Ergodic complex structures on hyperkähler manifolds: an erratum

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Abstract

Let M be a hyperkähler manifold, Γ its mapping class group, and Teich the Teichmüller space of complex structures of hyperkähler type. After we glue together birationally equivalent points, we obtain the so-called birational Teichmüller space Teich_b. Every connected component of Teich_b is identified with $\mathbb{P}\mathsf{er} = \frac{SO(3,b_2-3)}{SO(2)\times SO(1,b_2-3)}$ by global Torelli theorem. The mapping class group of M acts on $\mathbb{P}\mathsf{er}$ as a finite index subgroup of the group of isometries of the integer cohomology lattice, that is, satisfies assumptions of Ratner theorem. We prove that there are three classes of orbits, closed, dense and the intermediate class which corresponds to varieties with $\dim \operatorname{Re}(H^{2,0}(M)) \cap H^2(M,\mathbb{Q}) = 1$. The closure of the later orbits is a fixed point set of an anticomplex involution of $\mathbb{P}\mathsf{er}$. This fixes an error in the paper [V2], where this third class of orbits was overlooked. We explain how this affects the works based on [V2].

Contents

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1 Introduction

1.1 Orbits of monodromy action: an extra orbit

This paper is essentially an erratum to [V2]. For an introduction to Teichmüller theory, global Torelli theorem and Ratner theory as applied to the Teichmüller space, please see [V2].

Let M be a hyperkähler manifold, and Teich the Teichmüller space of complex structures of hyperkähler type. Fix a connected component Teich^I of Teich (there are finitely many of them, per [Hu]), and let Γ be the subgroup of mapping class group of M mapping Teich^I to itself. If we glue together the points of Teich^I which are non-separable, we obtain a Hausdorff manifold Teich_b, which is diffeomorphic to its period space $\mathbb{P}\mathsf{er} = \frac{SO(3,b_2-3)}{SO(1,b_2-3)\times SO(2)}$, as follows from the global Torelli theorem ([V1]). The mapping class group of M acts on $\mathbb{P}\mathsf{er}$ as an arithmetic lattice, that is, a finite index subgroup in $SO(H^2(M,\mathbb{Z}))$.

In [V2], we classified the orbits of Γ on Teich and proved that a general orbit of Γ is dense in Teich, and an orbit which corresponds to a manifold of maximal Picard rank is closed. However, one class of orbits was overlooked: the orbits of $I \subset \mathsf{Teich}$ such that $\operatorname{Re}(H^{2,0}(M,I)) \cap H^2(M,\mathbb{Q})$ has rank 1.

In the present erratum we give a complete classification of orbits of Γ-action on $\mathbb{P}\mathsf{er}$, and prove that the closure of the intermediate orbit is totally real: each of these orbits is a fixed point set of an anti-complex involution. In particular, it does not contain positive-dimensional complex subvarieties, and is not contained in any proper complex subvariety.

In the last section we check that the geometric applications obtained using the older version of [V2] remain valid with exception of one statement from [KLV] (not stated as a theorem, but implied in the abstract).

The third orbit was omitted in [V2] because here we actually classified the orbits of the group $H := SO(1,b_2-3)\times SO(2)$ on the space $SO(3,b_2-3)/\Gamma$. However, one cannot apply Ratner's theorem to the group H, because it is not generated by unipotents. To use Ratner's theorem, we need to consider orbits of the smaller group $H' := SO(1,b_2-3) \subsetneq H$.

Ratner's theorem implies that a closure of an orbit of H' is an orbit of an intermediate subgroup S, with $H' \subset S \subset SO(3,b_2-3)$. There are no proper intermediate subgroups between $SO(1,b_2-3)\times SO(2)$ and $SO(3,b_2-3)$; however, there is an intermediate subgroup $SO(2,b_2-3)$ between $SO(1,b_2-3)$ and $SO(3,b_2-3)$, and it corresponds to the extra orbit.

1.2 Preliminaries

Throughout this paper, **hyperkähler manifold** means a compact holomorphically symplectic manifold M of Kähler type, satisfying $\pi_1(M) =$

$0, H^{2,0}(M) = \mathbb{C}$. Its second cohomology are equipped with a primitive, bilinear symmetric, non-degenerate integral form q of signature $(3, b_2 - 3)$, called **Bogomolov-Beauville-Fujiki form**, or "BBF form". The corresponding **Teichmüller space** Teich is the space of all complex structures of hyperkähler type up to isotopy. We shall also speak of "Teichmüller spaces" for other geometric structures, defined in the same way. The **monodromy group** associated with a component of Teich is the subgroup of $O(H^2(M, \mathbb{Z})$. The **period space** $\mathbb{P}\mathsf{er}$ is the Grassmanian of positive, oriented 2-planes in $H^2(M, \mathbb{R})$. The **period map** $\mathsf{Per}: \mathsf{Teich} \longrightarrow \mathbb{P}\mathsf{er}$ takes a manifold (M, I) to the 2-plane $\operatorname{Re} H^{0,2}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$. "Picard rank" of a hyperkähler manifold is $\dim_{\mathbb{Q}}(H^{1,1}(M) \cap H^2(M, \mathbb{Q}))$. Global Torelli theorem ([V1]) states that the map Per is "the Hausdorff quotient": it is surjective, and satisfies $\mathsf{Per}(I) = \mathsf{Per}(I')$ if and only if I and I' are non-separable in Teich.

