_4 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & 1 & \ldots & \beta_{n+1} \end{vmatrix}$$

Subtracting the first column from the second column, the second column from the third column, ..., the n th column from the $(n + 1)$ th column, we obtain

$$\Delta_{n+1} = \begin{vmatrix} \beta_1 & 1-\beta_1 & 0 & \ldots & 0 & 0 \\ 1 & \beta_2 - 1 & 1 - \beta_2 & \ldots & 0 & 0 \\ 1 & 0 & \beta_3 - 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & \ldots & \beta_n - 1 & 1 - \beta_n \\ 1 & 0 & 0 & \ldots & 0 & \beta_{n+1} - 1 \end{vmatrix}$$
$$= (-1)^{n+2}(1 - \beta_1)(1 - \beta_2)\cdots(1 - \beta_n) + (\beta_{n+1} - 1)\Delta_n \tag{5}$$

When n is even, then both additives in equation (5) are positive, and when n is odd, they are both negative.

We have proven that for any n, the determinant of matrix $\tilde{S}$ and, hence, the determinant of matrix S are nonzero.

In the inner equilibrium, $k_i^* = k_i^s, i = 1, 2, \ldots, n$.

Remark 1.3. Notice that in general case (for incomplete networks), the theorem 1.1 is not true. As a counterexample, let us consider the chain of three nodes with matrix $\tilde{S}$ of the form

$$\tilde{S} = \begin{pmatrix} \beta_1 & 0 & 1 \\ 0 & \beta_2 & 1 \\ 1 & 1 & \beta_3 \end{pmatrix}$$

The determinant of this matrix

$$\beta_1\beta_2\beta_3 - \beta_1 - \beta_2$$

becomes 0 under

$$\beta_3 = \frac{\beta_1 + \beta_2}{\beta_1\beta_2}$$

which is possible under $0 < |\beta_i| < 1, i = 1, 2, 3$, for example, if $\beta_1 = 0.5$, $\beta_2 = -0.8$, and $\beta_3 = 0.75$. In this case, system (4) cannot have unique solution. Generally, in case of incomplete network, the solution of system (4) does not exist in all the cases when the vector $\tilde{b}^{-1}\bar{e}$ is not a linear combination of columns of matrix $\tilde{S}$.

The following theorem will serve as a tool for comparison of utilities.

Theorem 1.2. Let W, W** be networks with the same characteristic endowment e; i,j be, correspondingly, two of their nodes; b_i, b_j be productivities of the agents in these nodes; k_j^*, K_j^*, U_j^* and $k_j^{**}, K_j^{**}, U_j^{**}$ be equilibrium values of knowledge, environment, and utilities in these two nodes; and $k_i^* \in (0, e], k_j^{**} \in (0, e]$. In such case*

1) If $b_iK_i^* < b_jK_j^{**}$, then $U_i^* < U_j^{**}$.
2) If $b_iK_i^* \leq b_jK_j^{**}$, then $U_i^* \leq U_j^{**}$.
3) If $b_iK_i^* = b_jK_j^{**}$, then $U_i^* = U_j^{**}$.
4) If $k_i^* = 0, k_j^{**} > 0$, then $U_i^* = U(e, 0) < U_j^{**}$.

Proof. Let $b_iK_i^* < b_jK_j^{**}(b_iK_i^* \leq b_jK_j^{**})$. Since function $V_j(k_j, K_j^{**})$ achieves maximum at point k_j^{**}, we have $V_j(k_i^*, K_j^{**}) \leq V_j(k_j^{**}, K_j^{**})$. Since $D_2V(k, bK) > 0$ for any $k \neq 0$ and K, we obtain $V_i(k_i^*, K_i^*) < V_j(k_i^*, K_j^{**})$ (correspondingly, $V_i(k_i^*, K_i^*) \leq V_j(k_i^*, K_j^{**})$). It follows that $V_i(k_i^*, K_i^*) < V_j(k_i^*, K_j^{**}) \leq V_j(k_j^{**}, K_j^{**})$ (correspondingly, $V_i(k_i^*, K_i^*) \leq V_j(k_i^*, K_j^{**}) \leq V_j(k_j^{**}, K_j^{**})$). Hence, $U_i^* < U_j^{**}$ (correspondingly, $U_i^* \leq U_j^{**}$). Combining previous results, we see that if $b_iK_i^* = b_jK_j^{**}$, then $U_i^* = U_j^{**}$.

The last statement of the theorem is evident

$$U_j^{**} = V_j(k_j^{**}, K_j^{**}) > V_j(0, K_j^{**}) = V_i(0, K_i^*) = U_i^*$$

because for $k = 0$, function $V(0, K)$ does not depend on bK. □

Indication of agent's ways of behavior

The following statement plays the central role in analysis of equilibria.

Lemma 2.1. For unproductive agent, necessary and sufficient conditions of different ways of behavior in equilibrium are as follows:

1) Agent is passive iff

$$\tilde{K}_i \leq \frac{e(1-2a)}{b_i} \tag{6}$$

2) Agent is active iff

$$\frac{e(1-2a)}{b_i} < \tilde{K}_i < \frac{e(1-b_i)}{b_i} \tag{7}$$

3) Agent is hyperactive iff

$$\tilde{K}_i \geq \frac{e(1-b_i)}{b_i} \tag{8}$$

For productive agent, necessary conditions of different ways of behavior in equilibrium are as follows:

1) Agent may be passive only if

$$\tilde{K}_i \leq \frac{e(1-2a)}{b_i} \tag{9}$$

2) Agent may be active only if

$$\frac{e(1-b_i)}{b_i} < \tilde{K}_i < \frac{e(1-2a)}{b_i} \tag{10}$$

3) Agent may be hyperactive only if

$$\tilde{K}_i \geq \frac{e(1-b_i)}{b_i} \tag{11}$$

Proof. Equilibrium condition for passive agent is $D_1V(k_i, b_iK_i) \leq 0$ at point $k = 0$; hence, equation (3) implies

$$e(2a-1) - 2ak_i + b_iK_i|_{k_i=0} = b_iK_i - e(1-2a) \leq 0$$

which is equivalent to equations (6) and (9).

The agent is active iff

$$0 < \frac{e(2a-1) + b_i\tilde{K}_i}{2a - b_i} < e$$

Writing this relation in detail, we obtain equation (7) if agent is unproductive or equation (10) if agent is productive.

Equilibrium condition for hyperactive agent is $D_1V_i(k_i, K_i) \leq 0$ at point $k = e$; hence, equation (3) implies

$$e(2a-1) - 2ak_i + b_iK_i|_{k_i=e} = b_i\tilde{K}_i + b_ie - e \geq 0$$

which is equivalent to equations (8) and (11). □

Remark 2.1. Pure externality can be interpreted as social influence. Lemma 2.1 shows the values of social thresholds for agent i; achievement of a threshold is needed for a change in *behavior. The lemma implies that to turn from passive into active (or from active into hyperactive), an agent with a lower productivity needs a bigger social influence externality than an agent with a higher productivity.* We will denote by $\tilde{k}_i^s$ the root of the equation

$$D_1V_i(k_i, K_i) = (b_i - 2a)k_i + b_i\tilde{K}_i - e(1 - 2a) = 0$$

Thus

$$\tilde{k}_i^s = \frac{e(2a - 1) + b_i\tilde{K}_i}{2a - b_i}$$

where $\tilde{K}_i$ is the pure externality received by the agent. Evidently, if in equilibrium the agent is active, her investment is equal to $\tilde{k}_i^s$. In other cases, this value has only "informative" role. It is seen from the following corollary which describes the agents' ways of behavior in terms of $\tilde{k}_i^s$.

