Linear switched dynamical systems on graphs [*]

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Abstract

We consider linear dynamical systems with a structure of a multigraph. The vertices are associated to linear spaces and the edges correspond to linear maps between those spaces. We analyse the asymptotic growth of trajectories (associated to paths along the multigraph), the stability and the stabilizability problems. This generalizes the classical linear switching systems and their recent extensions to Markovian systems, to systems generated by regular languages, etc. We show that an arbitrary system can be factorized into several irreducible systems on strongly connected multigraphs. For the latter systems, we prove the existence of invariant (Barabanov) multinorm and derive a method of its construction. The method works for a vast majority of systems and finds the joint spectral radius (Lyapunov exponent). Numerical examples are presented and applications to the study of fractals, attractors, and multistep methods for ODEs are discussed.

Keywords: *constrained linear switching systems, joint spectral radius, multigraph, Markovian systems, regular languages, multinorm, polytope, fractal, attractor*

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1 Introduction

Linear switching systems draw much attention in the literature due to their applications in the electronic engineering, dynamical systems, control theory, etc. A discrete linear switching system (LSS) has the form

$$x(k) \; = \; A(k)\,x(k-1)\,, \quad k \in \mathbb{N}\,, \tag{1}$$

where $\{x(i)\}_{i=0}^{\infty}$ is a sequence of points (*trajectory*) in the Euclidean space $\mathbb{R}^d$, $\{A(k)\}_{k \in \mathbb{N}}$ is a sequence of matrices (*switching law*) chosen independently from a given compact set $\mathcal{A}$. The system is *stable* if $x(k) \to 0$ as $k \to \infty$ for every initial point $x(0) \in \mathbb{R}^d$ and for every switching law. This is actually the *asymptotic stability* property, but for the sake of simplicity, we omit the word "asymptotic".

It is well known that the stability is equivalent to the inequality $\rho(\mathcal{A}) < 1$, where

$$\rho(\mathcal{A}) = \lim_{k \to \infty} \max_{A(i) \in \mathcal{A},\, i=1,\ldots,k} \left\| A(k) \cdots A(1) \right\|^{1/k} \tag{2}$$

is the *joint spectral radius* (JSR) of the matrix family $\mathcal{A}$. This limit always exists and does not depend on the matrix norm. The joint spectral radius is equal to the Lyapunov exponent of the system, which is the maximal exponent of the asymptotic growth of trajectories: $\sup \frac{\log \|x_k\|}{\log k} = \rho$, where the supremum is over all trajectories with $\|x_0\| = 1$. If $\mathcal{A}$ consists of one matrix, then JSR becomes its usual spectral radius (the largest modulus of eigenvalues of a matrix).

The joint spectral radius originated in 1960 with Rota and Strang [47] and has found countless applications in various fields, from functional analysis to discrete mathematics and theory of formal languages (see [20, 30] for properties and applications). The computation of JSR is an algorithmically hard problem even for finite families $\mathcal{A}$. Nevertheless, there are several practically efficient methods for estimating the JSR [12, 19, 45, 46] or even for its precise computation [20, 35] for wide classes of matrices. The theoretical base of many methods is the so-called *invariant norm* called also the *Barabanov norm*. This is a norm in $\mathbb{R}^d$ such that $\|x\| = \rho(\mathcal{A}) \max_{A \in \mathcal{A}} \|Ax\|$ for all $x \in \mathbb{R}^d$. Every irreducible family of matrices (i.e., the matrices from $\mathcal{A}$ do not share a common nontrivial invariant subspace) possesses an invariant norm [3]. If $\mathcal{A}$ is reducible, then it can always be factorized, which makes the problem of JSR computation to be equivalent to several analogous problems in smaller dimensions [4, 41].

Recently many authors introduced and analysed *constrained switching systems*, where not all switching laws are possible but only those satisfying certain stationary constraints [11, 32, 36, 38, 39, 40, 49, 51]. The concept slightly varies in different papers, but in general can be described as follows: given a directed graph G with edges labelled by matrices from a family $\mathcal{A}$ (one matrix may correspond to several edges). We call a trajectory *admissible* if it is realized as a path along the graph. This leads to the concept of stability of constrained system (all admissible trajectories converge to zero) and the corresponding (constrained) joint spectral radius, where the maximum in (2) is computed only over admissible sequences. Methods for evaluating the lower bounds and upper bounds for the constrained JSR were presented in [32, 38, 40]. We shell discuss them in more detail in Section 12.

The aim of this paper is to develop a method of precise computation of JSR for constrained systems by evaluating a piecewise linear convex invariant Lyapunov function. That function is actually a norm defined by a special convex polytope. We first prove several theoretical results (Sections 2 – 7) based on which we derive Algorithm 1 for JSR computation and for constructing invariant polytopes (Section 9). As we shell see in examples (Section 9.1) and in statistics of numerical experiments (Section 9.1) the method is indeed

able to find precisely the joint spectral radius for a vast majority of constrained systems. For general systems, it works efficiently in dimensions at least up to 20, for positive systems, it works much faster and is applicable in higher dimensions (at least up to 100). For the classical (unconstrained) systems, the Invariant polytope algorithm was introduced and analyzed in [20]. That algorithms proved its efficiency for many problems, both numerical and theoretical. In particular, it helped solving several long-standing open problems in combinatorics, theory of formal languages, theory of wavelets and subdivisions [20, 21, 44]. However, a naive attempt to extend the Invariant polytope algorithm directly to constrained systems offers a strong resistance. The main difficulty is that a large part of constrained systems are *reducible*, i.e., possess common invariant subspaces. In this case, no results on invariant or extremal norms hold, and the polytope algorithm does not work. Indeed, all existence theorems for the extremal norms are formulated only for irreducible systems (see [3] for the scallassical case and [38] for constrained systems). For unconstrained systems this is not a serious problem, because the irreducible case is rare for them. However, for constrained systems, the reducible case becomes usual, or even generic for some graphs (see Remark 3).

Thus, an extension of results on extremal and Invariant norms and of the invariant polytope algorithm to all constrained systems needs a well-developed theoretical base. We derive this base in the following Sections 2 – 7. Everywhere, all the sets $\mathcal{A}_{ij}$ are arbitrary compact sets of linear operators from L_j to L_i. Only in Sections 9 and 10, where we introduce numerical algorithms, they are assumed to be finite.

First of all, we slightly generalize the concept of constrained systems. We consider a directed multigraph G with linear spaces $\{L_i\}_{i=1}^n$ (maybe, of different dimensions) associated to its vertices. For each pair of vertices g_i and g_j, there is a finite (maybe empty) collection of edges leading from g_i to g_j identified with linear operators acting from L_i to L_j. Thus we have a family of spaces and linear operators mapping them to each other according to the multigraph G. To every path starting at some vertex g_s and going along successive edges of G, and to every point $x_0 \in g_s$ we naturally associate a trajectory of the point x_0. Thus, we obtain a linear dynamical system on the multigraph G. We use the short terminology *system on a graph*, although it is actually defined on a multigraph. All definitions and properties of this construction are given in the next section. To study the stability, growth of trajectories, and JSR of such systems we realize the following plan:

1. First, we prove a factorization theorem that reduces a system on an arbitrary multigraph to several smaller systems on strongly connected multigraphs. This allows us to restrict the whole theory to the case of strongly connected multigraph (Section 4).

2. Then we introduce the concept of reducible and irreducible systems. Our concept is similar to that presented in [39], but different. An advantage of the new concept is that the reducibility can be easily proved. Then we show that every reducible system can be factorized to several irreducible ones of smaller dimensions (Section 6). Thus, we have the same situation as for the classical (unconstrained) systems. However, the notion of reducibility cannot be extended directly to constrained systems (see an example in [39]) and requires a significant modification. Several properties of irreducible systems are established. In particular, we show that every such a system is non-defective, i.e., its trajectories grow not faster than $\|x_k\| \leq C\rho^k$, $k \in \mathbb{N}$, where ρ is the joint spectral radius and $C > 0$ is

a constant (Section 5). Working with practical examples and with randomly generated matrices in Section 9.1 we observe a surprising phenomenon. Reducible systems, which are very exceptional in the usual (unconstrained) case, becomes usual for the constrained systems. According to our statistics given in Section 9.1 in dimension $d = 3$ about 20% of randomly generated systems are reducible, while for $d = 20$, this ratio growth to 60%. Thus, reducible systems become dominant in high dimensions. This makes our technique for factorizing reducible systems to be important in most practical cases. We discuss this issue in more detail in Remark 3.

3. We restrict the whole theory to irreducible systems on strongly connected multigraphs. This allows us to extend the concept of extremal multinorms [39] to all consrained systems. Moreover, we make the next step and introduce the concept of invariant multinorms. An invariant (Barabanov) multinorm is a collection of norms $\|\cdot\|_i$ in the spaces L_i, $i = 1, \ldots, n$, respectively such that $\max\limits_{A_{ji}} \|A_{ji}x\|_j = \rho\, \|x\|_i$ for every i and $x \in L_i$, where the maximum is over all operators associated to all outgoing edges from g_i. For the extremal multinorm, the definition is the same, but with inequality $\leq$ instead of equality (Section 7). We prove that an every irreducible system possesses an invariant multinorm (Theorem 3).

4. Based on these theoretical results we elaborate an algorithm that computes the joint spectral radius and constructs a piecewise-linear extremal mutlinorm (Section 9). A criterion of its convergence is provided by Theorem 6. The efficiency, even in relative high dimensions, is demonstrated in examples and in the statistics of numerical experiments (Section 9.1). Moreover, in Section 10 we show how to construct Barabanov piecewise-linear norm by that algorithm.

This is a summary of the main results. In addition, we consider several special cases of the general construction: Markovian systems, systems defined by regular languages, etc. (Section 3). We derive some corollaries, such as an improved Berger-Wang formula which sharpens the classical formula even in the usual (unconstrained) case; we estimate the rate of growth of defective (*marginally unstable*) systems (Section 8). In Section 11 we discuss some special cases, as the one of positive systems. Finally, in Section 12 we consider possible applications of our results to linear switching systems, automata, fractals, attractors of hyperbolic dynamical systems, consensus problems, and stability of multistep methods in ODEs.

We use the following notation. For two points $x, y \in \mathbb{R}^d$, we write $x \geq y$ ($x > y$) if the vector $x - y$ is nonnegative (respectively, strictly positive). As usual, the positive orthant $\mathbb{R}^d_+$ is the set of nonnegative vectors. For a given set $M \subset \mathbb{R}^d$ we denote by $\mathrm{co}(M)$ its convex hull and by $\mathrm{absco}(M) = \mathrm{co}\{M, -M\}$ the symmetrized convex hull. For $M \subset \mathbb{R}^d_+$ we denote

$$\mathrm{co}_-(M) \;=\; \Big\{\, x - y \;\Big|\; x \in \mathrm{co}\,(M),\; y \geq 0 \,\Big\}; \quad \mathrm{co}_+(M) \;=\; \Big\{\, x + y \;\Big|\; x \in \mathrm{co}\,(M),\; y \geq 0 \,\Big\}. \quad (3)$$

Note that $\mathrm{co}_+(M)$ is always unbounded, whenever M is nonempty. If M is finite then $\mathrm{co}_-(M)$ is called *infinite polytope*. The set $\mathrm{co}_+(M)$ in this case is a polytope (in a usual sense) containing the polytope $\mathrm{co}(M)$. The sign $\asymp$ denotes as usual the asymptopic equivalence of two values (i.e., equivalence up to multiplication by a constant).

2   The general construction

We have a directed multigraph G with n vertices $g_1, \ldots, g_n$. Sometimes, the vertices will be denoted by their numbers. To each vertex i we associate a linear space L_i of dimension $d_i < \infty$. If the converse is not stated, we assume $d_i \geq 1$. The set of spaces $L_1, \ldots, L_n$ is denoted by $\mathcal{L}$. For each vertices $i, j \in G$ (possibly coinciding), there is a set ℓ_{ji} of edges from i to j. Each edge from ℓ_{ji} is identified with a linear operator $A_{ji} : L_i \to L_j$. The family of those operators (or edges) is denoted by $\mathcal{A}_{ji}$. If $\ell_{ji} = \emptyset$, i.e., there are no edges from i to j, then $\mathcal{A}_{ji} = \emptyset$. Thus, we have a family of spaces $\mathcal{L}$ and a family of operators-edges $\mathcal{A} = \cup_{i,j} \mathcal{A}_{ji}$ that act between these spaces according to the multigraph G. This *triplet* $\xi = (G, \mathcal{L}, \mathcal{A})$ of the multigraph, spaces, and operators will be called *system*. A path α on the multigraph G is a sequence of connected subsequent edges, its length (number of edges) is denoted by $|\alpha|$. The length of the empty path is zero. To every path α along vertices $i_1 \to i_2 \to \cdots \to i_{k+1}$ that consists of edges (operators) $A_{i_{s+1} i_s} \in \mathcal{A}_{i_{s+1} i_s}$, $s = 1, \ldots, k$, we associate the corresponding product (composition) of operators $\Pi_\alpha = A_{i_{k+1} i_k} \cdots A_{i_2 i_1}$. Note that $|\alpha| = k$. Let us emphasize that a path is not a sequence of vertices but edges. If G is a graph, then any path is uniquely defined by the sequence of its vertices, if G is a multigraph, then there may be many paths corresponding to the same sequence of vertices. If the path is closed ($i_1 = i_{k+1}$), then Π_α maps the space L_{i_1} to itself. In this case Π_α is given by a square matrix, and possess eigenvalues, eigenvectors and the spectral radius $\rho(\Pi_\alpha)$, which is the maximal modulus of its eigenvalues. The set of all closed paths will be denoted by $\mathcal{C}(G)$. For an arbitrary $\alpha \in \mathcal{C}(G)$ we denote by $\alpha^k = \alpha \ldots \alpha$ the kth power of α. A closed path is called *simple* if it is not a power of a shorter path.

In what follows we assume all the sets ℓ_{ji} and the corresponding sets of operators $\mathcal{A}_{ji}$ are finite. This assumption is for the sake of simplicity; all our results are easily extended to the case of arbitrary compact sets $\mathcal{A}_{ji}$.

We make use of the concept of multinorm introduced in [38] and adapt it to arbitrary multigraph with arbitrary linear spaces:

Definition 1 *If every space L_i on the multigraph G is equipped with a norm $\| \cdot \|_i$, then the collection of norms $\| \cdot \|_i$, $i = 1, \ldots, n$, is called a multinorm. The norm of an operator $A_{ji} \in \mathcal{A}_{ji}$ is defined as* $\|A_{ji}\| = \sup\limits_{x \in L_i, \|x\|_i = 1} \|A_{ji} x\|_j.$

Note that the notation $\|x\|_i$ assumes that $x \in L_i$. In the sequel we suppose that our multigraph G is equipped with some multinorm $\{ \| \cdot \|_i \}_{i=1}^n$. We denote that multinorm by $\| \cdot \|$ and sometimes use the short notation $\|x\| = \|x\|_i$ for $x \in L_i$. Thus, we drop the index of the norm if it is clear to which space L_i the point x belongs to.

For a given $x_0 \in L_i$ and for an infinite path α starting at the vertex i, we consider the *trajectory* $\{x_k\}_{k \geq 0}$ of the system along this path. Here $x_k = \Pi_{\alpha_k} x_0$, where α_k is a prefix of α of length k.

Definition 2 *The system ξ is called stable if every its trajectory tends to zero as $k \to \infty$.*

As in the classical case of usual unconstrained discrete systems, the stability is decided in terms of the joint spectral radius, which in this case is modified as follows:

Definition 3 *The joint spectral radius (JSR) of a triplet $\xi = (G, \mathcal{L}, \mathcal{A})$ is*

$$\rho(\xi) \;=\; \lim_{k \to \infty} \max_{|\alpha|=k} \|\Pi_\alpha\|^{1/k} . \tag{4}$$

Note that the function $\varphi(k) = \max_{|\alpha|=k} \|\Pi_\alpha\|$ possesses the property

$$\varphi(k+l) \;\leq\; \varphi(k)\,\varphi(l), \quad k, l \in \mathbb{N},$$

hence by the well-known Fekete lemma [16] , the limit $\lim_{k \to \infty} \varphi(k)^{1/k}$ exists and coincides with $\inf_{k \in \mathbb{N}} \varphi(k)^{1/k}$. This ensures that the joint spectral radius is well defined. Moreover, for every $r, k \in \mathbb{N}$, we have the double inequality

$$\max_{\alpha \in \mathcal{C}(G),\, |\alpha|=r} \rho\big(\Pi_\alpha\big)^{1/r} \;\leq\; \rho(\xi) \;\leq\; \max_{|\alpha|=k} \|\Pi_\alpha\|^{1/k} . \tag{5}$$

The right hand side of this inequality tends to $\rho(\xi)$ as $k \to \infty$, this follows from the definition of the joint spectral radius. The upper limit of the left hand side as $k \to \infty$ is also equal to $\rho(\xi)$. In the classical case (m operators in one space) this fact is known as the *Berger-Wang formula* [4]. Recently it was generalized by Dai [11] and Kozyakin [32] to the Markovian systems (see subsection 3.2). We extend it to general systems $\xi = (G, \mathcal{L}, \mathcal{A})$ and establish an improved version of this formula in Theorem 4. The right hand side of inequality (5) for $k = 1$ implies that the joint spectral radius never exceeds the maximal norm of operators from $\mathcal{A}$, and this does not depend on the multinorm introduced for the system ξ. This leads to the alternative definition of JSR. In the classical case, this result is well-known and originated with Rota and Strang [47] and Elsner [15]. It is extended to systems on graphs in a straightforward manner, see, for instance [38, 40]. We give its proof for convenience of the reader.

