How market interactions shape the city structure*

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Abstract

We develop a spatial monopolistic competition model in which city structure formation is entirely driven by market interactions. When preferences and transport costs are described by real analytic functions, equilibrium land-use patterns are segregated. We completely solve the case of quasilinear quadratic preferences and quadratic transport costs. The city is monocentric when firms are few, duocentric when they are neither too few nor too many, and involves a residential central area bordered by two commercial clusters when firms are many. In the long-run equilibrium, the city size and its spatial structure may change swiftly in response to tiny variations in the opportunity cost of land. Our model captures spatial price dispersion without involving any search frictions.

Keywords: urban structure; love for variety; segregated spatial equilibrium; price dispersion.

JEL classification: R12, R14, D43, L13

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1 Introduction

Uncovering the nature of the dominant forces that shape the spatial structure of cities is probably the most important issue in urban economics. To a very large extent, the literature studying the formation of cities seeks to explain the variety of urban structures by assuming the existence of *spatial production externalities* whose intensity decays with distance. These externalities are *non-pecuniary* in the sense that they arise regardless of the prevailing market structure. Fujita (1988) and De Palma et al. (1985) area noticeable exceptions in that they focus on *imperfect competition* and stress the role of *pecuniary externalities* stemming from market interactions across firms and consumers. However, this line of research has been paid little attention later. We believe this is undeserved for at least two reasons.

First, the bulk of urban economics focuses on spatial production externalities to explain the structure of cities. For example, Ogawa and Fujita (1980), Fujita and Ogawa (1982), Lucas and Rossi-Hansberg (2002) and Berliant et al. (2002) all argue that firms benefit from proximity because of various non-market interactions which become more costly at longer distances. However, all these authors assume that spatial externalities exist without deriving them from more fundamental principles. We acknowledge that Duranton and Puga (2004) have developed several models that provide various microfoundations of agglomeration economies. However, as pointed out by Puga (2010), the empirical literature has not yet developed efficient tools for testing these alternative models against each other. Therefore, even though the existence of spatial externalities at the city level has a strong empirical support (Combes and Gobillon, 2015), we find it fair to say that the origin and nature of spatial externalities remain somewhat unclear. By contrast, as observed by Krugman (1991), pecuniary externalities have extremely clear origins since they are mediated by the market, which makes them less of a "black box" while delivering at the same time solid micro-foundations.[1]

Second, as stressed by Glaeser et al. (2001) and, more recently, by Schiff (2015), an easy access to a wide range of goods and services is definitely one of the most attractive features of urban life. In other words, variety-seeking behavior would be one of the key forces driving the formation of cities. In this context, it is reasonable to expect that "households are attracted to places where the density of firms is high because opportunities there are more numerous; and they are repulsed by places where the density of households is high because they dislike congestion. Firms are attracted to places where the density of consumers is high because there the expected volume of business is large; and they are repulsed by places where the density of sellers is high because of the stronger competition prevailing there" (Papageorgiou and Thisse, 1985). Moreover, recent works on monopolistic competition based on variable elasticity of substitution allow studying the impact of versatile variety-loving behavior on market competition and the geographical distribution of economic activities (Behrens and Murata, 2007; Zhelobodko et al., 2012; Bertoletti and Etro, 2017; Mrazova and Neary, 2017; Parenti et al., 2017). Therefore, we believe it is the right time to revisit the question of city structure formation along the following lines: *how does the demand side of the urban economy affect the city structure*?

[1] See Fujita and Thisse (2013) (Ch. 1, p. 13–15) for a detailed discussion.

To shed light on this issue, we propose a monopolistic competition model of city structure formation in which the urban landscape is entirely driven by market interactions between firms and consumers. The key factor that determines the city structure is the interaction between the two forces: (i) the desire of firms to locate as close as possible to the largest number of consumers, i.e. to supply the widest possible market area, and (ii) variety-loving consumers who wish to locate closer to retailing districts in order to reduce shopping costs. Unlike Fujita (1988), who uses linear transport costs and a very specific functional form of preferences, we work instead with a fairly broad family of demand and transport cost schedules, which are described by *real-analytic functions*, i.e. those portrayed precisely by Taylor series.[2] When preferences are quasilinear quadratic and transport costs are quadratic in distance, we obtain a fully analytically solvable model of spatial monopolistic competition with variable markups. Our model overcomes one drawback of Fujita (1988), where pricing behavior is independent of firms' location – a definitely counterfactual feature given the empirical evidence of spatial price dispersion (Sorensen, 2000; Lach, 2002).

One may question the value added of studying the quadratic model, which only differs from Fujita (1988) in parameterizations of preferences and transport costs. The reasons why we believe our analysis is a contribution in itself are similar to those why Ottaviano et al. (2002) shifted the frontier of New Economic Geography. These authors propose a model which, while retaining the key features of the core-periphery model by Krugman (1991), goes beyond the standard but restrictive assumptions about functional specifications of preferences (CES) and trade costs (iceberg). This research strategy leads to revealing new market effects triggered by economic interactions between countries. Likewise, our model, while retaining the main traits of the city structure formation model by Fujita (1988), explores the consequences of changing the assumptions about preferences and shopping costs. The new effects we obtain are equilibrium price dispersion and the possibility of discontinuous switching between multiple free-entry equilibria. None of these effects is present in the standard model. As for the absence of mixed zones in our model, we do acknowledge this result to be counterfactual. However, we argue in Section 4 that the presence of a mixed district in the model by Fujita (1988) is largely due to very specific choices of functional forms, and is not robust to arbitrarily small shocks in transport costs and/or consumer tastes. Thus, our message is that qualitative features of urban structures may vary a lot in response to tiny variations in the primitives of the model.

Our main findings can be summarized as follows. First, we provide a complete analytical solution to the model with quasilinear quadratic utility á la Ottaviano et al. (2002) and quadratic transport costs, as in d'Aspremont et al. (1979). The description of the land-use patterns is as follows. When firms are few relative to the mass of consumers, firms cluster at the city center to form a single commercial district, while consumers choose to reside at the city outskirts. By contrast, when firms are many, the city structure involves a residential central area and two commercial districts at the outskirts. Finally, when the mass of firms takes on intermediate values, the resulting spatial structure is more complex and involves two commercial districts and three residential areas. Note that Berliant et al. (2002) show that this "duocentric" configuration cannot be sustained in a

[2]The formal definition of a real-analytic function is given by equation (31). The class of real-analytic functions is only slightly less general than infinitely differentiable functions.

perfectly competitive equilibrium with knowledge spillovers. In contrast, this configuration does emerge in our model when firms are neither too few nor too many relative to the population of consumers.[3] This difference is due to imperfect competition. Toughness of competition and the urban structure are intertwined in the sense that a more dispersed firm distribution drives prices upwards because competition is relaxed. Since firms face a trade-off between market access and toughness of competition, there is a priori no reasons why this intermediate type of urban structure would not emerge.

Second, we show that, unlike De Palma et al. (1985) and Fujita (1988), the market outcome features *spatial price dispersion*. This result is unexpected as our framework does not involve any search costs, unlike in Wolinsky (1983) and Schulz and Stahl (1996), where incomplete information on the consumer side works as an agglomeration force. The only source of price dispersion in our model lies in firms' locational advantage of being more centrally located than other firms. That being said, spatial price dispersion should not come as a surprise because markups are variable. In particular, markups vary with firm distribution. In other words, in the wake of Hotelling (1929), our analysis stresses that locational advantages of firms operating in cities affect their pricing policies. We also study the impact of entry on price distribution and show that the average price decreases in response to a hike in the mass of firms, while price dispersion increases.

Third, we characterize free-entry equilibria and find that one of the three possibilities may occur. When fixed costs are sufficiently high, there is a unique stable free-entry equilibrium in which firms locate in the central area while consumers reside at the outskirts. When the level of fixed costs belongs to a range of intermediate values, there exist three free-entry equilibria, each corresponding to a specific urban configuration. Two of these equilibria – the one inviting few firms and the one inviting many firms – are stable, whereas the third one – which invites an intermediate mass of firms and generates a "duocentric" spatial structure – is unstable. In sum, the "duocentric" spatial structure is more fragile.[4] Finally, when fixed costs are sufficiently low, we fall back on the case of a unique and stable free-entry equilibrium, which gives rise to a city in which consumers reside in the central area while the peripheral districts host firms. More important, we show that an infinitesimal reduction in fixed costs may lead to a discontinuous increase in the mass of firms and to drastic changes in the spatial city structure. Among other things, this result highlights the fact that a small improvements in agricultural productivity may cause rapid urban growth and dramatic changes in the economic landscape.

Last, we discuss which of the above results still hold when preferences and/or transport costs are no longer quadratic. We demonstrate that, whenever preferences and transport costs are described by real-analytic functions, *all land-use equilibria are segregated*. In other words, consumers and firms never share land at the same location.[5] This result holds without assuming specific functional forms for preferences and transport costs. The only critical assumption is that

[3]We make the vague concept of "neither too few nor too many" more precise in Section 3. See Proposition 2.

[4]Interestingly, this concurs with the results of Berliant et al. (2002), who show in a different model that this pattern never arises in equilibrium.

[5]Reasons why they should are as follows (Fujita and Thisse, 2013, Ch.7): "Spatial separation may cease to be profitable, however, when firms sell a differentiated product. This is so because demand for a firm's variety now arises at each and every consumer location."

transport costs and preferences are described by real-analytic functions. Our conclusion stands in sharp contrast with Fujita (1988) who shows that there always exists an area where firms and consumers co-locate. The reason for this difference in results is as follows: Fujita assumes that shopping costs are linear in distance, that is, a function which is not real-analytic since it has a kink at the origin. This point is worth stressing because the assumption of linear transport costs is widely used in spatial economics. What our analysis shows is that this assumption is far from being innocuous.

We also differ from Picard and Tabuchi (2013) in three major respects: (i) in our model, individuals need not reside at their firm's location, whereas this assumption is critical for their result; (ii) we focus on the demand side as a dominant force shaping the urban space and disregard forward linkages; and (iii) we model explicitly competition between consumers and firms for land via a bidding process, while firms do not consume land in Picard and Tabuchi (2013). To sum up, it is no surprise that our conclusions about the urban structure differ from those obtained in the existing literature.

The paper is organized as follows. Section 2 describes a model (henceforth "quadratic model") of city structure formation under pecuniary externalities, in which functional specifications of preferences and transport costs differ from those used in Fujita (1988): preferences are quasilinear quadratic, while transport costs are quadratic in distance. Section 3 provides a complete solution of the quadratic model, and shows that spatial price dispersion occurs. In Section 4, we extend the model to the case when preferences and transport costs are described by non-specified real-analytic functions. We show that, under these conditions, only segregated land-use patterns exist. In Section 5, we discuss three other extensions of the baseline model: non-additive preferences, non-pecuniary externalities, and linear transport costs. Section 6 concludes. Simple proofs can be found in the Appendix, while long and technically involved arguments are in the Online Appendix.

2 An analytically solvable example: quadratic model

In this section, we look at the special case when preferences are quasilinear quadratic and transport costs are quadratic in distance. To the best of our knowledge, nobody has ever provided any examples of utility functions and/or transport costs, other than those used in Fujita (1988), under which Fujita's model would be analytically tractable. Developing alternative examples is important, as lack of these may create the false impression that city structure formation under pecuniary externalities can only be studied under very restrictive assumptions. More importantly, as shown in Section 3, our results differ a lot from those obtained by Fujita (1988).

2.1 Spatial structure

Consider a one-dimensional city X hosting a mass $N > 0$ of consumers, a mass $M > 0$ of firms, and absentee landlords. For conciseness, we will call a consumer located at $x \in X$ an *x-consumer*, and a firm located at $y \in X$ a *y-firm*.

The spatial distributions of firms and consumers across the city are described, respectively, by the densities $m(y), y \in X$, and $n(x), x \in X$, which are non-negative and satisfy the balance conditions:

$$\int_X m(y)\mathrm{d}y = M, \qquad \int_X n(x)\mathrm{d}x = N. \tag{1}$$

Both densities are endogenously determined by the market interactions between consumers and firms, which eventually shape the land-use pattern in the city. Although we postpone the detailed description of equilibrium land-use pattern formation until sub-section 2.4, two remarks are in order here. First, we follow the literature in urban economics by assuming that both $n(x)$ and $m(y)$ are *symmetric* with respect to the origin:

$$n(x) = n(-x), \text{ for all } x \in X; \qquad m(y) = m(-y), \text{ for all } y \in X. \tag{2}$$

Second, we borrow from Fujita (1988) the following terminology. Let $[a,b] \subset X$ be a non-degenerate interval. Call $[a,b]$

- a *residential district* if the equalities $n(x) = 1$ and $m(x) = 0$ hold for all $x \in [a,b]$, but fail to hold over an arbitrary open neighbourhood of $[a,b]$;
- a *commercial district* if the equalities $n(x) = 0$ and $m(x) = 1$ hold for all $x \in [a,b]$, but fail to hold over an arbitrary open neighbourhood of $[a,b]$;
- a *mixed district* if the inequalities $0 < m(x) < 1$ and $0 < n(x) < 1$ hold for all $x \in [a,b]$, but fail to hold over an arbitrary open neighbourhood of $[a,b]$.

The intuition behind these formal definitions is as follows: a district is residential (commercial) if all land there is rent by individuals (firms), while a district which accommodates both types of agents is a mixed one. We call a spatial equilibrium *segregated* if it does not involve mixed districts. Otherwise, we call a spatial equilibrium *mixed.* A mixed equilibrium is *pooled* if the city involves only one district.

2.2 Consumers

Consumers share identical quasilinear quadratic preferences

$$U(z; q(x,y), y \in X) = z + \alpha \int_X q(x,y)m(y)\mathrm{d}y - \frac{\beta}{2}\int_X [q(x,y)]^2 m(y)\mathrm{d}y, \tag{3}$$

where α and β are constants. The value of α captures the reservation price (common across varieties), while the value of β captures the degree of love for variety. Note that, unlike in Ottaviano et al. (2002), the utility function (3) does not involve any cross-effects. This implies that firms do compete for appealing locations, but do not compete directly for the consumer's wallet, acting like local monopolists. However, we show in Section 5 that our main results remain valid if we allow for non-zero substitutability across varieties.

Transport costs have the nature of *shopping costs* rather than shipping costs, i.e. they are fully borne by consumers. Furthermore, they vary quadratically with distance and enter additively the price. In other words, the full price of a y-good paid by an x-consumer equals $p(y)+t(x-y)^2$, where $p(y)$ is the mill price set by a y-firm.

Each x-consumer seeks to maximize utility (3) subject to the budget constraint:

$$\int_X \left[p(y)+t(x-y)^2\right] q(x,y)m(y)dy+R(x)+z=Y, \tag{4}$$

where Y is consumer's income, while $R(x)$ is the land rent at location x. Using (4), the utility function of an x-consumer may be represented as follows:

$$V(x)=\int_X \left[\alpha-p(y)-t(x-y)^2-\frac{\beta}{2}q(x,y)\right] q(x,y)m(y)dy-R(x)+Y, \tag{5}$$

The choke price α is assumed to be high enough to guarantee that the solution of the consumer's program is interior regardless of a consumer's location. We show in the Appendix that a sufficient condition for this to hold is as follows:

$$\sqrt{\frac{\alpha-c}{2t}}>M+N. \tag{6}$$

The intuition behind (6) is easy to comprehend. There are two potential hindrances for consumers to purchase all the available varieties: either their willingness-to-pay α is too low, or high transport costs restrain them from visiting remote shopping districts. Condition (6) rules out both these possibilities.

Given that (6) holds, the individual demands of an x-consumer are linear:

$$q(x,y)=\frac{\alpha-t(x-y)^2-p(y)}{\beta}. \tag{7}$$

As implied by (7), an increase in the distance between an x-consumer and a y-firm triggers a parallel downward shift of the corresponding individual demand curve. In other words, a larger distance means a lower willingness to pay, while the slope of the demand schedule is the same, both across consumers and varieties.

Combining (7) with (5) yields

$$V(x)=\frac{\beta}{2}\int_X [q(x,y)]^2 m(y)\mathrm{d}y-R(x)+Y, \tag{8}$$

which is a more parsimonious expression for the indirect utility of an x-consumer compared to (5).

2.3 Firms

It follows from (7) that the aggregate demand $Q(y)$ faced by a y-firm is given by

$$Q(y) = \frac{N}{\beta}\left[\alpha - t\delta(y) - p(y)\right], \tag{9}$$

where $\delta(y)$ is the *mean-squared distance from a y-firm to the whole mass of consumers* across the city:

$$\delta(y) \equiv \frac{1}{N}\int_X (x-y)^2 n(x)dx. \tag{10}$$

The intuition behind $\delta(y)$ is easy to grasp: it is a reverse measure of a *y-firm's market access*. Indeed, a lower $\delta(y)$ means that location y is "closer" to the whole mass of consumers, i.e. that a y-firm enjoys better access to the market. To further clarify this idea, we use the symmetry property (2) of the population distribution, and restate (10) as follows:

$$\delta(y) = y^2 + \delta_0, \tag{11}$$

where δ_0 is the *population dispersion*, defined as the mean-squared distance of the whole population of consumers from the central location $y = 0$:

$$\delta_0 \equiv \frac{1}{N}\int_X x^2 n(x)\mathrm{d}x. \tag{12}$$

As implied by (11), the further away location y is from the city center $x = 0$, the higher is $\delta(y)$. In other words, *firms located closer to the center have better market access*.

The profit of a y-firm is given by

$$\pi(y) = \frac{N}{\beta}\left[\alpha - t\delta(y) - p(y)\right]\left[p(y) - c\right] - R(y) - f. \tag{13}$$

Maximizing (13) with respect to $p(y)$ yields the profit-maximizing price of a y-firm:

$$p^*(y) = \frac{1}{2}\left[\alpha + c - t\delta(y)\right]. \tag{14}$$

It is worth noting that $\delta(y)$ captures the impact of new firms' entry on prices. This is due to the fact that changes in the mass M of firms may alter the spatial configuration of the city, which determines $\delta(y)$. See Section 3 for more details.