Corollary 2.1. For unproductive agent, necessary and sufficient conditions of different ways of behavior are as follows:

1) Agent is passive iff

$$\tilde{k}_i^s \leq 0 \tag{12}$$

2) Agent is active iff

$$0 < \tilde{k}_i^s < e \tag{13}$$

3) Agent is hyperactive iff

$$\tilde{k}_i^s \geq e \tag{14}$$

For productive agent, necessary conditions of different ways of agent's behavior are as follows:

4) Agent may be passive if

$$\tilde{k}_i^s \geq 0 \tag{15}$$

5) Agent may be active if

$$0 < \tilde{k}_i^s < e \tag{16}$$

6) Agent may be hyperactive if

$$\tilde{k}_i^s \leq e \tag{17}$$

Corollary 2.2. (Matveenko and Korolev,[9] corollary 3.4). Agents in a complete network of order $p(p > 1)$.

1) *are passive if* $b_i < \frac{1}{p}, i = 1, 2, \ldots, n$;
2) *are passive or hyperactive if* $b_i = \frac{1}{p}, i = 1, 2, \ldots, n$; and
3) *are passive, activ,e or hyperactive if* $b_i > \frac{1}{p}, i = 1, 2, \ldots, n$.

Remark 2.2. Evidently, $\tilde{k}_i^s$ can be presented as

$$\tilde{k}_i^s = \frac{b_iK_i - e(1 - 2a)}{2a} \tag{18}$$

Notice that formula (18), by itself, does not provide the equilibrium investment of ith agent, because $\tilde{k}_i^s$ enters K_i; however, this formula is very convenient for analysis.

Formula (18) implies that in equilibrium in complete network (where the environment is the same for all), any agent with a higher productivity invests more (and consumes in the first period less) than any agent with a lower productivity. Formula (18) and the expression for the payoff function, $V_i(k_i, K_i)$, also imply that the higher productivity of an agent is, the higher her utility is.

In lemma 2.1 and corollary 2.1, we provide description of agent's ways of behavior in terms of pure externality, $\tilde{K}_i$, and in terms of solution of equation (2), $\tilde{k}_i^s$. The following lemma gives a description of the ways of behavior in terms of environment, K_i.

Lemma 2.2. In equilibrium, i-th agent is passive iff

$$K_i \leq \frac{e(1 - 2a)}{b_i} \tag{19}$$

i-th agent is active iff

$$\frac{e(1 - 2a)}{b_i} < K_i < \frac{e}{b_i} \tag{20}$$

i-th agent is hyperactive iff

$$K_i \geq \frac{e}{b_i} \tag{21}$$

Proof. Since

$$D_1V_i(k_i, K_i) = -2ak_i + b_iK_i - e(1 - 2a)$$

the first-order conditions for the ith agent imply that

$$D_1V_i(k_i, K_i)|_{k_i=0} = b_iK_i - e(1 - 2a) \leq 0 \tag{22}$$

$$D_1V_i(k_i, K_i)|_{k_i\in(0,e)} = 0 \tag{23}$$

$$D_1V_i(k_i, K_i)|_{k_i=e} = b_iK_i - e \geq 0 \tag{24}$$

But equation (22) is equivalent to equation (19), and equation (24) is equivalent to equation (21). It follows from equation (23) that

$$0 < \tilde{k}_i^s < e$$

which is equivalent to equation (20). □

In any complete network, the environment is the same for all agents. This implies the following corollary.

Remark 2.3. *In complete network in equilibrium, agents with the same productivity make the same investments. If all agents have the same productivity, then a homophily takes place: Everyone behaves in the same way.*

Remark 2.4. *In complete network, there cannot be equilibrium in which an agent with a higher productivity is* *active while an agent with a lower productivity is hyperactive or when an agent with a higher productivity is passive while an agent with a lower productivity is active or hyperactive.*

Speaking about complete network, we will omit index i in notation for the i th agent's environment, because the environment in complete network is the same for all agents. In other words, K will denote the sum of investments of all agents of complete network.

Corollary 2.3. *In complete network, equilibrium with all hyperactive agents exists iff*

$$\min_i b_i \geq \frac{1}{n}$$

In this case

$$K \geq \frac{e}{\min_i b_i}$$

Equilibrium with all active agents exists iff

$$\frac{e(1-2a)}{\min_i b_i} < K < \frac{e}{\max_i b_i}$$

Equilibrium with all passive agents always exists. In this case, $K = 0$.

Proof. Assume that in a complete network, all agents are hyperactive. According to equation (24), it is possible iff

$$K = nb_i e \geq e$$

that is iff

$$b_i \geq \frac{1}{n}$$

Other statements of the corollary follow directly from lemma 2.2. □

Equilibria in complete network with two types of agents

Let a complete network consist of p agents with productivity b_1 (these agents will be referred as type 1) and q agents with productivity b_2 (type 2); $b_1 > b_2$. The following statement lists all possible equilibria and conditions of their existence. According to remark 2.4, only these six equilibria are possible.

Proposition 3.1. In complete network with two types of agents, the following equilibria exist.

1) Equilibrium with all hyperactive agents exists if

$$b_1 > b_2 \geq \frac{1}{p+q} \tag{25}$$

2) Equilibrium in which first type agents are hyperactive and second type agents are active exists if

$$0 < \frac{(1-2a-pb_2)}{qb_2 - 2a} < 1 \tag{26}$$

$$p + \frac{q(1-2a-pb_2)}{qb_2 - 2a} \geq \frac{1}{b_1} \tag{27}$$

3) Equilibrium in which first type agents are hyperactive and second type agents are passive exists if

$$b_1 \geq \frac{1}{p}, \quad b_2 \leq \frac{1-2a}{p} \tag{28}$$

4) Equilibrium in which first type agents are active and second type agents are passive exists if

$$b_1 > \frac{1}{p}, \quad b_2 \leq \frac{pb_1 - 2a}{p} \tag{29}$$

5) Equilibrium with all passive agents always exists.
6) Equilibrium in which agents of both types are active exists if

$$p(b_1 - b_2) < 2a, \quad 2ab_1(p+q) > 2a + q(b_1 - b_2)$$

Proof.

