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On normally hyperbolic inertial manifolds of evolutionary equations

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TO THE MEMORY OF MY TEACHER G. M. HENKIN

OUTLINE OF THE TALK

- ▲ Inertial manifolds of parabolic equations
- ▲ Sufficient conditions for the existence of an inertial manifold
- ▲ Necessary condition for the existence of an inertial manifold
- ▲ Normally hyperbolic inertial manifolds
- ▲ Reaction-diffusion system without a normally hyperbolic inertial manifold
- ▲ Inertial manifold without normal hyperbolicity
- ▲ Absolutely normally hyperbolic inertial manifolds

OBJECT OF STUDY

In a real separable Hilbert space $(X, \|\cdot\|)$, $\dim X = \infty$, consider the **semilinear parabolic equation**

$$u_t = -Au + F(u) \qquad (*)$$

Here

1. $A : D(A) \to X$ is a linear self-adjoint positive operator with compact inverse A^{-1}.
2. $F : H \to X$ is a smooth nonlinear function with domain

$$H = D(A^\theta),\ 0 \le \theta < 1,\ \|u\|_H = \|A^\theta u\|,$$

$$\|F(u) - F(v)\| \le K(r)\|u - v\|_H \text{ for } \|u\|_H \le r,\ \|v\|_H \le r.$$

3. There exists a smooth dissipative phase semiflow $\Phi_t : H \to H$.

Dissipativity means the existence of an **absorbing ball** in phase space H. Since the **nonlinearity exponent** $\theta < 1$, we see that the function F is "weaker" than the operator A, which means that Eq. (*) is **semilinear**. We have $H = X$ if $\theta = 0$. Nonlinearities $F : H \to X$ with the properties described above will be called **admissible** nonlinearities.

There exists compact **global attractor** $\mathcal{A} \subset H$: the collection of all complete bounded trajectories.

INERTIAL MANIFOLDS OF PARABOLIC EQUATIONS

The **inertial manifold** of the semilinear parabolic equation $(*)$ is a smooth **finite-dimensional** invariant surface $M \subset H$ that contains the global attractor and attracts all trajectories at large time **with exponential tracking**. Usually, M have globally Cartesian structure and M is diffeomorphic to R^n. The restriction of the parabolic equation to M is an **ordinary differential equation (inertial form) in** R^n **which completely describes the eventual dynamics of the system**.

The existence of an inertial manifold implies that the eventual behavior of an infinite-dimensional dynamical system is controlled by **finitely many** parameters.

CONCLUSION: a system with infinitely many degrees of freedom essentially has finitely many degrees of freedom as $t \to +\infty$**.**

HISTORY OF THE TOPIC

The term “inertial manifold” was introduced in the note [1]. The contemporary state of the topic: [2].

PARADOX: Nothing is known about inertial manifolds for a majority of equations of mathematical physics.

Namely, it has been possible to establish the existence of inertial manifolds for a narrow class of parabolic equations, while known examples [3,4] in which there is no inertial manifold seem to be artificial and are not related to practically important problems.

[1] C. Foias, G.R. Sell, R. Temam, *C. R. Acad. Sci. Paris I*, **301**:5, 1985.

[2] S. Zelik, *Proc. Roy. Soc. Edinburgh*, *Ser.* A, **144**:6, 2014.

[3] A.V. Romanov, *Math. Notes*, **68**:3-4, 2000.

[4] A. Eden, V. Kalantarov, S. Zelik, *Russian Math. Surveys*, **68**:2, 2013.

The main goals of the study:

to construct examples of parabolic equations of mathematical physics that do not have an inertial manifold

INERTIAL MANIFOLDS: SUFFICIENCY

The only *general* sufficient condition for the existence of an inertial manifold $M \subset H$ of the equation $u_t = -Au + F(u)$ **for an arbitrary admissible nonlinearity** F is **the spectrum sparseness condition** for the linear part of the equation:

$$\sup_{n\geq 1} \frac{\mu_{n+1} - \mu_n}{\mu_{n+1}^{\theta} + \mu_n^{\theta}} = \infty, \qquad (*)$$

where $\{0 < \mu_1 \leq \mu_2 < \ldots\} = \sigma(A)$. For the **reaction–diffusion** equation

$$\partial_t u = \Delta u + f(x,u)$$

in a bounded domain $\Omega \subset R^m$, one has $A = -\Delta,\ \theta = 0,\ \mu_n \sim cn^{2/m}$, so that the spectral sparseness condition $\sup_{n\geq 1} (\mu_{n+1} - \mu_n) = \infty$ holds only in one-dimensional and (rarely) two-dimensional problems. For the **reaction–diffusion-advection** equation

$$\partial_t u = \Delta u + f(x,u,u_x)$$

we have $\theta = 1/2$ and (*) is impossible

But for the Beltrami–Laplace operator on the sphere S^m **the spectral sparseness condition** $\sup_{n\geq 1} (\mu_{n+1} - \mu_n) = \infty$ **holds** $\forall m \geq 2$!

