# AFFINE TRANSFORMATIONS OF CIRCLE AND SPHERE 

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#### Abstract

A non-degenerate two-dimensional linear operator $\varphi$ transforms the unit circle into ellipse. Let $p$ be the length of its bigger axis and $q$ - the smaller. We can define the deformation coefficient $k(\varphi)$ as $q / p$. Analogously, if $\varphi$ is a non-degenerate three-dimensional operator, then it transforms the unit sphere into ellipsoid. If $p>q>r$ are lengths of its axes, then deformation coefficient $k(\varphi)$ will be defined as $r / p$. In this work we compute the mean value of deformation coefficient in two-dimensional case and give an estimation of the mean value in three-dimensional case.


## 1. Introduction

This work is a continuation of the work [1] where we study the deformation of angles under the action of a linear operator in $\mathbb{R}^{2}$. Here we study the deformation of the unit circle and also made some comments about three-dimensional case.

Let $\varphi$ be a non-degenerate linear operator in $\mathbb{R}^{2}$ and

$$
A=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right), \quad a d-b c \neq 0
$$

be its matrix in standard base. Operator $\varphi$ transforms unit circle $C$ into ellipse with canonical equation

$$
\frac{x^{\prime 2}}{p^{2}}+\frac{y^{\prime 2}}{q^{2}}=1, \quad p \geqslant q
$$

in appropriate coordinate system $\left(x^{\prime}, y^{\prime}\right)$. The number $q / p \leqslant 1$ will be called the deformation coefficient $k(\varphi)$ of operator $\varphi$.

In Section 2 we compute the deformation coefficient $k(\varphi)$.
Theorem 2.1.

$$
k(\varphi)=\frac{\sqrt{a^{2}+b^{2}+c^{2}+d^{2}-\sqrt{\left(a^{2}+b^{2}+c^{2}+d^{2}\right)^{2}-4(a d-b c)^{2}}}}{\sqrt{a^{2}+b^{2}+c^{2}+d^{2}+\sqrt{\left(a^{2}+b^{2}+c^{2}+d^{2}\right)^{2}-4(a d-b c)^{2}}}}
$$

In Section 3 compute the mean value $\bar{k}_{2}$ of coefficients $k(\varphi)$.
Theorem 3.1. $\bar{k}_{2}=3-4 \ln (2)$.
In Section 4 we demonstrate how to obtain a coarse upper bound for $\bar{k}_{2}$ : $\bar{k}_{2}<\frac{1}{2}$ (Theorem 4.1) with the aim to generalize this result to three dimensional case: $\bar{k}_{3}<\frac{1}{3}$ (Theorem 5.1).

## 2. Deformation coefficient in two-dimensional case

Let $\varphi: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ be a non-degenerate linear operator and

$$
A=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)
$$

be its matrix in the standard base. Operator $\varphi$ transforms unit circle into ellipse with axes $p$ and $q, p>q$.

## Theorem 2.1.

$$
k(\varphi)=\frac{q}{p}=\sqrt{\frac{a^{2}+b^{2}+c^{2}+d^{2}-\sqrt{\left(a^{2}+b^{2}+c^{2}+d^{2}\right)^{2}-4(a d-b c)^{2}}}{a^{2}+b^{2}+c^{2}+d^{2}+\sqrt{\left(a^{2}+b^{2}+c^{2}+d^{2}\right)^{2}-4(a d-b c)^{2}}}}
$$

Proof. If $\varphi^{*}$ is the conjugate operator, then $A^{t}$ is its matrix in the standard base. $p^{2}$ and $q^{2}$ are eigenvalues of operator $\varphi^{*} \varphi$ with matrix

$$
A^{t} A=\left(\begin{array}{cc}
a^{2}+c^{2} & a b+c d \\
a b+c d & b^{2}+d^{2}
\end{array}\right)
$$

Thus, $p^{2}$ and $q^{2}$ are roots of $A^{t} A$ characteristic polynomial

$$
\begin{gathered}
s(x)=x^{2}-\left(a^{2}+b^{2}+c^{2}+d^{2}\right) x+(a d-b c)^{2}: \\
p=\sqrt{\frac{a^{2}+b^{2}+c^{2}+d^{2}+\sqrt{\left(a^{2}+b^{2}+c^{2}+d^{2}\right)^{2}-4(a d-b c)^{2}}}{2}} \\
q=\sqrt{\frac{a^{2}+b^{2}+c^{2}+d^{2}-\sqrt{\left(a^{2}+b^{2}+c^{2}+d^{2}\right)^{2}-4(a d-b c)^{2}}}{2}}
\end{gathered}
$$

The change of variables simplifies these formulas. Put

$$
a:=x+y, d:=x-y, b:=z+t, c:=z-t
$$

In new variables

$$
k(\varphi)=\sqrt{\frac{x^{2}+y^{2}+z^{2}+t^{2}-\sqrt{\left(x^{2}+y^{2}+z^{2}+t^{2}\right)^{2}-\left(x^{2}+z^{2}-y^{2}-t^{2}\right)^{2}}}{x^{2}+y^{2}+z^{2}+t^{2}-\sqrt{\left(x^{2}+y^{2}+z^{2}+t^{2}\right)^{2}-\left(x^{2}+z^{2}-y^{2}-t^{2}\right)^{2}}} . . . . . . .}
$$

