Math. Res. Lett. **14** (2007), *no.* 1, 129–136

ON THE MULTIPLICATION MAP OF A MULTIGRADED ALGEBRA

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Abstract. Given a multigraded algebra A, it is a natural question whether or not for two homogeneous components A_u and A_v, the product $A_{nu}A_{nv}$ is the whole component A_{nu+nv} for n big enough. We give combinatorial and geometric answers to this question.

1. Statement and discussion of the results

In this note, we consider the multiplication map of a multigraded algebra and ask for its surjectivity properties on the homogeneous parts. More precisely, let A be an (associative, commutative), integral, finitely generated algebra (with unit) over an algebraically closed field $\mathbb{K}$, and suppose that A is graded by a lattice $M \cong \mathbb{Z}^d$, i.e., we have

$$A \quad = \quad \bigoplus_{u \in M} A_u.$$

By the weight cone of A we mean the convex, polyhedral cone $\omega(A) \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} M$ generated by all $u \in M$ with $A_u \neq 0$. We investigate the following problem: given $u, v \in \omega(A) \cap M$, does there exist an $m > 0$ such that for any $k > 0$ the multiplication map defines a surjection

$$\mu_{km} \colon A_{kmu} \otimes_{\mathbb{K}} A_{kmv} \; \to \; A_{km(u+v)}, \qquad f \otimes g \; \mapsto \; fg.$$

We call a pair $u, v \in \omega(A) \cap M$ *generating* if it has this property. Simple examples show that not every pair is generating. In our first result we provide combinatorial criteria for a pair to be generating, and in the second one, we give a geometric characterization for the case of a factorial algebra A.

To present the first result, let us recall from [3] the concept of the GIT-fan associated to A. The M-grading of A defines a (unique) action of the torus $T := \mathrm{Spec}(\mathbb{K}[M])$ on $X := \mathrm{Spec}(A)$ such that for any $u \in M$, the elements $f \in A_u$ are precisely the semiinvariants of the character $\chi^u \colon T \to \mathbb{K}^*$, i.e., each $f \in A_u$ satisfies

$$f(t \cdot x) \quad := \quad \chi^u(t) f(x).$$

Received by the editors July 31, 2006.
2000 *Mathematics Subject Classification.* 13A02, 14L24.
Supported by INTAS YS 05-109-4958.

The *orbit cone* of a (closed) point $x \in X$ is the convex, polyhedral cone $\omega(x) \subseteq \mathbb{Q}\otimes_{\mathbb{Z}} M$ generated by all $u \in \omega(A)$ admitting an $f \in A_u$ with $f(x) \neq 0$. The collection of orbit cones is finite, and thus one may associate to any element $u \in \omega(A)$ its, again convex, polyhedral, *GIT-cone*:

$$\lambda(u) \quad := \quad \bigcap_{\substack{x \in X,\\ u \in \omega(x)}} \omega(x).$$

These GIT-cones cover the weight cone $\omega(A)$, and by [3, Thm. 3.11], the collection $\Lambda(A)$ of all of them is a fan in the sense that if $\lambda \in \Lambda(A)$ then also every face of λ belongs to $\Lambda(A)$, and for $\tau, \lambda \in \Lambda(A)$, the intersection $\tau \cap \lambda$ is a face of both, λ and τ. Note that we allow here a fan to have cones containing lines.

Theorem 1.1. *Let $\mathbb{K}$ be an algebraically closed field, M a lattice, and A a finitely generated, integral, M-graded $\mathbb{K}$-algebra with GIT-fan $\Lambda(A)$.*

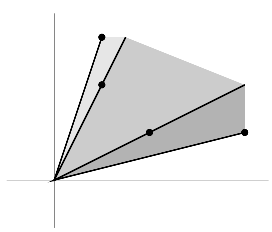
(i) *If $u, v \in \omega(A) \cap M$ is a generating pair, then the weights u, v lie in a common GIT-cone $\lambda \in \Lambda(A)$.*
(ii) *If $u, v \in \omega(A) \cap M$ lie in a common GIT-cone $\lambda \in \Lambda(A)$ and u belongs to the relative interior $\lambda^\circ \subseteq \lambda$, then u, v is a generating pair.*

If two weights $u, v \in \omega(A) \cap M$ lie on the boundary of a common GIT-cone $\lambda \in \Lambda(A)$, then no general statement in terms of the GIT-fan is possible: it may happen that u, v is generating, and also it may happen that u, v is not generating. For the first case there are obvious examples, and for the latter we present the following one.