Plugging (14) into (9) and (13), we obtain the y-firm's profit-maximizing output and the profit earned by a y-firm:

$$Q^*(y) = \frac{N}{2\beta}\left[\alpha - c - t\delta(y)\right], \tag{15}$$

$$\pi^*(y) = \frac{N}{4\beta}\left[\alpha - c - t\delta(y)\right]^2 - f - R(y). \tag{16}$$

Equations (14) and (16) show that a lower value of $\delta(y)$ implies a higher price and a higher operating profit for y-firms. As seen from (11), $\delta(y)$ achieves a minimum at the central point $y = 0$ of the city. Thus, we come to the following results.

Proposition 1. *More centrally located firms charge higher prices and earn higher operating profits. A reduction in transport cost t reduces spatial price differentials.*

Proof. See the Appendix. □

Proposition 1 has a far-fetched implication: unlike in Fujita (1988), our model allows for an impact of a firm's location on its pricing strategy. This impact is fully captured by the term $t\delta(y)$. Indeed, as shown by (56), higher transport cost increases the price differential between any two locations. Hence, *the market outcome shows spatial price dispersion*. Moreover, this result does not depend on the specificities of the equilibrium distribution of firms across space.

This finding is fairly intuitive: more centrally located firms have more market power. This feature distinguishes our results from those obtained in similar contexts (De Palma et al., 1985; Fujita, 1988) and echoes the industrial organization literature where imperfectly informed consumers incur search costs (Wolinsky, 1983; Schulz and Stahl, 1996). Proposition 1 also highlights that infrastructural improvements are beneficial for all firms.

That firms located toward the city center are associated with higher profit-maximizing markups, whence higher market power of firms over consumers, highlights the reasons why no mixed land-use patterns emerge in equilibrium. Under entropy-type preferences used by Fujita (1988), the equilibrium price schedule is flat, whence the degree of a firm's market power is the same regardless of where the firm is located. In other words, the interaction between the *market-crowding effect,* which stems from tougher competition in the central areas, and the *market-access effect* which makes central locations more appealing, fully outweigh each other. As a consequence, there are zones in the city in which neither consumers outbid firms nor firms outbid consumers. Such zones accommodate both types of land use. Very much in contrast, our model gives rise to an additional effect – the *price dispersion effect*, which reinforces the market-access effect and makes firms to bid more aggressively for locations which are closer to the CBD. We discuss in more detail the impact of entry on price dispersion in sub-section 3.2.

An object dual to $\delta(y)$ is the mean-squared distance $\sigma(x)$ from an x-consumer to the whole mass of firms across the city, defined by

$$\sigma(x) \equiv \frac{1}{M}\int_X (x-y)^2 m(y)\mathrm{d}y. \tag{17}$$

Because $m(y)$ is a symmetric density, we have $\sigma(x) = \sigma_0 + x^2$, where

$$\sigma_0 \equiv \frac{1}{M}\int_X y^2 m(y)\mathrm{d}y \tag{18}$$

is the mean-squared distance of the whole population of firms from the central location $x = 0$.

Observe that the dispersion measures δ_0 and σ_0 are not totally independent. It is readily verified that they satisfy the following identity:

$$M\sigma_0 + N\delta_0 = \frac{(M+N)^3}{12}. \tag{19}$$

Using (19) and (11), we can express profit-maximizing prices as follows:

$$p^*(y) = a(M) + \frac{tM}{2N}\sigma_0 - \frac{t}{2}y^2, \tag{20}$$

where $a(M) \equiv \frac{\alpha+c}{2} - \frac{t}{24N}(M+N)^3$. Equation (20) demonstrates that the impact of spatial distribution of firms on the prices is fully captured by the dispersion σ_0. Furthermore, (20) highlights the idea that *more dispersion means softer competition*: an increase in σ_0 shifts the whole price schedule upwards. We show in sub-section 5.2 how this result modifies if we augment the model with uncompensated knowledge spillovers á la Berliant et al. (2002).

2.4 Land-use pattern formation

The land market works in a standard way: absentee landlords choose between renting land to consumers and to firms, depending on whose bid is higher. We build on Fujita and Ogawa (1982) in introducing the consumer's *bid rent function* $\Psi(x, U^*)$, which is defined as the maximum rental price a consumer would agree to pay for locating at x, conditional on having the utility level at least as high as U^*. In other words, it must be that

$$\Psi[x, V(x)] = R(x), \tag{21}$$

where $V(x)$ is the utility level (5) gained by an x-consumer.

Similarly, $\Phi(y, \pi^*)$ stands for the firm's bid rent function, i.e. it shows the maximum rent $R(y)$, which guarantees $\pi(y) \geq \pi^*$. Equivalently, $\Phi(y, \pi^*)$ must satisfy the following equation:

$$\Phi[y, \pi^*(y)] = R(y). \tag{22}$$

Because landlords seek to maximize their income, the land rent at $x \in X$ is given by

$$R(x) = \max\{\Psi(x, U^*), \Phi(x, \pi^*), R_a\}, \qquad \text{for all } x \in X,$$

where R_a denotes the opportunity cost of land (e.g. agricultural rent).

The total number of households (firms) that occupy a unit of land are defined as the unit of households (firms). Our convention is that, whenever a landlord is indifferent whether to let a place or not, she always chooses to let. In accordance with this, we define a *spatial equilibrium* as a dyad $\{n(\cdot), m(\cdot)\}$, which satisfies the following conditions:

(i) for all $x, y \in X$,

$$\begin{aligned} n(x) > 0 &\implies \Psi(x, U^*) = R(x), \\ m(y) > 0 &\implies \Phi(y, \pi^*) = R(y); \end{aligned}$$

(ii) for all $x \in X$,

$$\max\{\Psi(x, U^*), \Phi(x, \pi^*)\} \geq R_a \quad \Longrightarrow \quad n(x) + m(x) = 1,$$
$$\max\{\Psi(x, U^*), \Phi(x, \pi^*)\} < R_a \quad \Longrightarrow \quad n(x) + m(x) = 0.$$

Condition (i) means that consumers (firms) locate at x if and only if the consumer's (firm's) bid for this location equals the highest bid. The meaning of condition (ii) is that, when at least one of the bids exceeds (or equals to) the opportunity cost of land in a particular location x, all the amount of land is rent (to either consumers, or firms, or both). Meanwhile, the sites where both consumer's and firm's bids are strictly lower than R_a are used for agriculture and lie beyond the scope of the city.

2.5 Bid-rent functions and market access

We now derive the consumer's and firm's bid-rent functions. As shown in the Appendix, these functions are given by:

$$\Phi(y, \pi^*) = \frac{Nt^2}{4\beta}\left(\frac{\alpha - c}{t} - \delta_0 - y^2\right)^2 - f - \pi^*, \tag{23}$$

$$\Psi(x, U^*) = \frac{Mt^2}{2\beta}\left[x^4 - \left(\frac{\alpha - c}{t} + \delta_0 - 5\sigma_0\right)x^2 + K\right] - U^* + Y, \tag{24}$$

where K is independent of x.

To better understand the mechanism of land pricing in our model, we now study the behavior of the bid-rent functions (24) – (23) in a bit more detail. This will help us to see how willingness to pay for land changes in response to better market access, which may stem from either relocation, or reduction in transport costs, or both. We focus on the firm's bid rent, since the main arguments apply to consumers as well.

The bid-rent gradients are given by

$$\frac{\partial \Psi}{\partial x} = -\frac{Mt^2 x}{\beta}\left(\frac{\alpha - c}{2t} - \frac{5\sigma_0 - \delta_0}{2} - x^2\right), \quad \frac{\partial \Phi}{\partial y} = -\frac{Nt^2 y}{2\beta}\left(\frac{\alpha - c}{t} - \delta_0 - y^2\right). \tag{25}$$

It is clear from (25) that, as $(\alpha - c)/t$ is assumed to be sufficiently large, both bid rents decrease with distance from the city center, at least when this distance is not too large. Furthermore, (25)) implies that both gradients depend non-trivially on the inverse market-access measures: the spatial dispersion of population and the transport cost t.

1. A change in the distribution of population. As discussed above, a relevant inverse measure for a y-firm's market access is the mean-squared distance $\delta(y)$ between the y-firm and the whole urban population. Assume that a reduction in δ_0 occurs, i.e. the distribution of consumers becomes more concentrated toward the city center $x = 0$. As implied by (11), this leads to a reduction of $\delta(y)$ for all $y \in X$, whence better market access for more (less) centrally located firms. To see how the bid-rent function (23) changes in this case, we differentiate Φ with respect to δ_0, which yields

$$\frac{\partial \Phi}{\partial \delta_0} = -\frac{Nt^2}{2\beta}\left(\frac{\alpha - c}{t} - \delta_0 - y^2\right) < 0.$$

In other words, *when population concentrates toward the center, firms are willing to pay higher (lower) rents*.

The impact of population concentration on the *slope* $|\partial\Phi/\partial y|$ of the bid rent function is captured by the cross derivative

$$\frac{\partial|\partial\Phi/\partial y|}{\partial \delta_0} = -\frac{Nt^2}{\beta} y < 0,$$

meaning that more dispersed population (i.e. a higher δ_0) renders the bid-rent function less steep. Figure 1 illustrates this effect.

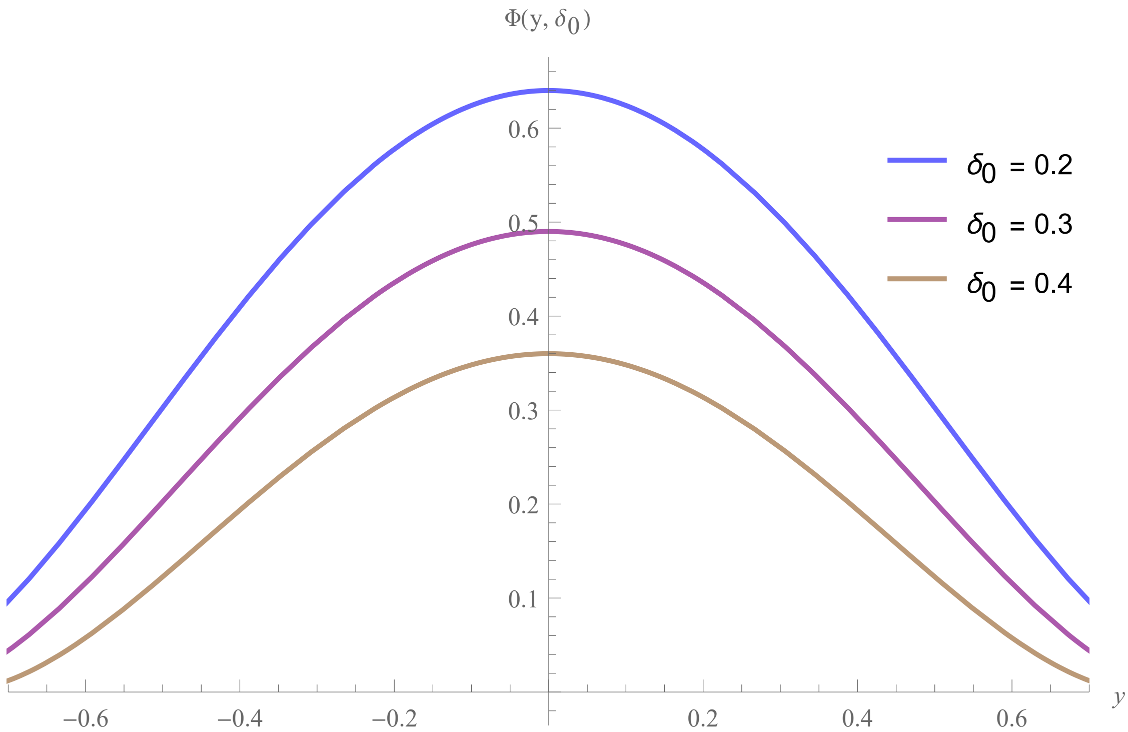

Figure 1. The plots of $\Phi(y, \delta_0)$ for different values of δ_0

The intuition behind this result is as follows: *locating closer to the central area has more marginal value for a firm when population is more concentrated toward the city center*.

2. A reduction in transport costs. A better market access may also stem from infrastructural improvements, i.e. a lower transport cost. Computing the cross-derivative in location y and transport cost t yields:

$$\frac{\partial|\partial\Phi/\partial y|}{\partial t} = \frac{Nty}{\beta}\left(\frac{\alpha - c}{2t} - \delta_0 - y^2\right).$$

This expression is positive when $(\alpha - c)/(2t)$ is sufficiently large, which concurs with the requirement that (6) holds. Conversely, a reduction in t makes the firm's bid-rent function less steep. This suggests the following interpretation: *an improvement in transportational infrastructure – i.e.*

a better market access – reduces the firm's marginal value of relocating closer to the center. This is in line with the standard intuition which states that infrastructural improvements downplay the importance for a firm to be centrally located.

3 Spatial equilibria in the quadratic model

In this section, we study the conditions for various spatial configurations of the city to emerge in the quadratic case. There are three major differences between our results and those obtained by Fujita (1988): our model (i) generates only segregated equilibria; (ii) captures equilibrium price dispersion; (iii) gives rise to multiple free-entry equilibria and the possibility of a discontinuous jump between them in response to small variations in the fixed costs of launching a firm and/or in the opportunity cost of land.

3.1 Short-run equilibria

By a short-run spatial equilibrium we understand an equilibrium in which the mass M of firms is given. We start with the following observation: as implied by (24) and (23), for any given values of M and N the bid rent functions $\Psi(x, U^*)$ and $\Phi(x, \pi^*)$ are polynomials of degree 4 with respect to x, which involve only even degrees of x. Hence, they may have either two, or four, or no intersection points over $\left[-\frac{M+N}{2}, \frac{M+N}{2}\right]$. Moreover, the latter case is impossible in equilibrium, since otherwise the whole city is either one big residential area or one big shopping area, whence at least one of the conditions (1) is violated. Therefore, it remains to consider four possibilities shown by Figure 2.

Fig. 2a: monocentric city

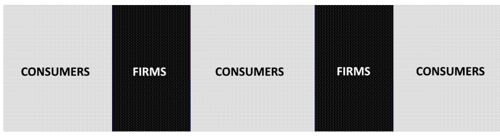

Fig. 2b: duocentric city

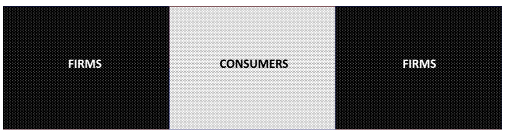

Fig. 2c: residential downtown area

Fig. 2d: tricentric city

Figure 2. Possible equilibrium configurations

We now provide a full characterization of short-run equilibria. Before stating the result mathematically, we describe it in intuitive terms. When firms are few, competition is relatively soft, hence the market access effect dominates the competition effect. This makes central locations very appealing for the firms. As a result, firms outbid consumers in the central area and form a commercial cluster, while consumers reside closer to the city borders. This results in a monocentric urban structure (Fig. 2a).

As more firms enter, competition in the central area becomes tougher, and at some point the cluster around the center breaks into two commercial districts. Firms located to the right acquire local market power over "East-side" consumers, while those to the left have local market power over "West-side" consumers. Meanwhile, since businesses get more dispersed across the city, while consumers wish to buy from all firms due to love for variety, consumer's willingness to pay for central locations increases. This leads to a formation of a residential area in the middle, but some consumers still choose to reside at the outskirts. This results in a duocentric urban structure (Fig. 2b).

When the mass of firms gets even larger, the two commercial areas diverge toward the outskirts of the city, while consumers keep moving closer to the geographical center of the city, which eventually becomes a residential downtown area bordered by two commercial clusters. This type of urban layout is illustrated by Fig. 2c. Finally, we will show that the spatial structure shown on Fig. 2d never emerges in equilibrium. The following Proposition provides a summary.

Proposition 2. *Assume that* (6) *holds. Then, for any values of M and N, a unique spatial equilibrium always exists. Moreover, there exist a lower bound $\underline{M}(N) < N$ and an upper bound $\overline{M}(N) > N$ of the number M of firms, such that*:

(i) *if $M \leq \underline{M}(N)$, then the spatial equilibrium is monocentric (Fig. 2a);*

(ii) *if $\underline{M}(N) < M < \overline{M}(N)$, then the spatial equilibrium is duocentric (Fig. 2b);*

(iii) *if $M \geq \overline{M}(N)$, then the residential center is bordered by two commercial districts (Fig. 2c);*

(iv) *a tricentric spatial structure (Fig. 2d) never emerges in equilibrium.*

Proof. See the Online Appendix. □

Figure 3 illustrates the results of Proposition 2 on the (N, M)-plane.