Proposition 1 *The joint spectral radius is the greatest lower bound of numbers $\lambda \geq 0$ for which there exists a multinorm $\|\cdot\| = \{\|\cdot\|_i\}_{i=1}^n$ on G such that $\|A_{ji}\| \leq \lambda$, $A_{ji} \in \mathcal{A}$.*

`Proof.` If $\|A_{ji}\| \leq \lambda$, $A_{ji} \in \mathcal{A}$, then from inequality (5) for $k = 1$, it follows that $\rho(\xi) \leq \lambda$. Conversely, assume $\rho(\xi) \leq \lambda$. We need to show that for every $\mu > \lambda$, there is a multinorm such that $\|A_{ji}\| \leq \mu$, $A_{ji} \in \mathcal{A}$. Let $\tilde{\mathcal{A}} = \mu^{-1}\mathcal{A}$. Clearly, all trajectories of the system $\tilde{\xi} = (G, \mathcal{L}, \tilde{\mathcal{A}})$ tend to zero, because $\rho(\tilde{\xi}) < 1$. Hence, those trajectories are uniformly bounded. Consequently, the function $f(x) = \sup_{|\alpha| \geq 0} \|\tilde{\Pi}_\alpha x\|$ (the supremum is taken over all paths α along G starting at x) is bounded for all $x \in L_i, i = 1, \ldots, n$. This function is positive, symmetric, and positively homogeneous. It is convex as a supremum of convex functions. Hence, f is a norm. For each operator $\tilde{A}_{ji}$, every path starting at the point $A_{ji}x \in L_j$ is a part of the corresponding path starting at x, hence, $f(x) \geq f(\tilde{A}_{ji}x) = \mu^{-1}f(A_{ji}x)$. Thus, $f(A_{ji}x) \leq \mu f(x)$ for all $x \in L_i$, hence the operator norm of A_{ji} does not exceed μ.
$\square$

The greatest lower bound in Proposition 1 is not necessarily attained. If it is, then the corresponding norm is called *extremal* (Definition 6). In Section 7 we are going to see that,

similarly to the classical case (unconstrained systems), an extremal norm exists at least for irreducible systems.

An immediate consequence of Proposition 1 is that the joint spectral radius is responsible for the simultaneous contractibility of all the operators A_{ij}.

Proposition 2 *The following properties of a system are equivalent:*

1) $\rho(\xi) < 1$;

2) There exists a multinorm $\|\cdot\|$ and a number $q < 1$ such that $\|A_{ji}\| < q$ for all $A_{ji} \in \mathcal{A}$.

Thus, $\rho(\xi) < 1$ precisely when each space L_i can be equipped with a norm $\|\cdot\|$ so that all operators A_{ji} are contractions. We use this property in Section 12 for applications to fractals and to dynamical systems.

`Proof.` If 2) holds, then obviously $\rho(\xi) \leq q < 1$. If 1) holds, then taking arbitrary $q \in (\rho, 1)$ and applying Proposition 1 for $\lambda = q$ we obtain a multinorm such that $\|A_{ji}\| < q$ for all $A_{ji} \in \mathcal{A}$.

$\square$

Similarly to the classical case, the joint spectral radius measures the stability of the system. The following fact was established in [11, Corollary 2.8] for the case of equal dimensions of the spaces L_i. The proof for the general case in the same.

Proposition 3 *A system is stable if and only if $\rho(\xi) < 1$.*

So, for systems on graphs we have the same situation as in the classical case. The proof is also similar, we give it in Section 7.

3 Special cases

Before we establish the main properties of systems on multigraphs, we spot several important special cases.

3.1 The classical (unconstrained) case: m operators in one space.

If G has one vertex and m edges (loops) connecting that vertex with itself, we obtain the classical case: m operators $A_1, \ldots, A_m$ act in one space $\mathbb{R}^d$. In this case the notions of trajectories, the joint spectral radius, invariant norms, etc., are the same as those elaborated in the extensive literature on asymptotics of matrix products (see [30, 20] for reviews).

3.2 Markovian systems

We are given a family $\mathcal{A} = \{A_1, \ldots, A_m\}$ of operators acting in the space $\mathbb{R}^d$. Let $\mathcal{D}$ be a subset of the set of m^2 ordered pairs $\{(j, i) \mid j, i = 1, \ldots, m\}$. The *Markovian system* consists of all *admissible* products of operators from $\mathcal{A}$, i.e., products that avoid subproducts $A_j A_i$, $(j, i) \in \mathcal{D}$. In other words, the operator A_j cannot follow the operator A_i in any

product, whenever $(j, i) \in \mathcal{D}$. The *Markovial joint spectral radius* originated in [11], it is defined as the usual joint spectral radius, but over a set of admissible matrix products.

The Markovian systems can be put in our framework as follows. We consider the graph G with m vertices, each vertex i is associated to the space $L_i = \mathbb{R}^d$; the edge $i \to j$ exists if and only if $(j, i) \notin \mathcal{D}$, this edge corresponds to the operator A_j. Thus, all edges from $\mathcal{D}$ are excluded from the graph, the remaining edges are arranged as follows: all incoming edges of the vertex j correspond to the same operator A_j, $j = 1, \ldots, m$.

In Theorem 4 we slightly improve the main results of Dai [11] and Kozyakin [32] and extend them from the Markovian systems to general systems.

3.3 Identifying several vertices

Consider the trivial Markovian system when the set of prohibited links $\mathcal{D}$ is empty. In this case the Marovian joint spectral radius coincides with the usual joint spectral radius. A question arises whether it is possible to treat this case without considering m copies of the space $\mathbb{R}^d$ as vertices of the graph G (which is a clique in this case) and to manage with one space $\mathbb{R}^d$ as in the classical case. The answer is affirmative. This can be done by the procedure of *identifying vertices that have the same sets of outgoing edges.* In many practical cases this significantly simplifies the analysis of general systems $\xi = (G, \mathcal{L}, \mathcal{A})$.

We consider a general system ξ with a multigraph G. If two its vertices i_1 and i_2 satisfy the following three conditions:

1. The associated spaces L_{i_1} and L_{i_2} have the same dimension;
2. They have the same set of outgoing edges, i.e., for every $j = 1, \ldots, n$, we have $\ell_{ji_1} = \ell_{ji_2}$;
3. $\mathcal{A}_{ji_1} = \mathcal{A}_{ji_2}$, i.e., there is a basis in the space L_{i_1} and a basis in the space L_{i_2} such that the operators from $\mathcal{A}_{ji_1}$ are written by the same matrices as the corresponding operators from $\mathcal{A}_{ji_2}$.

Then the vertices i_1 and i_2 can be identified. They are replaced by one vertex i. The sets $\mathcal{A}_{ji}$ of its outgoing vertices is the same as those of vertices i_1 and i_2. The set of incoming vertices is the union of those of i_1 and i_2. The norm in L_i can be chosen arbitrarily. For example, the pointwise maximum of norms in L_{i_1} and L_{i_2}. We obtain the multigraph G' with $n - 1$ vertices and the corresponding system ξ'.

Every path α on G is naturally identified with a path α' on G' by replacing both i_1 and i_2 by i. This establishes the correspondence between trajectories of ξ and ξ'. In particular, those trajectories have the same asymptotics as $k \to \infty$. Therefore, $\rho(\xi') = \rho(\xi)$.

For example, if the Markovian system has no prohibited links, then all m ist vertices can be identified, and we obtain the classical system, with one space and m operators.

3.4 Maximal growth of trajectories avoiding prohibited words

We have a family $\mathcal{A} = \{A_1, \ldots, A_m\}$ of operators acting in $\mathbb{R}^d$ and a finite set $\mathcal{D}$ of words of the m-ary alphabet $\{1, \ldots, m\}$. This is the dictionary of prohibited words. We consider a discrete system with operators from $\mathcal{A}$ and with trajectories avoiding those prohibited words. In particular, we are interested in the exponent of the maximal asymptotic growth

of those trajectories. This exponent is the joint spectral radius of the family $\mathcal{A}$ along the products that avoid words from $\mathcal{D}$. In particular, the system is stable if and only is this value is smaller than one.

These systems can be put in our framework as follows. Let $l \geq 2$ be the maximal length of words from $\mathcal{D}$. The vertices of the graph G are all m-ary words of length $l-1$ that avoid subwords from $\mathcal{D}$ (of lengths smaller than l, if they exist). There is an edge from the word (vertex) β to γ if and only if the following two conditions are satisfied:

1) the prefix of length $(l-2)$ of β is the suffix of γ;

2) the word $\gamma_1 \beta = \gamma \beta_{l-1}$ of length l is not from $\mathcal{D}$ (γ_1 and β_{l-1} are the first and the last letters of γ and β respectively).

If these conditions are fulfilled, then there is a unique edge $\beta \to \gamma$, it is associated to the operator A_{γ_1}. Note that if $l = 2$, then condition 1) is always fulfilled. If $\mathcal{D} = \emptyset$, i.e., there are no prohibited words, then the graph G has m^{l-1} vertices. Each vertex β has exactly m outgoing edges, to the vertices $1\beta, \ldots, m\beta$, and m incoming edges, from the vertices $\beta 1, \ldots, \beta_m$. Hence, for any set of prohibited words, the graph G has at most m^{l-1} vertices and at most m incoming and m outgoing edges for each vertex. So, the graph G has at most 2^l edges. All the spaces L_i at the vertices of G are copies of $\mathbb{R}^d$.

To every word $c = c_1 c_2 \ldots c_N$ of the m-ary alphabet avoiding subwords from $\mathcal{D}$, the corresponding path $\beta_1 \to \beta_2 \to \ldots \beta_N$ along G is naturally associated as follows: $\beta_k = c_k \ldots c_{k+l-2}$, $k = 1, \ldots, N+2-l$. This path corresponds to the product $A_{c_{N+2-l}} \cdots A_1$.

Thus, we have the triplet ξ. The joint spectral radius of the family $\mathcal{A}$ along the products avoiding words from $\mathcal{D}$ is equal to $\rho(\xi)$.

The Markovian systems is a special case of this construction when $l = 2$.

Example 1 Let us have two operators A_1 and A_2 and one prohibited word $A_1 A_2 A_1$. So, we are interested in the maximal asymptotic growth of products $A_1^{k_s} A_2^{r_s} \cdots A_1^{k_1} A_2^{r_1}$ with $r_i \geq 2$ for all i. In this case, $G^{(1)}$ has four vertices $A_1 A_1, A_1 A_2, A_2 A_1, A_2 A_2$ and eight edges. The edge $A_2 A_1 \to A_1 A_2$ associated to the operator A_1 is omitted because of the prohibited word $A_1 A_2 A_1$, all other seven edges are kept (see fig 1). The JSR along infinite paths of this graph equals to the JSR along those products.
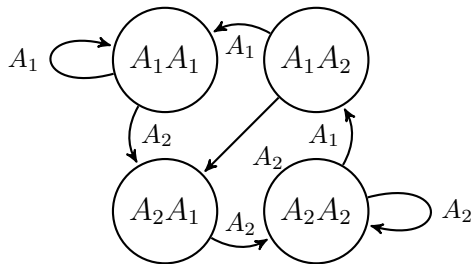

Figure 1: $G^{(1)}$ Graph

If $\mathcal{D} = \{A_1 A_2 A_1, A_1^2\}$, then we have the set of products $A_1 A_2^{r_s} \cdots A_1 A_2^{r_1}$ with $r_i \geq 2$ for all i. The graph $G^{(2)}$ has three vertices $A_1 A_2, A_2 A_1, A_2 A_2$ (the vertex $A_1 A_1$ has been

omitted) and four edges (one of the five edges, $A_2A_1 \to A_1A_2$, is omitted), see fig 2.
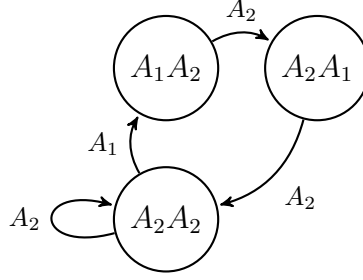

Figure 2: $G^{(2)}$ Graph

4   Strong connectivity

First of all, let us show that the analysis of general systems can be reduced to systems with strongly connected multigraphs G. Recall that a multigraph is strongly connected if for an arbitrary pair of vertices, there is a path from one to the other.

If G is not strongly connected, then there is a closed submultigraph $G_1 \subset G$, for which all vertices reachable from G_1 belong to G_1. Let $G_2 = G \setminus G_1$ be a complement of G_1 in G. Denote by ξ_i the restriction of the triplet ξ to the submultigraph $G_i, i = 1, 2$. The triplet ξ_1 contains only vertices from G_1 with the corresponding family of linear spaces $\mathcal{L}_1 \subset \mathcal{L}$, the edges of the multigraph G connecting vertices from G_1 and the family of operators $\mathcal{A}_1 \subset \mathcal{A}$ corresponding to those edges. The same with ξ_2.

Proposition 4 *If G is not strongly connected, then for every its closed submultigraph $G_1 \subset G$, we have $\rho(\xi) = \max\{\rho(\xi_1), \rho(\xi_2)\}$.*

`Proof`. If a path starts at a vertex of G_1, then it is contained in G_1. Therefore, any path α on the multigraph G is a concatenation $\alpha_1\alpha_2$, where α_k is on G_k, $k = 1, 2$. Hence, $\Pi_\alpha = \Pi_{\alpha_1} A_{ij} \Pi_{\alpha_2}$, where the edge the operator $A_{ji} \in \mathcal{A}_{ji}$ corresponds to an edge $l_{ji} \in \ell_{ji}$ connecting the two paths. Let $\rho_k = \rho(\xi_k), k = 1, 2$, and $\bar\rho = \max\{\rho_1, \rho_2\}$. For any $\varepsilon > 0$ we have $\|\Pi_{\alpha_k}\| \leq C(\rho_k + \varepsilon)^{|\alpha_k|}$, $k = 1, 2$. Since $|\alpha| = |\alpha_1| + |\alpha_2| - 1$ and $\|A_{ji}\| \leq C$, we have $\|\Pi_\alpha\| \leq C^3(\bar\rho + \varepsilon)^{|\alpha|-1}$. Hence $\rho \leq \bar\rho + \varepsilon$ for every $\varepsilon > 0$, and so $\rho \leq \bar\rho$. On the other hand, obviously $\rho \geq \rho_k$, for each $k = 1, 2$, and therefore $\rho \geq \bar\rho$.

$\square$

An elementary induction shows that for a not strongly connected multigraph, there is a disjoint partition of its vertices $G = \bigsqcup_{i=1}^r G_i$ such that each submultigraph G_i is strongly connected and G_i is not reachable from G_j, whenever $i > j$. Proposition 4 yields

Corollary 1 *If G is not strongly connected then $\rho(\xi) = \max\{\rho(\xi_1), \dots, \rho(\xi_r)\}$.*

Thus, the analysis of the asymptotic properties of an arbitrary system $\xi = (G, \mathcal{L}, \mathcal{A})$ is reduced to the same problem for the systems ξ_i that are characterized by strongly connected multigraphs Our next assumption concerns irreducibility of the system.

