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Realization of Morse–Smale Diffeomorphisms on 3-Manifolds

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Abstract—The paper presents a realization of an orientation-preserving Morse–Smale 3-diffeomorphism in each class of the topological conjugacy by means of an abstract scheme.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

Recall that a smooth dynamical system defined on an n-manifold M^n, $n \geq 1$, is called a *Morse–Smale system* if

(1) its nonwandering set consists of a finite number of fixed points and periodic orbits each of which is hyperbolic;

(2) the stable and unstable manifolds W^{s}_p and W^{u}_q intersect transversally for any nonwandering points p and q.

The crucial problem in the study of dynamical systems is that of determining a set of complete topological invariants, i.e., properties of a system that uniquely determine the decomposition of the phase space into trajectories up to topological equivalence (conjugacy). We recall that two diffeomorphisms f and f' on an n-manifold M^n are said to be *topologically conjugate* if there exists a homeomorphism $h\colon M^n \to M^n$ such that $f'h = hf$.

For Morse–Smale diffeomorphisms on a circle, A. Maier [15] found in 1939 a complete topological invariant (with respect to conjugation by orientation-preserving homeomorphisms) consisting of a triple of numbers: the number of periodic orbits, their periods, and the so-called ordinal number. The classification of Morse–Smale diffeomorphisms on surfaces required the search for new topological invariants in view of the existence of heteroclinic orbits that belong to the intersection of invariant manifolds of saddle periodic points. In the case when the number of heteroclinic orbits is finite, a topological invariant was obtained by V. Z. Grines [9], who used an invariant similar to the Peixoto graph [16] and applied heteroclinic permutations that describe the topological type of intersections of invariant manifolds of saddle periodic points. In the case of an infinite set of heteroclinic orbits, a topological classification was obtained by Ch. Bonatti and R. Langevin [6] with the use of the apparatus of topological Markov chains.

The topological classification of even the simplest Morse–Smale diffeomorphisms on 3-manifolds does not fit into the concept of singling out a skeleton consisting of stable and unstable manifolds of periodic orbits. The reason for this lies primarily in the possible "wild" behavior of separatrices of saddle points. More specifically, even though the closure of a separatrix may differ from a separatrix by only one point, it may fail to be even a topological submanifold. D. Pixton [17] in 1977 was the

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first to construct a Morse–Smale diffeomorphism with wild separatrices; to this end he employed the Artin–Fox curve [8] to realize the invariant manifolds of a saddle fixed point. The problem of topological classification of different classes of Morse–Smale diffeomorphisms on 3-manifolds was solved in a series of papers by Ch. Bonatti, V. Grines, F. Laudenbach, V. Medvedev, E. Pécou, and O. Pochinka [1–5] (see also the surveys [11, 13] and book [10]). Quite recently, Bonatti, Grines, and Pochinka have obtained a complete topological classification in the set $\mathrm{MS}(M^3)$ of orientation-preserving Morse–Smale diffeomorphisms defined on a smooth closed orientable 3-manifold M^3 (the complete text will appear soon). Let us describe the complete invariant.

Let $f \in \mathrm{MS}(M^3)$. For $q = 0,1,2,3$ denote by Ω_q the set of all periodic points of f with q-dimensional unstable manifolds. Let us represent the dynamics of f in a "source–sink" form in the following way.

Set $A_f = W^{\mathrm{u}}_{\Omega_0\cup\Omega_1}$, $R_f = W^{\mathrm{s}}_{\Omega_2\cup\Omega_3}$, and $V_f = M^3 \setminus (A_f \cup R_f)$. Then the set A_f (R_f) is a connected attractor (repeller)[1] of f with topological dimension less than or equal to 1, the set V_f is a connected 3-manifold, and $V_f = W^{\mathrm{s}}_{A_f\cap\Omega_f} \setminus A_f = W^{\mathrm{u}}_{R_f\cap\Omega_f} \setminus R_f$. Moreover, the quotient $\widehat{V}_f = V_f/f$ is a closed connected orientable 3-manifold, on which the natural projection $p_f\colon V_f \to \widehat{V}_f$ induces an epimorphism $\eta_f\colon \pi_1(\widehat{V}_f) \to \mathbb{Z}$ sending the homotopy class $[c] \in \pi_1(\widehat{V}_f)$ of a closed curve $c \subset \widehat{V}_f$ to an integer n such that the lift of c to V_f joins a point x with the point $f^n(x)$. Set $\widehat{L}^{\mathrm{s}}_f = p_f(W^{\mathrm{s}}_{\Omega_1} \setminus A_f)$ and $\widehat{L}^{\mathrm{u}}_f = p_f(W^{\mathrm{u}}_{\Omega_2} \setminus R_f)$.

The collection $S_f = \big(\widehat{V}_f, \eta_f, \widehat{L}^{\mathrm{s}}_f, \widehat{L}^{\mathrm{u}}_f\big)$ is called a *scheme* of the diffeomorphism $f \in \mathrm{MS}(M^3)$. The equivalence class (up to homeomorphisms preserving the components of the scheme) of the scheme S_f is a complete topological invariant.

A solution of the realization problem is based on three properties of the scheme S_f.

The first property, due to [7] (see also [12]), is that the quotient $\widehat{V}_f$ is a *prime manifold*; that is, it is either homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ or irreducible (any smooth 2-sphere bounds a 3-ball there).

The second property is that the sets $\widehat{L}^{\mathrm{s}}_f$ and $\widehat{L}^{\mathrm{u}}_f$ are neighborhood transversal intersecting s-lamination and u-lamination (see Definition 6) on $\widehat{V}_f$, respectively, each leaf of which is either a torus or a Klein bottle with empty, finite, or countable set of punctured points.

The third property is connected with the concept of surgery of a manifold along a lamination (see Section 3); namely, the result of such a surgery is a manifold each connected component of which is $\mathbb{S}^2 \times \mathbb{S}^1$.

It turns out that these three necessary properties are sufficient to distinguish a set of realizable abstract schemes (i.e., abstract schemes for each of which there exists a Morse–Smale diffeomorphism whose scheme is equivalent to this abstract scheme).

Definition 1. A collection $S = (\widehat{V}, \eta_{\widehat{V}}, \widehat{L}^{\mathrm{s}}, \widehat{L}^{\mathrm{u}})$ is called an *abstract scheme* if

(1) $\widehat{V}$ is a prime manifold whose fundamental group admits an epimorphism $\eta_{\widehat{V}}\colon \pi_1(\widehat{V}) \to \mathbb{Z}$;

(2) $\widehat{L}^{\mathrm{s}}$ and $\widehat{L}^{\mathrm{u}}$ are neighborhood transversal s-lamination and u-lamination, respectively, on the manifold $\widehat{V}$;

(3) each connected component of the manifold obtained from $\widehat{V}$ by a surgery along the s-lamination $\widehat{L}^{\mathrm{s}}$ (u-lamination $\widehat{L}^{\mathrm{u}}$) is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$.

Denote by $\mathcal{S}$ the set of abstract schemes.

Theorem 1. *For any abstract scheme $S \in \mathcal{S}$ there is a diffeomorphism $f \in \mathrm{MS}(M^3)$ whose scheme is equivalent to the scheme S.*

[1]A compact set $A \subset M^n$ is an *attractor* of a diffeomorphism $f\colon M^n \to M^n$ if there is a neighborhood U of A such that $f(U) \subset \operatorname{int} U$ and $A = \bigcap_{n\in\mathbb{N}} f^n(U)$. A set $R \subset M^n$ is called a *repeller* of f if it is an attractor of f^{-1}.

2. CANONICAL DIFFEOMORPHISMS AND ORBIT SPACES

More detailed information concerning the objects considered below can be found, for example, in [11, 10].

2.1. Canonical diffeomorphisms. For $q \in \{0, \dots, n\}$ and $\nu \in \{-1, +1\}$ denote by $a_{q,\nu}\colon \mathbb{R}^n \to \mathbb{R}^n$ a linear diffeomorphism given by the formula

$$a_{q,\nu}(x_1, \dots, x_n) = \Bigl(\nu \cdot 2x_1, 2x_2, \dots, 2x_q, \nu\frac{x_{q+1}}{2}, \frac{x_{q+2}}{2}, \dots, \frac{x_n}{2}\Bigr).$$

We call $a_{q,\nu}\colon \mathbb{R}^n \to \mathbb{R}^n$ a *canonical diffeomorphism*. Furthermore, we denote by $a^{\mathrm{u}}_{q,\nu}$ and $a^{\mathrm{s}}_{q,\nu}$ the restrictions of the canonical diffeomorphism to $Ox_1 \dots x_q$ and $Ox_{q+1} \dots x_n$ and call the diffeomorphisms $a^{\mathrm{u}}_{q,\nu}$ and $a^{\mathrm{s}}_{q,\nu}$ a *canonical expansion* and a *canonical contraction*, respectively.

For $q \in \{1, \dots, n-1\}$ and $t \in (0, 1]$ let

$$\mathcal{N}^t_q = \bigl\{(x_1, \dots, x_n) \in \mathbb{R}^n\colon\ (x_1^2 + \ldots + x_q^2)(x_{q+1}^2 + \ldots + x_n^2) < t\bigr\}$$

and $\mathcal{N}^1_q = \mathcal{N}_q$. Notice that the set $\mathcal{N}^t_q$ is invariant with respect to the canonical diffeomorphism $a_{q,\nu}$, which has the only fixed saddle point at the origin O, its unstable manifold being $W^{\mathrm{u}}_O = Ox_1 \dots x_q$ and its stable manifold being $W^{\mathrm{s}}_O = Ox_{q+1} \dots x_n$.

In the neighborhood $\mathcal{N}_q$, we define a pair of transversal foliations $\mathcal{F}^{\mathrm{u}}_q$ and $\mathcal{F}^{\mathrm{s}}_q$ in the following way:

$$\mathcal{F}^{\mathrm{u}}_q = \bigcup_{(c_{q+1}, \dots, c_n) \in Ox_{q+1} \dots x_n} \bigl\{(x_1, \dots, x_n) \in \mathcal{N}_q\colon\ (x_{q+1}, \dots, x_n) = (c_{q+1}, \dots, c_n)\bigr\},$$

$$\mathcal{F}^{\mathrm{s}}_q = \bigcup_{(c_1, \dots, c_q) \in Ox_1 \dots x_q} \bigl\{(x_1, \dots, x_n) \in \mathcal{N}_q\colon\ (x_1, \dots, x_q) = (c_1, \dots, c_q)\bigr\}.$$

Notice that the canonical diffeomorphism $a_{q,\nu}$ sends the leaves of the foliation $\mathcal{F}^{\mathrm{u}}_q$ ($\mathcal{F}^{\mathrm{s}}_q$) to the leaves of the same foliation.

2.2. Orbit spaces. In this subsection we consider the topology of an orbit space for some diffeomorphism $g\colon X \to X$ on a manifold X. We use the notation X/g for g-*orbits on* X and $p_{X/g}\colon X \to X/g$ for the natural projection. Recall that a *fundamental domain of the action of* g *on* X is a closed set $D_g \subset X$ such that there is a set $\widetilde{D}_g$ with the following properties:

(1) $\operatorname{cl} \widetilde{D}_g = D_g$;

(2) $g^k(\widetilde{D}_g) \cap \widetilde{D}_g = \varnothing$ for all $k \in \mathbb{Z} \setminus \{0\}$;

(3) $\bigcup_{k \in \mathbb{Z}} g^k(\widetilde{D}_g) = X$.

We say that g acts *discontinuously* on X if for each compact set $K \subset X$ the set of $k \in \mathbb{Z}$ such that $g^k(K) \cap K \neq \varnothing$ is finite. In the case of such an action, the projection $p_{X/g}$ is a cover (see Proposition 1 below) and then we can make the following construction. Suppose that the space X/g is connected and denote by n_X the number of connected components of X and by $p^{-1}_{X/g}(\widehat{x})$ the preimage of a point $\widehat{x} \in X/g$ with respect to the cover $p_{X/g}\colon X \to X/g$ (it is an orbit of some point $x \in p^{-1}_{X/g}(\widehat{x})$). Let $\widehat{c}$ be a loop in X/g such that $\widehat{c}(0) = \widehat{c}(1) = \widehat{x}$. Due to the monodromy theorem (see, for example, [14, Corollary 16.6]), there is a unique path c in X that starts at x ($c(0) = x$) and is the lift of $\widehat{c}$. Therefore, there is an element[2] $k \in n_X\mathbb{Z}$ such that $c(1) = g^k(x)$. Let $\eta_{X/g}\colon \pi_1(X/g) \to n_X\mathbb{Z}$ be the map sending $[\widehat{c}]$ to k.

