## AFFINE TRANSFORMATIONS OF CIRCLE AND SPHERE

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ABSTRACT. A non-degenerate two-dimensional linear operator  $\varphi$  transforms the unit circle into ellipse. Let p be the length of its bigger axis and q — the smaller. We can define the deformation coefficient  $k(\varphi)$  as q/p. Analogously, if  $\varphi$  is a non-degenerate three-dimensional operator, then it transforms the unit sphere into ellipsoid. If p > q > r are lengths of its axes, then deformation coefficient  $k(\varphi)$  will be defined as r/p. In this work we compute the mean value of deformation coefficient in two-dimensional case and give an estimation of the mean value in three-dimensional case.

## 1. INTRODUCTION

This work is a continuation of the work [1], where we study the deformation of angles under the action of a linear operator in  $\mathbb{R}^2$ . Here we study the deformation of the unit circle and also made some comments about three-dimensional case.

Let  $\varphi$  be a non-degenerate linear operator in  $\mathbb{R}^2$  and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc \neq 0$$

be its matrix in standard base. Operator  $\varphi$  transforms unit circle C into ellipse with canonical equation

$$\frac{x'^2}{p^2} + \frac{y'^2}{q^2} = 1, \quad p \ge q$$

in appropriate coordinate system (x', y'). The number  $q/p \leq 1$  will be called the deformation coefficient  $k(\varphi)$  of operator  $\varphi$ .

In Section 2 we compute the deformation coefficient  $k(\varphi)$ .

Theorem 2.1.

$$k(\varphi) = \frac{\sqrt{a^2 + b^2 + c^2 + d^2} - \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2}}{\sqrt{a^2 + b^2 + c^2 + d^2} + \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2}}$$

In Section 3 compute the mean value  $\overline{k}_2$  of coefficients  $k(\varphi)$ .

**Theorem 3.1.**  $\overline{k}_2 = 3 - 4 \ln(2)$ .

In Section 4 we demonstrate how to obtain a coarse upper bound for  $\overline{k}_2$ :  $\overline{k}_2 < \frac{1}{2}$  (Theorem 4.1) with the aim to generalize this result to three dimensional case:  $\overline{k}_3 < \frac{1}{3}$  (Theorem 5.1).

Let  $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$  be a non-degenerate linear operator and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be its matrix in the standard base. Operator  $\varphi$  transforms unit circle into ellipse with axes p and q, p > q.

## Theorem 2.1.

$$k(\varphi) = \frac{q}{p} = \sqrt{\frac{a^2 + b^2 + c^2 + d^2 - \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2}}{a^2 + b^2 + c^2 + d^2 + \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2}}}.$$

*Proof.* If  $\varphi^*$  is the conjugate operator, then  $A^t$  is its matrix in the standard base.  $p^2$  and  $q^2$  are eigenvalues of operator  $\varphi^*\varphi$  with matrix

$$A^{t}A = \begin{pmatrix} a^{2} + c^{2} & ab + cd \\ ab + cd & b^{2} + d^{2} \end{pmatrix}.$$

Thus,  $p^2$  and  $q^2$  are roots of  $A^t A$  characteristic polynomial

$$s(x) = x^{2} - (a^{2} + b^{2} + c^{2} + d^{2})x + (ad - bc)^{2}:$$

$$p = \sqrt{\frac{a^{2} + b^{2} + c^{2} + d^{2} + \sqrt{(a^{2} + b^{2} + c^{2} + d^{2})^{2} - 4(ad - bc)^{2}}}{2}};$$

$$q = \sqrt{\frac{a^{2} + b^{2} + c^{2} + d^{2} - \sqrt{(a^{2} + b^{2} + c^{2} + d^{2})^{2} - 4(ad - bc)^{2}}}{2}}.$$

The change of variables simplifies these formulas. Put

$$a := x + y, d := x - y, b := z + t, c := z - t.$$

In new variables

$$k(\varphi) = \sqrt{\frac{x^2 + y^2 + z^2 + t^2 - \sqrt{(x^2 + y^2 + z^2 + t^2)^2 - (x^2 + z^2 - y^2 - t^2)^2}}{x^2 + y^2 + z^2 + t^2 - \sqrt{(x^2 + y^2 + z^2 + t^2)^2 - (x^2 + z^2 - y^2 - t^2)^2}}}$$

The next change of variables

$$x := r \sin(\alpha), \, z := r \cos(\alpha), \, y := \rho \sin(\beta), \, t := \rho \cos(\beta)$$

allows us the further simplification:

$$k(\varphi) = \frac{|r-\rho|}{r+\rho}.$$

### 3. Computation of the mean value

**Theorem 3.1.**  $\overline{k}_2 = 3 - 4 \ln(2)$ .

*Proof.* Without loss of generality we can assume that |A| > 0, i.e. ad - bc > 0. In new variables this condition can be rewritten as

$$x^{2} + z^{2} - y^{2} - t^{2} > 0$$
 or  $r > \rho$ .