The **monodromy group** Γ is the subgroup of $O(H^2(M, \mathbb{Z}))$ generated by monodromy of all Gauss-Manin local systems for all complex deformations of (M, I) over a compact base. This group acts on the corresponding connected component of the Teichmüller space, denoted as Teich^I, and the quotient set $\mathsf{Teich}^I / \Gamma$ is precisely the set of all equivalence classes of complex structures on M in the same deformation class as I. In [V2] it was shown that Γ is a finite index subgroup of $O(H^2(M, \mathbb{Z}))$, and its action on $\mathbb{P}\mathsf{er}$ is ergodic.

We use the notation (M, I) to put emphasis on the complex structure I defined on the underlying smooth manifold M.

The connected component subgroup of the group $SO(p, q)$ is denoted as $SO^+(p, q)$.

2 Subgroups of $SO(a, b)$ and Ratner theory

2.1 Lie subalgebras $\mathfrak{so}(a-2, b) \subsetneq \mathfrak{g} \subsetneq \mathfrak{so}(a, b)$

In this subsection, we classify connected Lie subgroups $G \subset SO(a, b)$ containing $SO^+(a-2, b)$. To classify connected subgroups it suffices to classify Lie subalgebras $\mathfrak{so}(a-2, b) \subsetneq \mathfrak{g} \subsetneq \mathfrak{so}(a, b)$. This is done as follows.

Theorem 2.1: Let $V = \mathbb{R}^{p+q}$ be a vector space equipped with a quadratic form of signature (p, q), $V_1 \subset V$ a positive 2-dimensional subspace, and $V_0 = V_1^\perp$. The Lie algebra $\mathfrak{so}(V_0)$ is identified with a subalgebra of $\mathfrak{so}(V)$ acting trivially on V_1. Consider a proper Lie subalgebra $\mathfrak{g} \subsetneq \mathfrak{so}(V)$ such that $\mathfrak{so}(V_0) \subsetneq \mathfrak{g} \subsetneq \mathfrak{so}(V)$. Then either $\mathfrak{g} = \mathfrak{so}(V_2)$, where $V_2 \subset V$ is a $p+q-1$-dimensional subspace, $V_0 \subsetneq V_2 \subsetneq V$, or $\mathfrak{g} = \mathfrak{so}(V_0) \oplus \mathfrak{so}(V_1)$.

Proof. Step 1: We prove that $\mathfrak{g} = \mathfrak{so}(V_0) \oplus \mathfrak{so}(V_1)$ or $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(V_0) \oplus V_0$ as an $\mathfrak{so}(V_0)$-representation. Consider the decomposition of $\mathfrak{so}(V)$ as an $\mathfrak{so}(V_0)$-representation:

$$\mathfrak{so}(V) = \mathfrak{so}(V_0) \oplus (V_0 \otimes V_1) \oplus \mathfrak{so}(V_1).$$

Since $\mathfrak{g} \subset \mathfrak{so}(V)$ is $\mathfrak{so}(V_0)$-invariant, it is an $\mathfrak{so}(V_0)$-submodule in $\mathfrak{so}(V)$. Since $\mathfrak{so}(V_0) \oplus V_0 \otimes V_1 \subset \mathfrak{so}(V)$ generates the Lie algebra $\mathfrak{so}(V)$, one has $\mathfrak{g} \not\supset V_0 \otimes V_1$. Then either $\mathfrak{g} \cap V_0 \otimes V_1 = 0$, in which case $\mathfrak{g} = \mathfrak{so}(V_0) \oplus \mathfrak{so}(V_1)$ (the space $\mathfrak{so}(V_1)$ is 1-dimensional), or $\mathfrak{g} \cap V_0 \otimes V_1 = 0$ is a proper $\mathfrak{so}(V_0)$-invariant subspace of $V_0 \otimes V_1$. Since V_1 is a trivial 2-dimensional representation of the Lie algebra $\mathfrak{so}(V_0)$, the space $V_0 \otimes V_1$ is isomorphic (as an $\mathfrak{so}(V_0)$-module) to a direct sum of two copies of V_0, which is an irreducible representation of $\mathfrak{so}(V_0)$. Then any proper subrepresentation of $V_0 \otimes V_1$ is isomorphic to V_0.