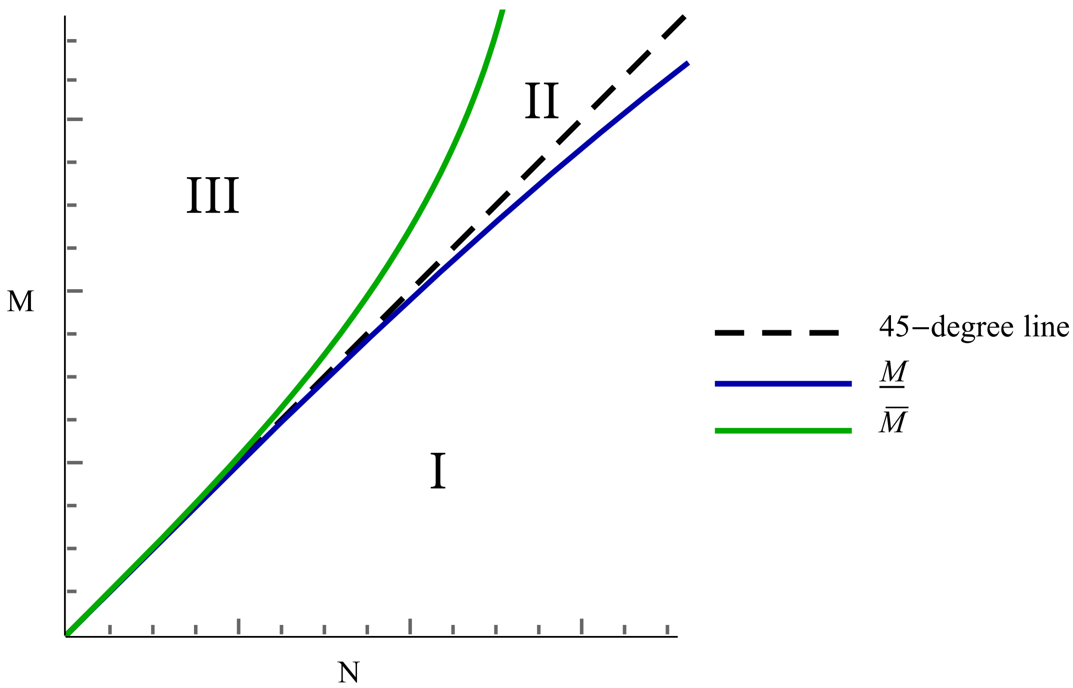

Figure 3. Short-run equilibria on the (N, M)-plane

Region I in Figure 3, which is delineated by the horizontal axis and the lower boundary curve $M = \underline{M}(N)$, corresponds to the case of a monocentric city (part (i) of Proposition 2). Region II, which lies between the two boundary curves,[6] comprises all combinations (N, M) for which the spatial equilibrium is duocentric (part (ii) of Proposition 2). Finally, region III, delineated by the vertical axis, and the upper boundary curve $M = \overline{M}(N)$, is the set of all (N, M), such that the city is composed by a residential central area bordered by two commercial clusters (part (iii) of Proposition 2).

3.2 The impact of entry on price dispersion

We are now equipped to study how entry affects price dispersion. Using the expression (14) for the profit-maximizing price of a y-firm, we find that the mean price is given by

$$\mathbb{E}(p) \equiv \frac{1}{M} \int\limits_{y \in X} p^*(y) m(y) \mathrm{d}y = \frac{\alpha + c}{2} - \frac{t}{2}(\delta_0 + \sigma_0). \tag{26}$$

As implied by 2.3 (see sub-section 4.2), varieties supplied closer to the city center fetch higher prices. To make this statement more precise, we combine (14) with (26) to compute the deviation of the profit-maximizing price of a y-firm from the average price:

$$p^*(y) - \mathbb{E}(p) = \frac{t}{2}\left(\sigma_0 - y^2\right). \tag{27}$$

Using (27), we observe that firms located at $y = \pm\sqrt{\sigma_0}$ price at the average level. Firms which are closer to the center charge prices above the mean, while those further away from the center set prices below the mean.

We choose to measure price dispersion by means of the standard deviation of the price distribution, which is given by

$$\sqrt{\mathbb{V}(p)} \equiv \sqrt{\frac{1}{M} \int\limits_{y \in X} \left[p^*(y) - \mathbb{E}(p)\right]^2 m(y) \mathrm{d}y} = \frac{t}{2}\sqrt{\frac{1}{M} \int\limits_{y \in X} y^4 m(y) \mathrm{d}y - \sigma_0^2}. \tag{28}$$

We now look at how the average price and the price dispersion measure respond to entry of new firms.

When firms are few ($M < \underline{M}(N)$), then it follows from Proposition 2 that firms are uniformly distributed over $[-M/2, M/2]$. In this case, we have $\sigma_0 = M^2/12$. As seen from (27), firms charge the price above the mean if and only if they are within the $M/(2\sqrt{3})$-radius of the city central point $x = 0$. The mean price is given by

$$\mathbb{E}(p) = \frac{\alpha + c}{2} - \frac{t}{2}\left(\frac{M^2}{3} + \frac{MN}{4} + \frac{N^2}{12}\right).$$

[6] The boundary curves $M = \underline{M}(N)$ and $M = \overline{M}(N)$ can be described in closed form by means of parametric equations. See formulas (92) – (99) in the Online Appendix.

Differentiating $\mathbb{E}(p)$ with respect to M yields

$$\frac{\partial \mathbb{E}(p)}{\partial M} = -t\left(\frac{M}{3} + \frac{N}{8}\right) < 0,$$

which implies that the mean price decreases with entry. This is in accordance with the standard intuition from industrial organization: an increase in the number of firms makes competition tougher and drives prices downwards.

As for the standard deviation of prices, it is given by

$$\sqrt{\mathbb{V}(p)} = \frac{t}{12\sqrt{5}} \cdot M^2.$$

Hence, price dispersion increases with entry. Moreover, it does so more than proportionally compared to the size M of the commercial district. The intuition behind this result is as follows. First, when more firms enter, the support of the price distribution expands, which leads to a higher price dispersion per se. Second, when new firms enter, they locate further away from the center than the incumbent firms, hence they charge lower prices. Because the profit-maximizing price decreases quadratically with the distance from the center, this leads to a further magnification of price dispersion.

It is readily verified that qualitatively the same results hold when firms are many, i.e. when $M > \overline{M}(N)$. In this case, the expressions for the mean price and the price dispersion are given, respectively, by:

$$\mathbb{E}(p) = \frac{\alpha + c}{2} - \frac{t}{2}\left(\frac{M^2}{12} + \frac{MN}{4} + \frac{N^2}{3}\right),$$

$$\sqrt{\mathbb{V}(p)} = \frac{t}{4}M\sqrt{\frac{1}{45}M^2 + \frac{1}{12}MN + \frac{1}{4}N^2}.$$

The only difference with the case of "few" firms is that now the radius within which firms price above the average depends not only on the mass M of firms, but also on the size N of urban population. This radius is given by

$$\frac{M}{2\sqrt{3}}\sqrt{1 + 3\frac{N}{M} + 3\left(\frac{N}{M}\right)^2}.$$

When the mass of firms takes on an intermediate value ($\underline{M}(N) \leq M \leq \overline{M}(N)$), the analysis becomes more complex. However, this analysis is of a limited interest, since, as shown below (see Proposition 3 in the next sub-section), free-entry equilibria in which the mass of firms is in this domain are always unstable. Therefore, we refrain from doing it.

It is worth stressing that price dispersion in our model has very different routes compared to those discussed in seminal papers by Varian (1980), Burdett and Judd (1983), etc. These authors stress the role of imperfect information on the consumer side, whereas we obtain spatial price dispersion just because both costs and market access differ across locations: more centrally located firms pay higher rents but have better access to the market (a lower $\delta(y)$).

3.3 Long-run equilibria

We now come to studying long-run equilibria where entry is endogenous. The mass M of firms is pinned down by the free entry condition given by the free-entry condition:

$$\Pi(M) = \phi, \tag{29}$$

where $\Pi(M)$ is the operating profit as a function of the mass of firms, while $\phi \equiv f + R_a$ is the *effective fixed cost*, which includes the per-firm entry cost f and the agricultural rent R_a. In other words, to launch a firm, one has to pay a startup cost f plus the opportunity cost R_a of not starting a farm instead of a firm.

Following the literature on market competition with endogenous entry, we call any positive solution M^* to (29) a *long-run equilibrium*, or a *free-entry equilibrium*. Moreover, we call M^* a *stable* equilibrium if $\Pi(M)$ is downward-sloping in the vicinity of M^*. Otherwise we call M^* an *unstable equilibrium*.[7] The intuition is as follows. Assume that $\Pi'(M^*) < 0$. Then, if a small mass of new firms enters, this will lead to a reduction in the operating profit, so that firms don't break-even any more. This, in turn, will lead to exit of some firms, and the market will be driven back to equilibrium. On the contrary, when $\Pi'(M^*) > 0$, entry of an arbitrarily small positive mass of new firms will drive profits upwards, which will create a snowball effect: more and more firms will start to enter, and the market will not get back to M^*.

Clearly, characterizing long-run equilibria amounts to studying how the operating profit $\Pi(M)$ varies with the mass M of firms. As shown in the Online Appendix, $\Pi(M)$ is non-monotone in M: it decreases in M when $M \leq \underline{M}(N)$, then starts increasing until M reaches $\overline{M}(N)$, and then decreases again when $M > \underline{M}(N)$, as illustrated by Figure 4.

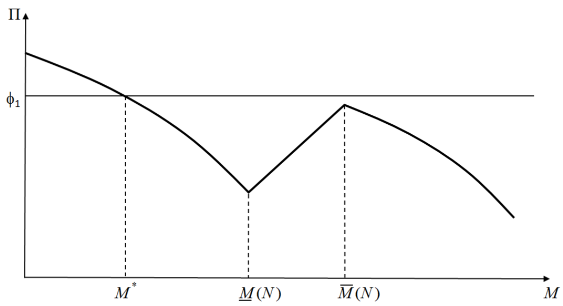

Fig. 4a. Unique equilibrium, $\phi_1 > \phi_{\text{high}}$

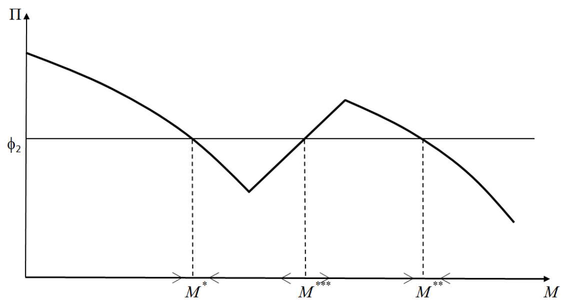

Fig. 4b. Multiple equilibria, $\phi_{\text{low}} < \phi_2 < \phi_{\text{high}}$

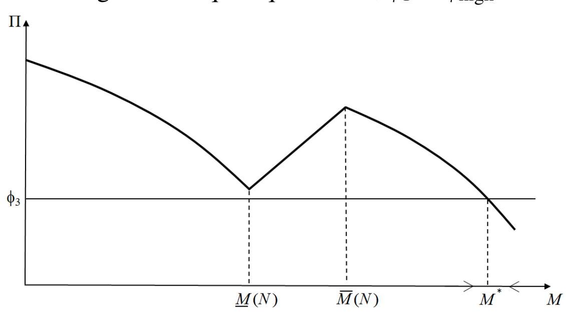

Fig. 4c. Unique equilibrium, $\phi_3 < \phi_{\text{low}}$

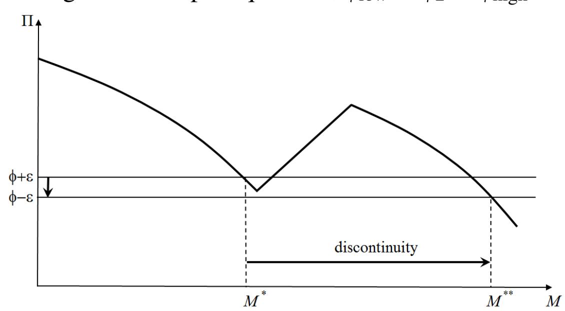

Fig. 4d. Discontinuity in response to a change in ϕ

Figure 4. Free-entry equilibria

[7] This concept of stability has been recently used by Ushchev et al. (2015) in a different context.

The intuition behind this shape of the operating profit function is fairly transparent. When firms are few ($M \le \underline{M}(N)$), we know from Proposition 2 that the city structure is monocentric (Fig. 2a). When more firms enter, the central area becomes broader, and consumers reside further away from the city center $x = 0$. This makes central locations less and less profitable for the incumbent firms. As a consequence, both because consumer demand gets more dispersed due to a broader product diversity and because consumers relocate further away from the incumbent firms, the operating profit decreases when new firms enter. This case is very typical in industrial organization (see Ch. 4 of Belleflamme and Peitz, 2015). However, when the mass of firms reaches an intermediate region ($\underline{M}(N) < M < \overline{M}(N)$), the spatial structure of the city switches from Fig. 2a to Fig. 2b. Hence, a new effect is put to work: *the incumbent firms start to relocate further away from the center in response to entry of new firms*. This "relocation effect" explains the atypical behavior of profit ($\Pi'(M) > 0$) in response to an increase in M. Finally, when firms are sufficiently many ($M \ge \overline{M}(N)$), the spatial structure involves a residential central area and two commercial districts at the outskirts. Because consumers no longer relocate in response to further increases of product diversity M, firms do not relocate either. These results can be summarized by the following Proposition.

Proposition 3. *Assume that* (6) *holds. There exist two threshold values,* $0 < \phi_{\text{low}} < \phi_{\text{high}} < \infty$*, of the effective fixed cost* ϕ*, such that:*

(i) *if* $\phi \ge \phi_{\text{high}}$*, then a unique free-entry equilibrium exists, and it gives rise to a monocentric spatial configuration (Fig. 2a);*

(ii) *if* $\phi_{\text{low}} < \phi < \phi_{\text{high}}$*, then there exist three distinct free-entry equilibria –* M^**,* M^{**}*, and* M^{***} *– such that* $M^* < \underline{M}(N) < M^{***} < \overline{M}(N) < M^{**}$*, which correspond to three different types of spatial configurations (Fig.2.a, 2.b and 2.c); furthermore,* M^* *and* M^{**} *are stable, while* M^{***} *– which gives rise to a duocentric city – is unstable;*

(iii) *if* $\phi \le \phi_{\text{low}}$*, then again a unique stable equilibrium exists, while the corresponding equilibrium spatial configuration involves a residential central area bordered by two commercial districts (Fig. 2c).*

Proof. See the Online Appendix.□

Cases (i) – (iii) of Proposition 3 are illustrated, respectively, by Fig. 4a – 4c.

We would like to highlight one implication of these results. Consider the case when the urban market for the differentiated good has just emerged, and that ϕ only slightly exceeds ϕ_{low} (ϕ_2 on Fig. 4c is only slightly above the left kink of the profit function). Then, firms start to enter, and the system ends up in M^*, the "left" stable free-entry equilibrium. However, a small reduction in ϕ leads to a discontinuous hike in the number of firms, as illustrated by Fig. 4d. Moreover, this tiny shock in ϕ leads to drastic changes in the spatial structure of the city. Indeed, observing that the equilibrium changes from $M^* < \underline{M}(N)$ to $M^{***} > \overline{M}(N)$ and taking Proposition 2 into account, we conclude that the downtown area becomes purely residential while firms fully relocate to the outskirts of the city in response to a small shock in ϕ. To sum up, our model demonstrates how small variations in either the fixed cost f or the agricultural rent R_a may lead to both rapid urban growth and dramatic changes in the city structure.

4 Mixed land use: almost impossible?

In this section, we extend the above model to the case of non-specified additive preferences. We show that, whenever both utility and transport cost belong to the broad class of real-analytic functions, only segregated spatial equilibria exist. Furthermore, even if a mixed land-use equilibrium happens to exist (like in Fujita (1988), where the transport cost is linear in distance, whence not real-analytic), such equilibria are never robust to arbitrarily small perturbations of demand and/or transport cost schedules.

4.1 Consumers

The individuals share identical quasilinear additive preferences. The utility function of an x-consumer is given by

$$U(z; q(x,y), y \in X) = z + \int_X u[q(x,y)]\, m(y) \mathrm{d}y \tag{30}$$

where z is the outside good consumption level, $q(x,y)$ stands for the x-consumer's consumption level of a variety supplied at y, while $u(q)$ is the sub-utility of consuming a single variety. The quadratic utility (3) is obtained as a special case of (30) by setting $u(q) = \alpha q - \beta q^2/2$. We impose the following assumptions on the sub-utility $u(q)$:

(**U1**) $u(q)$ is *strictly increasing* in q for all $q \geq 0$;

(**U2**) $u(q)$ is *strictly concave* in q for all $q \geq 0$;

(**U3**) $u(q)$ is *real-analytic* for all $q \in (0, (u')^{-1}(c))$, where $c > 0$ is the marginal cost (to be defined below).

Assumptions (**U1**) and (**U2**) are standard. Assumption (**U3**) means that $u(q)$ is infinitely differentiable, and for any $q_0 \in (0, (u')^{-1}(c))$ there exists a neighborhood where the Taylor series of $u(q)$ computed at $q = q_0$ converges to $u(q)$:

$$u(q) = \sum_{k=0}^{\infty} \frac{u^{(k)}(q_0)}{k!} (q - q_0)^k. \tag{31}$$

The class of real-analytic functions is broader than the class of polynomials, although it is slightly less general than the class of all infinitely differentiable functions. Is the assumption of real analyticity too restrictive? We believe it is not, for at least two reasons. First, the utility function (30) embraces a wide variety of preference specifications used in the literature satisfying this assumption, including (i) the quadratic utility $u(q) = \alpha q - \beta q^2/2$ with $\alpha, \beta > 0$; (ii) the CES utility under $u(q) = q^\rho$ with $0 < \rho < 1$, (iii) the CARA utility (Behrens and Murata, 2007) under $u(q) = 1 - \exp(-\gamma q)$ (with $\gamma > 0$), and (iv) the Stone-Geary utility giving rise to the linear expenditure system (Simonovska, 2015) under $u(q) = \ln(1 + \delta q)$ (with $\delta > 0$). Second, as implied by the Stone-Weierstrass theorem (Rudin, 1991), any twice continuously differentiable utility function can be uniformly approximated by a polynomial with an arbitrary degree of precision over any compact subinterval of the relevant domain $(0, (u')^{-1}(c))$ of consumption levels. A fortiori, the same can be done by means of real-analytic functions, a finite-order polynomial being a special case.

We assume that each individual consumes one unit of land. Given this, each x-consumer seeks to maximize utility (30) subject to the following budget constraint:

$$\int_X [p(y) * T(x,y)]\, q(x,y) m(y) dy + R(x) + z = Y. \tag{32}$$

In equation (32), Y is consumer's income, while $R(x)$ is the land rent at location x. As for $T(x,y)$, it is the *transport cost function*, whose value shows the cost (borne by a consumer) of transporting one unit of a good from location y to location x. The terms $p(y)$ and $p(y) * T(x,y)$ mean, respectively, the *mill price* charged by a y-firm, and the *full price* of a y-variety for an x-consumer. The sign $*$ stands for either addition ($* = +$) or multiplication ($* = \times$). Thus, the model captures the cases of both additive and multiplicative transport costs. In the former case, we set $T(x,x) = 0$ for all $x \in X$, while in the latter case $T(x,x) = 1$ for all $x \in X$.