[2] Here $n_X\mathbb{Z}$ denotes the set of integer multiples of n_X.

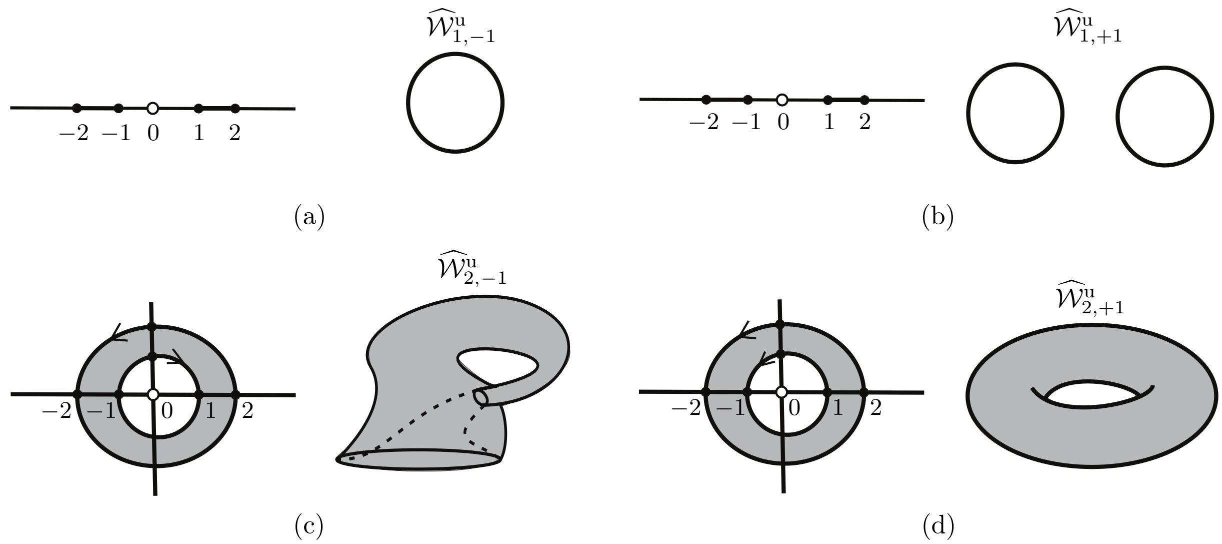

Fig. 1. Orbit spaces of the canonical expansion: (a) $q = 1$ and $\nu = -1$; (b) $q = 1$ and $\nu = +1$; (c) $q = 2$ and $\nu = -1$; (d) $q = 2$ and $\nu = +1$.

Proposition 1. *Let a diffeomorphism g act discontinuously on an n-manifold X. Then*

(1) *the natural projection $p_{X/g}\colon X \to X/g$ is a cover;*

(2) *the quotient X/g is an n-manifold;*

(3) *for a fundamental domain D_g of the action of g on X, the orbit spaces D_g/g and X/g are homeomorphic;*

(4) *the map $\eta_{X/g}\colon \pi_1(X/g) \to n_X\mathbb{Z}$ is an epimorphism.*

Proposition 2. *Let diffeomorphisms g and g' act discontinuously on manifolds X and X', respectively, and let X/g and X'/g' be connected. Then*

(1) *if $h\colon X \to X$ is a homeomorphism such that $hg = g'h$, then the map $\widehat h\colon X/g \to X'/g'$ given by the formula $\widehat h = p_{X'/g'} h p^{-1}_{X/g}$ is a homeomorphism and $\eta_{X/g} = \eta_{X'/g'}\widehat h_*$;*

(2) *if $\widehat h\colon X/g \to X'/g'$ is a homeomorphism such that $\eta_{X/g} = \eta_{X'/g'}\widehat h_*$, then for some $x \in X$ and $x' \in p^{-1}_{X'/g'}(\widehat h(p_{X/g}(x)))$ there is a unique homeomorphism $h\colon X \to X'$ that is a lift of $\widehat h$ and is such that $hg = g'h$ and $h(x) = x'$.*

Consider, for example, the orbit space $\widehat{\mathcal W}^{\rm u}_{q,\nu} = (\mathbb{R}^q \setminus O)/a^{\rm u}_{q,\nu}$ of the action of the canonical expansion $a^{\rm u}_{q,\nu}$ on $\mathbb{R}^q \setminus O$ for $q \in \{1,2,3\}$ and $\nu \in \{+1,-1\}$. It is obvious that this action is discontinuous and its fundamental domain is the annulus $\{(x_1,\dots,x_q) \in \mathbb{R}^q\colon 1 \le x_1^2 + \dots + x_q^2 \le 4\}$ (see Fig. 1), which implies the following list of possibilities:

- the space $\widehat{\mathcal W}^{\rm u}_{1,-1}$ is homeomorphic to the circle;
- the space $\widehat{\mathcal W}^{\rm u}_{1,+1}$ is homeomorphic to the pair of circles;
- the space $\widehat{\mathcal W}^{\rm u}_{2,-1}$ is homeomorphic to the Klein bottle;
- the space $\widehat{\mathcal W}^{\rm u}_{2,+1}$ is homeomorphic to the torus $\mathbb{T}^2$;
- the space $\widehat{\mathcal W}^{\rm u}_{3,+1}$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$.

The definition of the orbit space $\widehat{\mathcal W}^{\rm s}_{q,\nu} = (\mathbb{R}^{n-q} \setminus O)/a^{\rm s}_{q,\nu}$ of the canonical contraction for $q \in \{0,\dots,n-1\}$ and $\nu \in \{+1,-1\}$ is similar.

On the set $\mathcal N^{\rm u}_q = \mathcal N_q \setminus W^{\rm s}_O$ the action of the group $A_{q,\nu} = \{a^k_{q,\nu},\ k \in \mathbb{Z}\}$ is discontinuous. Then the orbit space $\widehat{\mathcal N}^{\rm u}_{q,\nu} = \mathcal N^{\rm u}_q/a_{q,\nu}$ is a smooth n-manifold. Since $a_{q,\nu}|_{W^{\rm u}_O\setminus O} = a^{\rm u}_{q,\nu}|_{W^{\rm u}_O\setminus O}$, the orbit

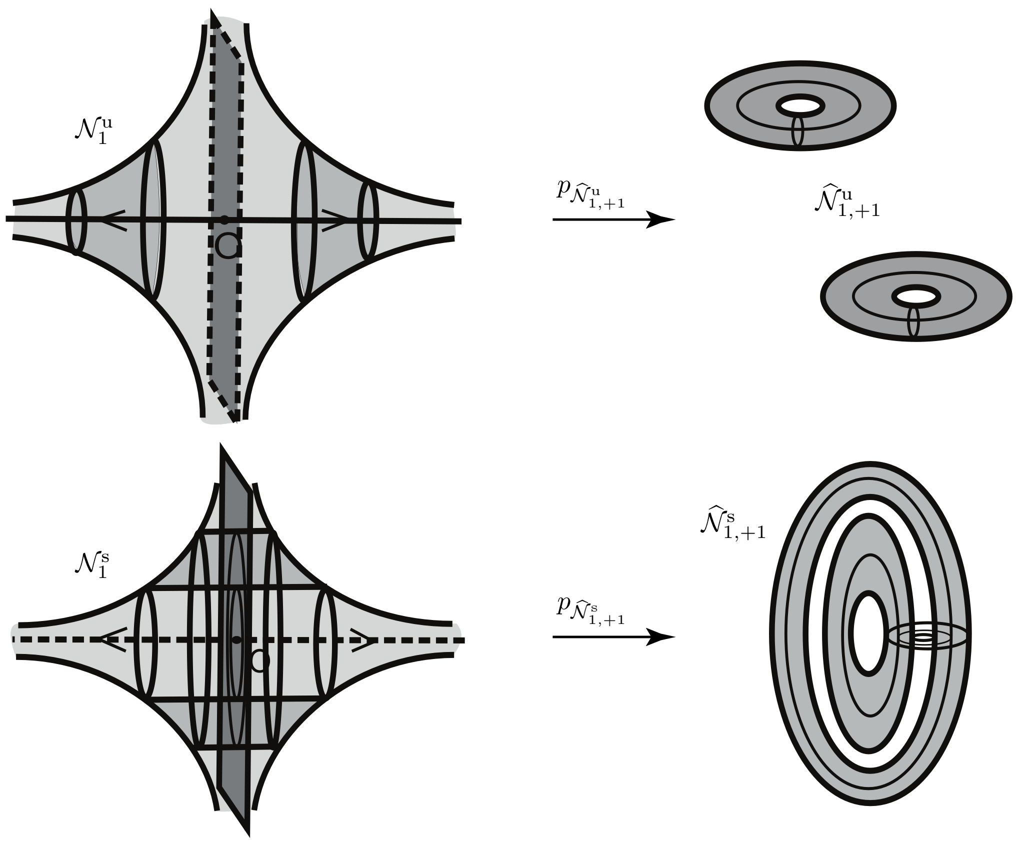

Fig. 2. Neighborhoods of the orbit spaces of the canonical contraction and expansion for $n = 3$.

space $\widehat{\mathcal{N}}^{\mathrm{u}}_{q,\nu}$ is a tubular neighborhood of the space $\widehat{\mathcal{W}}^{\mathrm{u}}_{q,\nu}$. Furthermore, $\widehat{\mathcal{W}}^{\mathrm{u}}_{q,+1}$ is homeomorphic to $\mathbb{S}^{q-1} \times \mathbb{S}^1 \times \{0\}$ and its tubular neighborhood $\widehat{\mathcal{N}}^{\mathrm{u}}_{q,+1}$ is homeomorphic to $\mathbb{S}^{q-1} \times \mathbb{S}^1 \times \mathbb{D}^{n-q}$. Since $a^2_{q,-1} = a^2_{q,+1}$ and the diffeomorphisms $a^2_{q,+1}$ and $a_{q,+1}$ are topologically conjugate, the manifold $\widehat{\mathcal{W}}^{\mathrm{u}}_{q,+1}$ is a two-fold cover for the manifold $\widehat{\mathcal{W}}^{\mathrm{u}}_{q,-1}$ and the manifold $\widehat{\mathcal{N}}^{\mathrm{u}}_{q,+1}$ is a two-fold cover for the neighborhood $\widehat{\mathcal{N}}^{\mathrm{u}}_{q,-1}$.

Similarly one defines the orbit space $\widehat{\mathcal{N}}^{\mathrm{s}}_{q,\nu} = \mathcal{N}^{\mathrm{s}}_q / a^{\mathrm{s}}_{q,\nu}$ (where $\mathcal{N}^{\mathrm{s}}_q = \mathcal{N}_q \setminus W^{\mathrm{u}}_O$), the covering map $p_{\widehat{\mathcal{N}}^{\mathrm{s}}_{q,\nu}} \colon \mathcal{N}^{\mathrm{s}}_q \to \widehat{\mathcal{N}}^{\mathrm{s}}_{q,\nu}$, and the map $\eta_{\widehat{\mathcal{N}}^{\mathrm{s}}_{q,\nu}}$ from the union of the fundamental groups of the connected components of the manifold $\widehat{\mathcal{N}}^{\mathrm{s}}_{q,\nu}$ into the group $\mathbb{Z}$.

Figure 2 shows these objects for $n = 3$, $q = 1$, and $\nu = +1$. To make the structure of the orbit spaces $\widehat{\mathcal{N}}^{\mathrm{s}}_{q,\nu}$ and $\widehat{\mathcal{N}}^{\mathrm{u}}_{q,\nu}$ more clear, we mark out the fundamental domains of the action of the canonical diffeomorphism $a_{q,\nu}$ on the sets $\mathcal{N}^{\mathrm{s}}_q$ and $\mathcal{N}^{\mathrm{u}}_q$.