1) Follows from lemma 2.2.
2) This equilibrium is possible iff inequality (26) is checked. According to equation (20), the equilibrium exists under equation (27).
3) Since in this case the environment is $K = pe$, according to equations (19) and (21), the equilibrium exists iff equation (28) is checked.
4) According to equations (19) and (20), the equilibrium exists iff

$$b_1 > \frac{1}{p}, \quad pk_1 \leq \frac{e(1-2a)}{b_2}$$

where

$$k_1 = \frac{e(1-2a)}{pb_1 - 2a}$$

5) Follows from lemma 2.1.
6) The system of equation (2) turns into

$$\begin{cases} (pb_1 - 2a)k_1 + qb_1 k_2 = e(1-2a) \\ pb_2 k_1 + (qb_2 - 2a)k_2 = e(1-2a) \end{cases}$$

The solution is

$$k_1^s = \frac{e(1-2a)(qb_2 - qb_1 - 2a)}{2a(2a - pb_1 - qb_2)}$$

$$k_2^s = \frac{e(1-2a)(pb_1 - pb_2 - 2a)}{2a(2a - pb_1 - qb_2)}$$

It is clear that $k_1^s > k_2^s$; hence, the necessary and sufficient conditions of existence of the inner equilibrium are

$$k_2^s > 0, \quad k_1^s < e$$

that is

$$p(b_1 - b_2) < 2a \tag{30}$$

$$2ab_1(p+q) > 2a + q(b_1 - b_2) \tag{31}$$

Under inequalities (30) and (31), the inner equilibrium is

$$k_1 = k_1^s, \quad k_2 = k_2^s. \square$$

Remark 3.1. The signs of the following derivatives show how a change in the types' productivities b_1, b_2 influences volumes of investments k_1, k_2:

$$(k_1)'_{b_1} = C_1(qb_2 - 2a) \quad (\text{where } C_1 > 0)$$

$$(k_1)'_{b_2} = C_2(-pqb_1 - q^2b_1) < 0 \quad (\text{where } C_2 > 0)$$

$$(k_2)'_{b_1} = C_3(-pqb_2 - p^2b_2) < 0 \quad (\text{where } C_3 > 0)$$

$$(k_2)'_{b_2} = C_4(pb_1 - 2a) > 0 \quad (\text{where } C_4 > 0)$$

Thus, with an increase in productivity of first type agents, their equilibrium investments increase if the second type consists of more than one agent ($q > 1$) or if $q = 1$ but this agent is productive and decrease if $q = 1$ and this agent is unproductive. The equilibrium investments of the second type agents always decrease.

With an increase in productivity of the second type agents, their equilibrium investments always increase, and the equilibrium investments of the first type agents always decrease.

Adjustment dynamics and dynamic stability of equilibria

Now, we introduce adjustment dynamics which may start after a small deviation from equilibrium or after junction of networks each of which was initially in equilibrium. We model the adjustment dynamics in the following way.

Definition 4.1. In the adjustment process, each agent maximizes her utility by choosing a level of investment; at the moment of decision-making, she considers her environment as exogenously given. Correspondingly, if $k_i^n = 0$ and $D_1V_i(k_i, K_i)|_{k_i=0} \le 0$, then $k_i^{n+1} = 0$, and if $k_i^n = e$ and $D_1V_i(k_i, K_i)|_{k_i=e} \ge 0$, then $k_i^{n+1} = e$; in all other cases, k_i^{n+1} solves the difference equation

$$-2ak_i^{n+1} + b_iK_i^n - e(1-2a) = 0$$

Definition 4.2. The equilibrium is called dynamically stable *if, after a small deviation of one of the agents from the equilibrium, dynamics starts which returns the equilibrium back to the initial state. In the opposite case, the equilibrium is called* dynamically unstable.

As before, let us consider complete network with p agents with productivity b_1 (type 1) and q agents with productivity b_2 (type 2). In initial time period, each first type agent invests k_{01} and each second type agent invests k_{02}. Correspondingly, the environment (common for all agents) in the initial period is $K = pk_{01} + qk_{02}$.

Assume that either $k_{01} = 0$ and $D_1V_1(k_1, K)|_{k_1=0} > 0$, or $k_{01} = e$ and $D_1V_1(k_1, K)|_{k_1=e} < 0$, or $k_{01} \in (0, e)$ and either $k_{02} = 0$ and $D_1V_2(k_2, K)|_{k_2=0} > 0$, or $k_{02} = e$ and $D_1V_2(k_2, K)|_{k_2=e} < 0$, or $k_{02} \in (0, e)$. Then, definition 4.1 implies that the dynamics is described by the system of difference equations

$$\begin{cases} k_1^{n+1} = \dfrac{pb_1}{2a}k_1^n + \dfrac{qb_1}{2a}k_2^n + \dfrac{e(2a-1)}{2a} \\ k_2^{n+1} = \dfrac{pb_2}{2a}k_1^n + \dfrac{qb_2}{2a}k_2^n + \dfrac{e(2a-1)}{2a} \end{cases} \tag{32}$$

with initial conditions

$$\begin{cases} k_1^0 = k_{01} \\ k_2^0 = k_{02} \end{cases} \tag{33}$$

Proposition 4.1. The general solution of the system of difference equation (32) has the form

$$k^n = C\left(\frac{pb_1 + qb_2}{2a}\right)^n \frac{1}{b_1 + b_2}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \quad n = 1, 2, \ldots \tag{34}$$

where $(D_1, D_2)^T$ is the steady state of equation (32)

$$D_1 = \frac{e(1-2a)(qb_2 - qb_1 - 2a)}{2a(2a - pb_1 - qb_2)} \tag{35}$$

$$D_2 = \frac{e(1-2a)(pb_1 - pb_2 - 2a)}{2a(2a - pb_1 - qb_2)} \tag{36}$$

The solution of the Cauchy difference problems (32) and (33) has the form

$$k^n = (pk_1^0 + qk_2^0 - \tilde{D})\left(\frac{pb_1 + qb_2}{2a}\right)^{n-1} \frac{1}{2a}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \quad n = 1, 2, \ldots \tag{37}$$

where

$$\tilde{D} = \frac{e(1-2a)(p+q)}{pb_1 + qb_2 - 2a} \tag{38}$$

Proof. The characteristic equation of system (32) is

$$\begin{vmatrix} \frac{pb_1}{2a}-\lambda & \frac{qb_1}{2a} \\ \frac{pb_2}{2a} & \frac{qb_2}{2a}-\lambda \end{vmatrix} = -\lambda\left(\frac{pb_1}{2a}+\frac{qb_2}{2a}\right)+\lambda^2=0$$

Thus, the eigenvalues are

$$\lambda_1=0, \quad \lambda_2=\frac{pb_1+qb_2}{2a}$$

An eigenvector corresponding λ_1 is

$$e_1=\begin{pmatrix} -q \\ p \end{pmatrix}$$

while an eigenvector corresponding λ_2 can be found as a solution of the system of equations

$$\begin{cases} \frac{-qb_2}{2a}x_1+\frac{qb_1}{2a}x_2=0 \\ \frac{pb_2}{2a}x_1-\frac{pb_1}{2a}x_2=0 \end{cases}$$