We assume $T(x,y)$ to be

(**T1**) *symmetric* in x and y: $T(x,y) = T(y,x)$ for all $x, y \in X$;

(**T2**) *increasing* in x for all $x > y$ (respectively, decreasing for all $x < y$);

(**T3**) *real-analytic* in both x and y over $X \times X$.

The symmetry assumption (**T1**) means that the shopping cost does not depend on the direction of the trip, while monotonicity assumption (**T2**) means that the further a trip, the higher it costs. Both these assumptions have been standard for spatial competition literature ever since Hotelling (1929). This is not the case with the assumption (**T3**), which formally means that (i) $T(x,y)$ is infinitely differentiable, and (ii) for any $(x_0, y_0) \in X \times X$, there exists a neighborhood where the Taylor series of $T(x,y)$, computed at $x = x_0$, $y = y_0$, converges to $T(x,y)$:

$$T(x,y) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l=0}^{k} (x - x_0)^l (y - y_0)^{k-l} \left. \frac{\partial^k T(x,y)}{\partial^l x\, \partial^{k-l} y} \right|_{x=x_0,\, y=y_0}. \tag{33}$$

Common across the fields of industrial organization and spatial economics, it is routinely assumed that the transport costs are linear in distance: $T(x,y) = t|x-y|$, where $t > 0$ is a constant transport cost per unit of distance. Clearly, (33) fails to hold for the linear transport cost $T(x,y) = t|x-y|$, which is not even differentiable when $x_0 = y_0$. By contrast, the quadratic transport cost, $T(x,y) = t(x-y)^2$, satisfies (**T1**) – (**T3**), and so does any polynomial of the form $T(x,y) = \sum_{k=0}^{d} t_k (x-y)^{2k}$, where t_k are positive reals and d is a positive integer.

One may wonder about the motivation of introducing the assumption (**T3**). It has been well known since d'Aspremont et al. (1979) that the functional form of transport costs may affect critically the equilibrium behavior in spatial competition models. By introducing (**T3**), we aim to show that using linear transport costs in urban models is equally far from being innocuous. And indeed, we find that even tiny infrastructural shocks may cause dramatic changes in the urban landscape (see Proposition 5 in Section 4).[8]

Using the budget constraint, the x-consumer's utility may be represented as follows:[9]

$$V(x) = \int_X u[q(x,y)]m(y)\mathrm{d}y - \int_X [p(y) * T(x,y)] \cdot q(x,y)m(y)dy - R(x) + Y. \tag{34}$$

Given that the solution of the consumer's problem is interior, the individual demand of an x-consumer for a y-variety is given by

$$q(x,y) = D[p(y) * T(x,y)], \tag{35}$$

where $D(\cdot) \equiv (u')^{-1}(\cdot)$ is the Marshallian demand for the good supplied at y. Due to concavity of $u(\cdot)$, the demand schedule (35) decreases in the full price $p(y) * T(x,y)$ paid by the x-consumer.

4.2 Firms

Production technology, which is the same across firms, exhibits a constant marginal cost c and a fixed cost f, both expressed in terms of the numéraire. In addition, a y-firm pays a land rent $R(y)$. The market demand $Q(y)$ faced by a y-firm can be obtained by aggregating (35) across X, and is given by

$$Q(y) \equiv \int_X q(x,y)n(x)dx = \int_X D[p(y) * T(x,y)]\,n(x)dx. \tag{36}$$

Each firm consumes one unit of land and seeks to maximize profit. The profit function of a y-firm is given by

$$\pi(y) = [p(y) - c] \int_X D[p(y) * T(x,y)]\,n(x)dx - R(y) - f, \tag{37}$$

while the y-firm's FOC can be stated as follows:

$$\int_X D[p(y) * T(x,y)]\,n(x)dx + [p(y) - c] \int_X D'[p(y) * T(x,y)] \frac{\partial [p(y) * T(x,y)]}{\partial p(y)} n(x)dx = 0.$$

Given the density $n(x)$ of consumers, the price pattern $\{p(y), y \in X\}$ is determined pointwise as an implicit solution of this FOC.

[8] As we will show, the substantial qualitative features of the urban structure under linear transport costs differ a lot from those generated by real-analytic transport costs. These differences do not vanish even when a real-analytic transport cost approximates the linear one extremely precisely. By the Stone-Weierstrass theorem, the linear transport cost $T(x,y) = t|x-y|$, which is not real-analytic, can be approximated with a polynomial transport cost $T(x,y) = \sum_{k=0}^{d} t_k (x-y)^{2k}$, which is real analytic, at an arbitrarily high degree of precision. This is what we mean by saying that tiny infrastructural shocks cause dramatic changes in the urban landscape.

[9] To be precise, such representation requires the standard assumption that each consumer's income Y is high enough to guarantee positive consumption level of the numéraire.

4.3 No mixed equilibria

Because both preferences and transport costs are non-specified, it may seem at first sight that one can hardly expect our model to generate unambiguous predictions about the urban structure. We will show, however, that when both the sub-utility $u(q)$ and the transport cost $T(x,y)$ are sufficiently regular functions – i.e. when they satisfy, respectively, (**U3**) and (**T3**), – there is *no equilibrium configuration involving mixed districts*.

Before proceeding, we briefly discuss the intuition of this result, which is as follows: the desire of both consumers and firms to be closer to each other, which may be viewed as a "co-agglomeration" force,[10] is suppressed by competition for land, which acts as a dispersion force. This competition never ends "in a draw": either firms outbid consumers, or vice versa. Hence, segregated land-use patterns are not extreme cases but rather an intrinsic property of our modeling strategy.

Proposition 4. *Assume that the sub-utility $u(q)$ satisfies* (**U1**) – (**U3**)*, while the transport cost $T(x,y)$ satisfies* (**T1**) – (**T3**)*. Assume also that $M \neq N$. Then, mixed spatial equilibria do not exist.*

Proof. See the Appendix. □

Proposition 4 highlights substantial differences of the spatial equilibrium patterns in our model with those in Fujita (1988), where the urban landscape becomes more and more homogeneous across space as M gets closer to N. In the limiting case of $N = M$, the whole city becomes a unified mixed district. Things work differently in our model, since consumers and firms never mix, except possibly for the case when $N = M$. However, this is a zero-measure case, hence this possibility may be disregarded as insignificant.[11]

To shed more light on the economic content of Proposition 4, recall that in Fujita (1988) the sub-utility $u(q)$ is real-analytic and is given by

$$u(q) = (1+\beta)\frac{q}{\alpha} - \frac{q}{\alpha}\ln\frac{q}{\alpha}, \tag{38}$$

while transport cost enters the price additively and is linear in distance, i.e. the full price of a y-good paid by an x-consumer is given by $p(y) + t|x-y|$. This function is not real-analytic (and not even differentiable) in x in the vicinity of $x = y$. As found by Fujita (1988)(p. 99, Figures 2 and 3), the spatial equilibrium in this setting always involves a mixed district. It is both legitimate and natural to as whether this result is robust to small perturbations of the transport cost schedule. Our next result gives a strongly negative answer.

[10] We follow Picard and Tabuchi (2013) in applying the term "co-agglomeration" to co-location of firms and consumers, although this concept is more frequently used in the context of co-location of firms belonging to different industries.

[11] One can show that the equilibrium is pooled under $M = N$ if the utility has the entropy form (38), like in Fujita (1988), while the transport cost is quadratic, like in Sections 2 and 3. However, the land-use swiftly becomes fully segregated in response to an arbitrarily small gap between M and N. Thus, pooled spatial configurations are not robust. As implied by Proposition 5, this is true for mixed land-use patterns in general.

Proposition 5. *Assume that the sub-utility $u(q)$ satisfies* **(U1) – (U2)** *and is twice continuously differentiable in q over $\left(0, (u')^{-1}(c)\right)$, while the transport cost $T(x,y)$ satisfies* **(T1) – (T2)** *and is continuous over $X \times X$. Assume also that a mixed spatial equilibrium exists. Then, for any $\varepsilon > 0$ there exist a perturbed sub-utility $u_\varepsilon(q)$ and a perturbed transport cost $T_\varepsilon(x,y)$ satisfying the following properties*:

(i) *$u_\varepsilon(q)$ is uniformly close to $u(q)$: $|u(q) - u_\varepsilon(q)| \le \varepsilon$ for all $q \in \left(0, (u')^{-1}(c)\right)$*;

(ii) *$T_\varepsilon(x,y)$ is uniformly close to $T(x,y)$: $|T(x,y) - T_\varepsilon(x,y)| \le \varepsilon$ for all $x,y \in X$*;

(iii) *in response to replacing $u(q)$ and $T(x,y)$ by, respectively, $u_\varepsilon(q)$ and $T_\varepsilon(x,y)$, a mixed spatial equilibrium ceases to exist.*

Proof. See the Appendix. □

An ε-shock which triggers the transformation $T(x,y) \to T_\varepsilon(x,y)$ may be viewed as a pattern of small infrastructural changes in different parts of the city (possibly non-uniformly distributed across the urban space). Intuitively, Proposition 5 means that salient qualitative features of the urban landscape may change dramatically even in response to tiny shocks in the demand schedules and/or minor transportation improvements here and there. To some extent, this result may be viewed as complimentary to discontinuous comparative statics obtained by Fujita and Ogawa (1982). Furthermore, Proposition 5 answers positively the following question posed by Lucas and Rossi-Hansberg (2002): "A remarkable feature of the examples we compute is an *extreme sensitivity* of the nature of equilibria to small changes in assumed travel costs. One wonders whether this feature may carry over to other spatial models."

5 Other extensions

In this section, we provide three extensions of the baseline model considered in Section 2. First, we go to non-additive preferences á la Ottaviano et al. (2002). By doing so, we achieve two purposes: (i) capture explicitly *pecuniary externalities* which stem from market interactions between firms, and (ii) demonstrate robustness of our main results.

Second, we show that our main results don't change much if we extend the model by introducing an additive quadratic non-pecuniary externality in the spirit of Berliant et al. (2002). Third, we discuss why our results on price dispersion may fail to hold if the transport cost schedule is linear.

5.1 Non-additive preferences

Until now, we have been assuming that preferences are additive. We now relax this assumption by introducing a cross-term into the utility function, like in Ottaviano et al. (2002):

$$U(z;\mathbf{q}) \equiv z + \alpha \int_X q(x,y)m(y)\mathrm{d}y - \frac{\beta}{2}\int_X [q(x,y)]^2 m(y)\mathrm{d}y - \frac{\gamma}{2}\left[\int_X q(x,y)m(y)dy\right]^2, \tag{39}$$

where $\gamma \in (0, \beta)$ captures substitutability across varieties of the differentiated good, hence the degree of competitive toughness in the market.

The inverse demand of an x-consumer for a variety produced by a y-firm is given by

$$p(y) = \alpha - \beta q(x,y) - \gamma \mathbb{Q}(x) - t(x-y)^2, \tag{40}$$

where $\mathbb{Q}(x) \equiv \int_X q(x,y)m(y)dy$ is the *consumption index* of an x-consumer, while $\sigma(x)$ is defined by (17) and captures the mean-squared distance from consumer's location x to the whole population of firms.

Multiplying both parts of (40) by $m(y)$ and integrating with respect to y across X, we obtain

$$P = M\alpha - (\beta + \gamma M)\mathbb{Q}(x) - Mt\sigma(x), \tag{41}$$

where P is the *price index* defined by

$$P \equiv \int\limits_X p(y)m(y)dy, \tag{42}$$

while $\sigma(x)$ stands for the mean-squared distance from an x-consumer to the whole mass of firms given by (17).

Using (41), we obtain the following expression for $\mathbb{Q}(x)$:

$$\mathbb{Q}(x) = \frac{M\alpha}{\beta + \gamma M} - \frac{P}{\beta + \gamma M} - \frac{Mt}{\beta + \gamma M}\sigma(x), \tag{43}$$

The first term in (43), $M\alpha/(\beta + \gamma M)$, shows that the x-consumer's aggregate willingness to pay increases with M. The second term, $-P/(\beta + \gamma M)$, captures the negative effect of an increase in the price index on an x-consumer's total consumption. Finally, the third term, $-Mt\sigma(x)/(\beta + \gamma M)$, keeps track of the impact of consumer's location x on the volume of consumption.

Solving (40) with respect to $q(x,y)$ and using (43), we find that the individual demand of an x-consumer for a variety produced by a y-firm is given by

$$q(x,y) = \frac{1}{\beta}\left[\frac{\alpha\beta}{\beta + \gamma M} + \frac{\gamma M}{\beta + \gamma M}\left(t\sigma(x) + \frac{P}{M}\right) - p(y) - t(x-y)^2\right]. \tag{44}$$

The aggregate demand faced by a y-firm is then given by

$$Q(y) = \frac{N}{\beta}\left[\frac{\alpha\beta}{\beta + \gamma M} + \frac{\gamma M}{\beta + \gamma M}\left(t\theta + \frac{P}{M}\right) - p(y) - t\delta(y)\right], \tag{45}$$

where $\delta(y)$ is the mean-squared distance between firm's location y and the population of consumers defined by (10), while θ is the overall mean-squared distance between consumers and firms:[12]

$$\theta \equiv \frac{1}{MN}\int\limits_X\int\limits_X (x-y)^2 n(x)m(y)\mathrm{d}x\mathrm{d}y. \tag{46}$$

[12]Using symmetry of both densities, it can be shown that $\theta = \delta_0 + \sigma_0$.

The presence of the price aggregate P in the right-hand side of (45) indicates that the market of the differentiated good is *no longer a collection of monopolists*, but that the market structure is now *truly monopolistic competition*. Indeed, the market demands faced by firms now depend directly on an aggregate of the choices of other players.[13] Because there is a continuum of firms, each firm is negligible to the market. Hence, firms lack the ability to strategically manipulate the value of the market aggregate P, which they treat parametrically.

As shown in the Appendix, the profit-maximizing price set by a y-firm is given by:

$$p^*(y) = c + \left[\frac{(\alpha - c)\beta}{2\beta + \gamma M} + \frac{\gamma M}{2\beta + \gamma M} \cdot \frac{\theta t}{2} - \frac{t}{2}\delta(y)\right]. \tag{47}$$

The bracketed term in (47) is the y-firm's markup. The first term in brackets, $(\alpha - c)\beta/(2\beta + \gamma M)$, captures the non-spatial component of the markup, which decreases in both the number M of firms and the degree γ of substitutability across varieties, reflecting the standard *competition effect*. The second term, $[\gamma M/(4\beta + 2\gamma M)] \cdot \theta t$, represents the *global shopping cost effect*, which increases with the transport cost but is independent of firm's location y. This effect is in line with the common wisdom of spatial competition theory: local monopoly power increases over the space when the shopping costs increase. Finally, the third term, $t\delta(y)/2$, captures the firm's *location effect*: moving further away from the center to a location with a higher $\delta(y)$ leads to the dilution of a firm's monopoly power.

It is legitimate to ask how markups change in response to (i) an increase in toughness of competition γ, and (ii) a reduction in transport cost t. The answer is given by the following result.

Proposition 6. *Assume that preferences are given by* (39). *Then*:

(i) *the markups of all firms decrease with γ if and only if the following inequality holds:*

$$\frac{\alpha - c}{t} > \theta; \tag{48}$$

(ii) *the markup of a y-firm increases in response to a reduction in t if and only if the following condition is satisfied:*

$$\delta(y) > \frac{\gamma M}{2\beta + \gamma M}\theta. \tag{49}$$

Proof. See the Appendix.□

Condition (48) has the same flavor as (6): either the willingness to pay is sufficiently high or transport cost is not too high. The intuition behind the condition (49) is as follows. When $\gamma > 0$, infrastructural improvements are always beneficial for peripheral firms (for which the $\delta(y)$s are high), but may be detrimental for more centrally located firms that face tougher competition. In other words, *higher substitutability across varieties damps the role of locational advantage*.

[13] See Anderson et al. (2015) for a discussion of the relationship between monopolistic competition and aggregative games.

How do spatial equilibria look like when $\gamma > 0$? In the same vein as in Section 2, it can be shown using (44) and (47) that the bid-rent functions $\Psi(x, U^*)$ and $\Phi(y, \pi^*)$ are still symmetric polynomials of degree 4 in location. Therefore, we again end up with *only segregated spatial equilibria*, either monocentric or duocentric.

5.2 Pecuniary vs non-pecuniary externalities

We now consider another extension of our model, in which preferences are again additive ($\gamma = 0$), but productivities of firms *now depend on their locations*. Along the lines of Berliant et al. (2002), we assume that the marginal cost $c(y)$ of a y-firm is now given by

$$c(y) = c_0 + \tau y^2 + \eta \sigma_0, \tag{50}$$

where $\eta > 0$ is a penalty on overall dispersion of firms, $\tau > 0$ stands for a penalty on the squared distance between the firm's location y and the mean site $x = 0$, while $c_0 > 0$ is a hypothetical value of the marginal cost when the whole mass of firms locates in the center.

When technological spillovers are specified by (50)), it is readily verified that the expression (15) for y-firm's profit-maximizing output remain valid, except that c must be replaced by $c(y)$. Furthermore, using (11) and the identity $N\delta_0 + M\sigma_0 = (M+N)^3/12$, we obtain after simplifications:

$$Q^*(y) = Q_0 - \lambda y^2 - \varepsilon \sigma_0, \tag{51}$$

where

$$Q_0 \equiv \frac{1}{2\beta}\left[N(\alpha - c_0) - t\frac{(M+N)^3}{12}\right],$$

$$\lambda \equiv \frac{N}{2\beta}(t+\tau), \qquad \varepsilon \equiv \frac{\eta N - tM}{2\beta}.$$

The way how y-firm's output (51) varies with the firm's individual location and the whole locational pattern is qualitatively similar to Berliant et al. (2002). There are two substantial differences. First, Berliant et al. (2002) use the square of the mean absolute distance from the center as a measure of overall dispersion, while we use the mean squared distance σ_0 since it's more compatible with our framework. Second, the dispersion penalty ε in (51) may be either positive (in which case it is truly a penalty) or negative (in which case it is rather a benefit). More precisely, we have $\varepsilon > 0$ if and only if $M < \eta N/t$, i.e. when firms are few. This reflects the two opposing effects: there are both losses from locational dispersion (due to lower spillovers) and gains from dispersion (because it reduces competition, as already discussed in sub-section 2.3). The former (latter) effect dominates when there are few (many) firms in the market. We refer the reader to Ballester et al. (2006) for a discussion why "global" spillovers can be negative.