5 Irreducibility

Consider an arbitrary triplet $\xi = (G, \mathcal{L}, \mathcal{A})$. Here the case of trivial spaces $L_i = \{0\}$ is allowed for some (but not all) i. A triplet $\xi' = (G, \mathcal{L}', \mathcal{A}')$ is *embedded* in ξ, if $L_i' \subset L_i$ for each i and every operator $A_{ji}' = A_{ji}|_{L_i'}$ maps L_i' to L_j', whenever $l_{ji} \in G$. The embedding is strict if L_i' is a proper subspace of L_i at least for one i. Thus, an embedded triplet has the same multigraph and smaller spaces at the vertices.

Remark 1 Actually, one could extend this definition allowing the embedded triplet ξ' to omit some edges from G, i.e., to have a submultigraph $G' \subset G$ instead of the same multigraph G. However, instead of eliminating an edge $l_{ij} \in \ell_{ji}$ we can set the corresponding A_{ji}' to be the null operator. In this case, all trajectories passing through the eliminated edges vanish and can therefore be ignored. So, for the sake of simplicity, we always assume that the embedded triplet ξ' has the same multigraph G.

Definition 4 *A triplet $\xi = (G, \mathcal{L}, \mathcal{A})$ is reducible if it has a strictly embedded triplet. Otherwise, it is called irreducible.*

There are three remarkable properties of irreducible triples:

    1) for an arbitrary initial vector $x \in L_j$, its images $\Pi_\alpha x$ span all the spaces $L_1, \ldots, L_n$;

    2) $\rho(\xi) > 0$, provided all the spaces L_i are nontrivial;

    3) if $\rho(\xi) = 1$, then all trajectories are uniformly bounded;

    4) there is an invariant (Barabanov) multinorm.

We prove these properties and then, in Theorem 2, show that a general triplet can be factorized to several irreducible ones of smaller dimensions. This will imply that the whole analysis can be focused on irreducible triples. Properties 1), 2), and 3) are established in this section, the proof of 4) is postponed to Section 7.

Remark 2 Properties 1) - 4) are well-known for the classical (unconstrained) systems. For constrained systems, in case when all spaces L_i have the same dimensions, the concept of reducibility was extended in [39] as follows: a system is reducible if there exists a proper nonzero subspace $L_i' \subset L_i$ (we use our terminology) such that for every cycle α passing though g_i, we have $\Pi_\alpha L_i' \subset L_i'$. That definition is not equivalent to Definition 4, although they become equivalent for strongly connected graphs. New Definition 4 has an advantage that it can easily be verified in practice: if we suspect that a triplet ξ' is embedded in ξ, we can easily check it. Moreover, evaluating embedded systems can be done within the framework of Invariant polytope algorithm (Section 9). On the other hand, to verify that a subspace L_i' causes reducibility in the sense of [39], one has to check all cycles passing through a given vertex.

    In [38, 40] properties 1) and 3) were proved for irreducible (in sense of [39]) systems with equal spaces L_i.

    We begin with the proof of Property 1), which is characteristic of an irreducible family: the orbits of any nonzero element $x \in L_i$ span all the spaces $L_1, \ldots, L_n$. Consider the set of

all paths α on G from a vertex i to j. For arbitrary $x \in L_i$ we denote by $\mathcal{O}_j(x) = \{\Pi_\alpha x \mid \alpha : i \to j\}$ the set of points from L_j to which all the corresponding products Π_α map the point x. Thus, $\mathcal{O}_j(x)$ is the complete *orbit* of the point $x \in L_i$ in the space L_j.

Proposition 5 *A triplet $\xi = (G, \mathcal{L}, \mathcal{A})$ is irreducible if and only if for every i and for an arbitrary point $x \in L_i, x \neq 0$, its orbits $\mathcal{O}_j(x)$, $j = 1, \ldots, n$, are all full-dimensional, i.e., each orbit $\mathcal{O}_j(x)$ spans the corresponding space L_j.*

`Proof.` Let $L'_j = \operatorname{span} \mathcal{O}_j(x)$. We have $A_{kj} L'_j \subset L'_k$ for all pairs $j, k \in \{1, \ldots, n\}$ and for all $A_{kj} \in \mathcal{A}_{kj}$. Besides, the space L'_i is nontrivial, since it contains x. Hence, $\xi' = (G, \mathcal{L}', \mathcal{A}|_{\mathcal{L}'})$ is an embedded triplet. By the irreducibility, it cannot be strictly embedded, consequently $L'_j = L_j$ for all j.

$\square$

Now we are ready to establish 2). The following fact was established in [40]. We include its proof for convenience of the reader.

Proposition 6 *If a triplet $\xi = (G, \mathcal{L}, \mathcal{A})$ is irreducible and all the spaces L_i are nontrivial, then $\rho(\xi) > 0$.*

`Proof.` By Proposition 5, for an arbitrary nonzero element $x \in L_1$, there are products $\Pi_i : L_1 \to L_1$ such that the vectors $\{\Pi_i x\}_{i=1}^{\dim L_1}$ span x, i.e, $x = \sum_i \alpha_i \Pi_i x$. On the other hand, Proposition 1 yields that for any $\varepsilon > 0$, there is a multinorm $\| \cdot \|$ such that $\|\Pi_i\| < \varepsilon$ for all i, and hence $\|x\| < \varepsilon \sum_i |\alpha_i|$. Taking ε small enough, we come to the contradiction.

$\square$

Definition 5 *A triplet ξ is* non-defective *if there is a constant C such that $\|\Pi_\alpha\| \leq C \rho^{|\alpha|}$ for all paths α along G.*

If all the spaces L_i are nontrivial, then $\rho > 0$ (Proposition 6). After normalization, when $\rho = 1$, the non-defectivity means that all admissible products of operators from $\mathcal{A}$ are uniformly bounded. The crucial fact about the non-defectivity of irreducible systems is well-known for the classical case (unconstrained systems). We are now extending it to arbitrary triples.

Theorem 1 *An irreducible triplet with nontrivial spaces is non-defective.*

`Proof.` After normalization it can be assumed that $\rho = 1$. For arbitrary i, we consider the unit sphere $S_i = \{x \in L_i \mid \|x\| = 1\}$ in the space L_i. For each $k \in \mathbb{N}$, we denote by $U_{i,k}$ the set of points $x \in L_i$ such that there exists a path α of length k starting at the vertex i such that $\|\Pi_\alpha x\| > 2$. Note that all those sets are open in S_i. Consider two possible cases.

If $\cup_{k \in \mathbb{N}} U_{i,k} = S_i$ for each $i = 1, \ldots, n$, then by the compactness of the spheres S_i, the open cover $\cup_{k \in \mathbb{N}} U_{i,k}$ admits a finite subcover $\cup_{k \leq N_i} U_{i,k} = S_i$. This implies that for every i and for every $x \in S_i$, there is a path $\alpha = \alpha(x)$ of length at most N_i such that $\|\Pi_\alpha x\| > 2$. Let $N = \max_{i=1,\ldots,n} N_i$. Then, for every $x \in L_i, x \neq 0$, there is a path $\alpha = \alpha(x)$, $|\alpha| \leq N$, such that

$\|\Pi_\alpha x\| > 2 \|x\|$. Starting with arbitrary $x_1 \in S_1$ we successively build a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that for every $j \in \mathbb{N}$, we have $\|x_{j+1}\| > 2\|x_j\|$ and there is a path of length at most N from x_j to x_{j+1}. Therefore, $\|x_{k+1}\| > 2^k$ and x_{k+1} is obtained from x_1 by multiplying with a product Π_k of length at most Nk. Hence, $\rho \geq \lim_{k\to\infty} \|\Pi_k\|^{1/kN} > 2^{1/N}$, which contradicts to the assumption $\rho = 1$.

Otherwise, if for some $q \in \{1, \ldots, n\}$, the union $\cup_{k\in\mathbb{N}} U_{q,k}$ does not cover the sphere S_q, then there exists $z \in S_q$ which does not belong to any of the sets $U_{q,k}$. This means that for every path α staring at the vertex q we have $\|\Pi_\alpha x\| \leq 2$. Hence, the point z has a bounded orbit. For every i, denote by M_i the set of points from L_i that have bounded orbits. Observe several properties of the sets $\mathcal{M} = \{M_i\}_{i=1,\ldots,n}$.

1. Every M_i is a linear subspace of L_i. Indeed, if $x, y \in M_i$, then for any linear combination $ax + by$, $a, b \in \mathbb{R}$, and for every product Π, we have $\|\Pi(ax+by)\| \leq |a|\|\Pi x\| + |b|\|\Pi y\|$. Hence, if x and y have bounded orbits, then so does $ax + by$.

2. If the set of edges ℓ_{ji} is nonempty, then $A_{ji}M_i \subset M_j$ for every $A_{ji} \subset \mathcal{A}_{ji}$. Indeed, if $x \in M_i$, then norms of elements of all trajectories starting at x are uniformly bounded by some constant. Hence, all trajectories starting at the point $A_{ji}x$ are also bounded by the same constant. Consequently, $A_{ji}x \in M_j$.

3. The subspace M_q is nontrivial, since it contains z.

Thus, we have a nontrivial triplet $(G, \mathcal{M}, \mathcal{A}|_{\mathcal{M}})$ embedded into $(G, \mathcal{L}, \mathcal{A})$. If $M_j \neq L_j$ at least for one j, then the triplet $(G, \mathcal{L}, \mathcal{A})$ is reducible, which is impossible. Otherwise, if $M_i = L_i$ for all i, then all points from these spaces have bounded orbits. Take some i and consider an orthonormal basis $e_1, \ldots, e_{d_i}$ of the space L_i. If the orbit of each e_s is bounded by a constant C, then for for every $x = \sum_s x^s e_s \in S_i$ we have $\sum_s |x^s| \leq \sqrt{d_i}$, and hence the orbit of x is bounded by the constant $C\sqrt{d_i}$. Hence, for all paths α starting at the vertex i, we have $\|\Pi_\alpha\| \leq C\sqrt{d_i}$. For $d = \max_i d_i$, we obtain that all admissible products of operators from $\mathcal{A}$ are bounded by norm by $C\sqrt{d}$.

$\square$

6 Factorization of reducible systems

We begin with one important comment.

Remark 3 For unconstrained systems, the reducible case is very exceptional. Indeed, reducibility means that all linear operators of the system possess a common invariant subspace, which is rare even for two operators. Rather surprisingly, for for constrained systems, the situation is fundamentally different. The reducibility becomes usual, and for some graphs G, it even becomes dominating. Consider the following simple example: a two-vertex directed graph $G = \{g_1, g_2\}$ with $L_1 = L_2 = \mathbb{R}^3$ and with two edges l_{12} and l_{21}. Then for *arbitrary* linear operators $A_{12}, A_{21} : \mathbb{R}^3 \to \mathbb{R}^3$, the system ξ is reducible. Indeed, assume for the sake of simplicity that both A_{12} and A_{21} non-degenerate. Since each linear operator in $\mathbb{R}^3$

possesses a real eigenvector, so does $A_{21}A_{12}$. Let $x_1 \in L_1$ be an eigenvector of this operator and $x_2 = A_{12}x_1$. Then the linear spans of vectors x_1 and x_2 support an embedded system ξ'.

Thus, for some graphs G, a large part (or even all) systems become reducible. Hence, neither extremal or invariant norms, nor the Invariant polytope algorithm can be extended for them. That is why, the problem of possible factorization to irreducible systems becomes crucial.

In the classical case (unconstrained systems), the main advantage of using irreducible systems is that the general case can be solved by several irreducible ones of smaller dimensions. This is done by a simultaneous factorization of all matrices of the family to an upper block-diagonal form. That is why the irreducible case can be considered as the basic one. What situation do we have for the general triplet $\xi = (G, \mathcal{L}, \mathcal{A})$? We are going to see that a reducible triplet can always be factorized, which splits the problem into several irreducible problems of smaller dimensions.

Let a triplet $(G, \mathcal{L}, \mathcal{A})$ be reducible and have an embedded triplet $(G, \mathcal{L}^{(1)}, \mathcal{A}^{(1)})$. For every i we take an arbitrary subspace $L_i^{(2)}$ that complements $L_i^{(1)}$ to L_i. Denote $\dim L_i^{(s)} = d_i^{(s)}$, $s = 1, 2$. Thus, $L_i = L_i^{(1)} \oplus L_i^{(2)}$ and $d_i^{(1)} + d_i^{(2)} = d_i$, $i = 1, \ldots, n$. Consider arbitrary vertices i, j and an operator A_{ji} from the family $\mathcal{A}_{ji}$. Let $A_{ji}^{(1)} = A_{ji}|_{L_i^{(1)}}$ be the operator from $L_i^{(1)}$ to $L_j^{(1)}$ and let $A_{ji}^{(2)} = P_j \circ A_{ji}|_{L_i^{(2)}}$ be the operator from $L_i^{(2)}$ to $L_j^{(2)}$, which is the composition of $A_{ji}|_{L_i^{(2)}}$ and of the operator P_j of projection of the space L_j to its subspace $L_j^{(2)}$ parallel to the subspace $L_j^{(1)}$. Each vector $x \in L_j$ has a unique representation $x = x^{(1)} + x^{(2)}$ with $x^{(s)} \in L_j^{(s)}$, $s = 0, 1$. The projection P_j is defined as $P_j x = x^{(2)}$.

For each j, we take arbitrary bases of subspaces $L_j^{(1)}$ and $L_j^{(2)}$, their union is a basis for L_j. In this basis, the projection P_j is written by $d_j \times d_j$-matrix

$$P_j \;=\; \begin{pmatrix} 0 & 0 \\ 0 & I_{d_j^{(2)}} \end{pmatrix} \tag{6}$$

with two square diagonal blocks: zero matrix of size d_j^1 and the identity matrix $I_{d_j^{(2)}}$ of size $d_j^{(2)}$. In the same basis, each matrix $A_{ji} \in \mathcal{A}_{ji}$ has the following block upper-triangular form:

$$A_{ji} \;=\; \begin{pmatrix} A_{ji}^{(1)} & D_{ji} \\ 0 & A_{ji}^{(2)}. \end{pmatrix} \tag{7}$$

Here $A_{ji}^{(1)}$ is a $d_j^{(1)} \times d_i^{(1)}$-matrix, $A_{ji}^{(2)}$ is a $d_j^{(2)} \times d_i^{(2)}$-matrix. It is easy to see that for any path $i_1 \to \ldots \to i_k$ along the multigraph G, any product $A_{i_k i_{k-1}} \cdots A_{i_2 i_1}$, for arbitrary choice $A_{i_{s+1} i_s} \in \mathcal{A}_{i_{s+1} i_s}$ for each $s = 1, \ldots, k-1$, also has block upper-triangular form (7) with the diagonal blocks of sizes $d_{i_k}^{(1)} \times d_{i_1}^{(1)}$ and $d_{i_k}^{(2)} \times d_{i_1}^{(2)}$ respectively. We denote $\xi^{(1)} = (G, \mathcal{L}^{(1)}, \mathcal{A}^{(1)})$, $\xi^{(2)} = (G, \mathcal{L}^{(2)}, \mathcal{A}^{(2)})$

Theorem 2 *Every reducible triplet ξ can be factorized into two triples $\xi^{(1)}$ and $\xi^{(2)}$ of smaller total dimensions. All matrices A_{ji} of the family $\mathcal{A}$ are factorized in the form (7). For the joint spectral radii, we have*

$$\rho(\xi) \;=\; \max\left\{\, \rho(\xi^{(1)})\,,\, \rho(\xi^{(2)}) \,\right\}. \tag{8}$$

`Proof.` We have proved all claims of the theorem except for the equality (8). Denote $\rho_i = \rho(\xi^{(i)}), i = 1, 2$, and $\bar{\rho} = \max\{\rho_1, \rho_2\}$. Observe that in the L_1-norm, all products of matrices from $\mathcal{A}$ are bigger than the corresponding products to their submatrices from $\mathcal{A}^{(1)}$. Consequently, $\rho \geq \rho_1$. Similarly, $\rho \geq \rho_2$, and hence $\rho \geq \max\{\rho_1, \rho_2\} = \bar{\rho}$. To establish the inverse inequality we take an arbitrary path $\alpha = i_1 \to \ldots \to i_{k+1}$. For each $s = 1, \ldots, k+1$ we denote $\alpha_s^- = i_1 \to \cdots \to i_s$ (if $s = 1$, the path is empty) and $\alpha_s^+ = i_{s+1} \to \cdots \to i_{k+1}$ (if $s = k$, the path is empty). Take arbitrary $\varepsilon > 0$. The product Π_α has the same upper triangular block form (7). In the upper diagonal block it has the product $\Pi_\alpha^{(1)}$ whose norm does not exceed $C(\rho_1 + \varepsilon)^k$. In the lower diagonal block it has the product $\Pi_\alpha^{(2)}$ whose norm does not exceed $C(\rho_2 + \varepsilon)^k$. Both these values do not exceed $C(\bar{\rho} + \varepsilon)^k$. Finally, the off-diagonal block is equal to

$$\sum_{s=1}^{k} \Pi_{\alpha_s^+}^{(1)} \; D_{i_{s+1}i_s} \; \Pi_{\alpha_s^-}^{(2)}. \tag{9}$$

the norm of the sth term is bounded below by $C(\rho_1+\varepsilon)^{s-1}\|D_{i_{s+1}i_s}\| C(\rho_2+\varepsilon)^{k-s}$. Estimating both ρ_1 and ρ_2 from above by $\bar{\rho}$ and all $\|D_{i_{s+1}i_s}\|$ by C, we obtain the upper bound $C^3(\bar{\rho} + \varepsilon)^{k-1}$. Hence, the norm of the the off-diagonal block in the product Π_α does not exceed $C^3 k\, (\bar{\rho} + \varepsilon)^{k-1}$. Thus, for every path α of length k, we have $\|\Pi_\alpha\| \leq C_0 k(\bar{\rho} + \varepsilon)^k$, where C_0 does not depend on α. Taking the power $1/k$ and a limit as $k \to \infty$, we see that $\rho \leq \bar{\rho} + \varepsilon$. Since this holds for every ε, we have $\rho \leq \bar{\rho}$.