We find

$$e_2=\begin{pmatrix} \frac{b_1}{b_1+b_2} \\ \frac{b_2}{b_1+b_2} \end{pmatrix}$$

The general solution of the homogeneous system of difference equations corresponding (32) has the form

$$(k^n)_g=C\left(\frac{pb_1+qb_2}{2a}\right)^n\begin{pmatrix} \frac{b_1}{b_1+b_2} \\ \frac{b_2}{b_1+b_2} \end{pmatrix}, \quad n=1,2,\ldots.$$

As a partial solution of the system (32), we take its steady state, that is, the solution of the linear system

$$\begin{cases} D_1=\frac{pb_1}{2a}D_1+\frac{qb_1}{2a}D_2+\frac{e(2a-1)}{2a} \\ D_2=\frac{pb_2}{2a}D_1+\frac{qb_2}{2a}D_2+\frac{e(2a-1)}{2a} \end{cases}$$

The solution is (35) and (36); hence, the general solution of the system (32) has the form (34). In solution of the Cauchy problems (32) and (33), constants of integration are defined from the initial conditions

$$\begin{pmatrix} k_1^0 \\ k_2^0 \end{pmatrix}=C_1\begin{pmatrix} -q \\ p \end{pmatrix}+C\frac{1}{b_1+b_2}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}+\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \quad (39)$$

However, since one of the eigenvalues is 0, we need only constant C to write the solution under $n>0$. Multiplying by (p,q), we obtain

$$pk_1^0+qk_2^0=C\frac{pb_1+qb_2}{b_1+b_2}+pD_1+qD_2$$

We denote $\tilde{D}=pD_1+qD_2$ and derive expression (38). Thus

$$C=\left(pk_1^0+qk_2^0-\tilde{D}\right)\frac{b_1+b_2}{pb_1+qb_2}$$

Substituting for C into equation (34), we obtain equation (37). □

Let us find conditions of dynamic stability/instability for the equilibria in complete network with two types of agents which are listed in proposition 3.1.

Proposition 4.2. The conditions of dynamic stability/instability of the equilibria listed in proposition 3.1 (in case of their existence) are as follows.

1. The equilibrium in which both types of agents are hyperactive is stable iff

$$b_1>\frac{1}{p+q}, \quad b_2>\frac{1}{p+q} \quad (40)$$

2. The equilibrium in which agents of first type are hyperactive and agents of second type are active is stable iff

$$\begin{aligned} & p+\frac{q(1-2a-pb_2)}{qb_2-2a}>\frac{1}{b_1} \\ & \frac{qb_2}{2a}<1 \end{aligned} \quad (41)$$

3. The equilibrium in which agents of first type are hyperactive and agents of second type are passive is stable iff

$$b_1>\frac{1}{p}, \quad b_2<\frac{1-2a}{p}$$

4. The equilibrium in which agents of first type are active and agents of second type are passive is always unstable.
5. The equilibrium with all passive agents is always stable.
6. The equilibrium with all active agents is always unstable.

Proof.

1. According to definition 4.1 and equation (3)

$$D_1V_1(k_1,K)|_{k_1=e}=b_1(p+q)e-e, \quad D_1V_2(k_2,K)|_{k_2=e}=b_2(p+q)e-e$$

Both derivatives are positive iff equation (40) is checked.

2. According to definition 4.1 and equations (3) and (27)

$$D_1V_1(k_1,K)|_{k_1=e} = b_1\left(pe + \frac{qe(1-2a-pb_2)}{qb_2-2a}\right)e - e \geq 0$$

However, for dynamic stability, the strict inequality is needed. Let equation (41) be checked and $k_1 = e$. The difference equation describing dynamics of each of the second group agents is

$$k_2^{n+1} = \frac{qb_2}{2a}k_2^n + \frac{peb_2}{2a} + \frac{e(2a-1)}{2a} \quad (42)$$

For stability, it is necessary and sufficient that $(qb_2/2a) < 1$.

3. According to definition 4.1 and equations (3) and (28)

$$D_1V_1(k_1,K)|_{k_1=e} = b_1e - e \geq 0$$
$$D_1V_2(k_2,K)|_{k_2=0} = e(2a-1) + b_2pe \leq 0$$

For stability, the strict inequalities are needed.

4. According to definition 4.1 and equations (3) and (20)

$$D_1V_2(k_2,K)|_{k_2=0} = e(2a-1) + b_2\frac{pe(1-2a)}{pb_1-2a} = \frac{e(1-2a)(pb_2-pb_1+2a)}{pb_1-2a} \leq 0$$

For stability, the strict inequalities are needed. Let the second inequality in equation (29) be satisfied strictly. The difference equation for any of the first group agents is

$$k_1^{n+1} = \frac{pb_1}{2a}k_1^n + \frac{e(2a-1)}{2a}$$

According to the first inequality in equation (29)

$$pb_1 > 1 > 2a$$

Hence, the equilibrium is unstable.

5. According to definition 4.1 and equation (3)

$$D_1V_1(k_1,K)|_{k_1=0} = e(2a-1) < 0$$
$$D_1V_2(k_2,K)|_{k_2=0} = e(2a-1) < 0$$

6. One of the eigenvalues of the system (32) is

$$\lambda_2 = \frac{pb_1+qb_2}{2a} > 1$$

Hence, the equilibrium is unstable. □

Junction of two complete networks

Let complete network 1 consists of p agents, each with productivity b_1, and let complete network 2 has q agents, each with productivity b_2. According to corollary 2.3, in initial equilibrium, each of the complete networks is in a homophily state: All (homogeneous) agents in the network make the same investments. Let these networks unify to create a common complete network with $p+q$ heterogeneous agents. Will the agents hold their initial behavior?

Proposition 5.1. After the junction, all agents hold their initial behavior (make the same investments as before the junction) in the following four cases.

1) If $b_1 \geq \frac{1}{p}$, $b_2 \geq \frac{1}{q}$, and initially agents in both networks are hyperactive.
2) If

$$b_2 \leq \frac{1-2a}{p}$$

and initially agents in the first network are hyperactive and agents in the second network are passive.

3) If

$$b_1 > \frac{1}{p}, \quad b_2 \leq b_1 - \frac{2a}{p}$$

and initially agents in the first network are active and agents in the second network are passive.

4) If initially agents in both networks are passive.

In all other cases, the equilibrium changes.

Proof.

1) According to corollary 2.4

$$b_1 \geq \frac{1}{p}, \quad b_2 \geq \frac{1}{q}$$

Substituting $k_1 = e$ and $k_2 = e$ into equation (3), we obtain, correspondingly

$$D_1V_1(k_1,K)|_{k_1=e} = b_1(p+q)e - e \geq 0$$
$$D_1V_2(k_2,K)|_{k_2=e} = b_2(p+q)e - e \geq 0$$

1) According to corollary 2.4

$$b_1 \geq \frac{1}{p}$$

Substituting $k_2 = 0$ and $k_1 = e$ into equation (3), we obtain

$$D_1V_1(k_1,K)|_{k_1=e} = b_1pe - e \geq 0$$
$$D_1V_2(k_2,K)|_{k_2=0} = b_2pe - e(1-2a) \leq 0$$

2) Substituting $k_1 = e(1-2a)/(pb_1-2a)$ and $k_2 = 0$ into equation (3), we obtain

$$D_1V_1(k_1,K) = \frac{e(1-2a)(2a-pb_1-2a+pb_1)}{pb_1-2a} = 0$$

$$D_2V_2(k_2,K) = \frac{e(1-2a)(pb_1-2a-pb_2)}{pb_1-2a} \leq 0$$

Substituting $k_1 = 0$ and $k_2 = 0$ into equation (3), we obtain

$$D_1V_1(k_1, K)|_{k_1=0} = D_1V_2(k_2, K)|_{k_2=0} = e(2a-1) \le 0$$

In all other cases, the initial values of investments of agents will not be equilibrium in the unified network, and the network will move to a different equilibrium. □

Proposition 5.1 shows, in particular, that passive agents (nonadopters), when connected with adopters, can remain nonadopters only if their productivity, b_2, is relatively low.