Step 2: Let $\mathfrak{g} = \mathfrak{so}(V_0) \oplus R$, where $R \subset V_0 \otimes V_1$ is a $\mathfrak{so}(V_0)$-invariant subspace. Then $R = V_0 \otimes R_1$, where $R_1 \subset V_1$ is a 1-dimensional subspace in a 2-dimensional subspace V_1. Denote by $S \subset V_1$ the orthogonal complement to S in V_1, and let $V_2 = V_0 \oplus S$. Then $\mathfrak{g}$ fixes S, which gives a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{so}(V_2)$. Comparing dimensions, we find that it is an isomorphism. ■

2.2 Ratner theory for $SO^+(a-2,b) \subset SO^+(a,b)$

To fix the notation, we state the Ratner orbit closure theorem in a particular situation where we needed it. For a general version, see [V2] or [Mor]. As usual, $SO^+(a,b)$ denotes the connected component of the special orthogonal group $SO(a,b)$.

Theorem 2.2: Consider an integer lattice $V_{\mathbb{Z}} = \mathbb{Z}^{a+b}$ equipped with an integer scalar product q of signature (a,b), $a > 2$, $b > 0$, $a+b > 4$. Let $\Gamma \subset SO(V_{\mathbb{Z}})$ be a finite index sublattice, $V_{\mathbb{R}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$, and $SO^+(V_{\mathbb{R}}, q)$ the connected component of its orthogonal group. Replacing Γ by a finite index subgroup if necessarily, we may assume that Γ belongs to the connected component $SO^+(V_{\mathbb{R}})$ of the group $SO(V_{\mathbb{R}})$. Let $H \subset SO^+(V_{\mathbb{R}}, q)$ be the group of all elements of $SO^+(V_{\mathbb{R}})$ acting trivially on a 2-dimensional positive subspace $V_1 \subset V_{\mathbb{R}}$. Consider the left action of H on G/Γ; let Hx be an orbit of $x \in G/\Gamma$, and $\overline{Hx}$ its closure in standard topology. Then there exists a rational point $y \in \overline{Hx}$ and a rational Lie subgroup H_1 generated by unipotents, $H \subset H_1 \subset G$, such that $\overline{Hx} = H_1 y$.

Proof: This is the statement of Ratner theorem from [Mor]; to apply it, we have to check that the group H is generated by unipotents. The universal cover of $SO^+(2,2)$ is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, hence it is generated by unipotents. For other values of (a,b), $a+b>4, a\geqslant 3, b>1$, the group $H=SO^+(a-2,b)$ is simple, and its subgroup generated by unipotents is normal, hence it is equal to H or empty. There are unipotent elements in $SO^+(a,b)$ for $a\geqslant 2, b\geqslant 1$, which gives the statement of Theorem 2.2. ■

Remark 2.3: In [V2, Example 4.4] the Ratner Theorem was stated in the same way and then applied to $H=SO(a-2,b)\times SO(2)$. However the group $SO(a-2,b)\times SO(2)$ is not generated by unipotents. Its smallest subgroup generated by unipotents is $SO(a-2,b)$. This is why in place of [V2, Example 4.4] we should use Theorem 2.2.

2.3 Ratner theory and complex geometry of the period space

Claim 2.4: Let $V=\mathbb{R}^{a,b}$ be a vector space equipped with a non-degenerate scalar product q of signature (a,b), and $\operatorname{Gr}_{++}=\frac{SO^+(a,b)}{SO^+(a-2,b)\times SO(2)}$ the Grassmannian of positive, oriented 2-planes. Then Gr_{++} is in bijective correspondence with

$$\mathbb{P}\mathsf{er} := \{l\in \mathbb{P}V_{\mathbb{C}} \ \mid\ q(l,l)=0, q(l,\bar{l})>0\}.$$

where $V_{\mathbb{C}}$ is the complexification of V.

Proof: See, for example, [V2, Proposition 2.12]. ■

Theorem 2.5: Consider an integral lattice $V_{\mathbb{Z}}=\mathbb{Z}^{a+b}$ equipped with an integer-valued scalar product q of signature (a,b), $a>2$, $b>0$, $a+b>4$, $V:=V_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{R}$, $\Gamma\subset SO(V_{\mathbb{Z}})$ a finite index sublattice, $l\in\operatorname{Gr}_{++}(V)$ a point, and $\Gamma l\subset \operatorname{Gr}_{++}(V)$ its orbit. Then one of the following three possibilities is true.

(i) Γl is closed. This happens when the 2-plane $l\subset V$ is rational.

(ii) Γl is dense in $\operatorname{Gr}_{++}(V)$. This happens when l contains no rational vectors.

(iii) The closure $\overline{\Gamma l}$ is the set of all 2-planes $V_1\in\operatorname{Gr}_{++}(V)$ containing a 1-dimensional rational subspace $v\subset V_{\mathbb{Q}}$.