$\square$

If we have a reducible triplet ξ, then applying Theorem 2 several times, we obtain

Corollary 2 *Every reducible triplet ξ can be factorized as a sum of $r \geq 2$ irreducible triples $\xi^{(1)}, \ldots, \xi^{(r)}$ of smaller total dimensions. All matrices A_{ji} of the family $\mathcal{A}$ are factorized in the form*

$$A_{ji} \;=\; \begin{pmatrix} A_{ji}^{(1)} & * & \ldots & * \\ 0 & A_{ji}^{(2)} & * & \vdots \\ \vdots & & \ddots & * \\ 0 & \ldots & 0 & A_{ji}^{(r)} \end{pmatrix}, \tag{10}$$

where the matrix $A_{ji}^{(s)}$ in the sth diagonal block represents the family $\mathcal{A}^{(s)}$ of the irreducible triplet $\xi^{(s)} = (G, \mathcal{L}^{(s)}, \mathcal{A}^{(s)})$. For the joint spectral radii, we have

$$\rho(\xi) \;=\; \max\left\{\, \rho(\xi^{(1)}), \,\ldots, \, \rho(\xi^{(r)}) \,\right\}. \tag{11}$$

Remark 4 Another concept of irreducibility of triplets was suggested in [38, 39], where it was shown that it also sufficient for non-defectivity, see Remark 2. Definition from [39] involves the set of all cycles of G, which is finite, but may be very large. This made it possible to prove theoretical decidability of irreducible systems, although its practical use is difficult for some graphs. Our concept has an advantage that it allows us to factorize an arbitrary system to several irreducible ones, exactly as in the classical case of usual (unconstrained) systems. This extends most of methods from irreducible systems to all systems. In particular, the problem of computing or estimating the joint spectral radius is completely reduced to that case by means of formula (11).

Remark 5 Note that the factorization procedure cannot be realized within the class of systems with equal dimensions of the spaces L_i. Indeed, even if the original system ξ belongs to this class, the embedded system ξ' may not. That is why, the factorization could not be done within the framework of previously known concepts of constrained systems.

Remark 6 Note that the problem of deciding irreducibility is algorithmically hard even for usual (unconstrained) systems of two matrices (see [2] and references therein). Nevertheless, for our method of JSR computation (Section 9) this problem usually does not offer any resistance. If the system is defective (Definition 5) and the Invariant polytope algorithm does not converge, then we can make a step-by-step construction of an embedded system, which reduces the problem to two similar problem of smaller total dimensions (Remark 8).

7 Extremal multinorms and invariant multinorms

The next crucial property of irreducible triplets is the existence of extremal and invariant multinorms. Again, for the classical case (with one space and an invariant norm instead of multinorm) this fact is well-known, it originated with Barabanov in [3] a dual fact was independently proved in [41]. We are going to extend Barabanov's theorem for all triplets $\xi = (G, \mathcal{L}, \mathcal{A})$.

Definition 6 A multinorm $\|\cdot\| = \{\|\cdot\|_i\}_{i=1}^n$ is extremal if for every i and $x \in L_i$, we have

$$\max_{A_{ji} \in \mathcal{A}_{ji}, \, j=1,\ldots,n} \|A_{ji}x\|_j \ \le \ \rho \, \|x\|_i \,. \tag{12}$$

A multinorm is called invariant, or Barabanov, if for every $i = 1, \ldots, n$ and $x \in L_i$, we have

$$\max_{A_{ji} \in \mathcal{A}_{ji}, \, j=1,\ldots,n} \|A_{ji}x\|_j \ = \ \rho \, \|x\|_i \,. \tag{13}$$

An invariant multinorm is also extremal. So, it will suffice to prove the existence results for invariant multinorms. On the other hand, extremal multinorms are sufficient to compute the JSR. The class of extremal multinorms is much wider and they are easier to find or to estimate in practice.

The invariance property of a Barabanov multinorm remains valid after multiplication of all operators $A_{ji} \in \mathcal{A}$ by the same constant. Hence, it can always be assumed than our system is normalized so that $\rho = 1$. The multinorm $\|\cdot\|$ is invariant if for every point $x \in L_i$, the maximal norm of its images $\|A_{ji}x\|_j$ over all edges going from the vertex i is equal to $\|x\|_i$. Let B_i and S_i be the unit ball and the unit sphere of the invariant norm in L_i. Then for every i and $x \in S_i$, all images $A_{ji}x$ lie inside the corresponding balls B_j and at least one of them lies on the sphere S_j. In what follows we work with a multinorm $\|\cdot\|$ and drop the index i of each concrete norm $\|\cdot\|_i$ of the space L_i (see the remark after Definition 1).

Theorem 3 *An irreducible triplet possesses an invariant multinorm.*

`Proof.` First, we omit all vertices with zero-dimensional spaces L_i, along with all their incoming and outgoing vertices. This does neither change irreducibility nor the JSR. Thus, we assume $d_i \geq 1$ for all i. By Proposition 6, $\rho(\xi) > 0$, hence, after normalization it can be assumed that $\rho = 1$. For every i and $x \in L_i$, we denote $f(x) = \limsup_{|\alpha| \to \infty} \|\Pi_\alpha x\|$. By Theorem 1, the function $f(x)$ is bounded. It is convex being a pointwise upper limit of convex functions. Obviously, f is symmetric and positively homogeneous. Furthermore, it possesses the invariance property: $\max_{l_{ji} \in \ell_{ji}, j=1,\ldots,n} f(A_{ji}x) = f(x)$, $x \in L_i$, $i = 1, \ldots, n$. It remains to show that f is a norm, i.e., that $f(x) > 0$ for all $x \neq 0$. Note that if an upper limit of a nonnegative sequence is zero, then that sequence tends to zero. Thus, $f(x) = 0$ implies $\lim_{|\alpha| \to \infty} \|\Pi_\alpha x\| = 0$. Let M_i be the set of points $x \in L_i$ satisfying this equality. It is shown easily that M_i is a linear subspace of L_i and moreover, $A_{ji}M_i \subset M_j$, for all i, j, and all $A_{ji} \in \mathcal{A}_{ji}$. Hence the irreducibility yields that either $M_i = L_i$ for all i or $M_i = \{0\}$ for all i. In the former case, for all x, we have $\Pi_\alpha x \to 0$ as $|\alpha| \to \infty$. Take some i and apply this assertion to all elements of an orthonormal basis $e_1, \ldots, e_{d_i}$ of the space L_i. If $\|\Pi_\alpha e_s\| < \varepsilon$ for all $s = 1, \ldots, d_i$, whenever $|\alpha| > N$, then $\|\Pi_\alpha\| < d_i \varepsilon$, whenever $|\alpha| > N$. Thus, $\|\Pi_\alpha\| \to 0$ as $|\alpha| \to \infty$, which contradicts to the assumption $\rho = 1$. Finally, in the latter case, when $M_i = \{0\}$ for all i, we have $f(x) > 0$ for all $x \neq 0$, i.e., f is a norm. Due to its invariance property, it is a desired invariant norm.

$\square$

Remark 7 If a system $\xi = (G, \mathcal{L}, \mathcal{A})$ has an invariant multinorm, then after possibly identifying two vertices i_1, i_2 of the multigraph G (see subsection 3.3) we set the norm in the new space L_i to be the maximum of "old" norms in L_{i_1} and L_{i_2}: $\|x\|_i = \max\{\|x\|_{i_1}, \|x\|_{i_2}\}$. It is checked easily that the new multinorm is also invariant.

We are now able to prove Proposition 3 on the stability of linear systems.

`Proof of Proposition 3.` The sufficiency is easy. If $\rho(\xi) < 1$, then for an arbitrary $q > \rho(\xi)$ and for an arbitrary multinorm, there exists a constant C such that $\|P_\alpha\| \leq C q^{|\alpha|}$ for all paths along G. Fix some $q < 1$. For every trajectory $\{x_k\}_{k \geq 0}$, we have $\|x_k\| \leq C q^{|\alpha|} \|x_0\| \to 0$ as $k \to \infty$, hence the system is stable.

To establish the necessity it suffices to show that if $\rho(\xi) = 1$, then there is at least one trajectory that does not converge to zero. One can show even more: there is a trajectory such that $\|x_k\| \geq C$, $k \in \mathbb{N}$, where $C > 0$ is some constant. In view of Corollary 1, the multigraph G has a strongly connected sub-multigraph G' with $\rho(\xi') = 1$. So, we can consider only trajectories along G'. Hence, without loss of generality we assume that G is strongly connected. Furthermore, if ξ is reducible, then it can be factorized to the form (10) with irreducible blocks $\mathcal{A}_{ji}^{(s)}$, $s = 1, \ldots, r$, and at least one of these blocks has joint spectral radius one (Corollary 2. If this block has a trajectory bounded away from zero, then the corresponding trajectory of the original system possesses the same property. Hence, it suffices to realize the proof for that irreducible block. This allows us to assume without loss of generality that ξ is irreducible, in which case Theorem 3 provides an invariant multinorm $\|\cdot\|$. Taking arbitrary $x_0 \in L_{i_0}$, $\|x_0\| = 1$, and applying recursively the invariance property we obtain a trajectory $\{x_k\}_{k \geq 0}$ such that $x_k \in L_{i_k}$ and $\|x_{k+1}\| = \max\limits_{A_{i_{k+1}i_k} \in \mathcal{A}_{i_{k+1}i_k}} \|A_{i_{k+1}i_k} x_k\| = \|x_k\| = 1$ for all k (by the strong connectivity, the maximum is taken over a nonempty set). Thus, $\|x_k\| = 1$ for all k which completes the proof. $\qquad\square$

8 Two corollaries

Before we turn into the algorithmic part we observe two corollaries of Theorem 3. The first one is Theorem 4 below that slightly improves the Berger-Wang formula that expresses the joint spectral radius by the spectral radii of products. To the best of our knowledge, this improved version is new even for the classical (unconstrained) case. The second corollary concerns the issue of marginal instability of a system, i.e., possible growth of trajectories in case $\rho(\xi) = 1$. Here we generalize some results that are known in the classical case.

8.1 An improved Berger-Wang formula

Applying Theorems 3 and 2 we can establish the following improved Berger-Wang formula that sharpens the results of Dai [11] and Kozyakin [32].

Theorem 4 *For any triplet we have*

$$\limsup_{k \to \infty} \ (\rho(\xi))^{-k} \max_{\alpha \in \mathcal{C}(G), |\alpha| = k} \rho(\Pi_\alpha) = 1 \, . \tag{14}$$

Clearly, this assertion is stronger than the Berger-Wang formula. If, for instance, $\rho(\xi) = 1$, then the upper limit of $\max\limits_{\alpha \in \mathcal{C}(G), |\alpha| = k} \rho(\Pi_\alpha)$ as $|\alpha| \to \infty$ is one. This, of course, implies that the upper limit of $\max\limits_{\alpha \in \mathcal{C}(G), |\alpha| = k} \left(\rho(\Pi_\alpha)\right)^{1/k}$ is one (the Berger-Wang formula), but not vice versa.

Let us emphasize that assertion (14) holds for all triplets, including reducible ones and those not strongly connected. An analogous statement for the norm $\|\Pi_\alpha\|$ instead of the spectral radius $\rho(\Pi_\alpha)$ holds only for special norms (i.e., for extremal norms). For reducible

triplets the upper limit of $(\rho(\xi))^{-k}\|\Pi_\alpha\|$ may be infinite, in which case there is no norm possessing property (14).

Proof. After normalization it can be assumed that $\rho(\xi) = 1$. Consider first the case when ξ is irreducible. By Theorem 3 there exists an invariant norm. Hence, there are infinite trajectories $\{x_k\}_{k\in\mathbb{N}}$ such that $\|x_k\| = 1$ for all k. Infinitely many points x_k belong to one space L_i and, due to compactness of the unit sphere, there is a convergent subsequence x_{k_s} as $s \to \infty$. Fix some $\varepsilon > 0$. For arbitrary $\delta > 0$, there is $N = N(\delta)$ such that $\|x_{k_s} - x_{k_{s+1}}\| < \delta$, whenever $s > N$. On the other hand, since the points x_{k_s} and $x_{k_{s+1}}$ belong to one trajectory, it follows that there is a product Π such that $x_{k_{s+1}} = \Pi x_{k_s}$ (see [15, Lemma 2]). Thus, $\|(\Pi - I)x_{k_s}\| < \delta$, $\|x_{k_s}\| = 1$ and $\|\Pi\| \le 1$. This implies that $\rho(\Pi) > 1 - \varepsilon$, whenever δ is small enough. Thus, there are closed paths of the multigraph G such that the spectral radii of the corresponding products are arbitrarily close to one. This proves (14) for irreducible triples.

If ξ is reducible, then by Corollary 2, ξ is a sum of irreducible triplets $\xi^{(1)}, \ldots, \xi^{(r)}$, and $\xi^{(s)} = 1$ for some of them. As shown above, assertion (14) holds for the triplet $\xi^{(s)}$. This means that for any $\varepsilon > 0$, there are arbitrarily long closed paths for which $\rho(\Pi_\alpha^{(s)}) > 1 - \varepsilon$. On the other hand, in factorization (10) we have $\rho(\Pi_\alpha) = \max\limits_{j=1,\ldots,r} \rho(\Pi_\alpha^{(j)}) \ge \rho(\Pi_\alpha^{(s)})$. Thus, $\rho(\Pi_\alpha) > 1 - \varepsilon$, which concludes the proof.

$\square$

8.2 Marginal instability

A system is called *marginally stable* if the corresponding normalized system, for which $\rho = 1$, has uniformly bounded trajectories. According to Theorem 1, an irreducible system is marginally stable. For reducible systems, the phenomenon of marginal instability may appear, even for the case of one matrix (when it has Jordan blocks corresponding to the largest by modulo eigenvalues). For general reducible systems, the growth of trajectories is at most polynomial and the power does not exceed the number of blocks in factorization (10) with the maximal joint spectral radius. The following theorem extends the results in [43, 10] from the classical case to arbitrary systems.

Theorem 5 *For any triplet $\xi = (G, \mathcal{L}, \mathcal{A})$, there is a constant $C_1 > 0$ such that for every $x_1 \in L_i$, $i = 1, \ldots, n$, there exists a trajectory $\{x_k\}_{k\in\mathbb{N}}$ such that*

$$\|x_k\| \ge C_1 \rho^k, \qquad k \in \mathbb{N}, \tag{15}$$

and there is a constant $C_2 > 0$ such that

$$\max_{|\alpha|=k} \|P_\alpha\| \le C_2 k^{r_1-1} \rho^k, \qquad k \in \mathbb{N}, \tag{16}$$

where r_1 is the total number of diagonal blocks $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(r)}$ in factorization (10) with $\rho(\xi^{(i)}) = \rho(\xi)$.