A pattern of transition process after the junction depends on initial conditions and parameters values. If adjustment dynamics of the unified complete network starts, it is described by the system of difference equation (32) with initial condition (33).

Proposition 5.2. Let the agents in the first network before junction be hyperactive (hence, $b_1 \ge (1/p)$ by corollary 2.2) and agents in the second network be passive. Then, the following cases are possible.

1) If $b_2 \le \left((1-2a)/p\right)$, then after junction all agents hold their initial behavior, and there is no transition process in the unified network. The unified network is in equilibrium $\{k_1 = e, k_2 = 0\}$.
2) If $b_2 > \left((1-2a)/p\right)$ and $b_2 \ge (2a/q)$, then the first group agents stay hyperactive; investments of the second group agents increase until they also become hyperactive. The unified network comes to equilibrium $\{k_1 = e, k_2 = e\}$ (notice that conditions $b_2 > \left((1-2a)/p\right)$ and $b_2 \ge (2a/q)$ imply $b_2 > \left(1/(p+q)\right)$, that is, condition of existence of equilibrium $\{k_1 = e, k_2 = e\}$).

If $(2a/p) > b_2 > \left((1-2a)/p\right)$, then the first group agents stay hyperactive; investments of the second group agents increase. The unified network comes to state $\{k_1 = e, k_2 = \left(e(pb_2 + 2a - 1)/(2a - qb_2)\right)\}$ if $b_2 < \left(1/(p+q)\right)$ and to state $\{k_1 = e, k_2 = e\}$ if $b_2 \ge \left(1/(p+q)\right)$.

In cases 2 and 3, utilities of all agents in the unified network increase. In case 1, the utilities do not change.

Proof.

1. Follows from proposition 5.1, point 2.
2. If for agents of the second group

$$D_1V_2(k_2, K)|_{k_2=0} = b_2pe - e(1-2a) > 0$$

they change their investments according to the difference equation (42). The general solution of equation (42) is

$$k_2^n = C\left(\frac{qb_2}{2a}\right)^n + D, \quad n = 0, 1, 2, \ldots. \tag{43}$$

where

$$D = \frac{e(pb_2 + 2a - 1)}{2a - qb_2} \tag{44}$$

and $pb_2 > 1 - 2a$. The initial conditions imply

$$C = k_2^0 - D = -D$$

The partial solution satisfying initial conditions is

$$k_2^n = D\left(1 - \left(\frac{qb_2}{2a}\right)^n\right), \quad n = 1, 2, \ldots.$$

If $b_2 > (2a/q)$, then $(qb_2/2a) > 1$, $D < 0$, and k_2^n converges to e. After the value e is achieved, $k_2^n = e$, since

$$D_1V_2(k_2, K)|_{k_2=e} = b_2(p+q)e - e \ge (1 - 2a + 2a)e - e = 0$$

If $b_2 < (2a/q)$, then $(qb_2/2a) < 1$, $D > 0$, and k_2^n converges to D if $D < e$, that is, if $b_2 < (1/(p+q))$. In the opposite case, if $b_2 \ge (1/(p+q))$, k_2^n converges to e.

It is clear that both the equilibria

$$\{k_1 = e, k_2 = e\} \text{ and } \left\{k_1 = e, k_2 = \frac{e(pb_2 + 2a - 1)}{2a - qb_2}\right\}$$

possible in result of junction are stable.

In the "resonance" case

$$b_2 = \frac{2a}{q}$$

We are looking for the partial solution of difference equation (42) not in form D, but in form nD. From equation (42), we have

$$(n+1)D = nD + \frac{peb_2}{2a} + \frac{e(2a-1)}{2a}$$

which implies

$$D = \frac{e(pb_2 + 2a - 1)}{2a}$$

Thus, the general solution of equation (42) has the form

$$k_2^n = C + n\frac{e(pb_2 + 2a - 1)}{2a}, \quad n = 0, 1, 2, \ldots.$$

It follows from initial conditions that $C = 0$, so the partial solution of equation (42) satisfying the initial conditions is

$$k_2^n = n\frac{e(pb_2 + 2a - 1)}{2a}, \quad n = 1, 2, \ldots.$$

Since $pb_2 > 1 - 2a$, the value of investment k_2^n converges to e and, since this value is achieved, stays equal to e, because

$$D_1V_2(k_2, K)|_{k_2=e} = b_2(p+q)e - e \geq (1-2a+2a)e - e = 0$$

The last statement (concerning utilities) follows directly from theorem 1.2. □

Proposition 5.3. Let agents of the first network before junction be hyperactive (which implies $b_1 \geq (1/p)$ by corollary 2.2) and agents of the second network be active (which implies $b_2 > (1/q)$). The unified network moves to the equilibrium with all hyperactive agents. The utilities of all agents increase.

Proof. The first group agents stay hyperactive, because, by equation (3),

$$D_1V(k_1, b_1K)|_{k_1=e} = e(2a-1) - 2ae + b_1pe + b_2qk_2 \geq 0$$

For the second group agents, we have equation (42). Its general solution is equation (43), where D is defined by equation (44). From the initial conditions, we find

$$C = k_2^0 - D = \frac{e(1-2a)}{qb_2 - 2a} - \frac{e(1-2a-pb_2)}{qb_2 - 2a} = \frac{pb_2}{qb_2 - 2a} > 0$$

Hence, k_2^n achieves the value e.

The statement concerning utilities follows directly from theorem 1.2. □

Proposition 5.4. If before junction agents of both networks are hyperactive (this implies $b_1 \geq (1/p)$ and $b_2 \geq (1/q)$ by corollary 2.2), they stay hyperactive after junction: There is no transition dynamics, and utilities of all agents do increase.

Proof. It follows from proposition 5.1, point 1. The increase of utilities follows from theorem 1.2. □

Proposition 5.5. If before junction agents of both networks are passive, they stay passive after junction: There is no transition dynamics, and agents' utilities do not change.

Proof. It follows from proposition 5.1, point 4. Utilities do not change according to theorem 1.2. □

The following two propositions show how, depending on the relation between the heterogeneous productivities, passive agents (nonadopters) may change their behavior (become adopters).