Example 1.2. Consider the polynomial ring $A := \mathbb{K}[T_1, T_2, T_3, T_4]$ over any field $\mathbb{K}$. Then one may define a $\mathbb{Z}^2$-grading of A by setting

$$\deg(T_1) := (4,1), \quad \deg(T_2) := (2,1), \quad \deg(T_3) := (1,2), \quad \deg(T_4) := (1,3).$$

Any cone in $\mathbb{Q}^2$ generated by a collection of these weights is actually an orbit cone, and the associated GIT-fan looks as follows.



The pair $u := (2,1)$ and $v := (1,2)$ is contained in a common GIT-cone but it is not generating: one directly checks that the monomials $T_1 T_2^{n-2} T_3^{n-1} T_4 \in A_{n(u+v)}$ can never be obtained by multiplying elements from A_{nu} and A_{nv}.

Remark 1.3. In order to compute the GIT-fan for concrete examples, one needs to know the orbit cones. Here comes a general recipe.

Let A be given by homogeneous generators and relations, i.e., we have a graded epimorphism $\mathbb{K}[T_1, \dots, T_r] \to A$ and generators $q_1, \dots, q_s$ for its kernel. With $w_i := \deg(T_i)$, the orbit cones are $\operatorname{cone}(w_i;\ i \in I)$, where $I \subseteq \{1, \dots, r\}$ satisfies

$$\prod_{i \in I} T_i \ \notin \ \sqrt{\langle q_1^I, \dots, q_s^I \rangle}, \qquad \text{with} \quad q_j^I \ := \ q_j(S_1, \dots, S_r), \quad S_l \ := \ \begin{cases} T_l & l \in I, \\ 0 & l \notin I. \end{cases}$$

So, finding the sets of weights generating an orbit cone, amounts to testing for radical ideal membership, which can be performed quite efficiently by appropriate computer algebra systems.

Remark 1.4. For the polynomial ring $A = \mathbb{K}[T_1, \dots, T_r]$, the property of being a generating pair can be formulated as follows in a purely combinatorial manner.

Let the grading arise from a linear map $Q\colon \mathbb{Z}^r \to M$, $e_i \mapsto \deg(T_i)$. Then the weight cone $\omega(A)$ is the Q-image of the positive orthant $\gamma \subseteq \mathbb{Q}^r$, and for any integral $u \in \omega(A)$, we have the polyhedron $\Delta_u := Q^{-1}(u) \cap \gamma$. A pair $u, v \in \omega(A) \cap M$ is generating if and only if there exists an $m > 0$ such that for any $k > 0$ one has

$$(\Delta_{kmu} \cap \mathbb{Z}^r) \ + \ (\Delta_{kmv} \cap \mathbb{Z}^r) \ \ = \ \ \Delta_{km(u+v)} \ \cap \ \mathbb{Z}^r.$$

In order to present the second result, we have to recall from [3, Sec. 2] some more facts concerning the GIT-fan. For any $u \in \omega(A) \cap M$, we have an associated nonempty set of semistable points:

$$X(u) \ := \bigcup_{\substack{f \in A_{nu},\\ n > 0}} X_f \ = \ \{x \in X;\ u \in \omega(x)\}.$$

We have $X(u) \subseteq X(v)$ if and only if the GIT-cone $\lambda(v)$ is a face of $\lambda(u)$. In particular, $u, v \in \omega(A) \cap M$ define the same set of semistable points if and only if they belong to the relative interior of a common GIT-cone.

Each set of semistable points $X(u)$ admits a good quotient $X(u) \to Y(u)$ for the action of T. For $X(u) \subseteq X(v)$, there is an induced projective morphism $Y(u) \to Y(v)$ of the quotient spaces. In particular, if u, v lie in a common GIT-cone, then we obtain a commutative diagram

$$(1.4.1) \qquad \begin{array}{ccccc} & & Y(u+v) & & \\ & \swarrow^{\kappa_u} & \big\downarrow \kappa & \searrow^{\kappa_v} & \\ Y(u) & & & & Y(v) \\ & \nwarrow_{\pi_u} & & \nearrow_{\pi_v} & \\ & & Y(u) \times Y(v) & & \end{array}$$

We denote the image of the downwards map κ by $Z(u,v) := \kappa(Y(u+v))$. Moreover, we consider the (open) set $W(A) := \{x \in X;\ \omega(x) = \omega(A)\}$ of points having a generic orbit cone. For a factorial A, we then obtain the following characterization of the generating property for a pair u, v in the relative interior $\omega(A)^\circ$ of $\omega(A)$.