3. SURGERIES

In this section, $\widehat{V}$ is a prime connected 3-manifold admitting a homomorphism $\eta_{\widehat{V}} \colon \pi_1(\widehat{V}) \to \mathbb{Z}$. The notation $(\widehat{V}, \eta_{\widehat{V}})$ means that the manifold $\widehat{V}$ is equipped with the homomorphism $\eta_{\widehat{V}}$.

Definition 2. Manifolds $(\widehat{V}, \eta_{\widehat{V}})$ and $(\widehat{V}', \eta_{\widehat{V}'})$ are said to be *equivalent* if there is a homeomorphism $\widehat{\varphi} \colon \widehat{V} \to \widehat{V}'$ such that $\eta_{\widehat{V}'} \widehat{\varphi}_* = \eta_{\widehat{V}}$.

Definition 3. Subsets $\widehat{a} \subset (\widehat{V}, \eta_{\widehat{V}})$ and $\widehat{a}' \subset (\widehat{V}', \eta_{\widehat{V}'})$ are said to be *equivalent* if there is a homeomorphism $\widehat{\varphi} \colon \widehat{V} \to \widehat{V}'$ realizing the equivalence of the manifolds $(\widehat{V}, \eta_{\widehat{V}})$ and $(\widehat{V}', \eta_{\widehat{V}'})$ and sending $\widehat{a}$ to $\widehat{a}'$.

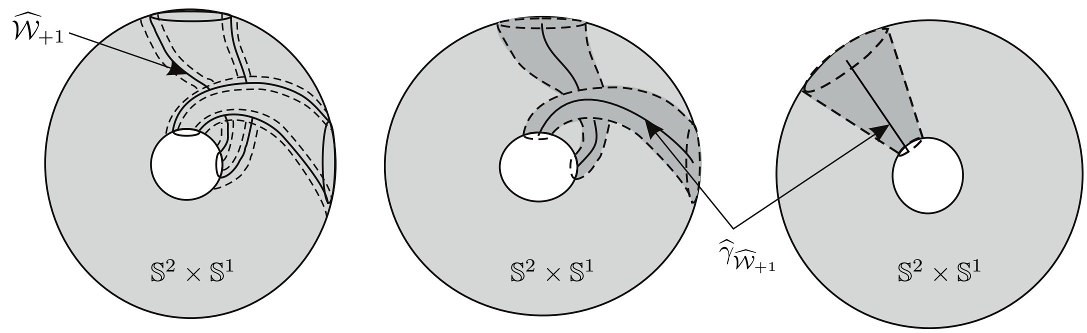

Fig. 3. Surgery on the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ along the torus.

Definition 4. A subset $\widehat{a} \subset (\widehat{V}, \eta_{\widehat{V}})$ is called $\eta_{\widehat{V}}$-*essential* if $\eta_{\widehat{V}}(e_{\widehat{a}*}(\pi_1(\widehat{a}))) \neq 0$, where $e_{\widehat{a}}\colon \widehat{a} \to \widehat{V}$ is the inclusion map.

3.1. Surgery along the torus and the Klein bottle. Let $\widehat{\mathcal{W}}_{+1} \subset (\widehat{V}, \eta_{\widehat{V}})$ be an $\eta_{\widehat{V}}$-essential tame torus and $N(\widehat{\mathcal{W}}_{+1}) \subset \widehat{V}$ be its tubular neighborhood. Then the manifold $N(\widehat{\mathcal{W}}_{+1}) \setminus \widehat{\mathcal{W}}_{+1}$ consists of two connected components each of which is homeomorphic to the manifold $\operatorname{int}\widehat{Y} \setminus \widehat{\gamma}_{\widehat{Y}}$, where $\widehat{Y} = \mathbb{D}^2 \times \mathbb{S}^1$ and $\widehat{\gamma}_{\widehat{Y}} = (\{O\} \times \mathbb{S}^1) \subset \widehat{Y}$. Let $\widehat{\beta}$ be a meridian of the solid torus $\widehat{Y}$ and

$$\zeta_{\widehat{\mathcal{W}}_{+1}}\colon\ \operatorname{cl} N(\widehat{\mathcal{W}}_{+1}) \setminus \widehat{\mathcal{W}}_{+1} \to (\widehat{Y} \setminus \widehat{\gamma}) \times \mathbb{S}^0$$

be a homeomorphism such that $\eta_{\widehat{V}}\big([\zeta^{-1}_{\widehat{\mathcal{W}}_{+1}}(\widehat{\beta} \times \{\pm 1\})]\big) = 0$.

Definition 5. We say that the space $\widehat{V}_{\widehat{\mathcal{W}}_{+1}} = (\widehat{V} \setminus \widehat{\mathcal{W}}_{+1}) \cup_{\zeta_{\widehat{\mathcal{W}}_{+1}}} (\operatorname{int}\widehat{Y} \times \mathbb{S}^0)$ is obtained from the manifold $\widehat{V}$ by a *surgery along the torus* $\widehat{\mathcal{W}}_{+1}$.

In a similar way one defines a surgery of the manifold $(\widehat{V}, \eta_{\widehat{V}})$ along an $\eta_{\widehat{V}}$-essential tame Klein bottle $\widehat{\mathcal{W}}_{-1}$, based on the fact that the tubular neighborhood $N(\widehat{\mathcal{W}}_{-1})$ of the Klein bottle $\widehat{\mathcal{W}}_{-1}$ without the Klein bottle is homeomorphic to the manifold $\operatorname{int}\widehat{Y} \setminus \widehat{\gamma}_{\widehat{Y}}$.

The structure of the manifolds $\widehat{V} \setminus \widehat{\mathcal{W}}_\nu$, $\nu \in \{+1, -1\}$, and $\widehat{Y}$ induces the structure of an orientable 3-manifold without boundary on the space $\widehat{V}_{\widehat{\mathcal{W}}_\nu}$ via the natural projection $p_{\widehat{\mathcal{W}}_\nu}\colon (\widehat{V} \setminus \widehat{\mathcal{W}}_\nu) \cup (\operatorname{int}\widehat{Y} \times \mathbb{S}^0) \to \widehat{V}_{\widehat{\mathcal{W}}_\nu}$. The surgery is well-defined; that is, it does not depend (up to homeomorphism) on the choice of a tubular neighborhood $N(\widehat{\mathcal{W}}_\nu)$ of the surface $\widehat{\mathcal{W}}_\nu$ and the homeomorphism $\zeta_{\widehat{\mathcal{W}}_\nu}$. The epimorphism $\eta_{\widehat{V}}$ induces a unique map $\eta_{\widehat{V}_{\widehat{\mathcal{W}}_\nu}}$ composed of nontrivial homomorphisms to the group $\mathbb{Z}$ on the fundamental group of each connected component of the manifold $\widehat{V}_{\widehat{\mathcal{W}}_\nu}$ with $\eta_{\widehat{V}_{\widehat{\mathcal{W}}_\nu}}([p_{\widehat{\mathcal{W}}_\nu}(c)]) = \eta_{\widehat{V}}([c])$ for every closed curve $c \subset \widehat{V} \setminus \widehat{\mathcal{W}}_\nu$.

Set $(\widehat{V}, \eta_{\widehat{V}})_{\widehat{\mathcal{W}}_\nu} = \big(\widehat{V}_{\widehat{\mathcal{W}}_\nu}, \eta_{\widehat{V}_{\widehat{\mathcal{W}}_\nu}}\big)$ and call the set $\widehat{\gamma}_{\widehat{\mathcal{W}}_\nu} = p_{\widehat{\mathcal{W}}_\nu}(\widehat{\gamma}_{\widehat{Y}} \times \mathbb{S}^0)$ the *trace of the surgery along the surface* $\widehat{\mathcal{W}}_\nu$. It is obvious that each connected component of the trace $\widehat{\gamma}_{\widehat{\mathcal{W}}_\nu}$ is an $\eta_{\widehat{V}_{\widehat{\mathcal{W}}_\nu}}$-essential knot.

Let us represent the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ as an orbit space of the action of the canonical contraction $a^{\mathrm{s}}_{3,+1}$ on $\mathbb{R}^3 \setminus O$. Figure 3 shows the surgery of the 3-manifold $\mathbb{S}^2 \times \mathbb{S}^1$ along an $\eta_{\mathbb{S}^2\times\mathbb{S}^1}$-essential torus $\widehat{\mathcal{W}}_{+1}$ for which $\eta_{\mathbb{S}^2\times\mathbb{S}^1}\big(i_{\widehat{\mathcal{W}}_{+1}*}(\pi_1(\widehat{\mathcal{W}}_{+1}))\big) = 2\mathbb{Z}$.

There is a natural generalization of the surgery to the case when the manifold $\widehat{V}$ consists of finitely many connected components $\widehat{V}^1, \dots, \widehat{V}^r$ and the map $\eta_{\widehat{V}}$ is composed of nontrivial homomorphisms $\eta_{\widehat{V}^1}\colon \pi_1(\widehat{V}^1) \to \mathbb{Z}, \dots, \eta_{\widehat{V}^r}\colon \pi_1(\widehat{V}^r) \to \mathbb{Z}$. The result of such a surgery is also denoted by $(\widehat{V}, \eta_{\widehat{V}})_{\widehat{\mathcal{W}}_\nu} = \big(\widehat{V}_{\widehat{\mathcal{W}}_\nu}, \eta_{\widehat{V}_{\widehat{\mathcal{W}}_\nu}}\big)$.

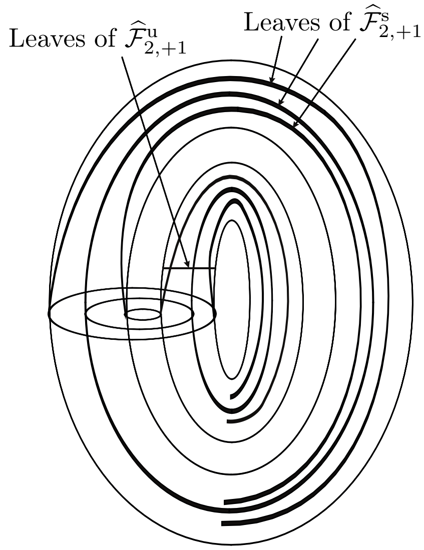

Fig. 4. Leaves of the foliations $\widehat{\mathcal{F}}^{\mathrm{s}}_{2,+1}$ and $\widehat{\mathcal{F}}^{\mathrm{u}}_{2,+1}$.

3.2. Surgery along a lamination. Let us generalize the surgery in the following way. Recall that $a_{2,\nu}\colon \mathbb{R}^3\to\mathbb{R}^3$ is the canonical diffeomorphism defined as $a_{2,\nu}(x_1,x_2,x_3)=(\nu\cdot 2x_1,2x_2,\nu x_3/2)$ and $a^{\mathrm{u}}_{2,\nu}=a_{2,\nu}|_{W^{\mathrm{u}}_O}$. Furthermore, the orbit space of the canonical expansion $\widehat{\mathcal{W}}^{\mathrm{u}}_{2,\nu}=(W^{\mathrm{u}}_O\setminus O)/a^{\mathrm{u}}_{2,\nu}$ is the 2-torus for $\nu=+1$ and the Klein bottle for $\nu=-1$. The set

$$\mathcal{N}_2=\bigl\{(x_1,x_2,x_3)\in\mathbb{R}^3\colon\ (x_1^2+x_2^2)x_3^2<1\bigr\}$$

is $a_{2,\nu}$-invariant, $\mathcal{N}^{\mathrm{u}}_2=\mathcal{N}_2\setminus W^{\mathrm{s}}_O$, and $\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu}=\mathcal{N}^{\mathrm{u}}_2/a_{2,\nu}$ is a tubular neighborhood of the surface $\widehat{\mathcal{W}}^{\mathrm{u}}_{2,\nu}$. The natural projection $p_{\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu}}\colon \mathcal{N}^{\mathrm{u}}_2\to\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu}$ is a cover, which induces an epimorphism $\eta_{\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu}}\colon \pi_1(\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu})\to\mathbb{Z}$.

Denote by $\widehat{\mathcal{F}}^{\mathrm{s}}_{2,\nu}$ and $\widehat{\mathcal{F}}^{\mathrm{u}}_{2,\nu}$ two transversal foliations on $\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu}$ whose leaves are the projections under $p_{\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu}}$ of the leaves of the foliations $\mathcal{F}^{\mathrm{s}}_2$ and $\mathcal{F}^{\mathrm{u}}_2$, respectively (see Fig. 4).