HOW CAN WE AVOID THE SPECTRUM SPASENESS CONDITION?

The principle of spatial averaging (PSA) for the Laplacian Δ in bounded domain $\Omega \subset R^m$ $(m \le 3)$ suggested in [1] sometimes permits one to construct inertial manifolds **avoiding the spectrum sparseness condition**. It is the following property: for $\forall h \in H^2(\Omega)$ operator $\Delta + h(x)$ can be *well approximated* by $\Delta + \bar{h}$ over "large segments" of $L^2(\Omega)$, where $\bar{h} = (\mathrm{vol}\,\Omega)^{-1}\int_\Omega h(x)dx$. **This property follows from the spectrum sparseness condition.** The corresponding method was used in [1] to prove the existence of an inertial manifold for the scalar reaction-diffusion equation

$$\partial_t u = \Delta u + f(x,u), \qquad f \in C^3,$$

in cube $\Omega = (0, 2\pi)^3$ and in rectangle $\Omega = (0,a)\times(0,b)$ with boundary conditions (N),(D) or (P). Analogical results were obtained [2] for some bounded domains $\Omega \subset R^m$ $(m = 2,3)$. The abstract scheme of **PSA** was suggested in [3] and successfully applied in [4,5] to the Cahn–Hilliard equation

$$\partial_t u = -\Delta(\Delta u - f(u)), \quad f \in C^3,$$

and to the modified Leray α-model of the Navier–Stokes equations on the 3D torus.

[1] J. Mallet-Paret, G.R. Sell, *J. Amer. Math. Soc.*, **1**:4, 1988.

[2] H. Kwean, *Int. J. Math. Math. Sci.*, **28**:5, 2001.

[3] S. Zelik, *Proc. Roy. Soc. Edinburgh, Ser.* A, **144**:6, 2014.

[4] A. Kostianko, S. Zelik, *Comm. Pure Appl. Anal.*, **14**:5, 2015.

[5] A. Kostianko, arXiv:1510.08936, 2015.

TRANSFORMATION OF THE EQUATION

The other way to avoid the spectral sparseness condition is to transform (*some change of variables*) the parabolic equation in order **to decrease** the nonlinearity exponent θ. **The summetry property of the linear part of the equation must be preserved.** In this way J. Vukadinovic has constructed [1, 2] inertial manifolds for a Smoluchowski equation – a nonlinear Fokker–Planck equation on S^m $(m=1,2)$ and [3] for a class of diffusive Burgers equations on torus $[0,2\pi]^m$ $(m=1,2)$. In paper [3] the Cole–Hopf transform has been employed. In paper [4] an inertial manifold for systems of 1D **reaction-diffusion-advection** equations is constructed after the proper nonlocal change of the dependent variable.

But the last two methods ("*spatial averaging" and "transformation"*) are not being general. In the present time we can avoid the spectrum sparseness condition in some special cases only.

[1] J. Vukadinovic, *Nonlinearity*, **21,** 2008.

[2] J. Vukadinovic, *Comm. Math. Phys*., **285**:3, 2009.

[3] J. Vukadinovic, *Discr. Cont. Dyn. Syst.*, **29**:1, 2011.

[4] A. Kostianko, S. Zelik, *arXiv*:1602.00301, 2016.

INERTIAL MANIFOLDS: NECESSITY

For a *fixed* admissible nonlinearity F, **there is only one known necessary condition** for the existence of an inertial manifold $M \subset H$ for the equation $u_t = -Au + F(u)$. For $u \in H$, we introduce the following notation:

1. $F'(u)$ is the Fréchet or Gâteaux derivative of the function F.
2. $\sigma(S_u)$ is the spectrum of the linear operator $S_u = F'(u) - A$ with compact resolvent.
3. E is the set of stationary points $u \in H : -Au + F(u) = 0$.
4. $l(u) < \infty$ is the number (counting algebraic multiplicity) of eigenvalues $\lambda > 0$ in $\sigma(S_u)$ for $u \in E$.
5. $E_- = \{u \in E : \sigma(S_u) \bigcap (-\infty, 0] = \phi\}$.