The next change of variables

$$
x:=r \sin (\alpha), z:=r \cos (\alpha), y:=\rho \sin (\beta), t:=\rho \cos (\beta)
$$

allows us the further simplification:

$$
k(\varphi)=\frac{|r-\rho|}{r+\rho}
$$

## 3. Computation of the mean value

Theorem 3.1. $\bar{k}_{2}=3-4 \ln (2)$.
Proof. Without loss of generality we can assume that $|A|>0$, i.e. $a d-b c>0$. In new variables this condition can be rewritten as

$$
x^{2}+z^{2}-y^{2}-t^{2}>0 \text { or } r>\rho
$$

We will compute the mean value of $k(\varphi)$ in the domain $a d-b c>0$, i.e. in the domain

$$
r>\rho>0,0 \leqslant \alpha \leqslant 2 \pi, 0 \leqslant \beta \leqslant 2 \pi
$$

We have

$$
\begin{gathered}
\bar{k}_{2}=\lim _{R \rightarrow \infty}\left(4 \pi^{2} \int_{0}^{R} r d r \int_{0}^{r} \rho \frac{r-\rho}{r+\rho} d \rho / 4 \pi^{2} \int_{0}^{R} r d r \int_{0}^{r} \rho d \rho\right)= \\
=\lim _{R \rightarrow \infty}\left(\left.\int_{0}^{R}\left(-\frac{1}{2} \rho^{2}+2 \rho r-2 r^{2} \ln (r+\rho)\right)\right|_{0} ^{r} r d r /-\frac{1}{8} r^{4}\right)= \\
=3-4 \ln 2 \approx 0.227411278
\end{gathered}
$$

## 4. Upper bound for deformation coefficient

Let $y$ and $z, y \geqslant z$, be lengths of vector-columns of matrix $A$ and $S \leqslant y z$ be the area of parallelogram, generated by these vectors. Characteristic polynomial of the matrix $A^{t} A$ can be written in the following way:

$$
s(x)=x^{2}-\left(y^{2}+z^{2}\right) x+S^{2}
$$

As

$$
p^{2}, q^{2}=\frac{y^{2}+z^{2} \pm \sqrt{\left(y^{2}+z^{2}\right)^{2}-4 S^{2}}}{2}
$$

then

$$
q^{2} \leqslant z^{2} \leqslant y^{2} \leqslant p^{2}, \text { and } k(\varphi)=\frac{q}{p} \leqslant \frac{z}{y}
$$

Theorem 4.1. $\bar{k}_{2}<\frac{1}{2}$.
Proof. We have

$$
\bar{k}_{2}<\int_{0}^{1} d y \int_{0}^{y} \frac{z}{y} d z / \int_{0}^{1} d y \int_{0}^{y} d z=\frac{1}{2}
$$

## 5. Three-dimensional case

Let $A$ be the matrix of linear operator $\varphi: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}, u, v, w, u \leqslant v \leqslant w$, be lengths vector-columns of this matrix, $S_{1}, S_{2}, S_{3}$ be areas of parallelograms, generated by pairs of vector-columns and $V$ be the volume of parallelepiped, generated by vectorcolumns. Then characteristic polynomial $s$ of the operator $\varphi^{*} \varphi$ is of form

$$
s(x)=x^{3}-\left(u^{2}+v^{2}+w^{2}\right) x^{2}+\left(S_{1}^{2}+S_{2}^{2}+S_{3}^{2}\right) x-V^{2}
$$

Proposition 5.1. Let $x_{1}, x_{2}, x_{3}, x_{1} \leqslant x_{2} \leqslant x_{3}$, be (real) roots of $s$. Then $x_{1} \leqslant$ $u^{2} \leqslant w^{2} \leqslant x_{3}$.
Proof. Computer assisted check.
Thus,

$$
k(\varphi)=\frac{x_{1}}{x_{3}} \leqslant \frac{u}{w},
$$

and we have the following estimation of the mean value $\bar{k}_{3}$ of three-dimensional deformation coefficient.
Theorem 5.1. $\bar{k}_{3}<\frac{1}{3}$.
Proof.

$$
\bar{k}_{3}<\int_{0}^{1} d w \int_{0}^{w} d v \int_{0}^{v} \frac{u}{w} d u / \int_{0}^{1} d w \int_{0}^{w} d v \int_{0}^{v} d u=\frac{1}{3}
$$

Remark 5.1. Actual value of $\bar{k}_{3}$ is quite difficult to compute. We have a rather coarse estimation: $\bar{k}_{3} \approx 0.15$.

## References

[1] Busjatskaja Irina, Kochetkov Yury, Affine transformations of the plane and their geometrical properties, arXiv 1603.02938 v 1 .
[2] Halmos Paul, Finite dimensional vector spaces, Springer, 1974.
[3] Lang Serge, Linear algebra, Springer, 1987.