Proof. Let $\rho(\xi) = 1$. Assume ξ is irreducible. Then it possesses an invariant norm (Theorem 3), for which is suffices to prove (15), since all norms in a finite-dimensional space are equivalent. From the definition of invariant norm it follows easily that there exists an infinite trajectory $\{x_k\}_{k \in \mathbb{N}}$ such that $\|x_k\| = 1$ for all $k \in \mathbb{N}$, which completes the proof for an irreducible triplet. If ξ is reducible, then consider its factorization (10) with irreducible systems $\xi^{(i)}$ corresponding to the diagonal blocks. By Corollary 2, at least one of them, say, $\xi^{(s)}$ has the joint spectral radius one. Hence, there is a trajectory $\{x_k^{(s)}\}_{k \in \mathbb{N}}$ of this system such that $\|x_k^{(s)}\| \geq C_1$ for all $k \in \mathbb{N}$. On the other hand, for the corresponding trajectory $\{x_k\}_{k \in \mathbb{N}}$ of the full system ξ, each element $x_k^{(s)}$ is a projection of x_k onto a subspace $L_i^{(s)}$ parallel to the other subspaces $L_i^{(t)}$, $t \neq s$. Hence $\|x_k\| \geq C_0 \|x_k^{(s)}\|$, where the constant C_0 does not depend on x_k. This proves (15).

We establish (16) for $r = 2$ blocks in factorization (10), the case of general r then follow by induction. We assume that the multinorms in both $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ are invariant. If $r_1 = 2$, i.e., $\rho_1 = \rho_2 = 1$, the norm of the sum (9) is bounded above by $\sum_{s=1}^{k} \|D_{i_{s+1}i_s}\| \leq k\,C$, and we arrive at (16) with $r_1 = 2$. If $\rho_1 = 1, \rho_2 = q < 1$, then this norm is bounded by $\sum_{s=1}^{k} q^{k-s} \|D_{i_{s+1}i_s}\| \leq \frac{1}{1-q}\,C$, and we arrive at (16) with $r_1 = 1$.

$\square$

9 The Invariant polytope algorithm

We give a short description of the Invariant polytope algorithm for exact computation of the constrained JSR and for constructing an extremal polytopic multinorm. The main approach is very similar to the classical single space case ($n = 1$) elaborated in detail in [20] (the basic idea traces back to the papers [41] and [22].) Then we make the formal description and provide a criterion for its convergence within finite time (Theorem 6). In this section all the operator sets $\mathcal{A}_{ij}$ are supposed to be finite.

A product of matrices Π_α corresponding to a path α is called *spectrum maximizing product* (in short, s.m.p.) if $[\rho(\Pi_\alpha)]^{1/|\alpha|} = \rho(\xi)$. Inequality (5) shows that we always have $[\rho(\Pi_\alpha)]^{1/|\alpha|} \leq \rho(\xi)$. So, an s.m.p. is a product for which this inequality becomes equality. Even in the classical (single-space) case an s.m.p. may not exist [6].

The idea of the algorithm is to select a *candidate s.m.p.* Π_α and prove that it is actually a real s.m.p. by constructing an extremal polytope multinorm for ξ.

Due to the numerical computation we make use of a tolerance `tol` in the computation which establishes whether a vector is internal or external to a polytope.

Given the triplet $\xi = (G, \mathcal{L}, \mathcal{A})$, where G is a graph with n nodes, $\mathcal{A} = \{A_{ji}^s\}$ where $i,\ j = 1,\ \ldots, n,\ s = 1,\ \ldots, N_{ji}$ and N_{ji} is the number of edges connecting a node i to the node j. We assume that

1. G is strongly connected

2. Π_α is a candidate s.m.p. for ξ of length $|\alpha| = N_\alpha$.

3. the largest by modulo eigenvalue of Π_α is real.

4. ξ is irreducible (this assumption is for the sake of simplicity and can be omitted, see Remark 8).

Algorithm 1: The algorithm for computing the constrained JSR for a triplet $\xi = (G, \mathcal{L}, \mathcal{A})$

Data: triplet $\xi = (G, \mathcal{L}, \mathcal{A})$, Π_α, tol

Result: $\rho(\xi)$, $\mathcal{V}$

1 Scale the set $\mathcal{A}$ and get $\widetilde{\mathcal{A}} = \{\widetilde{A}_{ji}^s = \rho(\Pi_\alpha)^{-1/|\alpha|} A_{ji}^s\}_{i,j,s}$ so that $\rho(\widetilde{\xi}) \geq 1$, with $\widetilde{\xi} = (G,\ \mathcal{L},\ \widetilde{\mathcal{A}},\ )$, and $\rho(\widetilde{\Pi}_\alpha) = 1$

2 Compute the leading eigenvectors v of $\widetilde{\Pi}_\alpha$ normalized with $\|v\|_2 = 1$ and of its $N_\alpha - 1$ cyclic permutations.

3 Set $k = 0$

4 Define $\mathcal{V}^{(0)} = \left\{V_i^{(0)}\right\}_{i=1}^n$ and $\mathcal{R}^{(0)} = \left\{R_i^{(0)}\right\}_{i=1}^n$ where, for all i, $R_i^{(0)} = V_i^{(0)} \subset L_i$ contains the eigenvectors $\{v_j\}$ which belong to the space L_i based on the path α^a.

   while $R_i^{(k)} \neq \emptyset$ *for at least one* $i = 1, \ldots, n$ **do**

5   Set $k = k + 1$

     for $i = 1, \ldots, n$ **do**

       Set $V_i^{(k)} = V_i^{(k-1)}$, $R_i^{(k)} = \emptyset$

       for all $v \in R_i^{(k-1)}$ **do**

         for all *edges from* i *to* j **do**

           for $s = 1, \ldots, N_{ji}$ **do**

             Set $P_j^{(k)} = \text{absco}\left\{V_j^{(k-1)}\right\}$

             if $\|\widetilde{A}_{ji}^s v\|_{P_j^{(k)}} \geq 1 - \text{tol}$ **then**

               add $\widetilde{A}_{ji}^s v$ to the sets $V_j^{(k)}$ and $R_j^{(k)}$

   Π_α is an s.m.p. for ξ, and $\rho(\xi) = \rho(\Pi_\alpha)^{1/|\alpha|}$

   $\mathcal{V} = \mathcal{V}^{(k)} = \left\{V_i^{(k)}\right\}_{i=1}^n$ is the set of vertices of the polytope extremal multinorm

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   ^{a}See the illustrative examples in Section 9.1

If Algorithm 1 terminates after Nth iteration, then we have the family of invariant polytopes $\{P_j^{(N)}\}_{j=1}^n$ such that $P_j^{(N)} = \text{co}\{\widetilde{A}_{ji} P_j^{(N)} \mid A_{ji} \in \mathcal{A}_{ji}, i = 1, \ldots, n\}$. The Minkowski norm defined by those polytopes is extremal, and hence $\rho(\widetilde{\xi}) = 1$, which proves that $\rho(\xi) = [\rho(\Pi_\alpha)]^{1/|\alpha|}$.

Regarding the assumptions, first of all we observe that if the graph is not strongly connected we can always find a disjoint partition of its vertices so that $G = \bigsqcup_{i=1}^r G_i$ where each submultigraph G_i is strongly connected and $\rho(\xi) = \max\{\rho(\xi_1), \ldots, \rho(\xi_r)\}$, ref. Corollary 1.

To identify a candidate s.m.p. Π_α for the triplet $\xi = (G, \mathcal{L}, \mathcal{A})$, we fix some number l_0

and look among all the simple closed path α on G, with $|\alpha| = l \leq l_0$, for the maximal value $\rho_\alpha = [\rho(\Pi_\alpha)]^{1/|\alpha|}$. As mentioned in the assumptions, in this work we assume the leading eigenvalue λ of Π_α, which is the largest by modulo eigenvalue, to be real. We observe that the ideas and the algorithm proposed in this paper extend as they are to the complex case.

Furthermore we assume that there exists a unique path α in G such that ρ_α is maximal. It is easy to construct graphs where such path is not unique by allowing the same sequence of matrices in different paths of the graph. However in order to handle such cases a proper balancing of the vectors involved is required. We plan to study such kind of problems in the future.

If we consider the scaled set $\widetilde{\mathcal{A}}$, as described in step 1 of Algorithm 1, the candidate s.m.p. becomes $\widetilde{\Pi}_\alpha$ with leading eigenvalue $\lambda = 1$. The eigenvector v_1 corresponding to λ is called *leading eigenvector*. Let $v_j = \widetilde{\Pi}_{\alpha_{j-1}} v_1$, $j = 2, \ldots, l$, where α_s is the prefix of the path α of length s (the first s edges of α). Thus, $\{v_j\}_{j=1}^{N_\alpha}$ are the leading eigenvectors of all cyclic permutations of $\widetilde{\Pi}_\alpha$.

Following the path α we can assign each eigenvector v_j to the corresponding space L_i. Examples of this procedure are given in Section 9.1.

Remark 8 Regarding the irreducibility, we observe that actually Algorithm 1 does not use it. If it produces full-dimensional invariant polytopes, then we are done. The only trouble may occur in the case when some polytopes $P_i^{(N)}$ are not full dimensional. In this case the triple ξ is reducible, and we can proceed as follows.

In kth iteration we compute the number $d^{(k)}$ which is the sum of dimensions of the linear spans $L_i^{(k)}$ of the sets $V_i^{(k)}$ over $i = 1, \ldots, n$. If $d^{(k)} = d^{(k-1)} < \sum_{j=1}^{n} \dim L_j$, then $L_i^{(k)} = L_i^{(k-1)}$ for all i. In this case, the triplet $\xi' = (G, \mathcal{L}^{(k)}, \mathcal{A}|_{\mathcal{L}^{(k)}})$ is strictly embedded into ξ, and hence ξ is reducible. Using Theorem 2 we make the reduction to two triples ξ' and ξ'' of smaller dimensions. We stop the algorithm and apply it to ξ' and ξ'' separately. If $d^{(k)} > d^{(k-1)}$ or if $d^{(k-1)} = \sum_{j=1}^{n} \dim L_j$, then we simply continue the iterations. A complete analysis of such procedure is out of the scope of this work.

If the algorithm terminates within finite time, then it proves that the chosen candidate is indeed an s.m.p. and gives the corresponding polytope extremal norm. Although there are simple examples, when Algorithm 1 does not terminate within finite time, this phenomenon is believed to be rare in practice. In the single-space case, all numerical experiments made with randomly generated matrices and with matrices from applications, Algorithm 1 did terminate in finite time providing an invariant polytope (see [20] for examples and statistics).

Remark 9 Although the Algorithm 1 works with polytopes, it does not involve the computation of the convex hull. Instead it deals with the set of vertices and iteratively complement this set by by solving corresponding linear programming problem. This is certainly an advantage, since the convex hull evaluation is notoriously hard. We are grateful to the anonymous Referee for rising this issue.

The following theoretical criterion ensures the convergence of Algorithm 1. It generalizes Theorem 4 from [20] proved for the single-space case and uses the notion of dominant product.

Definition 7 *Let $\xi = (G, \mathcal{L}, \mathcal{A})$ be an arbitrary triplet. A closed simple path α along G and the corresponding products Π_α are called dominant, if there is a constant $q < 1$ such that the spectral radii of products of the normalized family $\widetilde{\mathcal{A}} = [\rho(\Pi_\alpha)]^{-1/|\alpha|}\mathcal{A}$ corresponding to all other simple paths which are not cyclic permutations of α, are smaller than q.*

In view of Theorem 4, if the path α is dominant, then $\rho(\xi) = [\Pi_\alpha]^{1/|\alpha|}$. Thus, a dominant product is always an s.m.p., but, in general, not vice versa.

Theorem 6 *For a given triplet $\xi = (G, \mathcal{L}, \mathcal{A})$ and for a given initial path α (candidate s.m.p.), Algorithm 1 terminates within finite time if and only if α is dominant and the leading eigenvalue of Π_α is unique and simple.*

The proof is actually the same as the proof of the single-space case in [20, Theorem 4], and we omit it.

9.1 Illustrative examples

We start with two examples to show in details how the method works, whereas in Example 4 we run some statistics to show the performance of the algorithm.

Example 2 *Given the triplet $\xi = (G, \mathcal{L}, \mathcal{A})$ with $\mathcal{L} = \{L_i\}_{i \in \{1, \ldots, 4\}}$, $L_i = \mathbb{R}^2, i = 1, \ldots, 4$,*

$$\mathcal{A} = \{A_i\}_{i \in \{1, \ldots, 4\}} = \left\{ -I_2, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\}$$

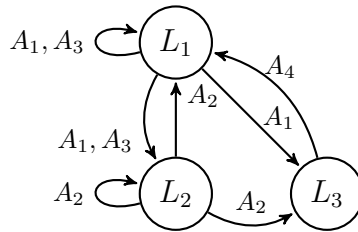*and G given in Figure 3*

Figure 3: Graph G

By an exhaustive search among all closed path of length $l_0 \leq 10$ we identify the candidate s.m.p. $\Pi = A_3 A_2 A_3 A_4 A_1 A_4 A_2$ corresponding to the closed path α shown in Figure 4, $|\alpha| = 7$.
We scale the set of matrices to get $\widetilde{\mathcal{A}} = \{\widetilde{A}_i = A_i \rho(\Pi)^{-\frac{1}{7}}\}_{i=1}^4$, so that $\rho(\widetilde{\xi}) \geq 1$, with $\widetilde{\xi} = (\widetilde{\mathcal{A}}, G, \mathcal{L})$, and $\rho(\widetilde{\Pi}) = 1$.

We denote the leading eigenvectors of the candidate s.m.p. $\widetilde{\Pi}$ and its cyclic permutations by $\{v_i\}_{i\in\{1,\dots7\}}$. Assuming that $v_1 = \widetilde{\Pi}v_1$, then $v_2 = \widetilde{A}_2 v_1$, $v_3 = \widetilde{A}_4 v_2$, $v_4 = \widetilde{A}_1 v_3$, $v_5 = \widetilde{A}_4 v_4$, $v_6 = \widetilde{A}_3 v_5$, $v_7 = \widetilde{A}_2 v_6$, $v_1 = \widetilde{A}_3 v_7$.

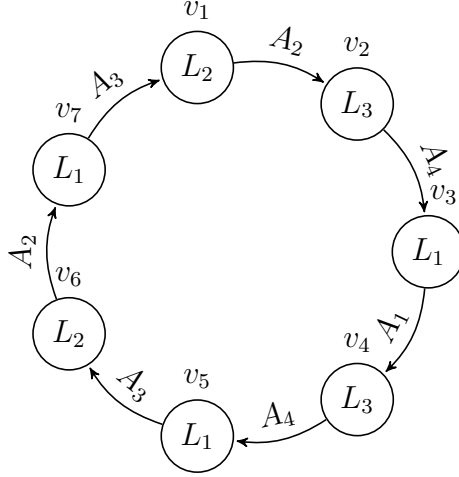We have v_3, v_5, $v_7 \in L_1$, v_1, $v_6 \in L_2$, and v_2, $v_4 \in L_3$, see fig. 5.

Figure 4: Closed path α of the candidate s.m.p. Π for the triplet ξ

After two steps of Algorithm 1 we have invariant polytopes which are plotted in Figure 7. So we can conclude that Π is an s.m.p. for ξ and $\rho(\xi) = \rho(\Pi)^{\frac{1}{7}} = 1.456846\dots$.

If we study the unconstrained problem using, for instance, the Matlab code presented in [9] and posted on MatlabCentral[1] we find the candidate s.m.p. $Q = A_4 A_3 A_4 A_4 A_2$. After scaling the set $\mathcal{A}$ by $\rho(Q)^{1/5}$ to get $\bar{\mathcal{A}}$, we can use the technique described in [20] to construct in 4 steps a extremal polytopic norm whose unit ball contains the following 7 vectors: v_0 which is the leading eigenvector of $\bar{Q}$, $\bar{A}_2 v_0$, $\bar{A}_4 v_0$, $\bar{A}_4 \bar{A}_2 v_0$, $\bar{A}_4 \bar{A}_4 \bar{A}_2 v_0$, $\bar{A}_2 \bar{A}_4 \bar{A}_2 v_0$, $\bar{A}_3 \bar{A}_4 \bar{A}_4 \bar{A}_2 v_0$. This allows to conclude that Q is an s.m.p. for $\mathcal{A}$ and that $\rho(\mathcal{A}) = \rho(Q)^{1/5} = 1.693476\dots$.

We observe that the matrix A_1, which in this example is the negative identity matrix, clearly does not count towards the computation of the unconstrained JSR. However the same matrix is fundamental for the computation of the constrained JSR. As a matter of fact it does appear in the constrained s.m.p. Π.