Let $X\subset\widehat{\mathcal{W}}^{\mathrm{u}}_{2,\nu}$ be an at most countable set of points and Z be the union of all leaves of the foliation $\widehat{\mathcal{F}}^{\mathrm{s}}_{2,\nu}$ that pass through the points of the set X. Set

$$\widehat{\mathcal{W}}^{\mathrm{u}}_{2,\nu,X}=\widehat{\mathcal{W}}^{\mathrm{u}}_{2,\nu}\setminus X,\qquad \widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu,X}=\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu}\setminus Z,\qquad \widehat{\mathcal{F}}^{\mathrm{u}}_{2,\nu,X}=\widehat{\mathcal{F}}^{\mathrm{u}}_{2,\nu}\setminus Z,\qquad \widehat{\mathcal{F}}^{\mathrm{s}}_{2,\nu,X}=\widehat{\mathcal{F}}^{\mathrm{s}}_{2,\nu}\setminus Z.$$

Definition 6. A compact set $\widehat{L}^{\mathrm{u}}\subset(\widehat{V},\eta_{\widehat{V}})$ is called a *u-lamination* if it consists of pairwise disjoint sets $\widehat{\ell}^{\mathrm{u}}_0,\dots,\widehat{\ell}^{\mathrm{u}}_n$ such that each connected component $\widehat{l}^{\mathrm{u}}_0$ of $\widehat{\ell}^{\mathrm{u}}_0$ is either a torus or a Klein bottle, each connected component $\widehat{l}^{\mathrm{u}}_i$ of $\widehat{\ell}^{\mathrm{u}}_i$ for $i>0$ is either a punctured torus or a punctured Klein bottle, and $\operatorname{cl}\widehat{\ell}^{\mathrm{u}}_i\setminus\widehat{\ell}^{\mathrm{u}}_i\subset\bigcup_{j=0}^{i-1}\widehat{\ell}^{\mathrm{u}}_j$ for $i>0$. Moreover, for each $i=0,\dots,n$ and for every connected component $\widehat{l}^{\mathrm{u}}_i$ of $\widehat{\ell}^{\mathrm{u}}_i$ there is a tubular neighborhood $N(\widehat{l}^{\mathrm{u}}_i)$ of $\widehat{l}^{\mathrm{u}}_i$ and there are numbers $m_{\widehat{l}^{\mathrm{u}}_i}\in\mathbb{N}$, $\nu_{\widehat{l}^{\mathrm{u}}_i}\in\{-1,+1\}$, a set $X_{\widehat{l}^{\mathrm{u}}_i}\subset\widehat{\mathcal{W}}^{\mathrm{u}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_i}}$, and a homeomorphism $\widehat{\mu}_{\widehat{l}^{\mathrm{u}}_i}\colon N(\widehat{l}^{\mathrm{u}}_i)\to\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_i},X_{\widehat{l}^{\mathrm{u}}_i}}$ with the following properties:

(1) $\widehat{\mu}_{\widehat{l}^{\mathrm{u}}_i}(\widehat{l}^{\mathrm{u}}_i)=\widehat{\mathcal{W}}^{\mathrm{u}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_i},X_{\widehat{l}^{\mathrm{u}}_i}}$ and every leaf of $\widehat{\mu}^{-1}_{\widehat{l}^{\mathrm{u}}_i}\bigl(\widehat{\mathcal{F}}^{\mathrm{u}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_i},X_{\widehat{l}^{\mathrm{u}}_i}}\bigr)$ and $\widehat{\mu}^{-1}_{\widehat{l}^{\mathrm{u}}_i}\bigl(\widehat{\mathcal{F}}^{\mathrm{s}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_i},X_{\widehat{l}^{\mathrm{u}}_i}}\bigr)$ is C^1-smooth;

(2) $\eta_{\widehat{V}}([c])=m_{\widehat{l}^{\mathrm{u}}_i}\eta_{\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_i}}}\bigl(\widehat{\mu}_{\widehat{l}^{\mathrm{u}}_i}([c])\bigr)$ for every closed curve $c\subset N(\widehat{l}^{\mathrm{u}}_i)$;

(3) for $j<i$ and for every leaf $\mathcal{D}$ of the foliation $\widehat{\mathcal{F}}^{\mathrm{u}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_i},X_{\widehat{l}^{\mathrm{u}}_i}}$, the intersection $\widehat{\mu}_{\widehat{l}^{\mathrm{u}}_j}\bigl(N(\widehat{l}^{\mathrm{u}}_j)\cap\widehat{\mu}^{-1}_{\widehat{l}^{\mathrm{u}}_i}(\mathcal{D})\bigr)$ is either an empty set or a union of leaves of the foliation $\widehat{\mathcal{F}}^{\mathrm{u}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_j},X_{\widehat{l}^{\mathrm{u}}_j}}$.

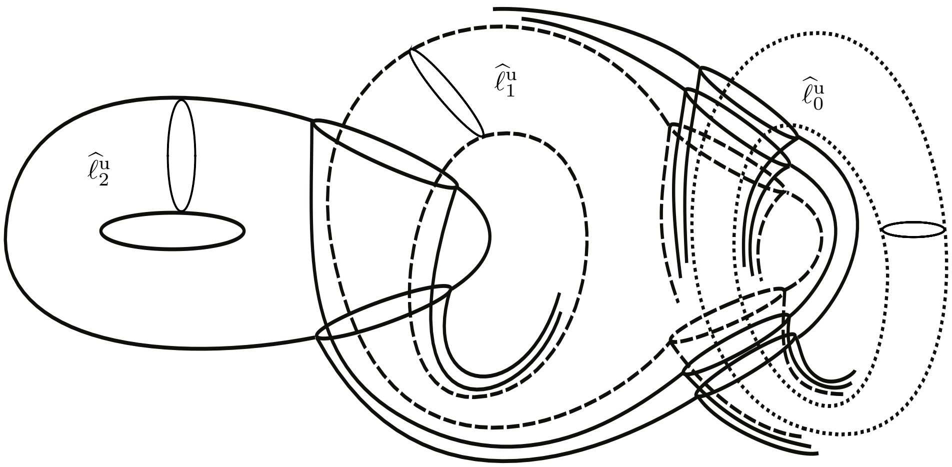

Fig. 5. A u-lamination from three tori.

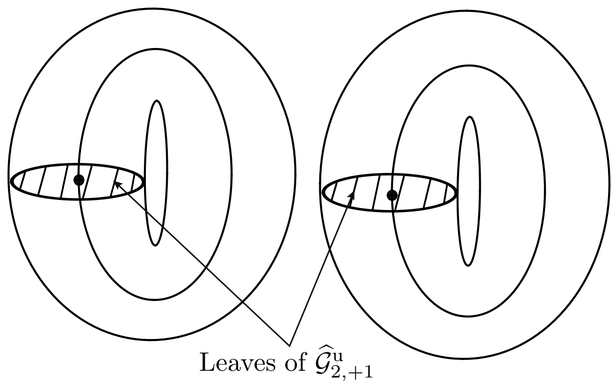

Fig. 6. Leaves of the foliation $\widehat{\mathcal{G}}^{\mathrm{u}}_{2,+1}$.

Figure 5 shows a u-lamination for which each path-connected component is a torus.

Let us consider the canonical contraction $a^{\mathrm{s}}_{2,\nu} = a_{2,\nu}|_{W^{\mathrm{s}}_O}$. Its orbit space $\widehat{\mathcal{W}}^{\mathrm{s}}_{2,\nu} = (W^{\mathrm{s}}_O \setminus O)/a^{\mathrm{s}}_{2,\nu}$ is a pair of knots for $\nu = +1$ and a knot for $\nu = -1$. The set $\widehat{\mathcal{N}}^{\mathrm{s}}_{2,\nu} = \mathcal{N}^{\mathrm{s}}_2/a_{2,\nu}$ is a tubular neighborhood of $\widehat{\mathcal{W}}^{\mathrm{s}}_{2,\nu}$, where $\mathcal{N}^{\mathrm{s}}_2 = \mathcal{N}_2 \setminus W^{\mathrm{u}}_O$. The natural projection $p_{\widehat{\mathcal{N}}^{\mathrm{s}}_{2,\nu}} \colon \mathcal{N}^{\mathrm{s}}_2 \to \widehat{\mathcal{N}}^{\mathrm{s}}_{2,\nu}$ is a cover, which induces the map $\eta_{\widehat{\mathcal{N}}^{\mathrm{s}}_{2,\nu}}$ composed of nontrivial homomorphisms to the group $\mathbb{Z}$ on the fundamental group of each connected component of the manifold $\widehat{\mathcal{N}}^{\mathrm{s}}_{2,\nu}$. Denote by $\widehat{\mathcal{G}}^{\mathrm{u}}_{2,\nu}$ a foliation on $\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu}$ whose leaves are the projections under $p_{\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu}}$ of the leaves of the foliation $\mathcal{F}^{\mathrm{u}}_2$ (see Fig. 6). Define a diffeomorphism $\zeta_{2,\nu} \colon \widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu} \setminus \widehat{\mathcal{W}}^{\mathrm{u}}_{2,\nu} \to \widehat{\mathcal{N}}^{\mathrm{s}}_{2,\nu} \setminus \widehat{\mathcal{W}}^{\mathrm{s}}_{2,\nu}$ by the formula $\zeta_{2,\nu} = p_{\widehat{\mathcal{N}}^{\mathrm{s}}_{2,\nu}}\bigl(p_{\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu}}\big|_{\widehat{\mathcal{N}}^{\mathrm{u}}_{2,\nu} \setminus \widehat{\mathcal{W}}^{\mathrm{u}}_{2,\nu}}\bigr)^{-1}$.

Let $\widehat{L}^{\mathrm{u}} = \bigcup_{i=0}^{n} \widehat{\ell}^{\mathrm{u}}_i$ be a u-lamination on the manifold $(\widehat{V}, \eta_{\widehat{V}})$. Each connected component $\widehat{l}^{\mathrm{u}}_0$ of $\widehat{\ell}^{\mathrm{u}}_0$ is a closed surface, and $\zeta_{\widehat{\ell}^{\mathrm{u}}_0} = \zeta_{2,\nu}\mu_{\widehat{l}^{\mathrm{u}}_0}\big|_{N(\widehat{l}^{\mathrm{u}}_0) \setminus \widehat{l}^{\mathrm{u}}_0}$ is a homeomorphism. The surgery of the manifold $(\widehat{V}, \eta_{\widehat{V}})$ along the surface $\widehat{l}^{\mathrm{u}}_0$ by means of the homeomorphism $\zeta_{\widehat{\ell}^{\mathrm{u}}_0} = \zeta_{2,\nu}\mu_{\widehat{l}^{\mathrm{u}}_0}\big|_{N(\widehat{l}^{\mathrm{u}}_0) \setminus \widehat{l}^{\mathrm{u}}_0}$ is called a *surgery along a compact u-lamination surface*. Denote by $G_{\widehat{l}^{\mathrm{u}}_0}$ the union of the leaves d of the foliation $\widehat{\mathcal{G}}^{\mathrm{u}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_0}}$ such that $p_{\widehat{l}^{\mathrm{u}}_0}(\widehat{L}^{\mathrm{u}}) \cap p_{\widehat{l}^{\mathrm{u}}_0}(d) \neq \varnothing$. We will perform such a surgery along all surfaces from $\widehat{\ell}^{\mathrm{u}}_0$ and denote by $p_{\widehat{\ell}^{\mathrm{u}}_0}$ the corresponding surgery projection. Set $G_{\widehat{\ell}^{\mathrm{u}}_0} = \bigcup_{\widehat{l}^{\mathrm{u}}_0 \subset \widehat{\ell}^{\mathrm{u}}_0} G_{\widehat{l}^{\mathrm{u}}_0}$. For $j = 1, \dots, n$ set $\check{\ell}^{\mathrm{u}}_j = p_{\widehat{\ell}^{\mathrm{u}}_0}\bigl(\widehat{\ell}^{\mathrm{u}}_{j+1} \cup G_{\widehat{\ell}^{\mathrm{u}}_0}\bigr)$. Set $\check{L}^{\mathrm{u}} = \bigcup_{j=0}^{n} \check{\ell}^{\mathrm{u}}_j$. It is obvious that the set $\check{L}^{\mathrm{u}}$ is again a u-lamination on the manifold $(\widehat{V}, \eta_{\widehat{V}})_{\widehat{\ell}^{\mathrm{u}}_0}$.