NECESSITY LEMMA [1]. *If the equation* $u_t = -Au + F(u)$ *admits an inertial manifold* $M \subset H$, *then the number* $l(u_1) - l(u_2)$ *is even for any two points* $u_1, u_2 \in E_-$.

The 1D parabolic equation **with nonlocal diffusion** without an inertial manifold has described in [2]. This example is much more realistic than the earlier-known ones but still is not completely natural.

NORMALLY HYPERBOLIC INERTIAL MANIFOLDS

For inertial manifolds with additional **normal hyperbolicity properties**, nonexistence examples can be constructed in the class of **reaction-diffusion systems**.

DEFINITION. An inertial manifold $M \subset H$ of the equation $u_t = -Au + F(u), H = D(A^\theta)$, is said to be *normally hyperbolic* if, for some (invariant with respect to the derivative Φ_t' of semiflow $\Phi_t : H \to H$) vector bundle $T_M H = TM \oplus N$, where TM is the tangent bundle of M, one has (for $t \geq 0$) the estimates

$$\begin{aligned} \|\Phi_t'(u)\xi\|_H &\geq C^{-1} e^{-\gamma_1 t} \|\xi\|_H \quad (\xi \in T_u M), \\ \|\Phi_t'(u)\xi\|_H &\leq C e^{-\gamma_2 t} \|\xi\|_H \quad (\xi \in N_u) \qquad (*) \end{aligned}$$

for $u \in M$ with constants $C \geq 1$ and $0 < \gamma_1 < \gamma_2$ depending on M and u.

It is well known [1] that normally hyperbolic invariant manifolds of dynamical systems are **structurally stable**.

If C, γ_1, γ_2 in relations $(*)$ are independent of u, then we say that the manifold M *is **absolutely normally hyperbolic***. If relations $(*)$ hold for $u \in E$ only, then we say that M is ***hyperbolic at the stationary points***.

[1] V. A. Pliss, G. R. Sell, *J. Diff. Equat.*, **169,** 2001.

NORMALLY HYPERBOLIC INERTIAL MANIFOLDS: SUFFICIENCY

THEOREM [1]. *The spectral sparseness condition*

$$\sup_{n\geq 1} \frac{\mu_{n+1}-\mu_n}{\mu_{n+1}^{\theta}+\mu_n^{\theta}} = \infty, \ \textit{where} \ \{0<\mu_1<\mu_2<\ldots\}=\sigma(A),$$

for the semilinear parabolic equation with the nonlinearity exponent $\theta\in[0,1)$

$$u_t = -Au + F(u) \qquad (*)$$

in Hilbert space X *implies the existence of an absolutely normally hyperbolic inertial manifold* M *in the phase space* $H = D(A^{\theta})$.

THEOREM [2]. *The scalar reaction-diffusion equation*

$$\partial_t u = \Delta u + f(x,u), \qquad f\in C^3,$$

in cube $\Omega=(0,2\pi)^3$ *and in rectangle* $\Omega=(0,a)\times(0,b)$ *with boundary condition* (N), (D) *or* (P) *has normally hyperbolic at the stationary points inertial manifold* $M\subset L^2(\Omega)$.

The principle of spatial averaging (PSA) holds in this case.

Recent progress: the technique in [3] permits one to derive the existence of a normally hyperbolic inertial manifold for Eq. (*) from abstract scheme of PSA**.**

[1] R. Rosa, R. Temam, *ACTA Applicandae Mathematicae*, **45,** 1996.

[2] J. Mallet-Paret, G.R. Sell, Z. Shao, *Indiana Univ. Math. J.*, **42**:3, 1993.

[3] A. Kostianko, S. Zelik, *Comm. Pure Appl. Anal.*, **14**:5, 2015.

NORMALLY HYPERBOLIC INERTIAL MANIFOLDS: NECESSITY

Let $M \subset H$ be an inertial manifold of the equation $u_t = -Au + F(u)$, let $\gamma \in \mathbb{R}$, and let $H(u,\gamma)$ be the finite-dimensional invariant subspace of the operator $S_u = F'(u) - A$ corresponding to the part of the spectrum $\sigma(S_u)$ with $\operatorname{Re}\lambda \ge \gamma$.