We will compute the mean value of  $k(\varphi)$  in the domain ad - bc > 0, i.e. in the domain

$$r > \rho > 0, \ 0 \leq \alpha \leq 2\pi, \ 0 \leq \beta \leq 2\pi$$

We have

$$\overline{k}_{2} = \lim_{R \to \infty} \left( 4\pi^{2} \int_{0}^{R} r \, dr \int_{0}^{r} \rho \, \frac{r-\rho}{r+\rho} \, d\rho \, \middle/ \, 4\pi^{2} \int_{0}^{R} r \, dr \int_{0}^{r} \rho \, d\rho \right) =$$
$$= \lim_{R \to \infty} \left( \int_{0}^{R} \left( -\frac{1}{2} \, \rho^{2} + 2\rho r - 2r^{2} \ln(r+\rho) \right) \Big|_{0}^{r} r \, dr \, \Big/ -\frac{1}{8} \, r^{4} \right) =$$

# $= 3 - 4\ln 2 \approx 0.227411278.$

## 4. Upper bound for deformation coefficient

Let y and z,  $y \ge z$ , be lengths of vector-columns of matrix A and  $S \le yz$  be the area of parallelogram, generated by these vectors. Characteristic polynomial of the matrix  $A^t A$  can be written in the following way:

$$s(x) = x^{2} - (y^{2} + z^{2})x + S^{2}.$$

 $\mathbf{As}$ 

$$p^2, q^2 = \frac{y^2 + z^2 \pm \sqrt{(y^2 + z^2)^2 - 4S^2}}{2},$$

then

$$q^2 \leqslant z^2 \leqslant y^2 \leqslant p^2$$
, and  $k(\varphi) = \frac{q}{p} \leqslant \frac{z}{y}$ .

## Theorem 4.1. $\overline{k}_2 < \frac{1}{2}$ .

*Proof.* We have

$$\overline{k}_2 < \int_0^1 dy \int_0^y \frac{z}{y} dz \Big/ \int_0^1 dy \int_0^y dz = \frac{1}{2}.$$

#### 5. Three-dimensional case

Let A be the matrix of linear operator  $\varphi : \mathbb{R}^3 \to \mathbb{R}^3, u, v, w, u \leq v \leq w$ , be lengths vector-columns of this matrix,  $S_1, S_2, S_3$  be areas of parallelograms, generated by pairs of vector-columns and V be the volume of parallelepiped, generated by vectorcolumns. Then characteristic polynomial s of the operator  $\varphi^*\varphi$  is of form

$$s(x) = x^{3} - (u^{2} + v^{2} + w^{2})x^{2} + (S_{1}^{2} + S_{2}^{2} + S_{3}^{2})x - V^{2}.$$

**Proposition 5.1.** Let  $x_1, x_2, x_3, x_1 \leq x_2 \leq x_3$ , be (real) roots of s. Then  $x_1 \leq x_2 \leq x_3$  $u^2 \leqslant w^2 \leqslant x_3.$ 

Proof. Computer assisted check.

Thus,

$$k(\varphi) = \frac{x_1}{x_3} \leqslant \frac{u}{w} \,,$$

and we have the following estimation of the mean value  $\overline{k}_3$  of three-dimensional deformation coefficient.

**Theorem 5.1.**  $\overline{k}_3 < \frac{1}{3}$ .

Proof.

$$\overline{k}_3 < \int_0^1 dw \int_0^w dv \int_0^v \frac{u}{w} du \Big/ \int_0^1 dw \int_0^w dv \int_0^v du = \frac{1}{3}.$$

*Remark* 5.1. Actual value of  $\overline{k}_3$  is quite difficult to compute. We have a rather coarse estimation:  $\overline{k}_3 \approx 0.15$ .

#### References

- [1] Busjatskaja Irina, Kochetkov Yury, Affine transformations of the plane and their geometrical properties, arXiv 1603.02938v1.
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- [3] Lang Serge, Linear algebra, Springer, 1987.

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