Proposition 5.6. Let agents of first network before junction be active (which implies $b_1 > (1/p)$ by corollary 2.2), $k_1^0 = \left(e(1-2a)/(pb_1 - 2a)\right)$, and agents of the second network be passive. Then, the following cases are possible.

1. Under $pb_1 \geq pb_2 + 2a$, all agents hold their initial behavior, and there is no transition process.

Let $pb_1 < pb_2 + 2a$. If $b_2 \geq (2a/q)$ and $\left((e - D_1 - k_1^0)/b_1\right) < \left((e - D_2)/b_2\right)$, then the network moves to the equilibrium with all hyperactive agents. If $b_2 < (2a/q)$ and $\left((e - D_1 - k_1^0)/b_1\right) < \left((e - D_2)/b_2\right)$, then the network moves to the equilibrium in which the first group agents are hyperactive and the second group agents are active

$$k_2 = \frac{e(1-2a-pb_2)}{qb_2 - 2a}$$

2. If $\left((e - D_1 - k_1^0)/b_1\right) \geq \left((e - D_2)/b_2\right)$, then the network moves to the equilibrium with all hyperactive agents.

In case 1, utilities of all agents do not change; in case 2, utilities of all agents increase.

Proof. For the second group agents initially

$$D_1V(k_2, b_2K) = e(2a-1) + b_2\frac{pe(1-2a)}{pb_1 - 2a} = \frac{e(1-2a)(pb_2 - pb_1 + 2a)}{pb_1 - 2a}$$

Thus, $D_1V(k_2, b_2K) \leq 0$ if $pb_1 \geq pb_2 + 2a$. In this case, the second group agents stay passive. The first group agents also hold their behavior unchanged, because their environment does not change.

Now, let $pb_1 < pb_2 + 2a$. The second group agents increase their investments, and so do agents of the first group, because their environment increases. Conditions $pb_1 < pb_2 + 2a$ and $b_1 > (1/p)$ imply $b_2 > (1 - 2a/p)$. Hence, by lemma 2.2, the equilibrium with hyperactive agents of one of the groups and active agents of another group is always possible, as well as the equilibrium with all hyperactive agents.

Agents of one of the groups may achieve the investment level e earlier than the agents of another group. Let it be the first group, that is, $\left((e - D_1 - k_1^0)/b_1\right) < \left((e - D_2)/b_2\right)$. The investment level of the second group agents in this moment is some $\tilde{k}_2^0$. After that investments of the second group agents follow equation (42). The general solution of equation (42) is equation (43), where D has the form (44). From the initial conditions, we have $C = \tilde{k}_2^0 - D$. Thus, if $b_2 > (2a/q)$, then $D < 0$, which implies $C > 0$; hence, investments of the second group agents will achieve level e. If $b_2 < (2a/q)$, then investments of agents of the second group will become equal to $D > 0$.

In the resonance case, $b_2 = (2a/q)$, as previously, the general solution of equation (42) has the form

$$k_2^n = C + n\frac{e(pb_2 + 2a - 1)}{2a}, \quad n = 0, 1, 2, \ldots .$$

It follows from initial conditions that $C = \tilde{k}_2^0$, so the partial solution of equation (42) satisfying the initial conditions is

$$k_2^n = \tilde{k}_2^0 + n\frac{e(pb_2 + 2a - 1)}{2a}, \quad n = 1, 2, \ldots .$$

Since $pb_2 > 1 - 2a$, the value of investments of the second group agents in this case also achieves e.

Suppose now, that the second group agents have received the investment level e first, that is, $\left((e - D_1 - k_1^0)/b_1\right) > \left((e - D_2)/b_2\right)$, while the investment level of the first group agents was equal to some $\tilde{k}_1^0$. It is possible only if $b_2 > b_1$. From that moment, the investments of the first group agents follow equation

$$k_1^{n+1} = \frac{pb_1}{2a}k_1^n + \frac{qeb_1}{2a} + \frac{e(2a-1)}{2a} \tag{45}$$

which general solution (45) is

$$k_1^n = C\left(\frac{pb_1}{2a}\right)^n + D, \quad n = 0, 1, 2, \ldots$$

where

$$D = \frac{e(1 - 2a - qb_1)}{pb_1 - 2a}$$

From the initial condition, we have $C = \tilde{k}_2^0 - D$. Moreover, $\tilde{k}_1^0 > k_1^0 = \left(e(1-2a)\right)/(pb_1 - 2a)$, which implies

$$C = \tilde{k}_1^0 - D > \frac{e(1-2a)}{pb_1 - 2a} - \frac{e(1 - 2a - qb_1)}{pb_1 - 2a} = \frac{qb_1}{pb_1 - 2a} > 0$$

Hence, investments of the first group agents achieve e.

In case when agents of both groups achieve investment level e simultaneously, that is, $\left((e - D_1 - k_1^0)/b_1\right) = \left((e - D_2)/b_2\right)$, the network, evidently, turns to the equilibrium with all hyperactive agents. □

Proposition 5.7. If before junction agents of both networks are active (this implies $b_1 > (1/p)$ and $b_2 > (1/q)$ by corollary 2.2), then after junction all agents become hyperactive; their utilities increase.

Proof. The initial conditions are $k_1^0 = \left(\left(e(1-2a)\right)/(pb_1 - 2a)\right), k_2^0 = \left(\left(e(1-2a)\right)/(qb_2 - 2a)\right)$. According to equation (3)

$$D_1V(k_1, b_1K) = e(2a-1) - 2a\frac{e(1-2a)}{pb_1 - 2a} + b_1p\frac{e(1-2a)}{pb_1 - 2a} + b_1q\frac{e(1-2a)}{qb_2 - 2a} > 0$$

$$D_1V(k_2, b_2K) = e(2a-1) - 2a\frac{e(1-2a)}{qb_2 - 2a} + b_2p\frac{e(1-2a)}{pb_1 - 2a} + b_2q\frac{e(1-2a)}{qb_2 - 2a} > 0$$

Thus, agents of both groups will increase their investments following equation (43). Agents of one of the groups will achieve investment level e first. Let it be the first group and let investments of the second group agents in this moment be $\tilde{k}_2^0$. Then, investments of the second group agents follow difference equation (5.7), which general solution is equation (5.8), where D has the form (5.9). The initial conditions imply $C = \tilde{k}_2^0 - D$, but

$$\tilde{k}_2^0 > k_2^0 = \frac{e(1-2a)}{qb_2 - 2a} > D$$

Hence, $C > 0$. Thus, investments of the second group agents will also achieve level e. Absolutely similar argument is for the case when the second group achieves the investment level e first. □

Remark 5.1. In all cases considered in propositions 5.1–5.7, agents' utilities in result of junction do increase or, at least do not change. Thus, all the agents have an incentive to unify, or at least have no incentive not to unify.