NECESSITY LEMMA [1]. *If M is normally hyperbolic on E, then*

$$\forall u \in E \;\; \exists \gamma = \gamma(u) < 0 : \dim H(u,\gamma) = \dim M .$$

Here $\gamma(u) = -(\gamma_1(u) + \gamma_2(u))/2$, the invariant subspaces $T_u M$ and N_u correspond to the parts of $\sigma(S_u)$ with $\operatorname{Re}\lambda \ge -\gamma_1(u)$ and $\operatorname{Re}\lambda \le -\gamma_2(u)$, respectively, and $0 < \gamma_1(u) < \gamma_2(u)$ in the definition of normal hyperbolicity. If M is *absolutely normally hyperbolic*, then the constants $\gamma, \gamma_1, \gamma_2$ are independent of $u \in M$.

THEOREM [1]. *There exists a real-analytic function f such that the reaction-diffusion equation*

$$\partial_t u = \Delta u + f(x,u), \quad \Omega = (0,\pi)^4, \; \partial_n u\big|_{\partial\Omega} = 0,$$

dissipative in $H = L^2(\Omega)$, does not admit a normally hyperbolic inertial manifold $M \subset H$.

The proof is based on the **necessity lemma** and uses the large multiplicity of the spectrum $\sigma(-\Delta)$ in $(0,\pi)^4$. The corresponding function $f : (0,\pi)^4 \times R \to R$ (polynomial in u) **is not constructed explicitly**.

[1] J. Mallet-Paret, G.R. Sell, Z. Shao, *Indiana Univ. Math.* J., **42**:3, 1993.

PROBLEM: Find 3D reaction-diffusion equations *with polynomial nonlinearity of degree* ≤ 3 that do not admit a normally hyperbolic inertial manifold

The restrictions on the dimension of the problem and the form of the nonlinearity are typical for the equations of chemical kinetics.

EXAMPLE OF NONEXISTENCE OF A NORMALLY HYPERBOLIC INERTIAL MANIFOLD

Consider the system

$$\partial_t u_1 = \Delta u_1 + f_1(u_1,u_2), \quad \partial_t u_2 = \Delta u_2 + f_2(u_1,u_2) \qquad (*)$$

in cube $\Omega = (0,\pi)^3$, under the condition $\partial_n u|_{\partial\Omega} = 0$ with the polynomial vector field

$$f_1(v_1,v_2) = kv_1(1 - av_1^2 + v_2^2), \quad f_2(v_1,v_2) = kv_2(1 - bv_2^2 - v_1^2),$$

where $a > 1$, $k, b > 0$, are constants. Dissipativity in $H = (L^2(\Omega))^2$ ("*vector sign condition*"): squares $|v_1| < r, |v_2| < r$ are positively invariant for ODE $v_t = f(v)$ when $r \geq r_0 > 0$.

PROPOSITION [1]. *There exist k, a, b such that this system does not have a normally hyperbolic inertial manifold $M \subset H$.*

The proof is based on the necessity condition of the normally hyperbolic inertial manifolds existing and on the "obstruction lemma" from [2] for the system (*).

[1] A.V. Romanov, *Dynamics of PDE*, **13**:3, 2016.

[2] A.V. Romanov, *Math. Notes*, **68**:3-4, 2000.

PROBLEM: Find 3D reaction-diffusion equations with polynomial nonlinearity of degree ≤ 3 with an inertial manifold that is not normally hyperbolic.

INERTIAL MANIFOLD THAT IS NOT NORMALLY HYPERBOLIC

Consider the system

$$\partial_t u_1 = \Delta u_1 + f_1(u_1, u_2), \quad \partial_t u_2 = \Delta u_2 + f_2(u_1, u_2), \qquad (*)$$

dissipative in $H = (L^2(\Omega))^2$, $\Omega = (0,\pi)^3$, with the boundary condition $\partial_n u\big|_{\partial\Omega} = 0$ and with the polynomial vector field

$$f_1(v_1, v_2) = v_1(a - v_1)(v_1 - b), \quad f_2(v_1, v_2) = v_2(c - v_2)(v_2 - d),$$

where a, b, c, d are constants. **THIS IS AN UNCOUPLED SYSTEM!**

PROPOSITION [1]. *For $a = 2, b = \sqrt{3}, c = \sqrt{6}, d = \sqrt{2}$, system $(*)$ has an inertial manifold $M \subset H$ but does not have a normally hyperbolic inertial manifold in H.*

The proof is based on:

1) the spatial averaging principle for scalar reaction-diffusion equations in $\Omega = (0,\pi)^3$;

2) the necessity condition of the normally hyperbolic inertial manifolds existing and on the obstruction lemma from [2] for the system (*).