It is also worth observing that the profit-maximizing price

$$p^*(y) = \frac{1}{2}\left[\alpha + c_0 - t\frac{(M+N)^3}{12N}\right] + \left(\eta + t\frac{M}{N}\right)\sigma_0 - \frac{t-\tau}{2}y^2 \tag{52}$$

is not always higher for more centrally located firms. This is only true when the transport cost per unit of distance exceeds the distance penalty ($\tau < t$). When the opposite occurs ($\tau > t$), prices increase toward the city outskirts. Finally, in the knife-edge case $t = \tau$ all firms charge the same price and price dispersion vanishes. This is because two opposing effects are at work: firms located towards the city center have reasons to price higher because of better market access, but they also have reasons to price lower because they are more technologically efficient.

Do technological spillovers specified by (50) generate any new equilibrium urban patterns compared to what we have seen in Section 2? The answer to this question is negative. To see this, observe that the profit-maximizing price (52) varies quadratically with the firm's location. Using this, one can show that the bid-rent functions $\Psi(y,U^*)$ and $\Phi(y,\pi^*)$ must be polynomials of degree 4, whence they cannot have more than four intersection points. As a consequence, *only segregated spatial equilibria* (either monocentric or duocentric) can emerge. [14]

5.3 Price dispersion and linear transport costs

Finally, we discuss the implications of assuming a *linear* shopping cost $t|x-y|$ while keeping quadratic preferences (3). In this case, it is readily verified that, given the density $n(\cdot)$ of consumers' distribution across the city, firms' profit-maximizing prices are given by

$$p^*(y) = \frac{1}{2}\left[\alpha + c - t\Delta(y)\right]. \tag{53}$$

where $\Delta(y)$ is the mean distance from firm's location y to the whole population of consumers:

$$\Delta(y) \equiv \frac{1}{N}\int_X |x-y|n(x)\mathrm{d}x.$$

It is well-known that the median of the distribution given by $n(\cdot)$ is a minimizer of $\Delta(y)$, and vice versa (Stroock, 2011). However, unlike the mean, a median need not be unique. This implies the following result.

Proposition 7. *Assume that the spatial equilibrium is segregated and involves a central shopping area. Then, all firms in this shopping area charge the same price.*

Proof. See the Appendix □

Proposition 7 shows that spatial price dispersion does not tend to show up in equilibrium under linear transport costs, even when preferences are no longer entropy-type, whence markups need not be a priori independent of firms locations.

[14] This result is, of course, driven by the assumption that spillovers enter the cost function as an additive term, τ, which varies quadratically with the distance.

We finish this section by discussing the essential differences between linear transport cost, under which the full price is $p(y)+t|x-y|$, and quadratic transport cost, under which the full price is $p(y)+t(x-y)^2$. One key difference is that the disutility of covering short distances (staying at home being an outside option) is much higher for consumers who face a linear transport cost schedule than for those who face a quadratic one. This is because there is a kink at $x=y$ in the linear case. In other words, under quadratic transport cost the compensating price differential is negligibly small compared to the distance covered, while both are of the same order of magnitude under linear costs. This aspect of the transport cost impact on location patterns of firms has been studied in theoretical work on spatial competition stressing the role of "convexity vs concavity" of transport cost schedules (De Frutos et al., 1999, 2002). One novel feature of our approach is that it allows to single out a very different (and, to the best of our knowledge, a new) channel of how the shape of transport cost function affects the qualitative features of spatial equilibria. Namely, what is key for the urban pattern in our model is not whether the transport cost schedule is concave or convex, but whether it is "sufficiently regular". This explains the central role of the assumptions (**T3**) and (**U3**) in most of the analysis.

6 Concluding remarks

We have developed a model that sheds light on the role of pecuniary externalities in endogenous city structure formation. We have shown that spatial distribution of firms may be fully driven by the demand-side factors, with no production externalities at work. Finally, we have demonstrated that imperfectly informed consumers and search costs are not inevitable ingredients for spatial price dispersion.

Our results imply that equilibrium patterns depend crucially on the assumptions imposed on consumers' variety-loving behavior. These considerations, emphasized by the growing literature on variable markups, suggest a new agenda for empirical urban economists, for little has been done by now to study the impact of these factors on urban structure.

We believe that our model is flexible enough to study industrial specialization of cities. Indeed, the approach proposed in this paper reveals the fundamental role of modeling assumptions for the type of resulting spatial equilibria (see the discussion at the end of Section 4). Therefore, we find it potentially interesting to blend our setting with that developed by Helsley and Strange (2014) in order to obtain further clear-cut theoretical results regarding the conditions of clustering. Another possible direction of further research is to study whether the array of city structures arising in equilibrium becomes richer if we allow for heterogeneities across consumers not only in locations, but also in tastes (like in Tarasov, 2014; Osharin et al., 2014) or incomes (Gaigné et al., 2017). We leave these tasks, as well as studying welfare implications of our results, for future research.

A last remark is in order. According to Kantor et al. (2014), "mixed commercial and residential land use is ubiquitous". We have shown, however, that mixed land-use patterns are neither easy to generate (Proposition 4), nor robust to small perturbations in the primitives of the model (Proposition 5) in cities driven by solely market interactions. This result, although counterfactual, is important for a better understanding of what can (and what cannot) be accomplished in urban economics by means of spatial models which are *free from the premise of Marshallian externalities*. One more direction for future research would be either introducing more sophisticated transportation behavior of individuals, like in an early work by Papageorgiou and Thisse (1985), or modeling more thoroughly the transport system (e.g. introducing potential congestion, like in Kantor et al., 2014). Whether accounting for these features of the real-world cities yields more realistic predictions of urban landscape without relying on Marshallian externalities is left for future research.

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Appendix

Derivation of (6)

We denote the support of a function $\phi : X \to \mathbb{R}$ by $\operatorname{supp}\phi(\cdot)$. Using (7), we find that

$$\alpha - t(x-y)^2 - p(y) > 0 \tag{54}$$

is necessary and sufficient to hold for all $x \in \operatorname{supp} n(\cdot)$ and for all $y \in \operatorname{supp} m(\cdot)$, for the solution of each consumer's program to be interior.

Using (14), we may restate (54) as follows:

$$\frac{\alpha - c}{t} > 2(x-y)^2 - \delta(y) \text{ for all } x,\, y \in \left[-\frac{M+N}{2}, \frac{M+N}{2}\right]. \tag{55}$$

Observe that, because $x,\, y \in [-\frac{M+N}{2}, \frac{M+N}{2}]$, we have (i) $(x-y)^2 < (M+N)^2$, and (ii) $\delta(y) \geq 0$, regardless of a particular shape of the population density $n(x)$. Hence, if (6) holds, then (55) holds. In other words, (6) is sufficient for each consumer's program to possess an interior solution. □

Proof of Proposition 1

Part (i) follows directly from (14) and (11). To prove part (ii), observe that, as implied by (14) and (11), the difference in prices between two firms located, respectively, at y_1 and y_2, such that $0 < y_1 < y_2$, equals

$$p^*(y_1) - p^*(y_2) = \frac{t}{2}\left[\delta(y_2) - \delta(y_1)\right] = \frac{t}{2}\left(y_2^2 - y_1^2\right), \tag{56}$$

which clearly decreases with t. □

Derivation of (23) and (24)

We start with the derivation of (24). Combining (14) with (7) implies that the quantity $q^*(x,y)$ of a variety supplied at y demanded by an x-consumer is given by

$$q^*(x,y) = \frac{1}{\beta}\left[\frac{\alpha - c}{2} - t(x-y)^2 + \frac{t}{2}\delta(y)\right].$$

Plugging $q^*(x,y)$ into (8), solving (8) for $R(x)$ and using 21, we obtain after rearranging:

$$\Psi(x, U^*) = \frac{t^2}{2\beta}\int_X \left[\frac{\alpha - c}{2t} + \frac{1}{2}\delta(y) - (x-y)^2\right]^2 m(y)\mathrm{d}y - U^* + Y.$$

Expand the integrand as follows:

$$\left[\frac{\alpha - c}{2t} + \frac{1}{2}\delta(y) - (x-y)^2\right]^2 = \left[\frac{\alpha - c}{2t} + \frac{1}{2}\delta(y)\right]^2 - 2(x-y)^2\left[\frac{\alpha - c}{2t} + \frac{1}{2}\delta(y)\right] + (x-y)^4,$$

and plug it back into $\Psi(x, U^*)$. Using (11) and setting

$$K \equiv \frac{1}{4M} \int\limits_X \left(\frac{\alpha - c}{t} + \delta_0 - y^2 \right)^2 m(y) \mathrm{d}y,$$

we obtain (24) after simplifications.

The expression (23) for the firm's bid rent function $\Phi(y, \pi^*)$ is derived along the same lines.□

Proof of Proposition 4

We start with the following technical result.

Lemma 1. *Assume that the sub-utility $u(q)$ satisfies* **(U1)** – **(U3)***, while the transport cost $T(x, y)$ satisfies* **(T1)** – **(T3)***. Then, the bid rent functions $\Psi(x, U^*)$ and $\Phi(x, \pi^*)$ are real-analytic in x.*

Lemma 2. *Assume that a pooled equilibrium exists. Then, it must be that $M = N$.*

As the proofs of Lemma 1 and Lemma 2 are long and tedious, we relegate it to the Online Appendix. Using the Lemma 1, Proposition 4 can be proven by contradiction. Assume that a mixed spatial equilibrium exists. A mixed district $[a, b]$ emerges in equilibrium if and only if $\Psi(x, U^*) - \Phi(x, \pi^*) = 0$ for all $x \in [a, b]$. By the Lemma 1, $\Psi(x, U^*)$ and $\Phi(x, \pi^*)$ are both real-analytic *in x over X*, hence $\Psi(x, U^*) - \Phi(x, \pi^*)$ is also real-analytic over X. It is known that either the zeros of a real-analytic function are isolated, or this function completely vanishes (Courant and Fritz, 2012, p. 545). Hence, it must be that $\Psi(x, U^*) = \Phi(x, \pi^*)$ over the whole city. In other words, any mixed spatial equilibrium is necessarily a pooled equilibrium. By Lemma 2, a pooled equilibrium can only emerge when $N = M$. Otherwise, only segregated equilibria can exist. This completes the proof of Proposition 4. □

Proof of Proposition 5

Given a small positive number ε, replace $T(x, y) = t|x - y|$ by some other transport cost function $T_\varepsilon(x, y)$, which is polynomial (whence real-analytic) in x and y over $X \times X$, and is ε-close to $T(x, y)$ in the sense that $|T(x, y) - T_\varepsilon(x, y)| \leq \varepsilon$ for all $x, y \in X$. By Stone-Weierstrass theorem (Rudin, 1991, Ch.5), such transport cost $T_\varepsilon(x, y)$ exists for any $\varepsilon > 0$. As implied by Proposition 4, only segregated equilibria exist under the sub-utility (38) and transport costs $T_\varepsilon(x, y)$. However, a mixed spatial equilibrium emerges suddenly when $\varepsilon = 0$, showing *discontinuous behavior* of urban configuration in response to small perturbations of the transport cost pattern. Furthermore, along thse lines we can uniformly approximate $u(q)$ by a polynomial (whence real-analytic function) $u_\varepsilon(q)$ over any compact subinterval of $\left(0, (u')^{-1}(c)\right)$. Then, non-existence of a mixed spatial equilibrium under transport cost $T_\varepsilon(x, y)$ and preferences (30) with a sub-utility $u_\varepsilon(q)$ ensues from Proposition 4. □

Derivation of (47)

The profit-maximizing price $\hat{p}(y,P)$ of a y-firm *conditional on the price index P* can be expressed as follows:

$$\hat{p}(y,P) = \frac{1}{2}\left[\frac{\alpha\beta}{\beta+\gamma M} + c + \frac{\gamma M}{\beta+\gamma M}\left(t\theta + \frac{P}{M}\right) - t\delta(y)\right]. \tag{57}$$

Integrating (57) with respect to y across X and using (42), we come to the following fixed-point condition for the average price-index P/M:

$$\frac{P}{M} = \frac{1}{2}\left[c + \frac{\beta(\alpha - t\theta)}{\beta+\gamma M} + \frac{\gamma M}{\beta+\gamma M}\frac{P}{M}\right],$$

solving which for P/M yields

$$\frac{P}{M} = c + \frac{\beta(\alpha - c - t\theta)}{2\beta+\gamma M}. \tag{58}$$

Plugging (58) into (57), we obtain (47). □

Proof of Proposition 6

To prove part (i), we differentiate the right-hand side of (47) with respect to γ, which yields after simplifications:

$$\frac{\partial p^*(y)}{\partial\gamma} = -\frac{\beta t M}{(2\beta+\gamma M)^2}\cdot\left(\frac{\alpha - c}{t} - \theta\right).$$

Hence, we have $\frac{\partial p^*(y)}{\partial\gamma} < 0$ if and only if (48) holds. This proves part (i).

To prove part (ii), we take the derivative of the right-hand side of (47) with respect to t:

$$\frac{\partial p^*(y)}{\partial t} = \frac{1}{2}\cdot\left[\frac{\gamma M}{2\beta+\gamma M}\theta - \delta(y)\right].$$

Clearly, we have $\frac{\partial p^*(y)}{\partial t} > 0$ if and only if (49) holds. This verifies part (ii) and completes the proof. □

Proof of Proposition 7

Assume that the city structure is monocentric, i.e. firms take the whole amount of land across $[-M/2, M/2]$. Then, each of the two intervals $[-(M+N)/2, -M/2]$ and $[M/2, (M+N)/2]$ supports the mass $N/2$ of consumers. Hence, any point $y \in [-M/2, M/2]$ is a median, which implies that $\Delta(y)$ is constant over $[-M/2, M/2]$. Combining this observation with the expression (53) for the profit-maximizing price, we conclude that under a monocentric city structure all firms will charge the same price, regardless of their locations.□

Online Appendix

Proof of the Lemma 1

If $u(q)$ is real-analytic over $\left(0,(u')^{-1}(c)\right)$, then so is $u'(q)$. Hence, by the inverse function theorem for real-analytic functions, $D(p) \equiv (u')^{-1}(p)$ is also real-analytic in p for all $p \in (c, u'(0))$. Because $T(x,y)$ is real-analytic in y, $D(p * T(x,y))$ is real-analytic in both y and p for any $x \in X$, regardless of whether $*$ denotes $+$ or $\cdot$.

Define the operating profit $\Pi(y,p)$ of a y-firm as

$$\Pi(y,p) \equiv (p-c)\int_X D[p * T(x,y)]n(x)\mathrm{d}x. \tag{59}$$

Since compositions, sums, and products of real-analytic functions are real-analytic, we conclude that $D[p * T(x,y)]$ is real-analytic in its three arguments (x, y, p), regardless of whether $*$ stands for addition or multiplication. We now show that the operating profit $\Pi(y,p)$ is also real-analytic in y and p. It suffices to prove that $\int_X D[p * T(x,y)]n(x)\mathrm{d}x$ is real-analytic in p and y. We begin with representing $D[p * T(x,y)]$ in a neighborhood of an arbitrary point $(x_0, y_0, p_0) \in (c, u'(0)) \times X \times X$ as a power series:

$$D[p * T(x,y)] = \sum_{l,r,s=0}^{\infty} a_{lrs}(x-x_0)^l (y-y_0)^r (p-p_0)^s. \tag{60}$$

What we need to show is that the identity

$$\int_X D[p * T(x,y)]n(x)\mathrm{d}x = \sum_{r,s=0}^{\infty} A_{rs}(y-y_0)^r (p-p_0)^s \tag{61}$$

holds over some open neighborhood $\mathscr{U}_0 \subseteq \mathbb{R}^2$ of (y_0, p_0), where A_{mn} are some reals.