Proof: Let $P:=SO(V)/SO(V_0)$, where $V_0\subset V$ is a codimension 2 subspace of signature $(a-2,b)$. Clearly, P is fibered over $\operatorname{Gr}_{++}(V)$ with the fiber $SO(2)=S^1$:

$$\pi:\ P\xrightarrow{/SO(2)}\frac{SO^+(a,b)}{SO^+(a-2,b)\times SO(2)}=\frac{P}{SO(2)}.$$

Since $SO(2)$ is compact, the closure of a Γ-orbit of $x\in P$ satisfies

$$\pi(\overline{\Gamma\cdot l})=\overline{\Gamma\cdot\pi(l)}. \qquad (2.1)$$

Notice that an H-invariant subset Z on G/Γ is closed if and only if its image is closed in the double quotient $H\backslash G/\Gamma$. Therefore, the closure of an orbit $H\cdot x$ in G/Γ can be obtained by taking the closure of $x\cdot\Gamma$ in the left quotient $H\backslash G$. Using Ratner's theorem (Theorem 2.2), we find that such closures correspond to intermediate subgroups $H\subset S\subset G$. For Γ, G and H as in Theorem 2.2, the list of possible S is provided by Theorem 2.1.

To obtain a classification of Γ-orbits in Gr_{++}, we consider the Γ-action on $\frac{SO^+(a,b)}{SO^+(a-2,b)}$, and use Theorem 2.1, (2.1) and Theorem 2.2. From (2.1) it is clear that it suffices to classify the orbits of Γ-action on $\frac{SO^+(a,b)}{SO^+(a-2,b)}$. From Theorem 2.2, it follows that the classes of orbits in Theorem 2.2 correspond to classes of intermediate rational subgroups $S\subset SO^+(a,b)$ generated by unipotents, with $SO^+(a-2,b)\subset H\subset SO^+(a,b)$. From Theorem 2.1, we obtain that S is either $SO^+(a-2,b)$, $SO^+(a-1,b)$, $SO^+(a,b)$ or $SO^+(a-2,b)\times SO(2)$; however, the later case is impossible, because S is generated by unipotents. The remaining three classes give the three cases (i)-(iii) of Theorem 2.5. ∎

The last class of orbits (Theorem 2.5, (iii)) is what is missing from [V2]. However, most applications of [V2], obtained since, remain valid, because the extra orbit has sufficiently high codimension. This is explained in more detail in Section 4. Also, the extra orbits are negligible from the complex-geometric point of view, which is implied by the following result. Recall that a subvariety $N\subset M$ of a complex manifold is called **totally real** if there exists an anti-holomorphic involution $\tau:\ M\longrightarrow M$ such that N is its fixed point set. Totally real subvarieties of smooth varieties are always smooth (the fixed point set of an involution is always smooth). From the definition it is easy to see that $TM\Big|_N = TN\otimes_{\mathbb R}\mathbb C$. Therefore, a tangent space T_xN to a totally real submanifold contains no positive-dimensional complex subspaces, and it is not contained in a proper complex subspace of T_xM.

This gives the following claim.

Claim 2.6: Let M be a complex manifold, and $N\subset M$ a totally real submanifold. Then N contains no positive-dimensional complex subvarieties of M, and is contained in no proper subvariety of M. ∎

Proposition 2.7: Let V be a vector space equipped with a scalar product of signature (a,b), $a>2$, and $\mathrm{Gr}_{++}:=\frac{SO^+(a,b)}{SO^+(a-2,b)\times SO(2)}$ the positive oriented Grassmannian, equipped with the complex structure as in Claim 2.4. Fix a vector $v\in V$,= with $(v,v)>0$, and let $\mathrm{Gr}_{++}(v)\subset\mathrm{Gr}_{++}$ be the set of all positive, oriented 2-planes containing v. Then $\mathrm{Gr}_{++}(v)$ is a fixed point set of an antiholomorphic involution of Gr_{++}. This means that $\mathrm{Gr}_{++}(v)$ is totally real, contains no positive-dimensional complex subvarieties of Gr_{++}, and is contained in no proper complex subvariety of Gr_{++}.

Proof: Let $\iota_v : V \longrightarrow V$ be the reflection

$$x \mapsto x - 2\frac{(x,v)}{(v,v)}v.,$$

and $\gamma_v : \mathrm{Gr}_{++} \longrightarrow \mathrm{Gr}_{++}$ be the composition of the map induced by ι_v and the orientation reversal. Clearly, γ_v fixes a 2-plane $W \subset V$ if and only if W contains v. Since the complex structure on Gr_{++} is replaced by its opposite when the orientation changes, γ_v is an anti-holomorphic map. ■

3 Mapping class group action on the Teichmüller space

3.1 Mapping class group action and non-Hausdorff points

The period space $\mathbb{P}\mathsf{er}$ is obtained by taking the Teichmüller space Teich and gluing together all non-separable points. It is not obvious how to relate closed or dense orbits on $\mathbb{P}\mathsf{er}$ to that on Teich. The answer to this problem, given in [V2], remains valid. Let us remind how it is done.