Thus, we have presented an inertial manifold of system $(*)$ without the normal hyperbolicity property.

[1] A.V. Romanov, *Dynamics of PDE*, **13**:3, 2016.

[2] A.V. Romanov, *Math. Notes*, **68**:3-4, 2000.

IS THE SPECTRUM SPARSENESS CONDITION EQUIVALENT TO THE ABSOLUTELY NORMALLY HYPERBOLIC INERTIAL MANIFOLD EXISTING?

ABSOLUTELY NORMALLY HYPERBOLIC INERTIAL MANIFOLDS

Let us discuss the existence of such manifolds for the semilinear parabolic equation

$$u_t = -Au + F(u) \qquad (*)$$

in Hilbert space X with the phase space $H = D(A^\theta), 0 \le \theta < 1.$

PROBLEM. *Find a relationship between the following properties*:

(A) *The spectrum sparseness condition for the linear part of Eq.* $(*)$:

$$\sup_{n\ge 1} \frac{\mu_{n+1} - \mu_n}{\mu_{n+1}^\theta + \mu_n^\theta} = \infty, \text{ where } \{0 < \mu_1 < \mu_2 < \ldots\} = \sigma(A);$$

(B) *For **any admissible** nonlinearity* F, *Eq.* $(*)$ *has an absolutely normally hyperbolic inertial manifold* $M \subset H$.

(C) *For **any admissible** nonlinearity* F, *Eq.* $(*)$ *has an inertial manifold* $M \subset H$ *absolutely normally hyperbolic at the stationary points.*

(D) *For **any admissible** nonlinearity* F, *Eq.* $(*)$ *has an inertial manifold* $M \subset H$.

PROPOSITION. *Properties* (A), (B), (C) *and* (D) *are equivalent.*

The implication (A)$\Rightarrow$(B) is known [1] and the implications (B)$\Rightarrow$(C), (C)$\Rightarrow$(D) are trivial. The implication (D)$\Rightarrow$(A) has been obtained in [2].

[1] R. Rosa, R. Temam, *ACTA Applicandae Mathematicae*, **45,** 1996.

[2] A. Eden, V. Kalantarov, S. Zelik, *Russian Math. Surveys*, **68**:2, 2013.

THE PARTICULAR CASE

One has slightly other picture for special classes of semilinear parabolic equations. Let us consider the scalar reaction-diffusion equation

$$\partial_t u = \Delta u + \eta f(u), \quad \eta > 0, \qquad (*)$$

in a bounded domain $\Omega \subset R^m$ $(m \leq 3)$ with the condition $\partial_n u\big|_{\partial\Omega} = 0$ and with smooth function f. We assume that Eq. $(*)$ is dissipative in $H = L^2(\Omega)$. Let

$$\{0 \leq \mu_1 < \mu_2 < \ldots\} = \sigma(-\Delta).$$

PROPOSITION [1]. *Let* $\mu_{n+1} - \mu_n \leq K, n \geq 0,$ *and* $f'(p_0) - f'(p_1) = a > 0$ *for some* $p_0, p_1 \in \mathbb{R},$ $f(p_0) = f(p_1) = 0.$ *Then Eq.* $(*)$ *with* $\eta > K/a$ *have no inertial manifold* $M \subset H$ *absolutely normally hyperbolic at the stationary points*.

For Eq. $(*)$ **we can affirm the equivalence of properties (A), (B), (C) only.**

[1] A.V. Romanov, *Math. Notes*, **68**:3-4, 2000.

POSSIBLE GOALS

1. **Construct an example of a reaction-diffusion system without an inertial manifold.**
2. **The study of the topic "inertial manifolds" for reaction-diffusion equations on close manifolds.**
3. **Successfully advancement in the principle of spatial averaging. The study of its relationship with existence and nonexistence of normally hyperbolic inertial manifolds.**

THANKS FOR ATTENTION