Definition 7. Let $\widehat{L}^{\mathrm{u}} = \bigcup_{i=0}^{n} \widehat{\ell}_i^{\mathrm{u}}$ be a u-lamination on the manifold $(\widehat{V}, \eta_{\widehat{V}})$. We say that a manifold $\widehat{V}_{\widehat{L}^{\mathrm{u}}}$ is obtained from the manifold $\widehat{V}$ by a *surgery along the u-lamination* $\widehat{L}^{\mathrm{u}}$ if it is obtained from $\widehat{V}$ by a series of n surgeries along compact surfaces of laminations.

Denote by $\eta_{\widehat{V}_{\widehat{L}^{\mathrm{u}}}}$ a map which is induced by this operation and consists of nontrivial homomorphisms to the group $\mathbb{Z}$ on the fundamental group of each connected component of the manifold $(\widehat{V}, \eta_{\widehat{V}})_{\widehat{L}^{\mathrm{u}}}$. Set $(\widehat{V}, \eta_{\widehat{V}})_{\widehat{L}^{\mathrm{u}}} = \big(\widehat{V}_{\widehat{L}^{\mathrm{u}}}, \eta_{\widehat{V}_{\widehat{L}^{\mathrm{u}}}}\big)$.

Similarly one can introduce an s-lamination $\widehat{L}^{\mathrm{s}} \subset (\widehat{V}, \eta_{\widehat{V}})$ using the canonical diffeomorphism $a_{1,\nu}\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the formula $a_{1,\nu}(x_1, x_2, x_3) = (\nu \cdot 2x_1, \nu x_2/2, x_3/2)$, the canonical contraction $a^{\mathrm{s}}_{1,\nu} = a_{1,\nu}|_{W^{\mathrm{s}}_O}$, the orbit space of the canonical contraction $\widehat{\mathcal{W}}^{\mathrm{s}}_{1,\nu} = (W^{\mathrm{s}}_O \setminus O)/a^{\mathrm{s}}_{1,\nu}$ and its tubular neighborhood $\widehat{\mathcal{N}}^{\mathrm{s}}_{1,\nu} = \mathcal{N}^{\mathrm{s}}_1/a_{1,\nu}$, where

$$\mathcal{N}_1 = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3\colon\ x_1^2(x_2^2 + x_3^2) < 1\big\}$$

and $\mathcal{N}^{\mathrm{s}}_1 = \mathcal{N}_1 \setminus W^{\mathrm{u}}_O$. Moreover, a surgery of the manifold $(\widehat{V}, \eta_{\widehat{V}})$ along an s-lamination $\widehat{L}^{\mathrm{s}}$ is defined in a similar way.

Definition 8. Let us call laminations $\widehat{L}^{\mathrm{s}}, \widehat{L}^{\mathrm{u}} \subset (\widehat{V}, \eta_{\widehat{V}})$ *neighborhood transversal* if

$$\widehat{\mathcal{F}}^{\mathrm{s}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_j}, X_{\widehat{l}^{\mathrm{u}}_i}, x} \cap N(\widehat{l}^{\mathrm{s}}_j) \subset \widehat{\mathcal{F}}^{\mathrm{s}}_{1,\nu_{\widehat{l}^{\mathrm{s}}_j}, X_{\widehat{l}^{\mathrm{s}}_j}, x} \qquad \text{and} \qquad \widehat{\mathcal{F}}^{\mathrm{u}}_{1,\nu_{\widehat{l}^{\mathrm{s}}_j}, X_{\widehat{l}^{\mathrm{s}}_j}, x} \cap N(\widehat{l}^{\mathrm{u}}_i) \subset \widehat{\mathcal{F}}^{\mathrm{u}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_i}, X_{\widehat{l}^{\mathrm{u}}_i}, x}$$

for the leaves

$$\widehat{\mathcal{F}}^{\mathrm{s}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_j}, X_{\widehat{l}^{\mathrm{u}}_i}, x}, \quad \widehat{\mathcal{F}}^{\mathrm{s}}_{1,\nu_{\widehat{l}^{\mathrm{s}}_j}, X_{\widehat{l}^{\mathrm{s}}_j}, x}, \quad \widehat{\mathcal{F}}^{\mathrm{u}}_{1,\nu_{\widehat{l}^{\mathrm{s}}_j}, X_{\widehat{l}^{\mathrm{s}}_j}, x}, \quad \widehat{\mathcal{F}}^{\mathrm{u}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_i}, X_{\widehat{l}^{\mathrm{u}}_i}, x}$$

of the foliations

$$\widehat{\mathcal{F}}^{\mathrm{s}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_j}, X_{\widehat{l}^{\mathrm{u}}_i}}, \quad \widehat{\mathcal{F}}^{\mathrm{s}}_{1,\nu_{\widehat{l}^{\mathrm{s}}_j}, X_{\widehat{l}^{\mathrm{s}}_j}}, \quad \widehat{\mathcal{F}}^{\mathrm{u}}_{1,\nu_{\widehat{l}^{\mathrm{s}}_j}, X_{\widehat{l}^{\mathrm{s}}_j}}, \quad \widehat{\mathcal{F}}^{\mathrm{u}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_i}, X_{\widehat{l}^{\mathrm{u}}_i}},$$

respectively, that pass through the point $x \in N(\widehat{l}^{\mathrm{u}}_i) \cap N(\widehat{l}^{\mathrm{s}}_j)$.

4. CONSTRUCTION OF A MODEL DIFFEOMORPHISM

In this section we prove Theorem 1 by showing that for every abstract scheme $S \in \mathcal{S}$ there is a diffeomorphism $f \in \mathrm{MS}(M^3)$ whose scheme is equivalent to the scheme S.

Proof of Theorem 1. Let $S = (\widehat{V}, \eta_{\widehat{V}}, \widehat{L}^{\mathrm{u}}, \widehat{L}^{\mathrm{s}})$ be an abstract scheme. Let us construct step by step a diffeomorphism $f \in \mathrm{MS}(M^3)$ such that the schemes S_f and S are equivalent.

Step 1. Denote by K the kernel of the epimorphism $\eta_{\widehat{V}}\colon \pi_1(\widehat{V}) \to \mathbb{Z}$. Then K is a normal subgroup of $\pi_1(\widehat{V})$ and the group $\pi_1(\widehat{V})/K$ is isomorphic to $\mathbb{Z}$. Since $\widehat{V}$ is a connected 3-manifold, there is a connected 3-manifold V and a lift $p_{\widehat{V}}\colon V \to \widehat{V}$ such that $p_{\widehat{V}*}(\pi_1(V)) = K$. Furthermore, the transformation group $G(V, p_{\widehat{V}}, \widehat{V})$ is isomorphic to $\mathbb{Z}$. Denote by $f_V\colon V \to V$ a diffeomorphism which is the positive generator of the group $G(V, p_{\widehat{V}}, \widehat{V})$; that is, the following condition holds for f_V: every path $c \subset V$ with beginning at a point x and end at the point $f_V(x)$ is projected to a loop $\widehat{c} = p_{\widehat{V}}(c) \subset \widehat{V}$ such that $\eta_{\widehat{V}}([\widehat{c}]) = 1$.

Step 2. It follows from the definition of an abstract scheme that $\widehat{L}^{\mathrm{u}} = \widehat{\ell}^{\mathrm{u}}_0 \cup \ldots \cup \widehat{\ell}^{\mathrm{u}}_n$ is a smooth u-lamination. According to Definition 6, every connected component $\widehat{l}^{\mathrm{u}}_0 \subset \widehat{\ell}^{\mathrm{u}}_0$ is either a torus or a Klein bottle and possesses a tubular neighborhood $N(\widehat{l}^{\mathrm{u}}_0)$ that intersects transversally the set $\widehat{\ell}^{\mathrm{u}}_1 \cup \ldots \cup \widehat{\ell}^{\mathrm{u}}_n$ along $\eta_{\widehat{V}}$-trivial closed curves. Moreover, to every connected component $\widehat{l}^{\mathrm{u}}_0 \subset \widehat{\ell}^{\mathrm{u}}_0$ there correspond numbers $m_{\widehat{l}^{\mathrm{u}}_0} \in \mathbb{N}$ and $\nu_{\widehat{l}^{\mathrm{u}}_0} \in \{-1, +1\}$. Then the set $W_{\widehat{l}^{\mathrm{u}}_0} = p^{-1}_{\widehat{V}}(\widehat{l}^{\mathrm{u}}_0)$ consists of

$m_{\widehat{l}_0^{\mathrm u}}$ cylinders, which we denote by $W_{\widehat{l}_0^{\mathrm u},0},\dots,W_{\widehat{l}_0^{\mathrm u},m_{\widehat{l}_0^{\mathrm u}}}$ (the set $N\big(W_{\widehat{l}_0^{\mathrm u}}\big)=p_{\widehat V}^{-1}(N(\widehat{l}_0^{\mathrm u}))$ is their tubular neighborhood).

Due to Proposition 2 there is a diffeomorphism $g_{\widehat{l}_0^{\mathrm u}}\colon \partial N\big(W_{\widehat{l}_0^{\mathrm u}}\big)\to\partial\mathcal N_2$ conjugating the diffeomorphism $f_V^{m_{\widehat{l}_0^{\mathrm u}}}|_{\partial N(W_{\widehat{l}_0^{\mathrm u}})}$ with the diffeomorphism $a_{2,\nu_{\widehat{l}_0^{\mathrm u}}}|_{\partial\mathcal N_2}$. Define a diffeomorphism $a_{2,\nu_{\widehat{l}_0^{\mathrm u}},m_{\widehat{l}_0^{\mathrm u}}}\colon \mathbb R^3\times\mathbb Z_{m_{\widehat{l}_0^{\mathrm u}}}\to\mathbb R^3\times\mathbb Z_{m_{\widehat{l}_0^{\mathrm u}}}$ by the formula

$$a_{2,\nu_{\widehat{l}_0^{\mathrm u}},m_{\widehat{l}_0^{\mathrm u}}}(x_1,x_2,x_3,j)=\Big(\nu_{\widehat{l}_0^{\mathrm u}}\cdot 2^{1/m_{\widehat{l}_0^{\mathrm u}}}x_1,\nu_{\widehat{l}_0^{\mathrm u}}\cdot 2^{-1/m_{\widehat{l}_0^{\mathrm u}}}x_2,2^{-1/m_{\widehat{l}_0^{\mathrm u}}}x_3,j+1\Big).$$

Then a diffeomorphism $G_{\widehat{l}_0^{\mathrm u}}\colon\partial N\big(W_{\widehat{l}_0^{\mathrm u}}\big)\to\partial\mathcal N_2\times\mathbb Z_{m_{\widehat{l}_0^{\mathrm u}}}$ given by the formula

$$G_{\widehat{l}_0^{\mathrm u}}(x)=a_{2,\nu_{\widehat{l}_0^{\mathrm u}},m_{\widehat{l}_0^{\mathrm u}}}\big(g_{\widehat{l}_0^{\mathrm u}}(f_V^{-j}(x))\big)$$

for every point $x\in f_V^j\big(\partial N\big(W_{\widehat{l}_0^{\mathrm u}}\big)\big)$ conjugates $f_V|_{\partial N(W_{\widehat{l}_0^{\mathrm u}})}$ with $a_{2,\nu_{\widehat{l}_0^{\mathrm u}},m_{\widehat{l}_0^{\mathrm u}}}\big|_{\partial\mathcal N_2\times\mathbb Z_{m_{\widehat{l}_0^{\mathrm u}}}}$.