Theorem 3.1: Let $\mathsf{Per} : \mathsf{Teich} \longrightarrow \mathbb{P}\mathsf{er}$ be the period map, and Γ the mapping class group. Assume that $b_2(M) \geqslant 5$. Then the period map commutes with taking closures: $\mathsf{Per}(\overline{\Gamma I}) = \overline{\Gamma\,\mathsf{Per}(I)}$, for all $I \in \mathsf{Teich}$.

The proof of Theorem 3.1 takes the rest of this section.

Recall that **the positive cone** $\mathrm{Pos}([I])$ is the connected component of the set of all real (1,1)-classes $v \in H^{1,1}([I])$ satisfying $q(v,v) > 0$, where q is the Bogomolov-Beauville-Fujiki form. There are two connected components, and we choose one which contains the Kähler classes. A subset $K \subset \mathrm{Pos}([I])$ is called **Kähler chamber** if it is a Kähler cone for some $I \in \mathsf{Teich}$ satisfying $\mathsf{Per}(I) = [I]$. The following fundamental result is due to Eyal Markman.

Proposition 3.2: Different Kähler chambers of $[I]$ do not intersect, and $\mathrm{Pos}([I])$ is a closure of their union. Moreover, there is a bijective correspondence between points of $\mathsf{Per}^{-1}([I])$ in the Teichmüller component of I and the set of Kähler chambers of $[I]$.

Proof: [Mar, Proposition 5.14]. ■

Consider the set Teich_K of pairs $(I \in \mathsf{Teich}, \omega \in \mathrm{Kah}(M, I))$, where $\mathrm{Kah}(M, I)$ denotes the Kähler cone, and $q(\omega, \omega) = 1$. Let Per_K be the set of all pairs

$$\{([I] \in \mathbb{P}\mathsf{er}, \omega \in \mathrm{Pos}([I])) \ \ | \ \ q(\omega, \omega) = 1\}.$$

Consider the period map $\mathsf{Per}_K : \ \mathsf{Teich}_K \longrightarrow \mathbb{P}\mathsf{er}_K$ mapping (I, ω) to $(\mathsf{Per}(I), \omega)$. By Proposition 3.2, Per_K is injective with dense image. This means that points of Teich can be identified with the space $\mathfrak{P}$ pairs (I, k), where $I \in \mathbb{P}\mathsf{er}$ and $k \subset H^{1,1}(M, I)$ is one of the Kähler chambers of $I \in \mathbb{P}\mathsf{er}$. The topology on this space of pairs is the quotient topology induced by the map $\tau : \ \mathbb{P}\mathsf{er}_K \longrightarrow \mathfrak{P}$.

Taking in account this picture, the closure of the monodromy orbit of $I \in \mathsf{Teich}$ can be understood as the closure of $\Gamma \cdot (\mathsf{Per}(I), k)$ in $\mathsf{Teich}_K \subset \mathbb{P}\mathsf{er}_K$, where k is the Kähler chamber $I \in \mathsf{Teich}$, and $\mathsf{Per}(I), k$ is considered as a subset in $\mathbb{P}\mathsf{er}_K$.

To prove that $\overline{\Gamma \cdot (\mathsf{Per}(I), k)}$ contains the positive cone $(I, \mathrm{Pos}(I)) \subset \mathbb{P}\mathsf{er}_K$, we use the Ratner's theorem again. We find an orbit R of a one-parametric group $H_0 \subset SO(H^2(M, \mathbb{R}))$ generated by unipotents and fixing $I \in \mathbb{P}\mathsf{er}$ with $R \subset \mathbb{P}\mathsf{er}_K$ lying within $(\mathsf{Per}(I), k)$, and show (using Ratner's theorem) that $\Gamma \cdot R$ is dense in $\mathbb{P}\mathsf{er}_K$.

The same strategy was used in [V2], however, the one-parametric group that we chose in [V2] was in fact not generated by unipotents. Here we fill this gap and give a corrected version of the proof.

3.2 Round segments on the boundary of the Kähler cone

In this subsection we shall speak of "round bits" on the boundary of the convex cone. The geometric intuition behind this terminology is that the Kähler cone is polyhedral at some points, and strictly convex with smooth boundary at other points; the points of the boundary where it's smooth and strictly convex constitute "round bits".