Example 3 We consider now the case where the dimensions of the spaces can be different each other and some matrices appear in edges corresponding to different spaces. For the sake of simplicity, we consider only one and two dimensional spaces L_j, however we recall that the proposed algorithm works with any dimension. The triplet is $\xi = (G, \mathcal{L}, \mathcal{A})$ with $\mathcal{L} = \{L_i\}_{i\in\{1, \dots, 3\}}$, $L_1 = L_3 = \mathbb{R}^2$, $L_2 = \mathbb{R}^1$, the operators

$$\mathcal{A} = \{A_i\}_{i\in\{1,\dots4\}} = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right\}$$

and the graph G depicted in Figure 8

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[1]http://www.mathworks.com/matlabcentral/fileexchange/36460-joint-spectral-radius-computation

L_1 L_2 L_3

Figure 5: Leading eigenvectors of $\widetilde{\Pi}$ and its cyclic permutations each of them assigned to the corresponding space.

L_1 L_2 L_3

Figure 6: Vertices and their symmetrized convex hulls after one step of the algorithm

L_1 L_2 L_3

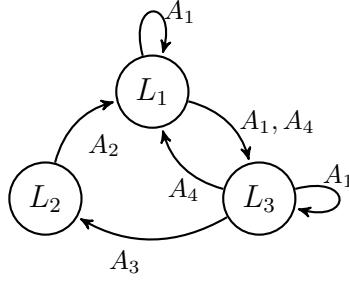Figure 7: Invariant polytopes for ξ

Figure 8: Graph G

By an exhaustive search among all closed path of length $l_0 \leq 10$ we identify the candidate s.m.p. $\Pi = A_3 A_4^3 A_2$ corresponding to the closed path α shown in Figure 9.
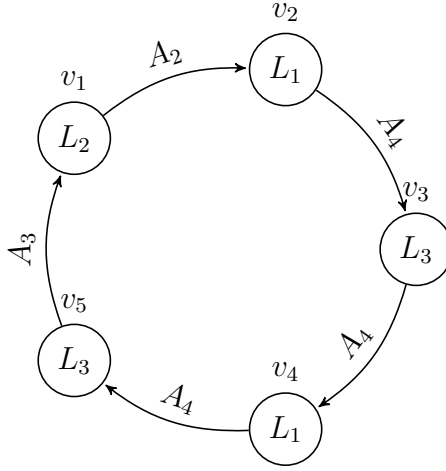

Figure 9: Closed path α of the candidate s.m.p. Π for the triplet ξ

We scale the set $\mathcal{A}$ and get $\widetilde{\mathcal{A}} = \{\widetilde{A}_i = A_i \rho(\Pi)^{-\frac{1}{5}}\}_i$, so that $\rho(\widetilde{\xi}) \geq 1$, with $\widetilde{\xi} = (\widetilde{\mathcal{A}}, G, \mathcal{L})$, and $\rho(\widetilde{\Pi}) = 1$.

We denote the leading eigenvectors of the candidate s.m.p. $\widetilde{\Pi}$ and its cyclic permutations by $\{v_i\}_{i \in \{1,\dots 5\}}$. Assuming that $v_1 = \widetilde{\Pi} v_1$, then $v_2 = \widetilde{A}_2 v_1$, $v_3 = \widetilde{A}_4 v_2$, $v_4 = \widetilde{A}_4 v_3$, $v_5 = \widetilde{A}_4 v_4$, $v_1 = \widetilde{A}_3 v_5$.

We have v_2, $v_4 \in L_1$, $v_1 \in L_2$, and v_3, $v_5 \in L_3$, see fig. 10.

After the first step of Algorithm 1 we have an extremal polytope multinorm whose unit balls (invariant polytopes) are plotted in Figure 11.

So we can conclude that Π is an s.m.p. for ξ and $\rho(\xi) = \rho(\Pi)^{\frac{1}{5}} = 1.515717\dots$.

In this case we cannot compute the unconstrained JSR since not all products of matrices are allowed.

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L_1 L_2 L_3

Figure 10: Leading eigenvectors of $\widetilde{\Pi}$ and its cyclic permutations each of them assigned to the corresponding space.

L_1 L_2 L_3

Figure 11: Invariant polytopes for ξ

Example 4 *In this example we provide some statistics on the performance of the proposed method. To this aim we consider two fixed graph structures $G^{(1)}$ and $G^{(2)}$, depicted in Figure 1 and 2 respectively, we fix the dimension d and generate $d \times d$ matrices A_1 and A_2, as either uniformly distributed random matrices, which we study in Case A, or normally distributed random ones, which is analyzed in Case B.*

First of all, we observe that focusing on cases in which the set of matrices contains only two elements may appear restrictive, however, as it has been shown for the computation of the joint spectral radius [8], already in this setting arises the complexity and variety of possible problems which we encounter in the computation of these kind of quantities.

Secondly we point out that, considering the two matrices A_1 and A_2 as letters of an alphabet, the graphs $G^{(1)}$ and $G^{(2)}$ correspond to two dictionaries with forbidden subword $A_1A_2A_1$ and subwords $A_1A_2A_1$ and A_1^2 respectively, as explained in Section 3.4, Example 1.

In producing the statistics we make the following assumptions:

- *We deal with irreducible triplets*

- *The s.m.p. is a dominant product*

- *The s.m.p. has a unique and simple real leading eigenvalue*

Case A *We fix the dimension $d \in \{3, 5, 8, 10, 15, 20\}$, and we produce random matrices whose entries are real numbers drawn from the standard normal distribution by means of the Matlab command* `randn`. *Then we scale each matrix so that they have spectral radii equal 1. In this way we produce 20 sets $\mathcal{A}^{(1)} = \{A_1, A_2\}$ and 20 sets $\mathcal{A}^{(2)}$. For both triplets $\xi^{(1)} = \left(G^{(1)}, \mathcal{L}^{(1)}, \mathcal{A}^{(1)}\right)$ and $\xi^{(2)} = \left(G^{(2)}, \mathcal{L}^{(2)}, \mathcal{A}^{(2)}\right)$, where sets $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ contain four and three spaces of dimension d respectively, we compute first a candidate s.m.p., by means of the method described in [9], then, using the algorithm presented in [20] we build an extremal polytopic norm computing the JSR of the unconstrained problem. Afterward, we identify a candidate constrained s.m.p. and we use the proposed method to find an extremal polytopic multinorm. We observe that for the triplet $\xi^{(1)}$ we can identify two vertices in the graph $G^{(1)}$. Hence we consider a set $\mathcal{L}^{(1)}$ which contains three spaces of dimension d.*

Mean values statistics on the performance are given in Table 1 and 2 for triplets $\xi^{(1)}$ and $\xi^{(2)}$ respectively.

We point out that, for the triplet $\xi^{(1)}$, we skip all cases in which both A_1 and A_2 are candidates s.m.p. In fact in this case the hypothesis of a dominant s.m.p. becomes clearly false. Such cases require an ad hoc balancing of the multinorms, as explained in Section 11.3, which is out of the scope of this example.

Furthermore we observe that, while in the unconstrained joint spectral radius problem it is non generic to deal with a reducible set of matrices, in the setting under study it is common to have reducibile triplets. The percentage of cases of reducible triplets out of all the tested cases goes from approximately 20% when $d = 3$ to more than 60% when $d = 20$, both for triplets $\xi^{(1)}$ and $\xi^{(2)}$.

We observe also that in the case of triplet $\xi^{(2)}$ the number of vertices contained in each extremal polytope tends to be equal for a fixed dimension in many cases, but not all the time,

Size	Steps	Length SMP	CPU time (s)	Ver. Pol. 1	Ver. Pol. 2	Ver. Pol. 3
3	8	6	5	7	7	8
5	14	5	35	26	25	28
8	18	7	148	83	83	89
10	21	6	349	139	139	151
15	26	5	18738	470	470	493
20	29	6	16331	933	938	978

Table 1: Example 4, Case A, mean values statistics on triplets $\xi^{(1)}$

Size	Steps	Length SMP	CPU time (s)	Ver. Pol. 1	Ver. Pol. 2	Ver. Pol. 3
3	9	2	3	6	6	6
5	13	2	13	14	14	14
8	15	2	72	34	34	34
10	19	2	315	62	62	62
15	22	2	21295	195	195	195
20	25	2	51076	308	308	308

Table 2: Example 4, Case A, mean values statistics on triplets $\xi^{(2)}$

thanks to the cyclic structure of the graph $G^{(2)}$. This is the reason why the mean values of the number of such vertices rounded to the closest integer, which we reported in the last three columns of Table 2, tend to be the same for a fixed dimension. In the case of triplet $\xi^{(1)}$, instead, the mean values of the number of vertices contained in each extremal polytope tends to be similar each other due to the averaging.

Finally we point out that during the numerical tests (which were performed with MATLAB R2011a installed on a 64–bit Windows 7 Professional computer equipped with a core i3-3227U processor and 8GB RAM) we run the proposed algorithm in some cases did not compute an extremal polytopic multinorm. Either because the initial guess for the candidate s.m.p. was wrong or because the number of iterations become bigger than a maximal number we set a priori, in our computation is set to 40. The percentage of such cases is ranging from 2% to 10%, as we increase the dimension of the matrices, for the triplets $\xi^{(1)}$, whereas is around 1% for any dimension of the matrices in the triplets $\xi^{(2)}$. In these cases we end up having anyway an interval of approximation for the constrained j.s.r. given by the spectral radius of the candidate s.m.p. and the maximal value of polytopic multinorm of the matrices in the set under study. The length of such intervals range from a maximum value of order 10^{-3} to a minimum of order 10^{-10}.

Case B *This time we consider $d \in \{5, 10, 20, 50, 100\}$, and by means of the Matlab command* `rand` *we produce matrices whose entries are uniformly distributed random real numbers in the interval $(0, 1)$. Then we scale each of them so that they have spectral radii equal to one. We repeat this process to produce 20 sets $\mathcal{A}^{(1)}$ and 20 sets $\mathcal{A}^{(2)}$. As for the*

previous case, for both triplets $\xi^{(1)}$ and $\xi^{(2)}$, we compute an s.m.p. and an extremal polytopic norm for the unconstrained problem, using both the method described in [9] and the algorithm presented in [20]. Afterward, we identify a candidate s.m.p. and we use the proposed method to find an extremal polytopic multinorm.

Mean values statistics for this second case are given in Table 3 and 4 for triplets $\xi^{(1)}$ and $\xi^{(2)}$ respectively.

As for case A, for the triplet $\xi^{(1)}$ we skip all cases in which both A_1 and A_2 are candidates s.m.p.

We point out also that in this case, since we produce matrices whose entries are uniformly distributed random number in the interval $(0, 1)$ as d increases, before the scaling, the average of each row and column of such matrices tends to 0.5 therefore the vector of all ones becomes an eigenvector corresponding to the eigenvalue 0.5. After the scaling the spectrum of such a matrix does contain an eigenvalue 1 and, the bigger the dimension d, the smaller all the other eigenvalues are going to be.

Size	Steps	Length SMP	CPU time (s)	Ver. Pol. 1	Ver. Pol. 2	Ver. Pol. 3
5	2	3	0.0764	2	1	1
10	3	3	0.1922	3	2	2
20	2	3	0.2092	2	2	2
50	1	3	0.2219	1	1	1
100	1	3	0.1079	1	1	1

Table 3: Example 4, Case B, mean values statistics on triplets $\xi^{(1)}$

Size	Steps	Length SMP	CPU time (s)	Ver. Pol. 1	Ver. Pol. 2	Ver. Pol. 3
5	3	2	0.0708	1	1	1
10	3	2	0.0726	1	1	2
20	3	2	0.0642	1	1	2
50	2	2	0.2083	1	1	1
100	2	2	0.0676	1	1	1

Table 4: Example 4, Case B, mean values statistics on triplets $\xi^{(2)}$

Finally we observe that also in these tests the proposed algorithm in some cases did not compute an extremal polytopic multinorm. The percentage of such cases is ranging from 0% to almost 30%, roughly as we decrease the dimension of the matrices, for the triplets $\xi^{(1)}$, whereas is always 0% for any dimension of the matrices in the triplets $\xi^{(2)}$. Also in these cases we end up having a length of such intervals which range from a maximum value of order 10^{-3} to a minimum of order 10^{-10}.

10   Construction of a polytope Barabanov multinorms

By Theorem 6 Algorithm 1 terminates within finite time if and only if the triplet ξ possesses a dominant product with a unique and simple leading eigenvalue. Moreover, the algorithm produces an invariant family of polytopes $\mathcal{P} = \{P_i\}_{i=1}^n$. The corresponding Minkowski multinorm $\|\cdot\|_{\mathcal{P}} = \{\|\cdot\|_{P_i}\}_{i=1}^n$ is extremal, i.e., $\|A_{ji}\|_{\mathcal{P}} \leq \rho(\xi)$ for all $A_{ji} \in \mathcal{A}$. However, it is not necessarily invariant. Nevertheless, the triplet ξ does have a polytope invariant norm, which can be constructively found. It turns out that Algorithm 1 applied to the dual triplet ξ^* also converges within finite time producing an invariant family of polytopes. The dual of those polytopes generate Barabanov norm for ξ. This is guaranteed by Theorem 7 below. To formulate it we need to define first the dual triplet.

Definition 8 *A triplet $\xi^* = (G^*, \mathcal{L}^*, \mathcal{A}^*)$ is dual to a triplet $\xi = (G, \mathcal{L}, \mathcal{A})$ if*
  1) the multigraph G^* has the same vertices as G^* and the reverses edges;
  2) L_i^* is a dual space for L_i, $i = 1, \ldots, n$;
  3) $\mathcal{A}_{ij}^*$ consists of operators adjoint to the operators of the family $\mathcal{A}_{ji}$.

Since all the spaces L_i are finite-dimensional, we identify L_i^* and L_i. Clearly, every path $\alpha : i_1 \to \cdots \to i_n$ along G corresponds to the reverse path $\alpha^* : i_n \to \cdots \to i_1$ on G^* and the corresponding matrix products are adjoint to each other. Hence, $\rho(\xi^*) = \rho(\xi)$. For a given polytope P we denote by $\mathcal{V}(P)$ the set of its vertices.

Theorem 7 *If Algorithm 1 applied for a triplet ξ converges within finite time, then it also does for the dual triplet ξ^*. Moreover, the multinorm*

$$\|x\|_i \;=\; \max_{u \in \mathcal{V}(P_i')} \big(u \, , \, x \big) , \qquad i = 1, \ldots, n , \tag{17}$$

where $\{P_i'\}_{i=1}^n$ is the invariant family of polytopes produced by Algorithm 1 for ξ^*, is a polytope Barabanov multinorm for ξ.

Proof. Assume $\rho(\xi) = 1$, and hence $\rho(\xi^*) = 1$. By Theorem 1, if Algorithm 1 converges within finite time, then the chosen candidate constrained s.m.p. product $\Pi \in \mathcal{C}(G)$ is dominant and has a unique simple eigenvalue. Therefore, the dual product Π^* (the product of adjoint operators in the inverse order) possess the same properties for the family ξ^*. Hence, Algorithm 1 applied for ξ^* with the candidate product Π^* converges as well producing some invariant family of polytopes $P_i' \subset L_i^*$, $i = 1, \ldots, n$. By the construction of the algorithm, for each i, the polytope P_i' coincides with the convex hull of images $A_{ij}' P_j'$, $A_{ij}' \in \mathcal{A}_{ij}^*$, taken over all incoming edges $l_{ij}^* \subset \ell_{ij}^*$, $i = 1, \ldots, n$. On the other hand, each operator A_{ij}' from the family $\mathcal{A}_{ij}^*$ is adjoint to the corresponding operator $A_{ji} \in \mathcal{A}_{ji}$, i.e., $A_{ij}' = A_{ji}^*$. Thus,

$$P_i' \;=\; \mathrm{co}\left\{ A_{ij}' \, P_j' \;\middle|\; A_{ij}' \in \mathcal{A}_{ij}^*, \, j = 1, \ldots, n \right\} \;=\; \mathrm{co}\left\{ A_{ji}^* \, P_j' \;\middle|\; A_{ji} \in \mathcal{A}_{ji}, \, j = 1, \ldots, n \right\}.$$

Consequently, the sets

$$\mathcal{V}(P_i') \qquad \text{and} \qquad \bigcup_{A_{ji} \in \mathcal{A}_{ji}, \, j=1,\ldots,n} A_{ji}^* \big(\mathcal{V}(P_j') \big) ,$$

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have the same convex hulls. Therefore, the multinorm $\{\|\cdot\|_i\}_{i=1}^n$ defined by (17) satisfies

$$\|x\|_i \quad = \quad \max\left\{ \left(v',\,x\right) \,\Big|\, v' \in \mathcal{V}(P_i') \right\} \quad =$$

$$\max\left\{ \left(v',\,x\right) \,\Big|\, v' \in A_{ji}^* \left(\mathcal{V}(P_j') \right),\; A_{ji} \in \mathcal{A}_{ji},\; j = 1,\ldots,n \right\} \quad =$$

$$\max_{A_{ji}\in\mathcal{A}_{ji},\, j=1,\ldots,n} \max\left\{ \left(A_{ji}^* u',\, x\right),\; u' \in \mathcal{V}(P_j') \right\} \quad =$$

$$\max_{A_{ji}\in\mathcal{A}_{ji},\, j=1,\ldots,n} \max\left\{ \left(w',\, A_{ji}x\right),\; w' \in \mathcal{V}(P_j') \right\} \quad = \quad \max_{A_{ji}\in\mathcal{A}_{ji},\, j=1,\ldots,n} \left\| A_{ji}x \right\|_j.$$

Thus, $\|x\|_i = \max\limits_{A_{ji}\in\mathcal{A}_{ji},\, j=1,\ldots,n} \left\| A_{ji}x \right\|_j$, hence $\{\|\cdot\|_i\}_{i=1}^n$ is an invariant multinorm. $\qquad \square$

Corollary 3 *If a triplet possesses a dominant product with a unique and simple leading eigenvalue, then it possesses a polytope Barabanov multinorm.*

The single-space version Theorem 7 was established in [23].