Remark 5.2. We have studied equilibria for complete networks formed in result of junction of two complete networks with p and q nodes. These equilibria do correspond to equilibria in some regular (equidegree) networks. For example, let W_1 be a regular network consisting of 20 nodes, each of which has 3 links, and let W_2 be regular network consisting of 15 nodes, each of which has 2 links. Let each node of W_1 establish three links with W_2 and each node of W_2 establish four links with W_1. A result of this junction is the regular network with 35 nodes, each of which has 6 links. This junction of the regular networks is "cognate" to the junction of complete networks with four and three nodes. It is rather evident that each statement concerning equilibria in junked complete networks corresponds to a similar statement concerning equilibria in junked cognate regular networks. Presumably, the regular networks may also have unsymmetrical equilibria which have no relation to the equilibria in their cognate complete networks.

Equilibria in star network with heterogeneous agents

Analysis of equilibria with heterogeneous agents in incomplete networks is much more complex than in complete networks, because there is no common for all agents environment.

Let us consider star network with ν peripheral nodes (rays). The agent in the center of the star has productivity b_0; each of the peripheral agents has productivity b. If the peripheral agents are unproductive ($b < 2a$), then investments of all peripheral agents in equilibrium are the same, because, according to lemma 2.1, the unproductive agent's investment is uniquely defined by her received externality. However, if the peripheral agents are productive ($b > 2a$), then their equilibrium investment levels can differ. In case when the central agent is active or hyperactive, it is possible that a part of peripheral agents are passive, while another part is active and the third part is hyperactive.

Example 6.1. In equilibrium under $b > 2a$, let the central agent be active with investment k_0, p peripheral agents be hyperactive, q peripheral agents be active with investment k, and r peripheral agents be passive. The values of investments of active agents satisfy the following system of linear equations:

$$\begin{cases} (b_0 - 2a)k_0 + qb_0k + pb_0e = e(1 - 2a) \\ bk_0 + (b - 2a)k = e(1 - 2a) \end{cases}$$

whose solution is

$$\tilde{k}_0^s = \frac{(1 - 2a)(qb_0 - b + 2a) + pb_0(b - 2a)}{(q - 1)b_0b + 2a(b + b_0) - 4a^2}e$$

$$\tilde{k}^s = \frac{(1 - 2a)(b - b_0 + 2a) - pb_0b}{(q - 1)b_0b + 2a(b + b_0) - 4a^2}e$$

For the equilibrium to exist, the conditions

$$0 < \tilde{k}_0^s < e,$$
$$0 < \tilde{k}^s < e$$

have to be checked. The inequality $\tilde{k}^s > 0$ is equivalent to $\tilde{k}_0^s = \tilde{K} < \Big(e(1 - 2a)/b\Big)$, and $\tilde{k}^s < e$ is equivalent to $\tilde{k}_0^s = \tilde{K} > \Big(e(1 - b)/b\Big)$.

For instance, let $b_0 = 3a$, $b = 6a$, $p = 1$, $q = 2$, and $r = 3$. Then, condition $\tilde{k}_0^s > 0$ reduces to $2a + 8a^2 > 0$ which is always true. Condition $\tilde{k}_0^s < e$ is equivalent to $2a + 8a^2 < 32a^2$ which, in turn, is equivalent to $a > 1/12$. Condition $\tilde{k}^s > 0$ is equivalent to $5a - 28a^2 > 0$, that is, $a < 5/28$. Condition $\tilde{k}^s < e$ reduces to

$$5a - 28a^2 < 32a^2$$

that is, $a > \frac{1}{12}$.

Thus, in a star with six peripheral nodes, under

$$\frac{1}{12} < a < \frac{5}{28},\ b_0 = 3a,\ b = 6a$$

there exists an equilibrium in which the central agent is active and invests

$$k_0 = \frac{1 + 4a}{16a}e$$

One of the peripheral agents is hyperactive; two peripheral agents are active and make investments

$$k = \frac{5 - 28a}{32a}e$$

and three peripheral agents are passive.

Let us consider inner equilibrium (in which all agents are active) in star network. The investment, k, of each peripheral agent is the same (because they receive the same externality). Let us see, how the values of investments, k_0, k, depend on productivities, b_0, b, and on the number of periphery agents, ν.

Theorem 6.1. For the star network, let the following inequalities be checked (if the number of peripheral nodes, ν, is sufficiently big, these three inequalities reduce to $2a > \max\{b_0 - b, 1 - b, b - 2b_0\}$).

$$\begin{aligned} &b_0 - 2a < b < 2b_0 + 2a \\ &\nu b_0 + 2a < b + 2a(\nu + 1)b_0 + (\nu - 1)b_0b \\ &b + 2a < b_0 + 4ab + (\nu - 1)b_0b \end{aligned} \tag{46}$$

Then, in the inner equilibrium

1) If

$$b < 2a\sqrt{\frac{\nu + 1}{\nu - 1}}$$

then k_0 decreases in b_0.

2) If

$$b = 2a\sqrt{\frac{\nu + 1}{\nu - 1}}$$

then k_0 does not depend on b_0.

3) If

$$b > 2a\sqrt{\frac{\nu + 1}{\nu - 1}}$$

then k_0 increases in b_0.

4) k_0 decreases in b.

5) k_0 decreases in ν if $b < 2a$, increases in ν if $b > 2a$, and converges to

$$\frac{e(1-2a)}{b}$$

as $\nu \to \infty$; the central agent's utility increases with respect to ν.

6) k decreases in b_0.

7) If $b_0 < 2a$, then k decreases in b.

8) If $b_0 > 2a$, then k increases in b.

9) k decreases in ν and converges to 0 as $\nu \to \infty$; the peripheral agent's utility decreases with respect to ν if $b_0 - b < 2a$; it does not change if $b_0 - b = 2a$; it increases if $b_0 - b > 2a$.

Proof. Equilibrium investment values, k_0 and k, solve the system of equations

$$\begin{cases} (b_0-2a)k_0 + \nu b_0 k = e(1-2a) \\ bk_0 + (b-2a)k = e(1-2a) \end{cases} \tag{47}$$

and satisfy inequalities

$$0 < k_0 < e$$
$$0 < k < e$$

The solution of equation (47) is

$$k_0 = \frac{e(1-2a)(\nu b_0 + 2a - b)}{(\nu-1)b_0 b + 2a(b_0+b) - 4a^2}$$
$$k = \frac{e(1-2a)(b + 2a - b_0)}{(\nu-1)b_0 b + 2a(b_0+b) - 4a^2}$$

Conditions

$$k_0 > 0, \quad k > 0$$

are satisfied iff

$$nb_0 + 2a - b > 0$$
$$b + 2a - b_0 > 0$$

that is

$$b_0 - 2a < b < \nu b_0 + 2a$$

Conditions

$$k_0 < e, \quad k < e$$

are satisfied iff

$$\nu b_0 + 2a < b + 2a(\nu+1)b_0 + (\nu-1)b_0 b$$
$$b + 2a < b_0 + 4ab + (\nu-1)b_0 b$$

It is defined by the sign of

$$(k_0)'_{b_0} = \frac{e(1-2a)\Big((\nu-1)b^2 - 4a^2(\nu+1)\Big)}{\Big((\nu-1)b_0 b + 2a(b_0+b) - 4a^2\Big)^2}$$

that is, by the sign of

$$(\nu-1)b^2 - 4a^2(\nu+1)$$

1. We obtain

$$(k_0)'_b = -\frac{e(1-2a)\Big(4a\nu b_0 + \nu(\nu-1)b_0^2\Big)}{\Big((\nu-1)b_0 b + 2a(b_0+b) - 4a^2\Big)^2} < 0$$

Thus, k_0 decreases in b.