Definition 3.3: Let (M, I) be a hyperkähler manifold, and $W \subset H^{1,1}_I(M, \mathbb{R})$ a 3-dimensional real subspace of signatue $(1, 2)$. We say that **the Kähler cone boundary has round bits on** W if the projectivization of the closure $\mathbb{P}\overline{\mathrm{Kah}(M, I) \cap W}$ contains an open subset of the boundary of the disk $\mathbb{P}\,\mathrm{Pos}(W)$.[1]

[1]Here, as elsewhere, $\mathrm{Pos}(W)$ denotes the positive cone, which is one of two components of the set $\{w \in W \ \ | \ \ q(w, w) > 0\}$. Notice that $\mathbb{P}\,\mathrm{Pos}(W)$ is a 2-dimensional disk in $\mathbb{P}(W) = \mathbb{R}P^2$, identified with one of the standard models of the hyperbolic plane.

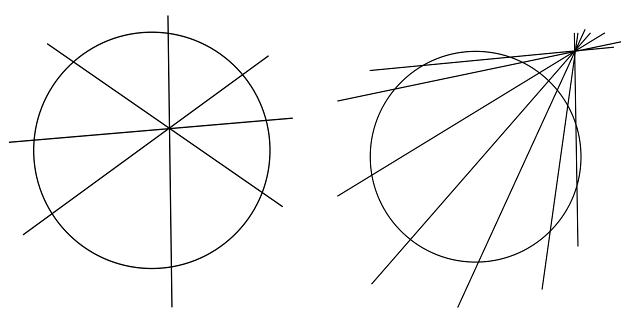
The following proposition is used to construct a one-parametric unipotent orbit which lies in $\operatorname{Kah}(M,I)$. The idea is to construct a horocycle in the hyperbolic segment $\mathbb{P}\operatorname{Pos}(W) \subset \mathbb{P}\operatorname{Kah}(M,I) = (I,k) \subset \mathbb{P}\mathsf{er}_K$. However, a horocycle exists only in subsets of $\overline{\operatorname{Pos}(W)}$ which contain open parts of the boundary.

Proposition 3.4: Let (M,I) be a hyperkähler manifold such that $b_2(M) \geqslant 5$ and the Picard rank of (M,I) is non-maximal. Denote by $\mathfrak{T}$ the orthogonal complement to $H^{1,1}_I(M,\mathbb{R}) \cap H^2(M,\mathbb{Q})$ in $H^{1,1}_I(M,\mathbb{R})$; since the Picard rank of (M,I) is non-maximal, the space $\mathfrak{T}$ has positive dimension. Then for each 3-dimensional space $W \subset H^{1,1}_I(M,\mathbb{R})$ of signature $(1,2)$ such that $\mathfrak{T} \cap W \neq 0$, the Kähler cone boundary has round bits on W.

Proof: By [AV1, Theorem 1.19], the Kähler cone $\operatorname{Kah}(M,I)$ is a connected component of $\operatorname{Pos}(M,I)\backslash S^\perp$, where $S^\perp = \bigcup_i s_i^\perp$ is a countable union of hyperplanes obtained as orthogonal complement to the so-called MBM classes $s_i \in H^{1,1}_I(M,\mathbb{Z})$. Clearly, all s_i are orthogonal to $\mathfrak{T}$. Therefore, all hyperplanes $s_i^\perp$ intersect the subspace $P := \mathfrak{T} \cap W \subset W$. The corresponding partition of the disk $\mathbb{P}\operatorname{Pos}(W)$ onto pieces by $S^\perp$ depends on the signature of P.

We may assume that P is one-dimensional; if it's 2-dimensional, one has $S_i^\perp \cap W = P$, and the disk is partitioned onto at most two pieces, hence each piece contains a round part of the boundary.

Notice that the set $s_i^\perp$ of walls of the Kähler chambers is locally finite ([HT, Proposition 10]). On the disk $\mathbb{P}\operatorname{Pos}(W)$ these lines trace geodesics which intersect in one point of the disk if $\mathfrak{T}$ has positive signature, and intersect in "imaginary" (negative) point outside of the disk if the signature is negative, We obtain that the hyperplanes $s_i^\perp$ partition the disk $\mathbb{P}\operatorname{Pos}(W)$ as follows.

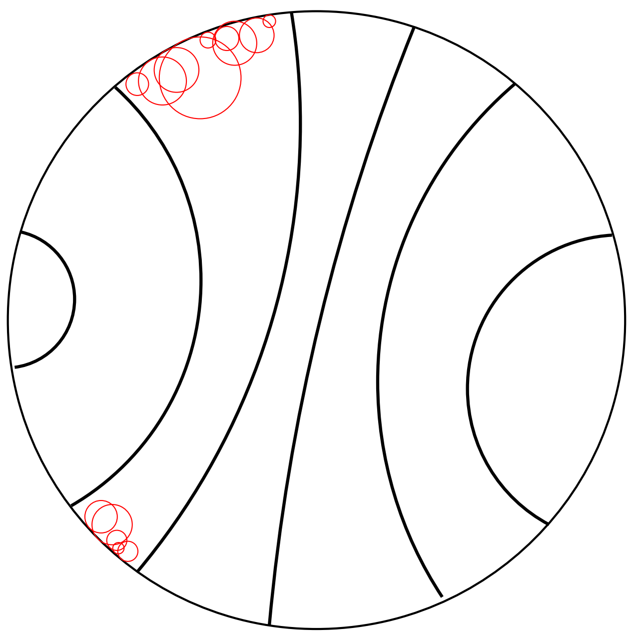


$\mathfrak{T}$ has positive signature **$\mathfrak{T}$ has negative signature**

In both cases, as seen from the picture, all connected components of $\mathbb{P}\operatorname{Pos}(W)\backslash S^\perp$ have round bits on the boundary of the disk. ∎