Let

$$A_{\widehat{l}_0^{\mathrm u}}=V\setminus N\big(W_{\widehat{l}_0^{\mathrm u}}\big),\qquad B_{\widehat{l}_0^{\mathrm u}}=\operatorname{cl}\mathcal N_2\times\mathbb Z_{m_{\widehat{l}_0^{\mathrm u}}},\qquad A'_{\widehat{l}_0^{\mathrm u}}=\partial N\big(W_{\widehat{l}_0^{\mathrm u}}\big),\qquad B'_{\widehat{l}_0^{\mathrm u}}=\partial\mathcal N_2\times\mathbb Z_{m_{\widehat{l}_0^{\mathrm u}}},$$

$$Q_{\widehat{l}_0^{\mathrm u}}=A_{\widehat{l}_0^{\mathrm u}}\cup_{G_{\widehat{l}_0^{\mathrm u}}}B_{\widehat{l}_0^{\mathrm u}},\qquad \overline Q_{\widehat{l}_0^{\mathrm u}}=A_{\widehat{l}_0^{\mathrm u}}\cup B_{\widehat{l}_0^{\mathrm u}}.$$

Denote by $p_{Q_{\widehat{l}_0^{\mathrm u}}}\colon\overline Q_{\widehat{l}_0^{\mathrm u}}\to Q_{\widehat{l}_0^{\mathrm u}}$ the natural projection. Then the projections $p_{A_{\widehat{l}_0^{\mathrm u}}}=p_{Q_{\widehat{l}_0^{\mathrm u}}}|_{A_{\widehat{l}_0^{\mathrm u}}}$ and $p_{B_{\widehat{l}_0^{\mathrm u}}}=p_{Q_{\widehat{l}_0^{\mathrm u}}}|_{B_{\widehat{l}_0^{\mathrm u}}}$ induce a structure of a smooth connected orientable separable 3-manifold without boundary. Let us show that $Q_{\widehat{l}_0^{\mathrm u}}$ is a Hausdorff space.

To this end it suffices to prove that the set

$$E_{Q_{\widehat{l}_0^{\mathrm u}}}=\Big\{(x,y)\in\overline Q_{\widehat{l}_0^{\mathrm u}}\times\overline Q_{\widehat{l}_0^{\mathrm u}}\colon\ p_{Q_{\widehat{l}_0^{\mathrm u}}}(x)=p_{Q_{\widehat{l}_0^{\mathrm u}}}(y)\Big\}$$

is closed in $\overline Q_{\widehat{l}_0^{\mathrm u}}\times\overline Q_{\widehat{l}_0^{\mathrm u}}$ (see, for example, [14]). This is equivalent to the fact that $(x,y)\in E_{Q_{\widehat{l}_0^{\mathrm u}}}$ for every sequence $(x_m,y_m)\in E_{Q_{\widehat{l}_0^{\mathrm u}}}$ converging in the space $\overline Q_{\widehat{l}_0^{\mathrm u}}\times\overline Q_{\widehat{l}_0^{\mathrm u}}$ to a point (x,y). Without loss of generality we can suppose that all points of the sequence x_m belong to the same connected component of $\overline Q_{\widehat{l}_0^{\mathrm u}}$ as x (otherwise we can consider a subsequence with this property) and that the same holds for the sequence y_m and y. Let us consider four possibilities:

(1) $x_m,y_m\in A_{\widehat{l}_0^{\mathrm u}}$;

(2) $x_m,y_m\in B_{\widehat{l}_0^{\mathrm u}}$;

(3) $x_m\in A_{\widehat{l}_0^{\mathrm u}}$ and $y_m\in B_{\widehat{l}_0^{\mathrm u}}$;

(4) $x_m\in B_{\widehat{l}_0^{\mathrm u}}$ and $y_m\in A_{\widehat{l}_0^{\mathrm u}}$.

In cases (1) and (2), $x_m=y_m$. Then $x=y$ and, hence, $(x,y)\in E_{Q_{\widehat{l}_0^{\mathrm u}}}$. In case (3), $x_m\in A_{\widehat{l}_0^{\mathrm u}}$, $y_m\in B_{\widehat{l}_0^{\mathrm u}}$, $y_m=G_{\widehat{l}_0^{\mathrm u}}(x_m)$, and there are two subcases:

(3a) $x\in\partial N\big(W_{\widehat{l}_0^{\mathrm u}}\big)$;

(3b) $x\notin\partial N\big(W_{\widehat{l}_0^{\mathrm u}}\big)$.

In subcase (3a), using the continuity of the map $G_{\widehat{l}_0^{\mathrm u}}$, we get the following chain of equalities:

$$y=\lim_{m\to\infty}y_m=\lim_{m\to\infty}G_{\widehat{l}_0^{\mathrm u}}(x_m)=G_{\widehat{l}_0^{\mathrm u}}\Big(\lim_{m\to\infty}x_m\Big)=G_{\widehat{l}_0^{\mathrm u}}(x).$$

Thus, $(x,y)\in E_{Q_{\widehat{l}^{\mathrm{u}}_0}}$. Let us show that subcase (3b) is impossible. Indeed, if $x\notin\partial N\big(W_{\widehat{l}^{\mathrm{u}}_0}\big)$, then $y=\lim_{m\to\infty}y_m=\lim_{m\to\infty}G_{\widehat{l}^{\mathrm{u}}_0}(x_m)\in\partial\mathcal{N}_2$. This contradicts the fact that the sequence $\{G(x_m)\}$ converges in $B_{\widehat{l}^{\mathrm{u}}_0}$.

In case (4), $x_m\in B_{\widehat{l}^{\mathrm{u}}_0}$, $y_m\in A_{\widehat{l}^{\mathrm{u}}_0}$, and $y_m=G^{-1}_{\widehat{l}^{\mathrm{u}}_0}(x_m)$. Since the closure of $B'_{\widehat{l}^{\mathrm{u}}_0}$ in $\overline{Q}_{\widehat{l}^{\mathrm{u}}_0}$ coincides with $B'_{\widehat{l}^{\mathrm{u}}_0}$, we have $x\in B'_{\widehat{l}^{\mathrm{u}}_0}$ and, as in subcase (3a), $y=G^{-1}_{\widehat{l}^{\mathrm{u}}_0}(x)$; hence, $(x,y)\in E_{Q_{\widehat{l}^{\mathrm{u}}_0}}$.

Thus $Q_{\widehat{l}^{\mathrm{u}}_0}$ is a smooth connected orientable 3-manifold without boundary. Set

$$f_{A_{\widehat{l}^{\mathrm{u}}_0}}=p_{A_{\widehat{l}^{\mathrm{u}}_0}}f_Vp^{-1}_{A_{\widehat{l}^{\mathrm{u}}_0}}\colon\, p_{A_{\widehat{l}^{\mathrm{u}}_0}}\big(A_{\widehat{l}^{\mathrm{u}}_0}\big)\to p_{A_{\widehat{l}^{\mathrm{u}}_0}}\big(A_{\widehat{l}^{\mathrm{u}}_0}\big),\qquad f_{B_{\widehat{l}^{\mathrm{u}}_0}}=p_{B_{\widehat{l}^{\mathrm{u}}_0}}a_{2,\nu_{\widehat{l}^{\mathrm{u}}_0},m_{\widehat{l}^{\mathrm{u}}_0}}p^{-1}_{B_{\widehat{l}^{\mathrm{u}}_0}}\colon\, p_{B_{\widehat{l}^{\mathrm{u}}_0}}\big(B_{\widehat{l}^{\mathrm{u}}_0}\big)\to p_{B_{\widehat{l}^{\mathrm{u}}_0}}\big(B_{\widehat{l}^{\mathrm{u}}_0}\big).$$

By construction the diffeomorphisms $f_{A_{\widehat{l}^{\mathrm{u}}_0}}$ and $f_{B_{\widehat{l}^{\mathrm{u}}_0}}$ coincide on the set $p_{A_{\widehat{l}^{\mathrm{u}}_0}}\big(A'_{\widehat{l}^{\mathrm{u}}_0}\big)=p_{B_{\widehat{l}^{\mathrm{u}}_0}}\big(B'_{\widehat{l}^{\mathrm{u}}_0}\big)$. Then a map $f_{Q_{\widehat{l}^{\mathrm{u}}_0}}\colon Q_{\widehat{l}^{\mathrm{u}}_0}\to Q_{\widehat{l}^{\mathrm{u}}_0}$ given by the formula

$$f_{Q_{\widehat{l}^{\mathrm{u}}_0}}(x)=\begin{cases} f_{A_{\widehat{l}^{\mathrm{u}}_0}}(x), & x\in p_{A_{\widehat{l}^{\mathrm{u}}_0}}\big(A_{\widehat{l}^{\mathrm{u}}_0}\big),\\ f_{B_{\widehat{l}^{\mathrm{u}}_0}}(x), & x\in p_{B_{\widehat{l}^{\mathrm{u}}_0}}\big(B_{\widehat{l}^{\mathrm{u}}_0}\big),\end{cases}$$

is a diffeomorphism of the manifold $Q_{\widehat{l}^{\mathrm{u}}_0}$. By construction the nonwandering set of $f_{Q_{\widehat{l}^{\mathrm{u}}_0}}$ consists of a unique saddle periodic orbit with Morse index 2.

Denote by $L^{\mathrm{u}}_{\widehat{l}^{\mathrm{u}}_0}$ the $f_{Q_{\widehat{l}^{\mathrm{u}}_0}}$-invariant u-lamination on the manifold $Q_{\widehat{\ell}^{\mathrm{u}}_0}$ which coincides with the lamination $p_{Q_{\widehat{l}^{\mathrm{u}}_0}}\big(p^{-1}_{\widehat{V}}(\widehat{\ell}^{\mathrm{u}}_1\cup\ldots\cup\widehat{\ell}^{\mathrm{u}}_n)\big)$ outside $p_{Q_{\widehat{l}^{\mathrm{u}}_0}}\big(N\big(W_{\widehat{l}^{\mathrm{u}}_0}\big)\big)$ and each connected component of which in $p_{Q_{\widehat{l}^{\mathrm{u}}_0}}\big(N\big(W_{\widehat{l}^{\mathrm{u}}_0}\big)\big)$ is a smooth 2-disc that is transversal to the leaves of the foliation $p_{Q_{\widehat{l}^{\mathrm{u}}_0}}\big(p^{-1}_{\widehat{V}}\big(\widehat{\mathcal{F}}^{\mathrm{s}}_{2,\nu_{\widehat{l}^{\mathrm{u}}_0}}\big)\big)$ and coincides with a leaf of the foliation $p_{Q_{\widehat{l}^{\mathrm{u}}_0}}(\mathcal{F}^{\mathrm{u}}_2)$ in some neighborhood $p_{Q_{\widehat{l}^{\mathrm{u}}_0}}(W^{\mathrm{s}}_O)$. The condition that the new lamination is transversal to the old one-dimensional foliation provides a homeomorphism between new and old laminations.

We will carry out the same operation with all connected components of the set $\widehat{\ell}^{\mathrm{u}}_0$ and will get a smooth connected closed orientable 3-manifold $Q_{\widehat{\ell}^{\mathrm{u}}_0}$ and a diffeomorphism $f_{Q_{\widehat{\ell}^{\mathrm{u}}_0}}\colon Q_{\widehat{\ell}^{\mathrm{u}}_0}\to Q_{\widehat{\ell}^{\mathrm{u}}_0}$ with a finite nonwandering set Σ_0 consisting of saddle periodic orbits with Morse index 2 and $f_{Q_{\widehat{\ell}^{\mathrm{u}}_0}}$-invariant u-lamination $L^{\mathrm{u}}_{\widehat{\ell}^{\mathrm{u}}_0}$. Continuing this process, we will get a smooth connected orientable noncompact 3-manifold Q_{u} without boundary and a diffeomorphism $f_{Q_{\mathrm{u}}}\colon Q_{\mathrm{u}}\to Q_{\mathrm{u}}$ whose nonwandering set consists of a finite set Ω_2 of saddle periodic hyperbolic points with Morse index 2.