Take $\varepsilon > 0$ so that $[p_0 - \varepsilon,\, p_0 + \varepsilon] \subset (c, u'(0))$, and relate to any point $(\xi, \gamma, \rho) \in X \times X \times [p_0 - \varepsilon,\, p_0 + \varepsilon]$ an open "brick" $\mathscr{U}(\xi, \gamma, \rho)$ with its sides being parallel to the axes, so that the power series representing $D[p * T(x,y)]$ converges uniformly in x over $\mathscr{U}(\xi, \gamma, \rho)$ (Courant and Fritz, 2012, p. 541). By doing so, we obtain an open cover $\mathfrak{I}_1$ of $X \times X \times [p_0 - \varepsilon,\, p_0 + \varepsilon]$. Because $X \equiv [-(M+N)/2,\, (M+N)/2]$, the set $X \times X \times [p_0 - \varepsilon,\, p_0 + \varepsilon]$ is a compact subset of $\mathbb{R}^3$. Hence, we can extract from $\mathfrak{I}_1$ a finite open subcover $\mathfrak{I}_2$. Furthermore, we can cut $X \times X \times [p_0 - \varepsilon,\, p_0 + \varepsilon]$ into small bricks by planes containing the edges of all elements of $\mathfrak{I}_2$. Let $\{\xi_1, \ldots, \xi_I\}$, $\{\gamma_1, ..., \gamma_J\}$ and $\{\rho_1, \ldots, \rho_K\}$ be the complete lists (in ascending order each) of points where these planes intersect, respectively, the x-axis, the y-axis and the p-axis. Here I, J and K are positive integers (maybe very large, but finite). Choose $j_0 \in \{1, ..., J-1\}$ and $k_0 \in \{1, \ldots, K-1\}$ so that $y_0 \in (\gamma_{j_0}, \gamma_{j_0+1})$ and $p_0 \in (\rho_{k_0}, \rho_{k_0+1})$. Set $\xi_0 \equiv -(M+N)/2$ and $\xi_{I+1} \equiv (M+N)/2$, and choose $x_j \in (\xi_{j-1}, \xi_j)$. Then, using (60), we can represent $\int_X D[p * T(x,y)]n(x)\mathrm{d}x$ in the neighborhood $(\gamma_{j_0}, \gamma_{j_0+1}) \times (\rho_{k_0}, \rho_{k_0+1})$ of (y_0, p_0) as follows:

$$\int_X D[p*T(x,y)]n(x)\mathrm{d}x = \sum_{i=1}^{I+1} \int_{\xi_{j-1}}^{\xi_j} \sum_{l,m,r=0}^{\infty} a_{lmn}^{(j)} (x-x_j)^l (y-y_0)^m (p-p_0)^r n(x)\mathrm{d}x, \tag{62}$$

where $a_{lmn}^{(j)}$ are the coefficients of the power series representing $D[p*T(x,y)]$ in the neighborhood $(\xi_{j-1},\xi_j)\times(\gamma_{j_0},\gamma_{j_0+1})\times(\rho_{k_0},\rho_{k_0+1})$ of (x_j, y_0, p_0). Furthermore, due to the way we have chosen $\xi_1,\ldots,\xi_I$, the series within each integral converges uniformly with respect to x. Hence, we can change the order of integration (see, e.g., Kolmogorov and Fomin, 2012, p. 303) and write:

$$\int_X D[p*T(x,y)]n(x)\mathrm{d}x = \sum_{m,n=0}^{\infty} \left[\sum_{l=0}^{\infty} \sum_{i=1}^{I+1} \int_{\xi_{i-1}}^{\xi_i} a_{lmn}^{(j)} (x-x_j)^l n(x)\mathrm{d}x \right] \cdot (y-y_0)^m (p-p_0)^n ,$$

for all $(y,p)\in(\gamma_{j_0},\gamma_{j_0+1})\times(\rho_{k_0},\rho_{k_0+1})$. Setting

$$A_{mn} \equiv \sum_{l=0}^{\infty} \sum_{i=1}^{I+1} \int_{\xi_{i-1}}^{\xi_i} a_{lmn}^{(j)} (x-x_j)^l n(x)\mathrm{d}x, \qquad \mathscr{U}_0 \equiv (\gamma_{j_0},\gamma_{j_0+1})\times(\rho_{k_0},\rho_{k_0+1}),$$

we obtain (61). Hence, $\int_X D[p*T(x,y)]n(x)\mathrm{d}x$ is real-analytic in p and y, and so is $\Pi(y,p)$. Moreover, the same is true about $\frac{\partial \Pi}{\partial p}(y,p)$, as the derivatives of real-analytic functions are also real-analytic,. The profit-maximizing price $p(y)$ charged by firm y is an implicit function of y described by the y-firm's FOC:

$$\frac{\partial \Pi}{\partial p}(y,p) = 0. \tag{63}$$

Because $\frac{\partial \Pi}{\partial p}(y,p)$ is real-analytic, the implicit function theorem for real-analytic functions implies that $p(y)$, which is defined as the solution to the firm's FOC (63), is real-analytic in y. Because $\Pi(y,p)$ is real-analytic in both p and y, and compositions of real-analytic functions are real-analytic, $\Pi(y,p(y))$ is also real-analytic in y. Since

$$\Phi(y,\pi^*) = \Pi(y,p(y)) - f - \pi^*,$$

we conclude that $\Phi(y,\pi^*)$ is real-analytic in y.

We now come to proving real-analyticity of the consumer's bid rent function $\Psi(x,U^*)$. The indirect utility of an x-consumer is given by

$$V(x) = \int_X u\left[D(p(y)*T(x,y))\right] m(y)\mathrm{d}y -$$

$$\int_X [p(y)*T(x,y)]\cdot D[p(y)*T(x,y)]m(y)dy - R(x) + Y$$

Combining this with (21), we get

$$\Psi(x,U^*) = \int_X u\left[D(p(y) * T(x,y))\right] m(y)\mathrm{d}y -$$

$$\int_X [p(y) * T(x,y)] \cdot D[p(y) * T(x,y)] m(y) dy - U^* + Y.$$

The integrands in this expression are real-analytic in x across X, which is implied by real analyticity of $u(\cdot)$ and $D(\cdot)$, as well as by real-analyticity of $T(\cdot,y)$ for any y. Using the same trick as in proving that $\Pi(p,y)$ is real-analytic, one can show that $\Psi(x,U^*)$ is also real-analytic in x. This completes the proof of Lemma 1. $\square$

Proof of Lemma 2

If a pooled equilibrium exists, it must be that $\Psi(x,U^*) = \Phi(x,\pi^*)$ for all $x \in X$. By Lemma 1, both $\Psi(x,U^*)$ and $\Phi(x,\Pi^*)$ are real-analytic in x. Combining equations (34) and (59) with the y-firm's FOC and using the land balance condition, $m(x)+n(x) = 1$, it can be shown along the same lines as in the proof of Lemma 1 that there exist non-zero coefficients $\{\Gamma_k\}_{k=0}^{\infty}$, such that

$$\sum_{k=0}^{\infty} \Gamma_k \int_X (2n(x)-1)\,(x-y)^{2k}\,\mathrm{d}x = 0, \quad \text{for all } y \in X.$$

This, in turn, is only possible when the function $2n(x)-1$ is identically zero over the whole urban space. In other words, it must be that $n(x) = 1/2$ for all $x \in [-(M+N)/2,\,(M+N)/2]$. Integrating both sides of this identity with respect to x over $[-(M+N)/2,\,(M+N)/2]$, we find that $N = (M+N)/2$, or, equivalently, that $N = M$. This completes the proof of Lemma 2. $\square$

Proof of Proposition 2

Using (12) and the fact that spatial equilibria are always segregated (Proposition 4), it can be shown that the following inequalities hold:

$$\frac{N^2}{12} \le \delta_0 \le \frac{N^2}{12} + \frac{M}{4}(M+N). \tag{64}$$

Finally, set $\xi \equiv x^2 \in [0,\,(M+N)^2/4]$. Combining this with (24) and (23) implies that the *bid-rent differential* is given (up to a positive affine transformation) by:

$$\Delta(\xi) \equiv 2M\left(\xi + \frac{5\sigma_0 - \delta_0}{2} - \frac{\alpha - c}{2t}\right)^2 - N\left(\xi + \delta_0 - \frac{\alpha - c}{t}\right)^2. \tag{65}$$

Proof of part (i). Consider first the case when the central area accommodates firms, while peripheral areas are residential (Fig. 2a). Such an equilibrium involves one business district, $[-\frac{M}{2},\,\frac{M}{2}]$, and two symmetric residential districts, $[-\frac{M+N}{2},\,-\frac{M}{2})$ and $(\frac{M}{2},\,\frac{M+N}{2}]$.

In such an equilibrium, the population density $n(x)$ is given by

$$n(x) = \begin{cases} 1, & \text{if } x \in [-\frac{M+N}{2}, -\frac{M}{2}) \cup (\frac{M}{2}, \frac{M+N}{2}], \\ 0, & \text{otherwise,} \end{cases} \tag{66}$$

while the density of firms takes the form:

$$m(y) = \begin{cases} 1, & \text{if } y \in [-\frac{M}{2}, \frac{M}{2}], \\ 0, & \text{otherwise.} \end{cases} \tag{67}$$

Combining (66) – (67) with (10), we find that the mean squared distance is given by

$$\delta(y) = y^2 + \frac{1}{12}\left(3M^2 + 3MN + N^2\right). \tag{68}$$

Given that a segregated equilibrium the consumer's program interior solution condition (6) can be considerably relaxed. Namely, each consumer purchases the whole range of available varieties if and only if the following inequality holds:

$$\frac{\alpha - c}{t} > M^2 + \frac{5}{4}MN + \frac{5}{12}N^2. \tag{69}$$

To show this, consider a spatial equilibrium in which the central area is a shopping district, while the outskirt areas are residential. In this case, we have $\operatorname{supp} n(\cdot) = [-\frac{M+N}{2}, -\frac{M}{2}] \cup [\frac{M}{2}, \frac{M+N}{2}]$ and $\operatorname{supp} m(\cdot) = [-\frac{M}{2}, \frac{M}{2}]$. Combining this with (68) and using symmetry, we find that (55) boils down to

$$\frac{\alpha - c}{t} > \lambda(x, y) - \frac{3M^2 + 3MN + N^2}{12}, \tag{70}$$

where $\lambda(x,y) \equiv 2x^2 - 4xy + y^2$. A necessary and sufficient condition for (70) to hold for all $x \in [\frac{M}{2}, \frac{M+N}{2}]$ and for all $y \in [-\frac{M}{2}, \frac{M}{2}]$ is

$$\frac{\alpha - c}{t} > \lambda(\hat{x}, \hat{y}) - \frac{3M^2 + 3MN + N^2}{12}, \tag{71}$$

where $(\hat{x}, \hat{y})$ is a global maximizer of $\lambda(x,y)$ over $[\frac{M}{2}, \frac{M+N}{2}] \times [-\frac{M}{2}, \frac{M}{2}]$. Because all admissible values of x are strictly positive, $\hat{y}$ has to be non-positive. Indeed, otherwise $\lambda(x, -\hat{y}) > \lambda(x, \hat{y})$ for any x, which is a contradiction. Moreover, for any given $x_0 \in [\frac{M}{2}, \frac{M+N}{2}]$, $\lambda(x_0, y)$ is a strictly decreasing function of y over $[-\frac{M}{2}, 0]$. Hence, we have $\hat{y} = -M/2$. As for $\hat{x}$, it is a maximizer for $\lambda(x, -M/2) = 2x^2 + 2Mx + M^2/4$ over $x \in [\frac{M}{2}, \frac{M+N}{2}]$. Because $\lambda(x, -M/2)$ is increasing in x, we have $\hat{x} = (M+N)/2$. Plugging $\hat{x}$ and $\hat{y}$ into (71) yields (69).

Plugging (68) into (23), we find that the firms' bid rent function takes the form:

$$\Phi(y, \pi^*) = \frac{Nt^2}{4\beta}\left[y^4 - 2\left(\frac{\alpha - c}{t} - \frac{3M^2 + 3MN + N^2}{12}\right)y^2 + \left(\frac{\alpha - c}{t} - \frac{3M^2 + 3MN + N^2}{12}\right)^2\right] - \pi^* - f. \tag{72}$$

The consumers' bid rent function may be expressed as follows:

$$\Psi(x,U^*) = \frac{Mt^2}{2\beta}\left[x^4 - \left(\frac{\alpha-c}{t} - \frac{2M^2-3MN-N^2}{12}\right)x^2\right] + k - U^* + Y, \tag{73}$$

where k depends on the parameters of the model, but not on x. To show this, we integrate the right-hand side of (17) to obtain

$$\sigma(x) = x^2 + \frac{M^2}{12}. \tag{74}$$

Furthermore, double integration in (46) yields

$$\theta = \frac{M^2}{3} + \frac{MN}{4} + \frac{N^2}{12}. \tag{75}$$

Plugging (66) – (68) and (74) – (75) into (24) and setting

$$\begin{aligned} k \equiv \frac{Mt^2}{8\beta}&\left[\left(\frac{\alpha-c}{t}\right)^2 + \frac{(M+N)(2M+N)}{6}\left(\frac{\alpha-c}{t}\right)\right] \\ +\frac{Mt^2}{8\beta}&\cdot\frac{24M^4+60M^3N+65M^2N^2+30MN^3+5N^4}{720}, \end{aligned} \tag{76}$$

we obtain (73).

When does a segregated spatial equilibrium with a commercial area in the center exist? To derive the necessary and sufficient conditions for that, we set

$$\widetilde{\Phi}(y) \equiv \frac{Nt^2}{4\beta}\left[\frac{\alpha-c}{t} - \frac{1}{12}\left(3M^2+3MN+N^2\right) - y^2\right]^2, \tag{77}$$

$$\widetilde{\Psi}(x) \equiv \frac{Mt^2}{2\beta}\left[x^4 - \left(\frac{\alpha-c}{t} - \frac{2M^2-3MN-N^2}{12}\right)x^2\right] + k, \tag{78}$$

where k is given by (76), we may state the existence conditions for a spatial equilibrium with a commercial central area and residential suburbs as follows:

$$\widetilde{\Psi}\left(\frac{M+N}{2}\right) - U^* + Y = R_a, \tag{79}$$

$$\widetilde{\Phi}\left(\frac{M}{2}\right) - \pi^* - f = \widetilde{\Psi}\left(\frac{M}{2}\right) - U^* + Y, \tag{80}$$

$$\widetilde{\Phi}(x) - \pi^* - f > \widetilde{\Psi}(x) - U^* + Y \quad \text{for all } x : 0 < x < \frac{M}{2}, \tag{81}$$

$$\widetilde{\Phi}(x) - \pi^* - f < \widetilde{\Psi}(x) - U^* + Y \quad \text{for all } x : \frac{M}{2} < x < \frac{M+N}{2}. \tag{82}$$

As seen from (77) – (78), $\widetilde{\Phi}(x)$ and $\widetilde{\Psi}(x)$ are both quadratic functions of solely x^2. Hence, (81) – (82) may be restated as follows:

$$\widetilde{\Phi}(0) - \pi^* - f > \widetilde{\Psi}(0) - U^* + Y, \tag{83}$$

$$\widetilde{\Phi}\left(\frac{M+N}{2}\right) - \pi^* - f < \widetilde{\Psi}\left(\frac{M+N}{2}\right) - U^* + Y. \tag{84}$$

>From (79) and (80) we obtain:

$$U^* = \widetilde{\Psi}\left(\frac{M+N}{2}\right) + Y - R_a, \tag{85}$$

$$\pi^* = \widetilde{\Phi}\left(\frac{M}{2}\right) - \widetilde{\Psi}\left(\frac{M}{2}\right) + \widetilde{\Psi}\left(\frac{M+N}{2}\right) - R_a - f. \tag{86}$$

Using (85) – (86), we find that (83) – (84) boil down to:

$$\widetilde{\Phi}(0) - \widetilde{\Psi}(0) > \widetilde{\Phi}(\frac{M}{2}) - \widetilde{\Psi}(\frac{M}{2}), \tag{87}$$

$$\widetilde{\Phi}\left(\frac{M+N}{2}\right) - \widetilde{\Psi}\left(\frac{M+N}{2}\right) < \widetilde{\Phi}\left(\frac{M}{2}\right) - \widetilde{\Psi}\left(\frac{M}{2}\right). \tag{88}$$

Restating (87) and (88) as

$$\begin{gathered}
\widetilde{\Phi}(0) - \widetilde{\Phi}\left(\frac{M}{2}\right) > \widetilde{\Psi}(0) - \widetilde{\Psi}\left(\frac{M}{2}\right), \\
\widetilde{\Phi}\left(\frac{M}{2}\right) - \widetilde{\Phi}\left(\frac{M+N}{2}\right) > \widetilde{\Psi}\left(\frac{M}{2}\right) - \widetilde{\Psi}\left(\frac{M+N}{2}\right),
\end{gathered}$$

and using (77) – (78), we find that the above inequalities hold iff

$$\begin{aligned}
24(N-M)\frac{\alpha - c}{t} &> -10M^3 + 15M^2N + 8MN^2 + 2N^3, \\
24(N-M)\frac{\alpha - c}{t} &> -16M^3 + 6M^2N + 8MN^2 + 5N^3.
\end{aligned}$$

A crucial role is played by total numbers of consumers and firms. When $N > M$, we get the following existence conditions:

$$\begin{cases} \dfrac{\alpha - c}{t} > \dfrac{-10M^3 + 15M^2N + 8MN^2 + 2N^3}{24(N-M)}, \\ \dfrac{\alpha - c}{t} > \dfrac{-16M^3 + 6M^2N + 8MN^2 + 5N^3}{24(N-M)}. \end{cases} \tag{89}$$

When $N < M$, the system of existence conditions are as follows:

$$\begin{cases} \dfrac{\alpha - c}{t} < \dfrac{-10M^3 + 15M^2N + 8MN^2 + 2N^3}{24(N-M)}, \\ \dfrac{\alpha - c}{t} < \dfrac{-16M^3 + 6M^2N + 8MN^2 + 5N^3}{24(N-M)}. \end{cases} \tag{90}$$

To sum up, a segregated spatial equilibrium with a shopping district in the center exists iff

(A) $\mu < 1$, where $\mu \equiv M/N$;

(B) (69) and (89) hold. Using μ, we can restate (B) as follows:

$$\frac{24}{N^2}\frac{\alpha - c}{t} > \frac{1}{1-\mu}\max\{G_1(\mu), G_2(\mu), G_3(\mu)\}, \quad \text{for all } \mu \in (0,1),$$

where

$$G_1(\mu) \equiv \left(24\mu^2 + 30\mu + 10\right)(1-\mu), \qquad G_2(\mu) \equiv -10\mu^3 + 15\mu^2 + 8\mu + 2,$$

$$G_3(\mu) \equiv -16\mu^3 + 6\mu^2 + 8\mu + 5.$$

Figure A1 plots $G_1(\mu)$, $G_2(\mu)$, and $G_3(\mu)$

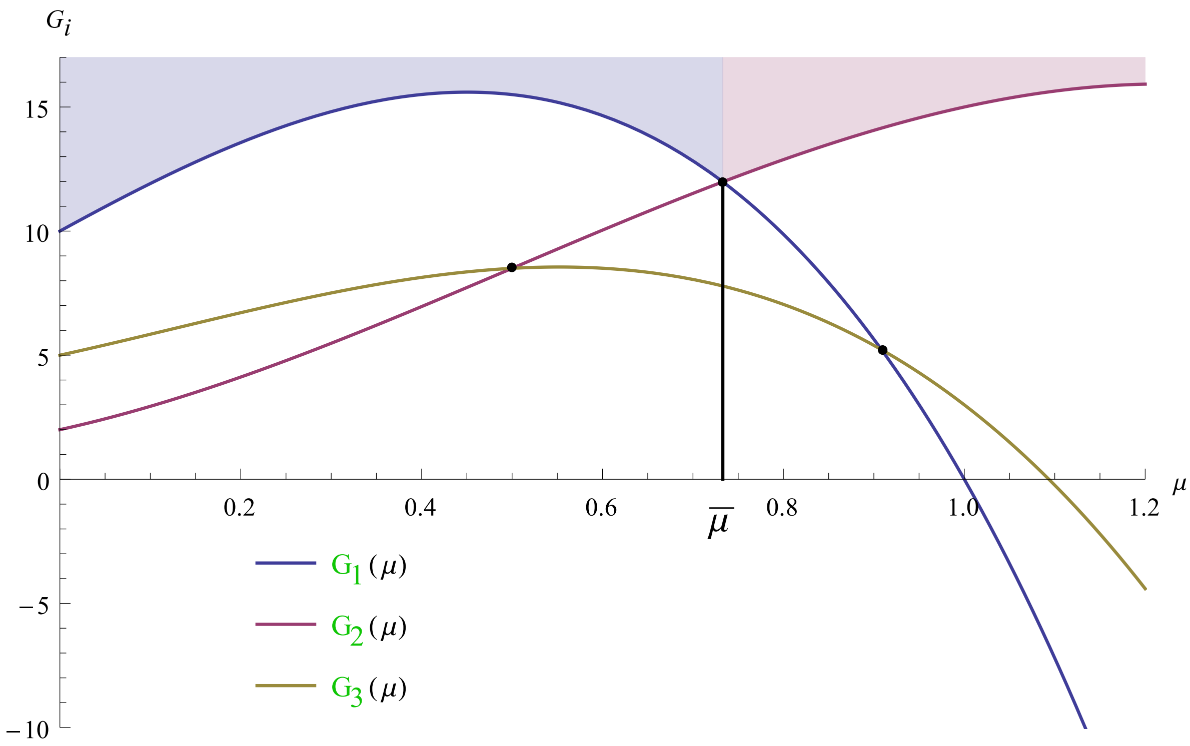

Figure A1. The plots of $G_1(\mu)$, $G_2(\mu)$, and $G_3(\mu)$: the case of $N > M$.