3.3 Horocycles in the Kähler cone

Let $\mathbb{P}\mathsf{er}_K$ the period space of Kähler classes (Subsection 3.1), and Γ the monodromy group acting on $\mathbb{P}\mathsf{er}_K$. We find a one-parametric group $H_0 \subset SO^+(H^2(M,\mathbb{R}))$ generated by unipotents such that for some $x \in \operatorname{Kah}(M,I)$, the orbit $H_0{\cdot}x$ is contained in $\operatorname{Kah}(M,I)$. We apply Proposition 3.4 and find $W \subset H^{1,1}(M,\mathbb{R})$ such that the Kähler cone boundary of (M,I) has round bits on W. The group H_0 acts on the Poincare disk $\Delta := \mathbb{P}\operatorname{Pos}(W)$ by parabolic isometries, and its orbits are horocycles. From the picture below it is clear that there are uncountably many horocycle subgroups with orbits in $\operatorname{Kah}(M,I)\cap W$ whenever the Kähler cone boundary of (M,I) has round bits on W.



Horocycles on a Poincare disc

The following proposition, applied together with Ratner's theorem, finishes the proof of Theorem 3.1, as indicated in Subsection 3.1.

By "general" we always mean "not contained in a union of countable number of Zariski closed subsets of positive codimension".

Proposition 3.5: Let $V_\mathbb{Q}$ be a rational vector space of signature $(3,d)$, $d \geqslant 2$, $V := V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{R}$, and $\Gamma \subset SO^+(V)$ an arithmetic lattice. Consider a subspace $V_1 \subset V$ of signature $(1,d)$, let $P \subset V_1$ be a 1-dimensional subspace, $W \supset P$ a general 3-dimensional subspace of V_1 of signature $(1,2)$ and containing P, and $H_0 \subset SO(W) \subset SO(V)$ a general one-parametric unipotent subgroup. Then the smallest rational Lie subgroup containing H_0 also contains $SO^+(V_1)$.

Proof: Denote by $\operatorname{Cl}(H_0)$ the smallest rational Lie subgroup containing H_0, and let $\mathfrak{I}_{H_0} \subset \coprod \mathbb{P}V^{\otimes i}$ be the set of all rational projective invariants of H_0. By Chevalley's theorem (see e.g. [Mor]), $\operatorname{Cl}(H_0)$ is the biggest Lie group fixing all points $x \in \mathfrak{I}_{H_0}$. Since the set $\mathfrak{I}_{H_0}$ is countable and semicontinuous in H_0, the group $\operatorname{Cl}(H_0)$ for generic H_0 is independent on H_0 and contains all unipotent sugroups of $SO^+(W)$, for all W of appropriate signature satisfying $P \subset W \subset V_1$. This implies $\operatorname{Cl}(H_0) \supset SO^+(V_1)$. ■

From Proposition 3.5, Theorem 3.1 follows immediately. Indeed, apply Proposition 3.5 to $V = H^2(M, \mathbb{R})$, $V_1 = H^{1,1}_I(M, \mathbb{R})$, and $P \subset \mathfrak{T}$, and choose generic $H_0 \subset SO(W)$ which has an orbit in $\operatorname{Kah}(M, I)$. By Ratner's theorem and Proposition 3.5, the closure of this orbit in $\mathbb{P}\mathsf{er}_K /\Gamma$ contains an orbit of $SO^+(V_1)$ which contains the set of unit vectors in the positive cone of I considered as a subset of $\mathbb{P}\mathsf{er}_K$. However, the closure of this set in $\mathbb{P}\mathsf{er}_K /\Gamma$ is mapped surjectively to the closure of I in $\mathbb{P}\mathsf{er} /\Gamma$, because $\mathbb{P}\mathsf{er}_K$ is fibered over $\mathbb{P}\mathsf{er}$ with fiber $\operatorname{Pos}(M, I)$, and this fibration is Γ-equivariant.

4 Applications to hyperkähler geometry: Hyperbolicity and Kobayashi pseudometric

Let Γ be the mapping class group of a complex manifold. Recall that a complex structure is called **ergodic** if its Γ-orbit is dense in its connected component of the Teichmüller space of complex structures.