Step 3. By construction the manifold $Q_{\mathrm{u}}\setminus W^{\mathrm{s}}_{\Sigma^2_{f_{Q_{\mathrm{u}}}}}$ is diffeomorphic to the manifold V, the diffeomorphism $f_{Q_{\mathrm{u}}}|_{Q_{\mathrm{u}}\setminus W^{\mathrm{s}}_{\Sigma^2_{f_{Q_{\mathrm{u}}}}}}$ is topologically conjugate to the diffeomorphism f_V, and hence, by Proposition 2, the orbit spaces $\widehat{V}$ and $\big(Q_{\mathrm{u}}\setminus W^{\mathrm{s}}_{\Sigma^2_{f_{Q_{\mathrm{u}}}}}\big)/f_{Q_{\mathrm{u}}}$ are diffeomorphic. Thus we will identify the objects described. Then, repeating the construction of step 2 with appropriate modifications for the s-lamination $\widehat{L}^{\mathrm{s}}\subset Q_{\mathrm{u}}$, we get a smooth connected orientable noncompact 3-manifold Q_{s} without boundary and a diffeomorphism $f_{Q_{\mathrm{s}}}\colon Q_{\mathrm{s}}\to Q_{\mathrm{s}}$ whose nonwandering set consists of a finite set Ω_1 of saddle periodic hyperbolic points with Morse index 1 and a finite set Ω_2 of saddle periodic hyperbolic points with Morse index 2.

Step 4. Set $C_{\mathrm{s}}=Q_{\mathrm{s}}$ and $C'_{\mathrm{s}}=Q_{\mathrm{s}}\setminus W^{\mathrm{s}}_{\Omega_{f_{Q_{\mathrm{s}}}}}$. Denote by $\widehat{C}'_{\mathrm{s}}$ the orbit space of the action of $f_{Q_{\mathrm{s}}}$ on C'_{s} and by $p_{\widehat{C}'_{\mathrm{s}}}\colon C'_{\mathrm{s}}\to\widehat{C}'_{\mathrm{s}}$ the natural projection. Then the space $\widehat{C}'_{\mathrm{s}}$ is homeomorphic to a manifold which is obtained from the manifold $\widehat{V}$ by a surgery along the s-lamination $\widehat{L}^{\mathrm{s}}$ and is hence homeomorphic to a finite number (which we denote by l_{s}) of copies of $\mathbb{S}^2\times\mathbb{S}^1$. On the fundamental group of each connected component $\widehat{C}'_{\mathrm{s},i}$, $i = 1, \dots, l_{\mathrm{s}}$, of the space $\widehat{C}'_{\mathrm{s}}$, such a surgery induces an epimorphism to the group $r_i^{\mathrm{s}}\mathbb{Z}$, $r_i^{\mathrm{s}} \in \mathbb{N}$. Then the set $C'_{\mathrm{s},i} = p^{-1}_{\widehat{C}'_{\mathrm{s},i}}(\widehat{C}'_{\mathrm{s},i})$ consists of r_i^{s} connected components $C'_{\mathrm{s},i,0}, \dots, C'_{\mathrm{s},i,r_i^{\mathrm{s}}-1}$. Due to Proposition 2, there is a diffeomorphism $\rho_{\mathrm{s},i,0}\colon C'_{\mathrm{s},i,0} \to \mathbb{R}^3 \setminus O$ that conjugates $f_{Q_{\mathrm{s}}}^{r_i^{\mathrm{s}}}|_{C'_{\mathrm{s},i,0}}$ and $a^{\mathrm{s}}_{0,+1}|_{\mathbb{R}^3\setminus O}$.

Define a diffeomorphism $a^{\mathrm{s}}_{0,+1,r_i^{\mathrm{s}}}\colon \mathbb{R}^3 \times \mathbb{Z}_{r_i^{\mathrm{s}}} \to \mathbb{R}^3 \times \mathbb{Z}_{r_i^{\mathrm{s}}}$ by the formula

$$a^{\mathrm{s}}_{0,+1,r_i^{\mathrm{s}}}(x_1, x_2, x_3, j) = \bigl(2^{-1/r_i^{\mathrm{s}}}x_1, 2^{-1/r_i^{\mathrm{s}}}x_2, 2^{-1/r_i^{\mathrm{s}}}x_3, j+1\bigr).$$

Then a diffeomorphism $\rho_{\mathrm{s},i}\colon C'_{\mathrm{s},i} \to (\mathbb{R}^3 \setminus O) \times \mathbb{Z}_{r_i^{\mathrm{s}}}$ given by the formula

$$\rho_{\mathrm{s},i}(x) = a^{\mathrm{s}}_{0,+1,r_i^{\mathrm{s}}}\bigl(\rho_{\mathrm{s},i,0}\bigl(f_{Q_{\mathrm{s}}}^{-j}(x)\bigr)\bigr)$$

for $x \in f^j(C'_{\mathrm{s},i,0})$ conjugates the diffeomorphism $f_{Q_{\mathrm{s}}}|_{C'_{\mathrm{s},i}}$ with the diffeomorphism $a^{\mathrm{s}}_{0,+1,r_i^{\mathrm{s}}}\big|_{(\mathbb{R}^3\setminus O)\times\mathbb{Z}_{r_i^{\mathrm{s}}}}$.

Set

$$D_{\mathrm{s}} = \coprod_{i=1}^{l_{\mathrm{s}}}(\mathbb{R}^3 \times \mathbb{Z}_{r_i^{\mathrm{s}}}), \qquad D'_{\mathrm{s}} = \coprod_{i=1}^{l_{\mathrm{s}}}\bigl((\mathbb{R}^3 \setminus O) \times \mathbb{Z}_{r_i^{\mathrm{s}}}\bigr)$$

and denote by $d_{\mathrm{s}}\colon D_{\mathrm{s}} \to D_{\mathrm{s}}$ a map composed of the diffeomorphisms $a^{\mathrm{s}}_{0,+1,r_1^{\mathrm{s}}}, \dots, a^{\mathrm{s}}_{0,+1,r_{l_{\mathrm{s}}}^{\mathrm{s}}}$ and by $\rho_{\mathrm{s}}\colon C'_{\mathrm{s}} \to D'_{\mathrm{s}}$ a map composed of the diffeomorphisms $\rho_{\mathrm{s},1}, \dots, \rho_{\mathrm{s},l_{\mathrm{s}}}$. Set $R_{\mathrm{s}} = C_{\mathrm{s}} \cup_{\rho_{\mathrm{s}}} D_{\mathrm{s}}$ and $\overline{R}_{\mathrm{s}} = C_{\mathrm{s}} \cup D_{\mathrm{s}}$ and denote by $p_{R_{\mathrm{s}}}\colon \overline{R}_{\mathrm{s}} \to R_{\mathrm{s}}$ the natural projection. As above, proving that the topological space R_{s} is a smooth connected orientable 3-manifold without boundary reduces to checking that the set

$$E_{R_{\mathrm{s}}} = \bigl\{(x,y) \in \overline{R}_{\mathrm{s}} \times \overline{R}_{\mathrm{s}}\colon\ p_{R_{\mathrm{s}}}(x) = p_{R_{\mathrm{s}}}(y)\bigr\}$$

is closed in $\overline{R}_{\mathrm{s}} \times \overline{R}_{\mathrm{s}}$. Let a sequence $(x_m, y_m) \in E_{R_{\mathrm{s}}}$ converge in $\overline{R}_{\mathrm{s}} \times \overline{R}_{\mathrm{s}}$ to a point (x,y), and let us show that the point (x,y) belongs to $E_{R_{\mathrm{s}}}$.

Consider four cases:

(1) $x_m, y_m \in C_{\mathrm{s}}$;

(2) $x_m, y_m \in D_{\mathrm{s}}$;

(3) $x_m \in C_{\mathrm{s}}$ and $y_m \in D_{\mathrm{s}}$;

(4) $x_m \in D_{\mathrm{s}}$ and $y_m \in C_{\mathrm{s}}$.

In cases (1) and (2), $x_m = y_m$. Then $x = y$ and, hence, $(x,y) \in E_{R_{\mathrm{s}}}$. In case (3), $x_m \in C'_{\mathrm{s}}$, $y_m \in D'_{\mathrm{s}}$, $y_m = \rho_{\mathrm{s}}(x_m)$, and there are two possibilities:

(3a) $x \in C'_{\mathrm{s}}$ or

(3b) $x \notin C'_{\mathrm{s}}$.

In subcase (3a), as above, $y = \rho_{\mathrm{s}}(x)$ and, hence, $(x,y) \in E_{R_{\mathrm{s}}}$. Let us show that subcase (3b) is impossible.

Since $C_{\mathrm{s}} \setminus C'_{\mathrm{s}} = \bigcup_{p\in\Omega_{f_{Q_{\mathrm{s}}}}} W^{\mathrm{s}}_p$, we have $x \in W^{\mathrm{s}}_p$ for some saddle point $p \in \Omega_{f_{Q_{\mathrm{s}}}}$. Then there is a subsequence $\{x_{m_j}\}$, a sequence of integers $k_{m_j} \to +\infty$, and a point $z \in W^{\mathrm{u}}_p$ such that the sequence $\bigl\{z_{m_j} = f_{Q_{\mathrm{s}}}^{k_{m_j}}(x_{m_j})\bigr\}$ converges to the point z. Set $w_{m_j} = \rho_{\mathrm{s}}(z_{m_j})$ and $w = \rho_{\mathrm{s}}(z)$. Then the sequence $\{w_{m_j}\}$ converges to $w \in D'_{\mathrm{s}}$. Since $y_m = \rho_{\mathrm{s}}(x_m)$, we have $y_{m_j} = \rho_{\mathrm{s}}(x_{m_j}) = \rho_{\mathrm{s}}\bigl(f_{Q_{\mathrm{s}}}^{-k_{m_j}}(z_{m_j})\bigr)$. Since the diffeomorphism ρ_{s} conjugates $f_{Q_{\mathrm{s}}}|_{C'_{\mathrm{s}}}$ and $d_{\mathrm{s}}|_{D'_{\mathrm{s}}}$, we have $y_{m_j} = d_{\mathrm{s}}^{-k_{m_j}}(\rho_{\mathrm{s}}(z_{m_j})) = d_{\mathrm{s}}^{-k_{m_j}}(w_{m_j})$. Hence the sequence $\bigl\{d_{\mathrm{s}}^{-k_{m_j}}(w_{m_j})\bigr\}$ has no limit in D_{s}, a contradiction.

In case (4), $x_m \in D'_{\mathrm{s}}$, $y_m \in C'_{\mathrm{s}}$, $y_m = \rho_{\mathrm{s}}^{-1}(x_m)$, and there are two possibilities:

(4a) $x \in D'_{\mathrm{s}}$ or

(4b) $x \notin D'_{\mathrm{s}}$.

In subcase (4a), as in subcase (3a), $y = \rho_{\mathrm{s}}^{-1}(x)$ and, hence, $(x, y) \in E_{R_{\mathrm{s}}}$. Let us show that subcase (4b) is impossible.

Since $x_m \in D'_{\mathrm{s}}$ and $x = O$, the sequence $y_m = \rho_{\mathrm{s}}^{-1}(x_m)$ has no limit in C_{s}, a contradiction.

Thus R_{s} is a smooth connected orientable 3-manifold without boundary. Set $p_{C_{\mathrm{s}}} = p_{R_{\mathrm{s}}}|_{C_{\mathrm{s}}}$, $p_{D_{\mathrm{s}}} = p_{R_{\mathrm{s}}}|_{D_{\mathrm{s}}}$, and

$$f_{C_{\mathrm{s}}} = p_{C_{\mathrm{s}}} f_{Q_{\mathrm{s}}} p_{C_{\mathrm{s}}}^{-1}\colon\ p_{C_{\mathrm{s}}}(C_{\mathrm{s}}) \to p_{C_{\mathrm{s}}}(C_{\mathrm{s}}), \qquad f_{D_{\mathrm{s}}} = p_{D_{\mathrm{s}}} d_{\mathrm{s}} p_{D_{\mathrm{s}}}^{-1}\colon\ p_{D_{\mathrm{s}}}(D_{\mathrm{s}}) \to p_{D_{\mathrm{s}}}(D_{\mathrm{s}}).$$

As above, we can prove that a map $f_{R_{\mathrm{s}}}\colon R_{\mathrm{s}} \to R_{\mathrm{s}}$ given by the formula

$$f_{R_{\mathrm{s}}}(x) = \begin{cases} f_{C_{\mathrm{s}}}(x), & x \in p_{C_{\mathrm{s}}}(C_{\mathrm{s}}), \\ f_{D_{\mathrm{s}}}(x), & x \in p_{D_{\mathrm{s}}}(D_{\mathrm{s}}), \end{cases}$$

is a diffeomorphism of the manifold R_{s} whose nonwandering set consists of n_{s} saddle periodic hyperbolic orbits with Morse index 1, of n_{u} saddle periodic hyperbolic orbits with Morse index 2, and of l_{s} sink periodic hyperbolic orbits.