Solving the equation $G_1(\mu) = G_2(\mu)$ numerically yields a unique solution $\overline{\mu} \approx 0.732982$ over $(0,1)$. As seen from Figure A1, when $0 < \mu < \overline{\mu}$, we have

$$G_1(\mu) = \max\{G_1(\mu), G_2(\mu), G_3(\mu)\},$$

hence, (B) boils down to (69). However, Figure A1 also shows that, when $\overline{\mu} < \mu < 1$, (B), we have

$$G_2(\mu) = \max\{G_1(\mu), G_2(\mu), G_3(\mu)\}.$$

In this case, (B) amounts to (91).

When we have $N < M$, using $\mu \equiv M/N$, (69) and (90) deliver the following conditions on the existence of the equilibrium:

$$\frac{24}{N^2}\frac{\alpha - c}{t} > \frac{1}{1-\mu} G_1(\mu),$$

$$\frac{24}{N^2}\frac{\alpha - c}{t} < \frac{1}{1-\mu} \max\{G_2(\mu), G_3(\mu)\},$$

for all $\mu \geq 1$.

Figure A2 illustrates that there is no solution for the case when $\mu \geq 1$:

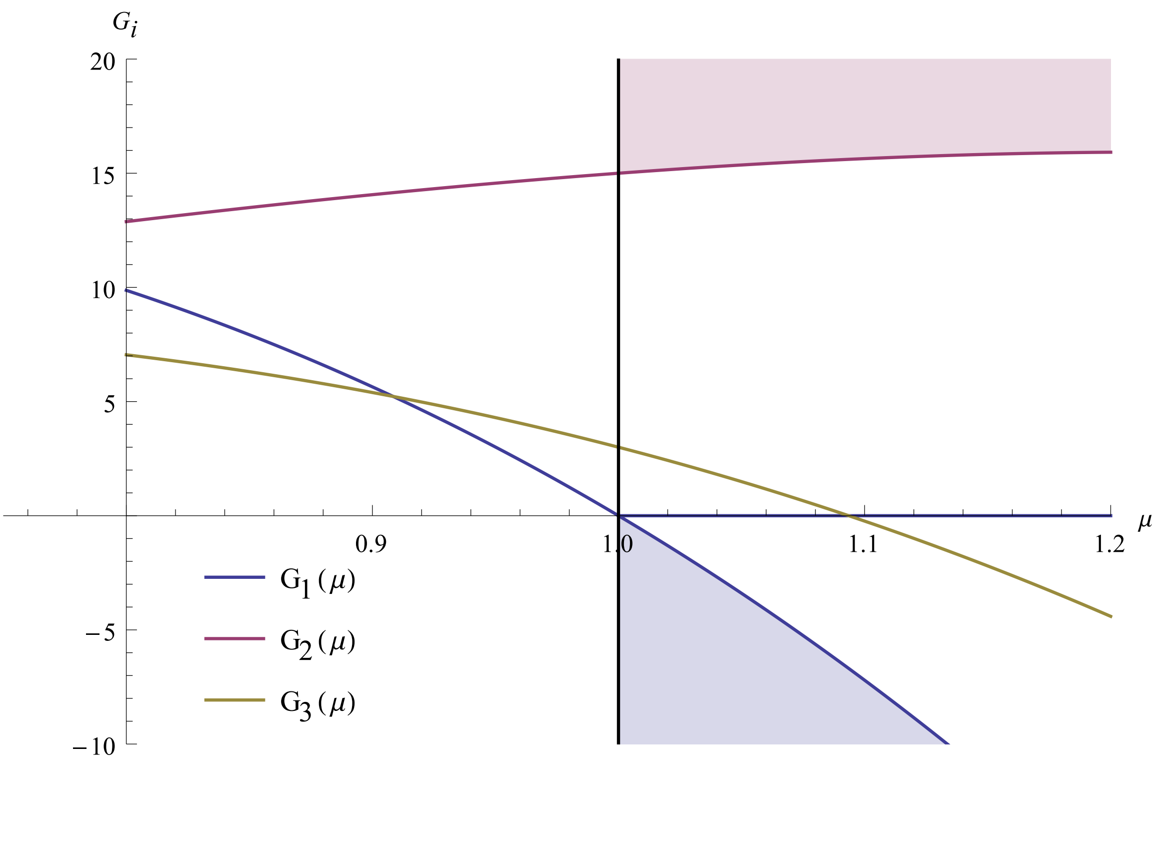

Figure A2. The plots of $G_1(\mu)$, $G_2(\mu)$, and $G_3(\mu)$: the case of $N < M$.

To sum up, a monocentric equilibrium configuration (Fig. 2a) exists if and only if at least one of the following conditions holds: either $\mu \leq \overline{\mu} \approx 0.732982$, or when the following inequality holds:

$$\frac{\alpha-c}{t}>\frac{N^2}{24}\frac{2+8\mu+15\mu^2-10\mu^3}{1-\mu}. \tag{91}$$

It can be shown that, whenever (6) holds, (91) is a weaker condition than $\mu \leq \overline{\mu}$. Whence, (91) is a *necessary and sufficient* condition for a monocentric equilibrium to exist. Furthermore, (91) may be equivalently reformulated as follows: a monocentric equilibrium configuration exists if and only if the masses N and M of consumers and firms are such that the corresponding point of the (N,M)-plane lies below the curve described by the following parametric equations:

$$N_1(\mu)=\sqrt{24\frac{\alpha-c}{t}\frac{1-\mu}{-10\mu^3+15\mu^2+8\mu+2}}, \tag{92}$$

$$M_1(\mu)=\mu\sqrt{24\frac{\alpha-c}{t}\frac{1-\mu}{-10\mu^3+15\mu^2+8\mu+2}}, \tag{93}$$

where $\mu\in(0,1)$. As shown by Figure 3 in the paper, the curve $(N_1(\mu),M_1(\mu))$ described by (92) – (93) is upward-sloping, hence it can be viewed as a graph of an increasing function. Denote this function by $\underline{M}(N)$ and observe that $\underline{M}(N)<N$ (because $\mu<1$). Whence, condition (91) may be reformulated as $M<\underline{M}(N)$. This completes the proof of part (i).

Proof of part (ii). Consider now a duocentric equilibrium configuration (Fig. 2b). Let $[b,b+M/2]$ be the business district to the right of the origin $x=0$. Then, the dispersion δ_0 of consumers is given by:

$$\delta_0=\frac{2}{N}\int_0^b x^2n(x)\mathrm{d}x+\frac{2}{N}\int_{b+M/2}^{(N+M)/2} x^2n(x)\mathrm{d}x.$$

Integration yields:

$$\delta_0=\frac{(M+N)^3}{12N}-\frac{2}{3N}\left(\left(b+\frac{M}{2}\right)^3-b^3\right),$$

simplifying which and using (19), we find the following expressions for σ_0 and δ_0:

$$\sigma_0=\frac{M^2}{12}+\frac{M}{2}b+b^2, \tag{94}$$

$$\delta_0=\frac{3M^2+3MN+N^2}{12}-\frac{M^2}{2N}b-\frac{M}{N}b^2. \tag{95}$$

For an equilibrium to exist, it must be that there is a $b\in(0,N/2)$which satisfies the following conditions:

$$\Delta\left(b^2\right)=\Delta\left(\left(b+\frac{M}{2}\right)^2\right), \qquad \Delta(\xi)<\Delta(b)\Longleftrightarrow b^2<\xi<\left(b+\frac{M}{2}\right)^2.$$

This, in turn, is equivalent to the following:

$$\arg \min_{\xi \in [0,(M+N)^2/4]} \Delta(\xi) = b^2 + \frac{M}{2}b + \frac{M^2}{8}.$$

Put differently, $b^2 + \frac{M}{2}b + \frac{M^2}{8}$ must solve the following FOC and SOC:

$$\Delta'\left(b^2 + \frac{M}{2}b + \frac{M^2}{8}\right) = 0, \qquad \Delta''\left(b^2 + \frac{M}{2}b + \frac{M^2}{8}\right) > 0,$$

or, equivalently,

$$2M > N,$$

$$(2M-N)\left(b^2 + \frac{M}{2}b + \frac{M^2}{8}\right) + 5M\sigma_0 - (M+N)\delta_0 + (N-M)\frac{\alpha - c}{t} = 0. \tag{96}$$

Plugging the expressions for σ_0 and δ_0 into (96), we obtain after simplifications:

$$\left(8M - N + \frac{M^2}{N}\right)\left(b^2 + \frac{M}{2}b + \frac{M^2}{12}\right) + \frac{M^2}{24}(2M-N) - \frac{(M+N)^4}{12N} + (N-M)\frac{\alpha - c}{t} = 0,$$

which is a quadratic equation with respect to b. For this equation to have a positive root between 0 and $N/2$, it is necessary and sufficient that the following two inequalities hold simultaneously:

$$\left(8M - N + \frac{M^2}{N}\right)\frac{M^2}{12} + (N-M)\frac{\alpha - c}{t} + (2M-N)\frac{M^2}{24} - \frac{(M+N)^4}{12N} < 0,$$

$$\left(8M - N + \frac{M^2}{N}\right)\left(\frac{N^2}{4} + \frac{MN}{4} + \frac{M^2}{12}\right) + (N-M)\frac{\alpha - c}{t} + (2M-N)\frac{M^2}{24} - \frac{(M+N)^4}{12N} > 0.$$

This, in turn, is equivalent to:

$$-16\mu^3 - 39\mu^2 - 34\mu + 8 < \frac{24}{N^2}\frac{\alpha - c}{t}(1-\mu) < -10\mu^3 + 15\mu^2 + 8\mu + 2, \tag{97}$$

where $\mu \equiv M/N$. Note that (97) always holds when $\mu = 1$ (i.e. when $M = N$). When $1/2 < \mu < 1$, then (97) amounts to:

$$N > \sqrt{24\frac{\alpha - c}{t}\frac{1-\mu}{-10\mu^3 + 15\mu^2 + 8\mu + 2}}.$$

Hence, when $1/2 < \mu < 1$, we can determine a curve on the (N,M)-plane which separates the domain where equilibrium is duocentric (Fig. 2b) from the one which gives rise to a monocentric equilibrium (Fig. 2a). This boundary curve is described by the parametric equations (92) – (93).

Finally, when $\mu > 1$, (97) amounts to:

$$\frac{16 + 39\nu + 34\nu^2 - 8\nu^3}{1-\nu} > \frac{24}{M^2}\frac{\alpha - c}{t} > \frac{10 - 15\nu - 8\nu^2 - 2\nu^3}{1-\nu},$$

where $\nu \equiv 1/\mu \in (0,1)$. Hence, another curve separating the duocentric city (Fig. 2b) from a city with a residential center (Fig. 2c) is given by

$$N_2(\nu) = \nu\sqrt{24\frac{\alpha-c}{t}\frac{1-\nu}{16+39\nu+34\nu^2-8\nu^3}}, \tag{98}$$

$$M_2(\nu) = \sqrt{24\frac{\alpha-c}{t}\frac{1-\nu}{16+39\nu+34\nu^2-8\nu^3}}, \tag{99}$$

where $\nu \in (0,1)$. As shown by Figure 3 in the paper, the curve $(N_2(\nu), M_2(\nu))$ described by (98) – (99) is upward-sloping in the region defined by (6). Hence, $(N_2(\nu), M_2(\nu))$ can be viewed as a graph of an increasing function. Denote this function by $\overline{M}(N)$. Since the graph of $M = \overline{M}(N)$lies above the 45°-line (see Fig. 3), we have $\overline{M}(N) > N > \underline{M}(N)$. Hence, the condition that the (N,M) point lies between the curves $(N_1(\nu), M_1(\nu))$ and $(N_2(\nu), M_2(\nu))$, can be reformulated as $\underline{M}(N) < M < \overline{M}(N)$. This completes the proof of part (ii).

Proof of part (iii). Consider now an urban configuration with a residential downtown area (Fig. 2c). In this case the city involves one residential district, $[-\frac{N}{2}, \frac{N}{2}]$, and two symmetric business districts, $[-\frac{M+N}{2}, -\frac{N}{2})$ and $(\frac{N}{2}, \frac{M+N}{2}]$. In an equilibrium of this type, the population density $n(x)$ is given by

$$n(x) = \begin{cases} 1, & \text{if } x \in \left[-\frac{N}{2}, \frac{N}{2}\right], \\ 0, & \text{otherwise,} \end{cases} \tag{100}$$

while the density of firms takes the form

$$m(y) = \begin{cases} 1, & \text{if } y \in [-\frac{M+N}{2}, -\frac{N}{2}) \cup (\frac{N}{2}, \frac{M+N}{2}], \\ 0, & \text{otherwise.} \end{cases} \tag{101}$$

Plugging (105) into (23), we find that the firms' bid rent function takes the form:

$$\Phi(y,\pi^*) = \frac{Nt^2}{4\beta}\left[y^4 - 2\left(\frac{\alpha-c}{t} - \frac{N^2}{12}\right)y^2\right] + \frac{Nt^2}{4\beta}\left(\frac{\alpha-c}{t} - \frac{N^2}{12}\right)^2 - \pi^* - f. \tag{102}$$

The consumer's bid rent function boils down to

$$\Psi(x,U^*) = \frac{Mt^2}{2\beta}\left[x^4 - \left(\frac{\alpha-c}{t} - \frac{5M^2+15MN+14N^2}{12}\right)x^2\right] + \kappa - U^* + Y, \tag{103}$$

where κ is independent on x. To show this, we integrate the right-hand sides of (17) and (10) across X, which yields:

$$\sigma(x) = x^2 + \frac{M^2+3MN+3N^2}{12}, \tag{104}$$

$$\delta(y) = y^2 + \frac{N^2}{12}. \tag{105}$$

Double integration of (46) across $X \times X$ yields

$$\theta = \frac{M^2}{12} + \frac{MN}{4} + \frac{N^2}{3}. \tag{106}$$

Plugging (101) – (105) and (104) – (106) into (24) and setting

$$\begin{aligned}\kappa \equiv \frac{Mt^2}{8\beta}\left[\left(\frac{\alpha-c}{t}\right)^2 - \frac{(M+N)(2M+N)}{6}\left(\frac{\alpha-c}{t}\right)\right] \\ +\frac{Mt^2}{8\beta}\cdot\frac{9M^4+45M^3N+80M^2N^2+60MN^3+20N^4}{720},\end{aligned} \tag{107}$$

we obtain (103). Set

$$\begin{aligned}\widetilde{\Phi}(y) &= \frac{Nt^2}{4\beta}\left[y^4 - 2\left(\frac{\alpha-c}{t} - \frac{N^2}{12}\right)y^2\right] + \frac{Nt^2}{4\beta}\left(\frac{\alpha-c}{t} - \frac{N^2}{12}\right)^2 \\ &= \frac{Nt^2}{4\beta}\left(\frac{\alpha-c}{t} - \frac{N^2}{12} - y^2\right)^2,\end{aligned} \tag{108}$$

$$\widetilde{\Psi}(x) = \frac{Mt^2}{2\beta}\left[x^4 - \left(\frac{\alpha-c}{t} - \frac{5M^2+15MN+14N^2}{12}\right)x^2\right] + \kappa. \tag{109}$$

The conditions for the existence of an equilibrium with consumers in the center are as follows:

$$\widetilde{\Phi}\left(\frac{M+N}{2}\right) - \pi^* = R_a, \tag{110}$$

$$\widetilde{\Phi}\left(\frac{N}{2}\right) - \pi^* = \widetilde{\Psi}\left(\frac{N}{2}\right) - U^*, \tag{111}$$

$$\widetilde{\Phi}(0) - \pi^* < \widetilde{\Psi}(0) - U^*, \tag{112}$$

$$\widetilde{\Phi}\left(\frac{M+N}{2}\right) - \pi^* > \widetilde{\Psi}\left(\frac{M+N}{2}\right) - U^*, \tag{113}$$

which results in:

$$\pi^* = \widetilde{\Phi}\left(\frac{M+N}{2}\right) - R_a = \frac{Nt^2}{4\beta}\left(\frac{\alpha-c}{t} - \frac{N^2}{12} - \frac{(M+N)^2}{4}\right)^2 - R_a,$$

$$U^* = \widetilde{\Psi}\left(\frac{N}{2}\right) - \widetilde{\Phi}\left(\frac{N}{2}\right) + \widetilde{\Phi}\left(\frac{M+N}{2}\right) - R_a.$$

Combining these two equations with (112) – (113), we get:

$$\widetilde{\Psi}(0) - \widetilde{\Phi}(0) > \widetilde{\Psi}\left(\frac{N}{2}\right) - \widetilde{\Phi}\left(\frac{N}{2}\right) > \widetilde{\Psi}\left(\frac{M+N}{2}\right) - \widetilde{\Phi}\left(\frac{M+N}{2}\right). \tag{114}$$

Using (114), we derive the existence conditions for a segregated spatial equilibrium with a residential central area. Deriving (108) –(109), it can be shown that (114) holds if the following inequalities hold simultaneously:

$$24(M-N)\frac{\alpha - c}{t} > 10M^3 + 30M^2N + 34MN^2 - 5N^3, \tag{115}$$

$$24(M-N)\frac{\alpha - c}{t} > 16M^3 + 39M^2N + 34MN^2 - 8N^3. \tag{116}$$

Doing so, we assume implicitly that each individual's consumer optimum is interior. Very much in the spirit of proving (69), it can be shown that

$$\frac{\alpha - c}{t} > \frac{1}{2}M^2 + 2MN + \frac{5}{3}N^2 \tag{117}$$

is a necessary and sufficient condition for interior consumers' choices under a duocentric urban configuration (Fig. 2c).