11 Generalizations and special cases

11.1 Positive systems

A triplet ξ is called *positive* if one can introduce a basis in each space L_i, $i = 1,\ldots,n$, such that all operators from $\mathcal{A}$ are written by nonnegative matrices. Dealing with positive systems we will assume that such a collection of bases is fixed and identify the operators with their matrices. It was observed in the literature that some of methods of computing of the joint spectral radius in the classical (unconstrained) case work more efficiently for nonnegative matrices. For instance, the Invariant polytope algorithm works effectively for nonnegative matrices of dimensions $d = 100$ and higher (see examples and statistics in [20] along with the discussion of this phenomenon). That is why the positive systems deserve a special analysis. First of all, the irreducibility assumption can be relaxed to positive irreducibility. This notion is directly extended from the single-space case, where it is well-known. To define it we need some extra notation.

A coordinate subspace of $\mathbb{R}^d$ is a subspace spanned by several basis vectors. A triplet $\xi' = (G, \mathcal{L}', \mathcal{A}')$ is *positively embedded* in ξ, if both these triplets are positive, L_i' is a coordinate subspace of L_i for each i, and every operator $A_{ji}' = A_{ji}|_{L_i'}$ maps L_i' to L_j', whenever $l_{ji} \in G$. The embedding is strict if L_i' is a proper subspace of L_i at least for one i.

Definition 9 *A triplet $\xi = (G, \mathcal{L}, \mathcal{A})$ is positively reducible if it is positive and has a strictly embedded triplet. Otherwise, it is called positively irreducible.*

The factorization of a positively reducible triplet to positive triplets of smaller dimensions is realized in precisely the same way as in Section 6. Thus, analysing positive triplet we can concentrate on the positively irreducible case.

The following analogue of Theorem 1 holds for positive triples.

Theorem 8 *An positively irreducible triplet is non-defective.*

`Proof` actually repeats the proof of Theorem 1 with several different points. First, we denote by S_i not the unit sphere in L_i but its intersection with the positive orthant. The sets $U_{i,k}$ are defined in the same way, the case $\cup_{k \in \mathbb{N}} U_{i,k} = S_i$ is considered in the same way as for Theorem 1. In the converse case, there exists $z \in S_q$ which does not belong to any of the sets $U_{q,k}$, i.e., for every path α staring at the vertex q we have $\|\Pi_\alpha x\| \leq 2$. Hence, the point z has a bounded orbit. For every i, denote by M_i the set of nonnegative points from L_i that have bounded orbits. The linear span of each M_i is a linear subspace of L_i. Moreover, it is a coordinate subspace. Otherwise, there is $a \in M_i, a > 0$. For every $x \in L_i, x \geq 0$, there is a number $\lambda > 0$ such that $\lambda x \leq a$. Hence the orbit of x is bounded as well. Thus, M_i coincides with the positive orthant of L_i. Now the positive irreducibility implies that M_j is the positive orthant of L_j for all j which implies non-defectivity. The remainder of the proof is the same as for Theorem 1.

$\square$

A norm is called *monotone* if $\|x\| \geq \|y\|$, whenever $x \geq y \geq 0$. A monotone multinorm is a connection of monotone norms. The following theorem sharpens Theorem 3 for the case of positive systems. Its single-space version was established in [20].

Theorem 9 *A positively irreducible triplet possesses a monotone invariant multinorm.*

`Proof` is actually the same as for Theorem 3, with two modifications. First of all, we take an arbitrary initial monotone multinorm. Then the function $f(x) = \limsup\limits_{|\alpha| \to \infty} \|\Pi_\alpha x\|$ defined for nonnegative x is obviously monotone. To show that $f(x) > 0$ for all $x \neq 0$, we note that $f(x) = 0$ implies $\lim\limits_{|\alpha| \to \infty} \|\Pi_\alpha x\| = 0$. Let M_i be the set of nonegative points $x \in L_i$ satisfying this equality. The linear span of M_i is a linear subspace of L_i. The collection $\{M_i\}_{i=1}^n$ possesses the invariance property. If some M_i contains a strictly positive point a, then it coincides with the positive orthant of L_i. Indeed, an arbitrary point $x \in L_i, x \geq 0$, satisfies $\lambda x \leq a$ for some $\lambda > 0$, hence $\lim\limits_{|\alpha| \to \infty} \|\Pi_\alpha x\| = 0$, and so $x \in M_i$. Thus, the linear span of M_i is a coordinate subspace of L_i. Then we repeat the proof of Theorem 3 and come to the contradiction with the positive irreducibility.

$\square$

Algorithm 1 is modified for positive case as follows. First of all, we omit all extra assumptions on the candidate constrained s.m.p. $\tilde{\Pi}_\alpha$. Indeed, since this matrix is now positive, then by the Perron-Frobenius theorem its leading eigenvalue is real and positive. Second, in the kth iteration of Algorithm 1 for the general case, we check whether $\tilde{A}_{ji}v$ is an interior point of the set absco $\left(\mathcal{V}_j^{(k-1)}\right)$. Now we replace this set by co$_-\left(\mathcal{V}_j^{(k-1)}\right)$. Thus, the

version of Algorithm 1 for a positive system constructs a collection of monotone polytopes $\{P_i\}_{i=1}^n$ instead of symmetric polytopes. This version works much faster than the algorithm for general matrices, because for any set $M \subset \mathbb{R}_+^d$, the set co$_-(M)$ is bigger than absco(M) in the positive orthant (usually it is much bigger). Hence, in the positive case, each iteration of Algorithm 1 sorts our more vertices than in general case.

A complete analogue of Theorem 7 holds for positive case and gives a monotone invariant polytope norm.

11.2   Stabilizability, lower spectral radius and antinorms

The notion of stailizability well-known for dynamical systems is also extended for systems on graphs in a direct manner.

Definition 10 *The system ξ is called stabilizable if it has at least one infinite path such that all corresponding trajectories converge to zero.*

If the multigraph G is strongly connected and has an edge associated to a zero operator, then every path going through this edge produce vanishing trajectories. Hence, the system is stabilizable in this case. We see that analysing stabilizability one cannot identify an empty edge with a zero operator, otherwise, all systems will be stabilizable. In the sequel we assume that G has at least one infinite path, i.e., has a cycle.

See [5, 13, 17, 33, 48] for properties of stabilizable systems in the classical (unconstrained) case and for criteria of stabilizability. Most of those properties and criteria are extended to systems on graphs. In particular, the stabilizability is expressed in terms of the lower spectral radius.

Definition 11 *The lower spectral radius (LSR) of a triplet $\xi = (G, \mathcal{L}, \mathcal{A})$ is*

$$\check{\rho}(\xi) \;=\; \lim_{k \to \infty} \min_{|\alpha|=k} \|\Pi_\alpha\|^{1/k}. \tag{18}$$

Proposition 7 *The system is stabilizable if and only if $\check{\rho} < 1$.*

This fact in the single-space case is well-known and originated with [13]. The proof for general systems on graphs is different.

   `Proof.` The sufficiency is obvious. To prove the necessity we assume that there is a path α such that $\|P_{\alpha_j} x_0\| \to 0$ as $j \to \infty$ for every $x \in L_{i_0}$. Here α_j denotes the prefix of α of length j. There is at least one vertex through which α passes infinitely many times. Without loss of generality we assume that this is the starting vertex x_0. For each basis vector e_s of the space L_{i_0} we have $\|P_{\alpha_j} e_s\| \to 0$, hence $\|P_{\alpha_j}\| \to 0$ as $j \to \infty$. Therefore, there exist arbitrarily long closed paths α_j such that $\|P_{\alpha_j}\| < 1$. Take one of them and denote $q = \left[P_{\alpha_j}\right]^{1/|\alpha_j|} < 1$. Since $\alpha_j \in \mathcal{C}(G)$ it follows that all powers $(\alpha_j)^k$ are well-defined. Consequently, $\check{\rho} \le \left[\rho(P_{\alpha_j}^k)\right]^{1/k|\alpha_j|} = q < 1$.

<div align="right">□</div>

From the computational point of view, the lower spectral radius is still worse than JSR. For instance, it is, in general, a discontinuous function of matrices. Nevertheless, some algorithms of approximate computing of LSR for the classical case (unconstrained systems) exist [46]. In [20] an algorithm of exact computation of LSR for positive systems (i.e., with all matrices from $\mathcal{A}$ nonnegative) was presented. Under some mild assumptions, it gives the exact value of LSR for a vast majority of families of nonnegative matrices. The idea is similar to Algorithm 1, with some modificaions. First of all, it uses the notion of *antinorm* instead of norm. An antinorm is a nonnegative, nontrivial (not identical zero), positively homogeneous, and concave function defined on the positive orthant. In a sense, this is a "concave norm" on the positive orthant. A collection of antinorms $\{f_i\}_{i=1}^n$ can be called "multi-antinorm", but we drop the prefix and call it just antinorm. The notions of extremal and invariant antinorms are the same as for norms (Definition 6), with the replacement of max by min. See [20] for the existence results for extremal and invariant antinorms in the single-space case. In particular, every positive system has a monotone extremal antynorm [20, Theorem 5]. This result is extended to general systems on graphs without any change.

The LSR version of Algorithm 1 constructs an extremal polytope antinorm. We begin with exhaustion of all closed paths of lengths bounded by a given number l_0 and find the candidate Π for *spectrum minimizing product*, i.e., a product for which the value $\rho_\alpha = \|\Pi_\alpha\|^{1/|\alpha|}$ is minimal. Then we repeat the routine of Algorithm 1, replacing the symmetrized convex hull $\mathrm{absco}(M)$ by $\mathrm{co}_+(M)$. So, the LSR version of Algorithm 1 deals with infinite polytopes $P_j^{(k)} = \mathrm{co}_+\big(\mathcal{V}_j^{(k)}\big)$ (see Introduction for the definition). The algorithm halts when no new vertices appear. In this case the constructed infinite polytopes $\big\{P_j^{(k)}\big\}_{j=1}^n$ in invariant and generate an extremal antinorm, and the lower spectral radius is found: $\check\rho(\xi) = \rho_\alpha$.

11.3 The case of several spectral maximizing products. Balancing method

In applications of the Invariant polytope algorithm in the single-space case, we sometimes meet the following trouble: the constrained s.m.p. candidate product is not unique. There are several products $\Pi_1, \ldots, \Pi_r$ (not powers or cyclic permutations of each other) that have the same maximal value $\rho(\Pi_j)^{1/|\alpha_j|}$, $j = 1, \ldots, r$. In this case, the system ξ does not have a dominant product because the dominant product must be (by definition!) unique! Hence, by Theorem 6, Algorithm 1 cannot converge. Of course, the situation when spectral radii of some products coincide is not generic. Nevertheless, it is sometimes emerges in applications, when the operators from $\mathcal{A}$ have some symmetries or relations to each other. In the classical (unconstrained) case the method of *balancing* is presented in [21] to extend Algorithm 1 to this case. The idea is to multiply the leading eigenvectors of the products $\Pi_1, \ldots, \Pi_r$ by certain positive coefficients $a_1, \ldots, a_n$ respectively, and then apply Algorithm 1 starting with all those multiplied initial vectors simultaneously. The coefficients $\{a_j\}_{j=1}^r$ realizing the balancing of eigenvectors can be found as solutions of an optimization problem. See [21] for details, examples, and for the criterion of convergence. This method is extended to the systems on graphs without any change. The optimization problem and the way of computing

of the balancing coefficients remain the same.

12   Applications

We elaborated a factorization procedure of an arbitrary system on graph to several irreducible systems of smaller dimensions, proved the theorem of existence of invariant multinorm for irreducible system, and presented the Invariant polytope algorithm (Algorithm 1) for computing that antinorm and the value of JSR. Now let us discuss possible areas of applications of our results.

12.1   Constrained linear switching systems

One of the main applications is in the study of constrained linear switching systems. The constraints are usually generated by graphs, regular languages, or finite automata. Philippe and Jungers in [38] considered systems that have finitely many stages; a transfer from one stage to another is either impossible or is realized by a given linear operator. Thus, we have a graph and a system of linear operators corresponding to its edges. This is a special case of a triplet $\xi(G, \mathcal{L}\,\mathcal{A})$, with a graph (not mutigraph) G and with the same linear spaces $L_1 = \cdots = L_n = \mathbb{R}^d$ in all its vertices. Sufficient conditions for the stability of such systems presented in [38] establish the stability by constructing an ellipsoidal multinorm, in which all operators from $\mathcal{A}$ become contractions. As a rule, those conditions are not necessary. The existence of such an antinorm can be verified by solving the corresponding s.d.p. problem. Algorithm 1 (Section 9) makes it possible to find a precise value of JSR, which gives the criterion of stability and the corresponding Barabanov multinorm.

Special cases of the constrained switching systems from [38] are Markovian systems, when each operator $A_i \in \mathcal{A}$ has ist own list of operators from $\mathcal{A}$ that are allowed to follow A_i [11, 32, 51]. In [49] this concept was extended to arbitrary graphs and interpreted as "systems with memory". Another extension was studied in [39] as systems with switching sequences defined by a regular language generated by a finite automata, see also [36]. We considered Markovian systems and its generalization, with an arbitrary dictionary of prohibited words, in subsections 3.2 and 3.4.

The results of subsections 11.1 and 11.2 are also directly applicable to the constrained switching systems. The modification of Algorithm 1 to nonnegative matrices (§ 11.1) makes it more efficient for positive systems. The results on the lower spectral radius and invariant antinorms (§ 11.2) give a criterion of stabilizability of constrained switching systems.

12.2   Fractals

We consider a nonlinear generalization of our construction, to a set of arbitrary metric spaces and arbitrary maps between them. Thus, we have a triplet $\xi = (G, \mathcal{M}, \mathcal{F})$ with a multigraph G with vertices $\{g_i\}_{i=1}^n$, complete metric spaces $\{M_i\}_{i=1}^n$ associated to them, and finite sets $\mathcal{F}_{ji}$ of maps $F_{ji} : M_i \to M_j$. A collection of compact sets $\mathcal{K} = \{K_i\}_{i=1}^n$, $K_i \subset$

M_i, $i = 1, \ldots, n$, is called a fractal if it possesses the property

$$K_j \;=\; \bigcup_{F_{ji} \in \mathcal{F}_{ji}, \, i=1,\ldots,n} F_{ji} \, K_i \,, \qquad j = 1, \ldots, n \,.$$

Thus, each set K_j is the union of images of sets $\{K_i\}_{i=1}^n$ by the operators associated to all incoming edges of the vertex g_j. This is a straightforward generalization of the classical concept of fractal by J.Hutchinson [29] in case of a single metric space. Similarly to the classical situation, one can show that if all maps F_{ji} are contractions, then the system has a unique fractal. If all the spaces M_i and all the maps F_{ji} are affine, then we have an *affine system (triplet)* ξ. In this case the fractal (if it exists) is called affine. In most cases the contraction property of an affine system is too restrictive. Indeed, an affine map may be a non-contraction, but become contraction in a different norm introduced in the space. That is why, a more general existence and uniqueness result is the following:

A system possesses a unique affine fractal provided there exists a multinorm in the affine spaces M_i in which all the maps F_{ji} become contractions, i.e., the operator norm of linear parts of all those maps are smaller than one.