Step 5. Set $C_{\mathrm{u}} = R_{\mathrm{s}}$ and $C'_{\mathrm{u}} = R_{\mathrm{s}} \setminus W^{\mathrm{u}}_{\Omega_{f_{R_{\mathrm{s}}}}}$. Denote by $\widehat{C}'_{\mathrm{u}}$ the orbit space of the action of $f_{R_{\mathrm{s}}}$ on C'_{u} and by $p_{\widehat{C}'_{\mathrm{u}}}\colon C'_{\mathrm{u}} \to \widehat{C}'_{\mathrm{u}}$ the natural projection. Then the space $\widehat{C}'_{\mathrm{u}}$ is homeomorphic to a manifold which is obtained by a surgery of $\widehat{V}$ along $\widehat{L}^{\mathrm{u}}$ and, hence, is homeomorphic to a finite number (denote it by l_{u}) of copies of $\mathbb{S}^2 \times \mathbb{S}^1$. On the fundamental group of each connected component $\widehat{C}'_{\mathrm{u},i}$, $i = 1, \dots, l_{\mathrm{u}}$, of $\widehat{C}'_{\mathrm{u}}$, this surgery induces an epimorphism to the group $r_i^{\mathrm{u}}\mathbb{Z}$, $r_i^{\mathrm{u}} \in \mathbb{N}$. Then the set $C'_{\mathrm{u},i} = p^{-1}_{\widehat{C}'_{\mathrm{u},i}}(\widehat{C}'_{\mathrm{u},i})$ consists of r_i^{u} connected components $C'_{\mathrm{u},i,0}, \dots, C'_{\mathrm{u},i,r_i^{\mathrm{u}}-1}$. Due to Proposition 2, there is a diffeomorphism $\rho_{\mathrm{u},i,0}\colon C'_{\mathrm{u},i,0} \to \mathbb{R}^3 \setminus O$ that conjugates the diffeomorphisms $f_{R_{\mathrm{s}}}^{r_i^{\mathrm{u}}}|_{C'_{\mathrm{u},i,0}}$ and $a^{\mathrm{u}}_{3,+1}|_{\mathbb{R}^3\setminus O}$.

Define a diffeomorphism $a^{\mathrm{u}}_{3,+1,r_i^{\mathrm{u}}}\colon \mathbb{R}^3 \times \mathbb{Z}_{r_i^{\mathrm{u}}} \to \mathbb{R}^3 \times \mathbb{Z}_{r_i^{\mathrm{u}}}$ by the formula

$$a^{\mathrm{s}}_{3,+1,r_i^{\mathrm{u}}}(x_1, x_2, x_3, j) = \bigl(2^{-1/r_i^{\mathrm{u}}} x_1, 2^{-1/r_i^{\mathrm{u}}} x_2, 2^{-1/r_i^{\mathrm{u}}} x_3, j+1\bigr).$$

Then a diffeomorphism $\rho_{\mathrm{u},i}\colon C'_{\mathrm{u},i} \to (\mathbb{R}^3 \setminus O) \times \mathbb{Z}_{r_i^{\mathrm{u}}}$ given by the formula

$$\rho_{\mathrm{u},i}(x) = a^{\mathrm{u}}_{3,+1,r_i^{\mathrm{u}}}\bigl(\rho_{\mathrm{u},i,0}\bigl(f^{-j}_{Q_{\mathrm{u}}}(x)\bigr)\bigr)$$

for $x \in f^j(C'_{\mathrm{u},i,0})$ conjugates $f_{Q_{\mathrm{u}}}|_{C'_{\mathrm{u},i}}$ and $a^{\mathrm{u}}_{3,+1,r_i^{\mathrm{u}}}\big|_{(\mathbb{R}^3\setminus O)\times \mathbb{Z}_{r_i^{\mathrm{u}}}}$.

Set

$$D_{\mathrm{u}} = \coprod_{i=1}^{l_{\mathrm{u}}} (\mathbb{R}^3 \times \mathbb{Z}_{r_i^{\mathrm{u}}}), \qquad D'_{\mathrm{u}} = \coprod_{i=1}^{l_{\mathrm{u}}} \bigl((\mathbb{R}^3 \setminus O) \times \mathbb{Z}_{r_i^{\mathrm{u}}}\bigr).$$

Denote by $d_{\mathrm{u}}\colon D_{\mathrm{u}} \to D_{\mathrm{u}}$ a map consisting of the diffeomorphisms $a^{\mathrm{u}}_{3,+1,r_1^{\mathrm{u}}}, \dots, a^{\mathrm{u}}_{3,+1,r_{l_{\mathrm{u}}}^{\mathrm{u}}}$ and by $\rho_{\mathrm{u}}\colon C'_{\mathrm{u}} \to D'_{\mathrm{u}}$ a map consisting of the diffeomorphisms $\rho_{\mathrm{u},1}, \dots, \rho_{\mathrm{u},l_{\mathrm{u}}}$. Set $R_{\mathrm{u}} = C_{\mathrm{u}} \cup_{\rho_{\mathrm{u}}} D_{\mathrm{u}}$ and $\overline{R}_{\mathrm{u}} = C_{\mathrm{u}} \cup D_{\mathrm{u}}$ and denote by $p_{R_{\mathrm{u}}}\colon \overline{R}_{\mathrm{u}} \to R_{\mathrm{u}}$ the natural projection. As above, proving that R_{u} is a smooth connected orientable 3-manifold without boundary reduces to checking that the set

$$E_{R_{\mathrm{u}}} = \bigl\{(x, y) \in \overline{R}_{\mathrm{u}} \times \overline{R}_{\mathrm{u}}\colon\ p_{R_{\mathrm{u}}}(x) = p_{R_{\mathrm{u}}}(y)\bigr\}$$

is closed in $\overline{R}_{\mathrm{u}} \times \overline{R}_{\mathrm{u}}$. To check this, let us show that if a sequence $(x_m, y_m) \in E_{R_{\mathrm{u}}}$ converges in $\overline{R}_{\mathrm{u}} \times \overline{R}_{\mathrm{u}}$ to a point (x, y), then (x, y) belongs to $E_{R_{\mathrm{u}}}$.

Consider four cases:

(1) $x_m, y_m \in C_{\rm u}$;

(2) $x_m, y_m \in D_{\rm u}$;

(3) $x_m \in C_{\rm u}$ and $y_m \in D_{\rm u}$;

(4) $x_m \in D_{\rm u}$ and $y_m \in C_{\rm u}$.

In cases (1) and (2), $x_m = y_m$. Then $x = y$ and, hence, $(x, y) \in E_{R_{\rm u}}$. In case (3), $x_m \in C'_{\rm u}$, $y_m \in D'_{\rm u}$, $y_m = \rho_{\rm u}(x_m)$, and there are two possibilities:

(3a) $x \in C'_{\rm u}$ or

(3b) $x \notin C'_{\rm u}$.

In subcase (3a), as above, $y = \rho_{\rm u}(x)$ and, hence, $(x, y) \in E_{R_{\rm u}}$. Let us show that subcase (3b) is impossible.

Since $C_{\rm u} \setminus C'_{\rm u} = \bigcup_{p\in\Omega_{f_{R_{\rm s}}}} W^{\rm u}_p$, we have $x \in W^{\rm u}_p$ for some saddle or sink point $p \in \Omega_{f_{R_{\rm s}}}$. Then, as above, we arrive at a contradiction, because the sequence y_m has no limit in $D_{\rm u}$.

In case (4), $x_m \in D'_{\rm u}$, $y_m \in C'_{\rm u}$, $y_m = \rho_{\rm u}^{-1}(x_m)$, and there are two possibilities:

(4a) $x \in D'_{\rm u}$ or

(4b) $x \notin D'_{\rm u}$.

In subcase (4a), as in subcase (3a), $y = \rho_{\rm u}^{-1}(x)$ and, hence, $(x, y) \in E_{R_{\rm s}}$. Let us show that subcase (4b) is impossible.

Since $x_m \in D'_{\rm u}$ and $x = O$, the sequence $y_m = \rho_{\rm u}^{-1}(x_m)$ has no limit in $C_{\rm u}$, a contradiction.

Set $p_{C_{\rm u}} = p_{R_{\rm u}}|_{C_{\rm u}}$, $p_{D_{\rm u}} = p_{R_{\rm u}}|_{D_{\rm u}}$, and

$$f_{C_{\rm u}} = p_{C_{\rm u}} f_{R_{\rm s}} p_{C_{\rm u}}^{-1}\colon\, p_{C_{\rm u}}(C_{\rm u}) \to p_{C_{\rm u}}(C_{\rm u}), \qquad f_{D_{\rm u}} = p_{D_{\rm u}} d_{\rm u} p_{D_{\rm u}}^{-1}\colon\, p_{D_{\rm u}}(D_{\rm u}) \to p_{D_{\rm u}}(D_{\rm u}).$$

As above, we can prove that a map $f_{R_{\rm u}}\colon R_{\rm u} \to R_{\rm u}$ given by the formula

$$f_{R_{\rm u}}(x) = \begin{cases} f_{C_{\rm u}}(x), & x \in p_{C_{\rm u}}(C_{\rm u}), \\ f_{D_{\rm u}}(x), & x \in p_{D_{\rm u}}(D_{\rm u}), \end{cases}$$

is a diffeomorphism of the manifold $R_{\rm u}$ whose nonwandering set consists of $n_{\rm s}$ saddle periodic hyperbolic orbits with Morse index 1, of $n_{\rm u}$ saddle periodic hyperbolic orbits with Morse index 2, of $l_{\rm s}$ sink periodic hyperbolic orbits, and of $l_{\rm u}$ source periodic hyperbolic orbits.

Step 6. In this step we show that the manifold $M^3 = R_{\rm u}$ is compact and, hence, the diffeomorphism $f = f_{R_{\rm u}}$ belongs to the class $\mathrm{MS}(M^3)$ and its scheme is equivalent to the abstract scheme S by construction.

To prove the compactness of $R_{\rm u} = M^3$, it is enough to show that any sequence $\{x_n\} \in M^3$ has a converging subsequence. If infinitely many members of $\{x_n\}$ belong to Ω_f, the fact is obvious. Consider the opposite case. By construction $M^3 = \bigcup_{p\in\Omega_f} W^{\rm s}_p = \bigcup_{p\in\Omega_f} W^{\rm u}_p$. Up to passing to a subsequence there is a point $p_1 \in \Omega_f$ such that $\{x_n\} \subset W^{\rm s}_{p_1} \setminus p_1$. Denote by K the fundamental domain of the restriction of f to $W^{\rm s}_{p_1} \setminus p_1$. Then for each member x_n of the sequence $\{x_n\}$ there is an integer k_n such that $y_n = f^{k_n}(x_n) \in K$. Without loss of generality we can suppose that the sequence $\{y_n\} = \{f^{k_n}(x_n)\}$ converges to a point $y \in K$ (otherwise we can consider a subsequence with this property). For the sequence $\{k_n\}$ there are two possibilities:

(1) $\{k_n\}$ is bounded;

(2) $\{k_n\}$ is not bounded.

In case (1), up to passing to a subsequence, the sequence $\{k_n\}$ converges to an integer k. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^{-k_n}(y_n) = f^{-k}(y)$. Thus a subsequence of $\{x_n\}$ converges to $f^{-k}(y) \in W^{\mathrm{s}}_{p_1}$.

In case (2), up to passing to a subsequence, $\{k_n\}$ converges to $+\infty$ or $-\infty$. If $k_n \to -\infty$, then a subsequence of $\{x_n = f^{-k_n}(y_n)\}$ converges to p_1. If $k_n \to +\infty$, then, up to passing to a subsequence, there is a point $p_2 \in \Omega_f$ such that $\{x_n\} \subset W^{\mathrm{u}}_{p_2} \setminus p_2$ and, hence, a subsequence of $\{x_n = f^{-k_n}(y_n)\}$ converges to p_2. $\square$

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