In [V2], we proved that a hyperkähler manifold is not Kobayashi hyperbolic. To prove this, one needs to produce a deformation which is not Kobayashi hyperbolic and has ergodic complex structure.

As shown by F. Campana in [Cam], for any hyperkähler structure, the corresponding twistor deformation has at least one non-hyperbolic fiber. The twistor deformation can be interpreted in terms of the period space as follows. Let $W \subset H^2(M, \mathbb{R})$ be the 3-dimensional space generated by three Kähler forms $\omega_I, \omega_J, \omega_K$, associated with the quaternionic triple of complex structures. Consider the space S_W of oriented 2-planes in W. We can consider S_W as a subset in Gr_{++}. This subset is precisely the set of all complex structures in the twistor deformation associated with I, J, K.

Let now $W_0 \subset W$ be the 2-plane corresponding to the non-hyperbolic complex structure constructed in [Cam]. If we chose W such that it contains no rational vectors, $W_0 \cap H^2(M, \mathbb{Q}) = 0$, hence the associated Γ-orbit is dense in $\mathbb{P}\mathsf{er}$ (Theorem 2.5). The corresponding complex structure is ergodic. This proves the non-hyperbolicity of hyperkähler manifolds.

In [KLV, Theorem 1.1], it was shown that a hyperkähler manifold M with $b_2(M) \geqslant 7$ has vanishing Kobayashi pseudometric if the "SYZ conjecture" holds. SYZ conjecture, which is formulated further on in this paper, is the same as Kawamata's abundance conjecture for hyperkähler manifolds. It is known for the Hilbert scheme of K3 surfaces and all its deformations ([BM]) and for the generalized Kummer varieties ([Y]).

By a theorem of Matsushita (see [Mat1]), any holomorphic map $\pi : M \longrightarrow B$ is a Lagrangian fibration if $0 < \dim B < \dim M$. The space B is known to be a Fano manifold, and it has the same rational cohomology as $\mathbb{C}P^n$ if it is normal ([Mat2]). The pullback $\eta := \pi^*\omega$ of an ample class $\omega \in H^2(B, \mathbb{Z})$ is an integer class on the boundary of the Kähler cone of M which satisfies $\int_M \eta^{\dim_{\mathbb{C}} M} = 0$. Such cohomology classes are known as **parabolic**[1], and SYZ conjecture claims that, conversely, any parabolic class $\eta \in H^2(M, \mathbb{Z})$ is obtained this way.

Consider a manifold M which has two transversal Lagrangian fibrations (possibly rational). Since the fibers of these fibrations are tori, the Kobayashi pseudometric on M vanishes.

To prove that any manifold (M, I) has vanishing Kobayashi pseudometric, we need to show that in the closure $\overline{\Gamma I}$ of its Γ-orbit there exists a manifold with vanishing Kobayashi pseudometric. Since the diameter of the Kobayashi pseudometric is upper semicontinuous ([KLV]), this would imply that diameter of Kobayashi pseudometric on (M, I) also vanishes.

Meyer's theorem states that any integer lattice which is not sign-definite and of rank $\geqslant 5$ represents 0, that is, contains an integer (1,1)-class with square 0. Applying this result and global Torelli theorem to a hyperkähler manifold with $b_2(M) \geqslant 5$, we can find a complex structure with any Picard rank $\leqslant b_2(M) - 2$ and a vector with square 0, which would lie in a boundary of the Kähler cone for some complex structure ([AV2]), and give a Lagrangian fibration if SYZ conjecture is true.

Using the same argument for two non-proportional nef vectors, we would get two transversal Lagrangian fibrations. If we want to have a point with the two Lagrangian fibrations on a positive-dimensional orbit $\mathrm{Gr}_{++}(v)$ (Proposition 2.7), we would need to find two Lagrangian fibration (that is, two nef classes) on a manifold with $\mathrm{Re}(H^{2,0}) \ni v$, which is done using the same argument. This argument takes care of [KLV, Theorem 1.1].

A special case of this result, [KLV, Corollary 2.2], says that any non-algebraic K3 surface has vanishing Kobayashi metrics (for algebraic K3 it was already known). The idea is to show that an algebraic complex structure can be found in the closure of the mapping class orbit of a non-algebraic

[1] In more generality, parabolic classes can be defined as nef classes of volume 0.

complex structure and use the semicontinuity of the diameter of Kobayashi metric. This argument remains true. However, we should take care about an extra orbit and notice that the closure of this orbit is fixed point set of a rational anticomplex involution on the period space. It is easy to see that this fixed point set intersects the divisor $\mathsf{Teich}_g \subset \mathsf{Teich}$ representing polarized K3 of any given genus. This implies that there is a dense set of algebraic complex structures in any intermediate orbit.

The rest of results of [KLV] remain the same, with the same proofs.

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