In view of Proposition 2, we obtain the following tight and affinely-invariant sufficient condition:

Proposition 8 *An affine system possesses a unique fractal whenever the joint spectral radius of the associated linear system is smaller than one.*

Thus, the stability of a linear system implies the existence of a fractal for the affine system.

12.3 Attractors of hyperbolic dynamical systems

Systems of maps of metric spaces along edges of a graph arise naturally in the study of attractors of dynamical systems. A hyperbolic dynamical system satisfying *A Axiom* admits the so-called *Markov partition* to sets $\{M_i\}_{i=1}^n$ with a collection of diffeomorphisms $F_{ji} : M_i \to M_j$ (see [7, 31]). An attractor of that dynamical system is defined similarly to the definition of a fractal above. See [14, 28] for the analysis of general hyperbolic attractors and [34] for extensions to Lipschitz maps. If all those diffeomorphisms are affine, then one can use Proposition 8 to prove the existence of an attractor. Applying Algorithm 1 one can construct a polytope multinorm in which all F_{ji} are contractions (in case $\rho < 1$). Moreover, the corresponding operator norms $\|F_{ji}\|$ can be used to estimate the dimension of the attractor [34].

12.4 Application to numerical ODEs

An important application of our approach is the possibility to determine sharp bounds for the stepsize ratio in the zero stability analysis of k–step BDF–formulas for the numerical approximation of initial value problems for ODEs, on grids with variable stepsize.

k–step BDF–method with variable stepsize

For the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0,$$

consider the grid $\Delta = \{t_0, t_1, \ldots, t_n, \ldots\}$ characterized by the stepsizes $h_j = t_{j+1} - t_j$, $j = 0, 1, \ldots$, and the k–step BDF–method defined by

$$y_{j+k} = \alpha_{j,k-1} y_{j+k-1} + \ldots + \alpha_{j,1} y_{j+1} + \alpha_{j,0} y_j + h_j \beta_j f(t_{j+k}, y_{j+k}).$$

The coefficients $\alpha_{j,s}$, $s = 0, 1, k - 1$, depend on the step ratios

$$\omega_{j,1} = \frac{h_{j+1}}{h_j}, \ldots, \omega_{j,k-1} = \frac{h_{j+k-1}}{h_{j+k-2}}. \tag{19}$$

It is well–known (see e.g. [27]) that zero–stability is equivalent to the uniform boundedness of all the solutions of the homogeneous linear difference equation

$$u_{j+k} = \alpha_{j,k-1} u_{j+k-1} + \ldots + \alpha_{j,1} u_{j+1} + \alpha_{j,0} u_j, \tag{20}$$

which, in turn, is equivalent to the uniform boundedness of the sequence of products

$$A_\nu \ldots A_0, \quad \nu = 0, 1, \ldots,$$

where the $k \times k$–matrices

$$A_j = \begin{pmatrix} \alpha_{j,k-1} & \alpha_{j,k-2} & \cdots & \alpha_{j,1} & \alpha_{j,0} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & & \cdots & 0 & 1 & 0 \end{pmatrix}, \quad j = 0, 1, \ldots,$$

are the companion matrices associated to the difference equation (20).

The presence of the common eigenvector $[1, 1, \ldots, 1]^T$, corresponding to the common eigenvalue $\lambda = 1$, makes it possible to reduce the dimension of one unit by using a suitable similarity transformation,

$$T^{-1} A_j T = \begin{pmatrix} \gamma_{j,k-2} & \gamma_{j,k-3} & \cdots & \gamma_{j,0} & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & & \cdots & 0 & 1 & 1 \end{pmatrix}, \quad j = 0, 1, \ldots.$$

Letting

$$C_j = \begin{pmatrix} \gamma_{j,k-2} & \gamma_{j,k-3} & \cdots & \gamma_{j,0} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ 0 & \cdots & 1 & 0 \end{pmatrix},$$

where $\gamma_{j,0}, \ldots, \gamma_{j,k-2}$ are certain coefficients depending on $\alpha_{j,0}, \ldots, \alpha_{j,k-1}$, the zero–stability of the BDF–method is guaranteed if

$$\lim_{\ell \to \infty} C_\ell \ldots C_0 = O.$$

It can be shown that the matrix C_j can be written in terms of the ratios

$$\omega_{j,1}, \ldots, \omega_{j,k-1}$$

so that a natural question is that of finding sharp bounds for such ratios, which guarantee zero–stability of the formula.

This would determine an infinite dimensional family, whose analysis is difficult in general. For this reason, in order to simplify the analysis, we allow here only a finite number of values for the ratios $\omega_{j,t}$, say – for example –

$$\omega_{j,t} \in \{\theta, 1, 1/\theta\} \tag{21}$$

(with $\theta > 1$). This is a sort of discretization of the set of possible stepsize variations.

Our aim is to prove that for a certain given θ, every sequence of matrices $\{C_\theta^{(j)}\}_{j \geq 0}$ of the family

$$\mathcal{F}_\theta = \{C(\omega_1, \ldots, \omega_{k-1})\}_{\omega_1, \ldots, \omega_{k-1}}, \tag{22}$$

with $\omega_1, \ldots, \omega_{k-1} \in \{\theta, 1, 1/\theta\}$, and under the constraint:

$$\omega_1^{(\ell+1)} = \omega_2^{(\ell)}, \qquad \omega_2^{(\ell+1)} = \omega_3^{(\ell)}, \ldots, \omega_{k-2}^{(\ell+1)} = \omega_{k-1}^{(\ell+1)} \tag{23}$$

is such that

$$\lim_{\ell \to \infty} C_\theta^{(\ell)} \ldots C_\theta^{(0)} = O. \tag{24}$$

Moreover finding the supremum Θ such that (24) holds for any $\theta \leq \Theta$ would provide a sharp limit for uniform boundedness of the solutions of the variable stepsize BDF formula with the assigned values for stepsize ratios.

In view of the constraint (23) this leads to a Markovian joint spectral radius problem.

The 3–step BDF formula

This case has been extensively studied in the literature (see e.g. [26]) but it gives a natural benchmark for our approach. In this case we make use of the similarity transform determined by the matrix

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

which gives

$$C_j = \begin{pmatrix} \gamma_{j,1} & \gamma_{j,0} \\ 1 & 0 \end{pmatrix},$$

$$\gamma_{j,0} = -\alpha_{j,0} \quad \text{and} \quad \gamma_{j,1} = -1 + \alpha_{j,2}.$$

Straightforward calculations give:

$$C(\omega_1, \omega_2) = \begin{pmatrix} \gamma_1(\omega_1, \omega_2) & \gamma_0(\omega_1, \omega_2) \\ 1 & 0 \end{pmatrix}, \tag{25}$$

where

$$\gamma_0(\omega_1, \omega_2) = -\frac{\omega_1^3 \omega_2^2 (\omega_2 + 1)^2}{(\omega_1 + 1)(3\omega_1\omega_2^2 + 4\omega_1\omega_2 + \omega_1 + 2\omega_2 + 1)} \tag{26}$$

and

$$\gamma_1(\omega_1, \omega_2) = \frac{\omega_2^2(\omega_1^2\omega_2^2 + 4\omega_1^2\omega_2 + 2\omega_1\omega_2 + 3\omega_1^2 + 3\omega_1 + 1)}{(\omega_1 + 1)(3\omega_1\omega_2^2 + 4\omega_1\omega_2 + \omega_1 + 2\omega_2 + 1)}. \tag{27}$$

The constraint is now

$$\omega_1^{(j+1)} = \omega_2^{(j)}.$$

In the considered illustrative case (21), we have to determine the constrained JSR of the family of nine matrices

$$\{C_i\}_{i=1,\dots,9} \tag{28}$$

where

$$\begin{array}{lll} C_1 = C(1/\theta, 1/\theta) & C_2 = C(1/\theta, 1) & C_3 = C(1/\theta, \theta) \\ C_4 = C(1, 1/\theta) & C_5 = C(1, 1) & C_6 = C(1, \theta) \\ C_7 = C(\theta, 1/\theta) & C_8 = C(\theta, 1) & C_9 = C(\theta, \theta) \end{array}$$

Based on the constraint (21) we build the graph of admissible products which is depicted in Figure 12.

Numerical results

We apply Algorithm 1 to the 3–step BDF formula when $\omega_{j,t}$ may assume only three values: θ, 1, and $1/\theta$. In particular we deal with the triplet $\xi = (G_3, \mathcal{L}, \mathcal{C})$ where G_3 is the graph of admissible products plotted in Figure 12, $\mathcal{L} = \{L_i\}_{i=1}^9$ is a set of linear spaces in $\mathbb{C}^{2\times 2}$, and $\mathcal{C}$ is the set of the nine matrices C_i described in (28).

To run the calculation we use as candidate s.m.p. the matrix $C_9 = C(\theta, \theta)$ whose spectral radius is given by the modulus of a couple of complex conjugate eigenvalues for values of θ in the interval $[1, 2]$.

Following what described in Section 3.3, it is possible to reduce the number of vertices of the graph G_3 to three.

Furthermore, if we set $\theta = \Theta = \frac{1+\sqrt{5}}{2}$, then $C_9 = C(\theta, \theta)$ has spectral radius equal to 1 and, using the proposed algorithm, after three steps we construct an invariant complex polytope multinorm which contains two vertices in each linear space L_i. For values $1 < \theta < \Theta$ we prove that C_9 is still an s.m.p. and the joint spectral radius of the family $\{C_i\}_{i=1,\dots,9}$ is smaller than 1.
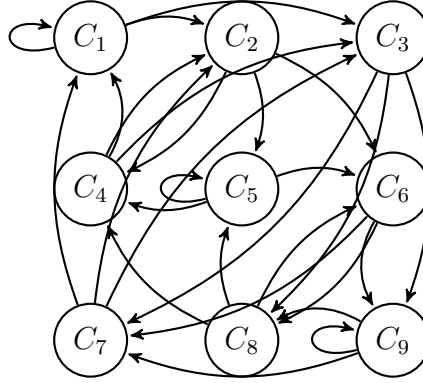
Figure 12: Graph G_3 of admissible products corresponding to the 3–step BDF formula when only three values for the ratios $\omega_{j,t}$ are allowed

The 4–step BDF formula

This case is unexplored in the literature so that our results give an indication about zero stability of such formula, which might be useful in a code implementing it.

For this formula a suitable transformation is determined by the matrix

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which gives

$$C(\omega_1, \omega_2, \omega_3) = \begin{pmatrix} \gamma_2(\omega_1, \omega_2, \omega_3) & \gamma_1(\omega_1, \omega_2, \omega_3) & \gamma_0(\omega_1, \omega_2, \omega_3) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The coefficients $\gamma_2, \gamma_1, \gamma_0$ are the following:

$\gamma_2(\omega_1, \omega_2, \omega_3) =$

$$\frac{(\omega_3+1)^2(\omega_3\omega_2+\omega_2+1)^2(\omega_1(\omega_3\omega_2+\omega_2+1)+1)^2}{(\omega_2+1)(\omega_2\omega_1+\omega_1+1)\left(3\omega_2\omega_3^2+4\omega_2\omega_3+2\omega_3+\omega_2+\omega_1\left(\omega_2^2(4\omega_3+1)(\omega_3+1)^2+2\omega_3+2\omega_2\left(3\omega_3^2+4\omega_3+1\right)+1\right)+1\right)} - 1$$

$\gamma_1(\omega_1, \omega_2, \omega_3) =$

$$-\frac{\omega_3^2(\omega_3\omega_2+\omega_2+1)^2(\omega_1(\omega_3\omega_2+\omega_2+1)+1)^2}{(\omega_1+1)\left(3\omega_2\omega_3^2+4\omega_2\omega_3+2\omega_3+\omega_2+\omega_1\left(\omega_2^2(4\omega_3+1)(\omega_3+1)^2+2\omega_3+2\omega_2\left(3\omega_3^2+4\omega_3+1\right)+1\right)+1\right)} +$$

$$\frac{(\omega_3+1)^2(\omega_3\omega_2+\omega_2+1)^2(\omega_1(\omega_3\omega_2+\omega_2+1)+1)^2}{(\omega_2+1)(\omega_2\omega_1+\omega_1+1)\left(3\omega_2\omega_3^2+4\omega_2\omega_3+2\omega_3+\omega_2+\omega_1\left(\omega_2^2(4\omega_3+1)(\omega_3+1)^2+2\omega_3+2\omega_2\left(3\omega_3^2+4\omega_3+1\right)+1\right)+1\right)} - 1$$

$\gamma_0(\omega_1, \omega_2, \omega_3) =$

$$\frac{\omega_1^4\omega_2^3\omega_3^2(\omega_3+1)^2(\omega_3\omega_2+\omega_2+1)^2}{(\omega_1+1)(\omega_2\omega_1+\omega_1+1)\left(3\omega_2\omega_3^2+4\omega_2\omega_3+2\omega_3+\omega_2+\omega_1\left(\omega_2^2(4\omega_3+1)(\omega_3+1)^2+2\omega_3+2\omega_2\left(3\omega_3^2+4\omega_3+1\right)+1\right)+1\right)}$$

The constraint are now

$$\omega_1^{(j+1)} = \omega_2^{(j)}, \qquad \omega_2^{(j+1)} = \omega_3^{(j)} \tag{29}$$

In the illustrative case (21) we have to determine the constrained JSR of the family

$$\{C_i\}_{i=1,\dots,27}$$

where

$$
\begin{array}{lll}
C_1 = C(1/\theta, 1/\theta, 1/\theta) & C_2 = C(1/\theta, 1/\theta, 1) & C_3 = C(1/\theta, 1/\theta, \theta) \\
C_4 = C(1/\theta, 1, 1/\theta) & C_5 = C(1/\theta, 1, 1) & C_6 = C(1/\theta, 1, \theta) \\
\dots & \dots & \dots
\end{array}
\tag{30}
$$

Numerical results

As for the case of the 3–step BDF, we consider first the illustrative case (21) where $\omega_{j,t}$ is allowed to attain only three values: θ, 1, $1/\theta$, for some $\theta \geq 1$.

To run the calculation we use as candidate s.m.p. the matrix $C_{27} = C(\theta, \theta, \theta)$ whose spectral radius is given by the modulus of a pair of complex conjugate eigenvalues for values of θ in the interval $[1, 2]$.

Similarly to the case studied for the 3–step BDF formula, we can construct the graph of admissible products of matrices $\{C_i\}_{i=1}^{27}$ following the constraint (29). Furthermore, also in this case, we can reduce the number of vertices of such graph to four, applying the procedure described in Section 3.3.

If we set $\theta = \Theta \approx 1.2807$, such that C_{27} has spectral radius equal to 1, then after seven steps of Algorithm 1 we construct an invariant complex polytope multinorm which contains six vertices in each of the four linear spaces.

We tested also the case of five possible values of $\omega_{j,t}$: θ, $\sqrt{\theta}$, 1, $1/\theta$, $1/\sqrt{\theta}$. The set $\{C_i\}$ contains now 125 matrices and, based on Section 3.3, the graph of admissible products can be reduced to a graph containing only 25 vertices. Using as candidate s.m.p. the matrix corresponding to the case of ω's all equal to θ, which is $C_{125} = C(\theta, \theta, \theta)$. If we set $\theta \approx 1.2807$ we construct an invariant complex polytope multinorm which contains eight vertices, except two of them that contain nine vertices. The same procedure works for $1 < \theta < \Theta$.

Therefore on the basis of the numerical experiments we conjecture that the 4–step BDF formula is zero stable for $\omega_{j,t} \leq 1.2807\ldots$, i.e. the value associated to a maximal constant increase of the stepsize ratio.

13   Conclusion

We define and investigate asymptotic properties of general linear dynamical systems on multigraphs. They generalize the constrained linear switching systems explored in the literature. Out main result is the factorization of an arbitrary system to an irreducible strongly connected system. This allows us to extend basic results on extremal multinorms and on sharp asymptotics of the trajectories from irreducible systems to general ones. This extension is crucial, because for constrained systems, the reducible case is usual and even becomes dominant for some multigraphs. To realize this factorization we modify the concept of irreducible systems and extend the definition of a constrained system by allowing all the spaces to have different dimensions (without it, the factorization is impossible). We introduce the concept of invariant (Barabanov) multinorm and prove its existence for irreducible systems. Using those results we develop the Invariant polytope algorithm, which for a vast majority of systems, compute the sharp value of the joint spectral radii and construct the invariant polytope norms. We discuss some faworable cases, in particular, positive systems, and apply our results to problems of combinatorics, functional and numerical analysis, differential equations.

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