2. Differentiating k_0 with respect to ν (taking ν as continuous variable), we obtain

$$(k_0)'_\nu = \frac{e(1-2a)b_0(b-2a)(b+2a-b_0)}{\Big((\nu-1)b_0 b + 2a(b_0+b) - 4a^2\Big)^2}$$

According to equation (46), $b + 2a - b_0 > 0$; hence, $(k_0)'_\nu < 0$ if $b < 2a$ and $(k_0)'_\nu > 0$ if $b > 2a$. The equilibrium value k_0 converges to

$$\frac{e(1-2a)}{b}$$

as $\nu \to \infty$.

The value

$$b_0 K_0 = \frac{b_0 e(1-2a)[(2a+b)\nu + 2a - b]}{(\nu-1)b_0 b + 2a(b_0+b) - 4a^2}$$

increases in ν

$$(b_0 K_0)_\nu{}' = \frac{2ab_0 e(1-2a)(2a+b)(b_0+b-2a)}{[(\nu-1)b_0 b + 2a(b_0+b) - 4a^2]^2} > 0$$

Hence, utility of the central agent increases if the number of peripheral agents rises.

3. We obtain

$$k'_{b_0} = -\frac{e(1-2a)\Big(2a(\nu+1)b + (\nu-1)b^2\Big)}{\Big((\nu-1)b_0 b + 2a(b_0+b) - 4a^2\Big)^2} < 0$$

Thus, the equilibrium value k decreases in b_0.

4. We obtain

$$k'_b = \frac{e(1-2a)\Big((\nu-1)b_0^2 - 2a(\nu-3)b_0 - 8a^2\Big)}{\Big((\nu-1)b_0 b + 2a(b_0+b) - 4a^2\Big)^2}$$

Hence, the sign of derivative is defined by the sign of the quadratic trinomial

Table 1. The dependence of investment of the central agent, k_0, on parameters.

Dependence of k_0 on b_0			Dependence of k_0 on b	Dependence of k_0 on ν	
If $b < 2a\sqrt{\frac{\nu+1}{\nu-1}}$	If $b = 2a\sqrt{\frac{\nu+1}{\nu-1}}$	If $b > 2a\sqrt{\frac{\nu+1}{\nu-1}}$		If $b < 2a$	If $b > 2a$
k_0 decreases	k_0 is independent	k_0 increases	k_0 decreases	k_0 decreases	k_0 increases

Table 2. The dependence of investment of each peripheral agent, k, on parameters.

Dependence of k on b_0	Dependence of k on b		Dependence of k on ν
	If $b_0 < 2a$	If $b_0 > 2a$	
k decreases	k decreases	k increases	k decreases

$$(\nu - 1)b_0^2 - 2a(\nu - 3)b_0 - 8a^2$$

The roots of the trinomial are

$$(b_0)_1 = -\frac{2a}{\nu - 1} < 0, \quad (b_0)_2 = 2a$$

Thus, if $b_0 < 2a$, then the equilibrium value k decreases with respect to b.

1. If $b_0 > 2a$, then the equilibrium value k increases in b.
2. Differentiating k with respect to ν (taking ν as a continuous variable), we obtain

$$k'_\nu = -\frac{e(1 - 2a)(b + 2a - b_0)b_0 b}{\left((\nu - 1)b_0 b + 2a(b_0 + b) - 4a^2\right)^2} < 0$$

by equation (46). Thus, the equilibrium value k decreases in ν. It is easily seen that it converges to 0 as $\nu \to \infty$.

The derivative of

$$bK = \frac{be(1 - 2a)(\nu b_0 + 4a - b_0)}{(\nu - 1)b_0 b + 2a(b_0 + b) - 4a^2}$$

with respect to ν is

$$(bK)_\nu' = \frac{2ab_0 be(1 - 2a)(b_0 - b - 2a)}{[(\nu - 1)b_0 b + 2a(b_0 + b) - 4a^2]^2}$$

Thus, utility of peripheral agents decreases under growth of the star if $b_0 - b < 2a$, does not change if $b_0 - b = 2a$, and increases if $b_0 - b > 2a$. □

One important result of theorem 6.1 is that the utility of the central agent, U_0, increases in the number of peripheral agents, ν; thus, the central agent is always interested in a growth of the star network.

Tables 1–3 summarize other results of theorem 6.1.

We see that, in the inner equilibrium, investment of any of the two counterparts, center and periphery, depends negatively on the productivity of the other counterpart. The own productivity also plays role, but in dependence of investment on the productivity of the counterpart. Investment of any agent, central or peripheral, depends positively (negatively) on the own productivity if the productivity of the counterpart is sufficiently high (sufficiently low, correspondingly).

Table 3. The dependence of the utility of the peripheral agent, U, on the number of peripheral agents, ν.

If $b_0 - b < 2a$	If $b_0 - b = 2a$	If $b_0 - b > 2a$
U decreases	U is independent	U increases

Conclusion

Research on the role of heterogeneity of actors/agents in social and economic networks is rather new in the literature. In our model, we assume presence of two types of agents possessing different productivities. On the first stage (in time period 1), each agent in network may invest some resource (money or time) to increase her gain on the second stage (in period 2). The gain depends on her own investment and productivity as well as on investments of her neighbors in the network. Such situations are typical not only for social systems but also for various economic, political, and organizational systems. In framework of the model, we consider relations between network structure, incentives, agents' behavior, and the equilibrium state in terms of welfare (utility) of the agents.

We prove that agent's utility depends monotonously on her environment (the sum of her own investment and her neighbors' investments) and provide description of agent's behavior in terms of pure externalities and in terms of environments. We show that in inner equilibrium, the agent's behavior is completely defined by her generalized α-centrality which depends not only on her position in the network but also on her relative productivity.

We touch some questions of network formation and identify agents potentially interested in particular ways of enlarging the network. In star network, the central agent is always interested in enlarging the networks, while the peripheral agents are interested in this only if their productivity is sufficiently low in comparison with the central agent's productivity.

We introduce adjustment dynamics which may start after a deviation from equilibrium or after a junction of networks initially being in equilibrium.

We study behavior of agents with different productivities in two complete networks after junction. In particular, we are interested how nonadopters (passive agents) change their behavior (become adopters). If a network consisting of nonadopters (passive agents) does unify with a network consisting of adopters (active or hyperactive agents), and the nonadopters possess a low productivity, then there is no transition process, and the nonadopters stay passive. Under somewhat higher productivity, the nonadopters become adopters (come to active state), and under even higher productivity, they become hyperactive.

Agents who are initially active in equilibrium in complete network (which implies that their productivities are sufficiently high) may also increase their level of investment in result of unification with another complete network with hyperactive or active agents. The unified network comes into equilibrium in which all agents are hyperactive.

A natural task for future research is to expand the results to broader classes of networks.

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