If (117), (115), and (116) simultaneously hold, it must be that $\nu < 1$, where $\nu \equiv N/M$.

For a duocentric equilibrium (Fig. 2c) to exist, (117), (115), and (116) must hold simultaneously. When $\nu < 1$ (where $\nu \equiv N/M$), this is equivalent to

$$(1-\nu)\frac{\alpha - c}{t} > \frac{M^2}{24}\max\{H_1(\nu), H_2(\nu), H_3(\nu)\},$$

where

$$H_1(\nu) \equiv \left(40\nu^2 + 48\nu + 12\right)(1-\nu), \qquad H_2(\nu) \equiv -5\nu^3 + 34\nu^2 + 30\nu + 10,$$

$$H_3(\nu) \equiv -8\nu^3 + 34\nu^2 + 39\nu + 16.$$

When $\nu > 1$, the conditions (117), (115), and (116) hold simultaneously iff

$$\frac{M^2}{24}\max\{H_2(\nu), H_3(\nu)\} < (1-\nu)\frac{\alpha - c}{t} < \frac{M^2}{24}H_1(\nu).$$

Fig. A3 shows that there is no solution for the case when $\nu \geq 1$:

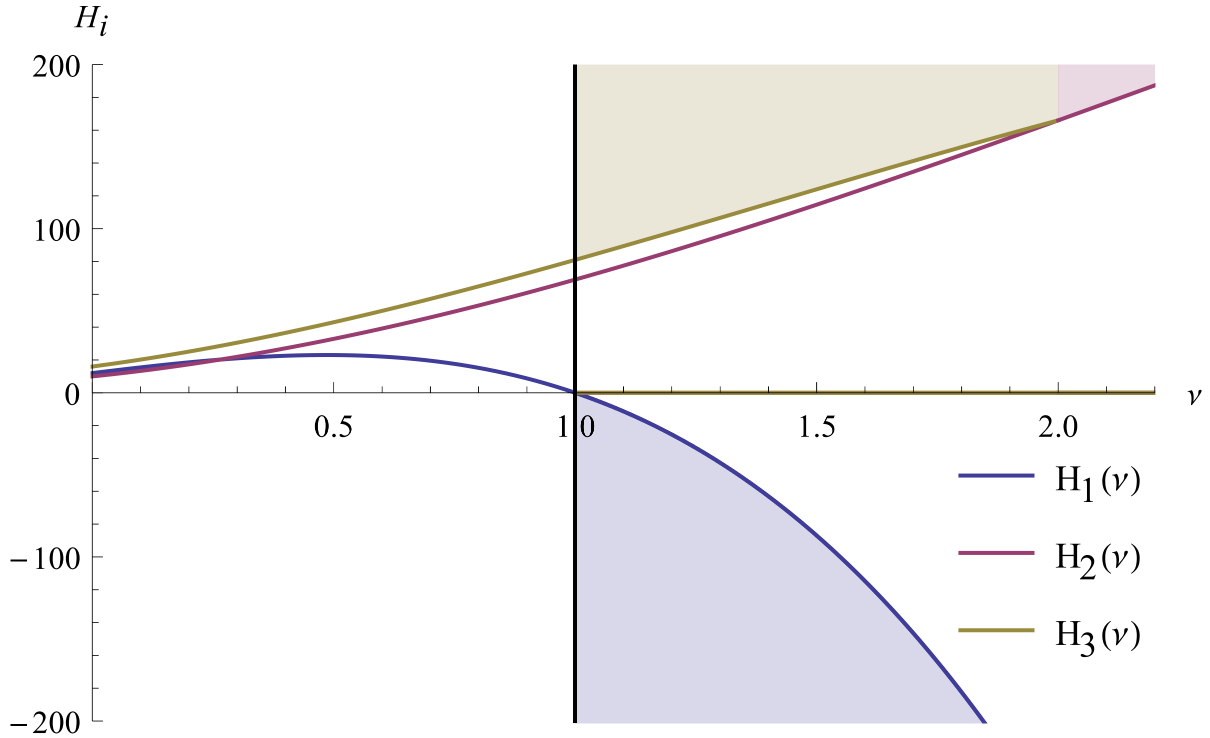

Figure A3. The graphs of $H_1(\nu)$, $H_2(\nu)$, and $H_3(\nu)$.

Using Figure A3 and setting $\nu \equiv N/M$ to be the relative number of consumers, we obtain that a three-district segregated spatial equilibrium with a residential area in the center and shopping areas at the outskirts exists if and only if the following inequalities hold:

$$\nu > 1, \qquad \frac{\alpha - c}{t} > \frac{M^2}{24} \frac{16 + 39\nu + 34\nu^2 - 8\nu^3}{1 - \nu}. \tag{118}$$

Using the same type of reasoning as in the proofs of parts (i) and (ii), condition can be equivalently reformulated as $M > \overline{M}(N)$, where $\overline{M}(N)$ has been defined at the end of the proof of part (ii). This completes the proof of part (iii).

Proof of part (iv). Finally, consider a duocentric urban configuration with residential outskirts (Fig. 2d). Let $[a, a + N/2]$ be the residential district to the right of the origin $x = 0$. Then, it is readily verified that the expressions of δ_0 and σ_0 are as follows:

$$\delta_0 = a^2 + \frac{N}{2} a + \frac{N^2}{12}$$

$$\sigma_0 = \frac{(M+N)^3}{12M} - \frac{N}{M}\delta_0 = \frac{1}{12}\left(M^2 + 3MN + 3N^2\right) - \frac{N^2}{2M} a - \frac{N}{M} a^2$$

For an equilibrium configuration of the type shown on Fig. 2d to exist, it must be that a value of $a \in (0, M/2)$ exists which satisfies the following conditions:

$$\Delta\left(a^2\right) = \Delta\left(\left(a + \frac{N}{2}\right)^2\right), \qquad \Delta(\xi) > \Delta\left(a^2\right) \Longleftrightarrow a^2 < \xi < \left(a + \frac{N}{2}\right)^2.$$

This, in turn, is equivalent to the following:

$$\arg\max_{\xi\in[0,(M+N)^2/4]} \Delta(\xi) = a^2+\frac{N}{2}a+\frac{N^2}{8}.$$

In other words, $a^2+\frac{N}{2}a+\frac{N^2}{8}$ must satisfy the FOC and the SOC, which are given by

$$\Delta'\left(a^2+\frac{N}{2}a+\frac{N^2}{8}\right)=0, \qquad \Delta''\left(a^2+\frac{N}{2}a+\frac{N^2}{8}\right)<0.$$

The SOC amounts to $2M-N<0$ (or, equivalently, $\mu<1/2$), while the FOC can be recast as follows:

$$(9N-M)a^2+N(7N-M)a-2(N-M)\frac{\alpha-c}{t}-\frac{5}{6}M^3-\frac{5}{2}M^2N-\frac{17}{6}MN^2+\frac{5}{12}N^3=0.$$

This is a quadratic equation with respect to a. For this equation to have a positive solution which does not exceed $M/2$, the left-hand side must be negative for $a=0$ and positive for $a=M/2$. This is equivalent to the following system of inequalities:

$$\frac{24}{N^2}\frac{\alpha-c}{t}>\frac{-10\mu^3-30\mu^2-34\mu+5}{1-\mu},$$

$$\frac{24}{N^2}\frac{\alpha-c}{t}<\frac{-13\mu^3-9\mu^2+8\mu+10}{1-\mu}.$$

Since we also have $\mu<1/2$, these inequalities imply

$$\frac{\alpha-c}{t}<\frac{81}{96}N^2,$$

which is not compatible with (6). Hence, an equilibrium configuration depicted by Fig. 2d never exists. This completes the proof of part (iv).

Observing that at least one equilibrium always exists, and the domains where equilibria of different kinds exist are disjoint, we obtain existence and uniqueness. This completes the proof of Proposition 2.□

Proof of Proposition 3

We first prove that the operating profit $\Pi(M)$ is non-monotone.

Case 1: $M<\underline{M}(N)$. In this case the operating profit function is given by

$$\Pi(M)=\widetilde{\Phi}\left(\frac{M}{2}\right)-\widetilde{\Psi}\left(\frac{M}{2}\right)+\widetilde{\Psi}\left(\frac{M+N}{2}\right),$$

where

$$\widetilde{\Phi}(y)\equiv\frac{Nt^2}{4\beta}\left[\frac{\alpha-c}{t}-\frac{1}{12}\left(3M^2+3MN+N^2\right)-y^2\right]^2, \tag{119}$$

$$\widetilde{\Psi}(x) \equiv \frac{Mt^2}{2\beta}\left[x^4 - \left(\frac{\alpha - c}{t} - \frac{2M^2 - 3MN - N^2}{12}\right)x^2 + K\right], \tag{120}$$

where K is independent of x.

Combining these expressions yields after simplifications:

$$\Pi(M) = \frac{Nt^2}{4\beta}\left[\left(\frac{\alpha - c}{t} - \frac{1}{12}N^2 - \frac{1}{4}MN - \frac{1}{2}M^2\right)^2 - \left(M^2 + \frac{1}{2}MN\right)\left(\frac{\alpha - c}{t} - \frac{1}{6}N^2 - \frac{1}{4}MN - \frac{2}{3}M^2\right)\right].$$

Differentiating $\pi(M)$ with respect to M and simplifying, we obtain:

$$\frac{\mathrm{d}\Pi}{\mathrm{d}M} = \frac{Nt^2}{4\beta}\left[-\frac{\alpha - c}{t}(4M + N) + \frac{N^3}{8} + \frac{7N^2}{8}M + \frac{5N}{2}M^2 + \frac{11}{3}M^3\right].$$

Using the interior consumer optimum condition (6), we estimate the derivative from above as follows:

$$\frac{\mathrm{d}\Pi}{\mathrm{d}M} < \frac{Nt^2}{4\beta}\left[-2(M+N)^2(4M+N) + \frac{N^3}{8} + \frac{7N^2}{8}M + \frac{5N}{2}M^2 + \frac{11}{3}M^3\right] =$$

$$= \frac{Nt^2}{4\beta}\left(-8M^3 + \frac{11}{3}M^3 - 18NM^2 + \frac{5N}{2}M^2 - 8N^2M - 4N^2M + \frac{7N^2}{8}M - 2N^3 + \frac{N^3}{8}\right) =$$

$$= -\frac{Nt^2}{4\beta}\left(\frac{13}{3}M^3 + \frac{31}{2}NM^2 + \frac{89}{8}N^2M + \frac{15}{8}N^3\right) < 0.$$

Hence, $\frac{\mathrm{d}\Pi}{\mathrm{d}M} < 0$ for $M < \underline{M}(N)$.

Case 2: $\underline{M}(N) < M < \overline{M}(N)$. In this case, the spatial equilibrium conditions take the form:

$$\Phi(b) = \Psi(b), \tag{121}$$

$$\Phi\left(b + \frac{M}{2}\right) = \Psi\left(b + \frac{M}{2}\right), \tag{122}$$

$$\Psi\left(\frac{M+N}{2}\right) = R_a, \tag{123}$$

where b is a solution to the following quadratic equation:

$$b^2 + \frac{M}{2}b - \frac{\frac{M^2}{24}(N - 2M - 2) + \frac{(M+N)^4}{12N} + (M-N)\frac{\alpha - c}{t}}{8M - N + \frac{M^2}{N}} = 0. \tag{124}$$

Plugging the bid-rent functions into (121) – (123) yields

$$\frac{Mt^2}{2\beta}\left[\left(\frac{M+N}{2}\right)^4-\left(\frac{\alpha-c}{t}+\delta_0-5\sigma_0\right)\left(\frac{M+N}{2}\right)^2+K\right]-U^*+Y=R_a.$$

Using (123), we get:

$$U^*=\frac{Mt^2}{2\beta}\left[\left(\frac{M+N}{2}\right)^4-\left(\frac{\alpha-c}{t}+\delta_0-5\sigma_0\right)\left(\frac{M+N}{2}\right)^2+K\right]+Y-R_a$$

Plugging this expression for U^* into (122), we get after simplifications:

$$\begin{aligned}\Pi(M)&=\frac{Nt^2}{4\beta}\left(\frac{\alpha-c}{t}-\delta_0-b^2\right)^2\\&+\frac{Mt^2}{2\beta}\left[\left(\frac{M+N}{2}\right)^4-b^4-\left(\frac{\alpha-c}{t}+\delta_0-5\sigma_0\right)\left(\left(\frac{M+N}{2}\right)^2-b^2\right)\right],\end{aligned}\tag{125}$$

where b, σ_0 and δ_0 satisfy, respectively, (124), (94) and (95). It is readily verified that b, σ_0 and δ_0 vary with M as follows:

$$\frac{\partial b}{\partial M}>0,\quad \frac{\partial \delta_0}{\partial M}<0,\quad \frac{\partial \sigma_0}{\partial M}>0.$$

Furthermore, we have:

$$\frac{\mathrm{d}\Pi}{\mathrm{d}M}=\frac{\partial\Pi}{\partial M}+\frac{\partial\Pi}{\partial b}\frac{\partial b}{\partial M}+\frac{\partial\Pi}{\partial\delta_0}\frac{\partial\delta_0}{\partial M}+\frac{\partial\Pi}{\partial\sigma_0}\frac{\partial\sigma_0}{\partial M}.$$

To show that $\frac{\mathrm{d}\Pi}{\mathrm{d}M}>0$, it remains to verify the following inequalities:

$$\frac{\partial\Pi}{\partial b}>0,\quad \frac{\partial\Pi}{\partial M}>0,\quad \frac{\partial\Pi}{\partial\delta_0}<0,\quad \frac{\partial\Pi}{\partial\sigma_0}>0.$$

To prove the first inequality, observe that, as $[-b,b]$ is a residential area, the consumer's bid-rent function falls steeper than the firm's bid-rent function in the vicinity of $x=b$. Combining this with (122), we have:

$$\frac{\partial\Pi}{\partial b}=\left(\frac{\partial\Phi(y,\pi^*)}{\partial y}-\frac{\partial\Psi(x,U^*)}{\partial x}\right)\Bigg|_{x=y=b}>0.$$

The other three inequalities follow immediately from (125). Hence, when $\underline{M}(N)<M<\overline{M}(N)$, we have $\frac{\mathrm{d}\Pi}{\mathrm{d}M}>0$.

Case 3. When $M>\overline{M}(N)$, the operating profit $\Pi(M)$ takes the following form:

$$\Pi(M)=\frac{Nt^2}{4\beta}\left(\frac{\alpha-c}{t}-\frac{N^2}{12}-\frac{(M+N)^2}{4}\right)^2.$$

Differentiating $\Pi(M)$ w.r.t. M yields:

$$\frac{\mathrm{d}\Pi}{\mathrm{d}M}=-\frac{Nt^2}{4\beta}(M+N)\left(\frac{\alpha-c}{t}-\frac{N^2}{12}-\frac{(M+N)^2}{4}\right)<0,$$

due to (6). Hence, $\Pi(M)$ decreases in M for $M > \overline{M}(N)$. To sum up, $\Pi(M)$ varies with M just like it is shown by Figure 4.

Furthermore, it is readily verified that the following inequalities hold:

$$\Pi(0) > \Pi\left(\overline{M}(N)\right) > \Pi\left(\underline{M}(N)\right) > \Pi\left(\sqrt{\frac{\alpha - c}{2t}} - N\right).$$

The free-entry condition (29) has no solutions when $\phi > \Pi(0)$. When either $\Pi\left[\overline{M}(N)\right] < \phi < \Pi(0)$ or $\Pi\left(\sqrt{(\alpha - c)/(2t)} - N\right) < \phi < \Pi\left[\underline{M}(N)\right]$, then (29) has a single solution M^*, such that $\Pi'(M^*) < 0$. Finally, when $\Pi\left[\underline{M}(N))\right] < \phi < \Pi\left[\overline{M}(N)\right]$ has three distinct solutions, $M^* < M^{**} < M^{***}$, such that $\Pi'(M^*) < 0$, $\Pi'(M^{**}) > 0$ and $\Pi'(M^{***}) < 0$. Setting

$$\phi_{\text{high}} \equiv \Pi\left(\overline{M}(N)\right), \qquad \phi_{\text{low}} \equiv \Pi\left(\underline{M}(N)\right)$$

completes the proof of Proposition 3. □