Theorem 1.5. *Let $\mathbb{K}$, M and A be as in 1.1. Moreover, suppose that A is factorial and that $X \setminus W(A)$ is of codimension at least two in X. Then, for any two $u, v \in \omega(A)^\circ$ belonging to a common GIT-cone, the following statements are equivalent.*

(i) *The pair u, v is generating.*
(ii) *The variety $Z(u, v)$ is normal.*

Remark 1.6. Under slightly sharper conditions on the algebra A as posed in Theorem 1.5, one may view A as the "Cox ring" of certain varieties, see [2]. Theorem 1.5 then tells about surjectivity properties of the multiplication map for global sections of divisors.

2. Proof of the results

The setup is the same as in the first section. In particular, M is a lattice, and A is a finitely generated, integral algebra over an algebraically closed field $\mathbb{K}$. We consider again the corresponding affine variety $X := \operatorname{Spec}(A)$, and the action of the torus $T := \operatorname{Spec}(\mathbb{K}[M])$ on X defined by the M-grading of A.

In a first step, we give a more algebraic characterization of the GIT-fan. For $u, v \in \omega(A) \cap M$, we will work in terms of the following subalgebras:

$$A(u) \ := \ \bigoplus_{n \in \mathbb{Z}_{\ge 0}} A_{nu}, \qquad A(u,v) \ := \ \bigoplus_{n \in \mathbb{Z}_{\ge 0}} A_{nu} \cdot A_{nv}.$$

Clearly, $A(u, v)$ is contained in $A(u + v)$. We call $A(u, v)$ *large* in $A(u + v)$, if the ideals $A(u, v)_+ \subseteq A(u, v)$ and $A(u + v)_+ \subseteq A(u + v)$ generated by the homogeneous parts of strictly positive degree satisfy

$$\sqrt{\langle A(u,v)_+ \rangle} \ = \ A(u+v)_+ \ \subseteq \ A(u+v).$$

Proposition 2.1. *Let M be a lattice, and A an M-graded, finitely generated, integral $\mathbb{K}$-algebra. Then, for any two $u, v \in \omega(A)$, the following statements are equivalent.*

(i) *There is a GIT-cone $\lambda \in \Lambda$ satisfying $u, v \in \lambda$.*
(ii) *We have $X(u) \cap X(v) = X(u + v)$.*
(iii) *The algebra $A(u, v)$ is large in $A(u + v)$.*

Proof. We begin with the equivalence of (i) and (ii). If (i) holds, then every orbit cone $\omega(x)$ containing $u + v$ must contain u and v as well. This gives

$$\begin{aligned} x \ \in \ X(u) \cap X(v) \quad & \Longleftrightarrow \quad u, v \ \in \ \omega(x) \\ & \Longleftrightarrow \quad u + v \ \in \ \omega(x) \\ & \Longleftrightarrow \quad x \in \ X(u+v). \end{aligned}$$

Conversely, if (ii) holds, then we see that $\lambda(u)$ and $\lambda(v)$ are faces of $\lambda(u + v)$. Thus, we have $u, v \in \lambda(u + v)$.

For the equivalence of (ii) and (iii) note that for any $w \in \omega(A) \cap M$ the complement $X \setminus X(w)$ equals the zero set $V(A(w)_+)$. Thus, setting $w := u + v$, we obtain

$$\begin{aligned} X(u) \cap X(v) \ = \ X(w) \quad & \Longleftrightarrow \quad V(A(u)_+) \cup V(A(v)_+) \ = \ V(A(w)_+) \\ & \Longleftrightarrow \quad V(A(u)_+ \cdot A(v)_+) \ = \ V(A(w)_+). \end{aligned}$$

The latter property holds if and only if the ideals generated by $A(u)_+ \cdot A(v)_+$ and $A(w)_+$ have the same radical in A. This holds if and only if they generate the same radical ideal in $A(w)$, which eventually is equivalent to $A(u,v)$ being a large subalgebra of $A(w)$. □

This observation enables us to decide whether or not two weights u, v belong to a common GIT-cone by just looking at $A(u)$, $A(v)$ and $A(u+v)$. As a consequence, we may produce examples of nontrivial affine varieties with simple variation of GIT-quotients.

Recall that a point $x \in X(u)$ in a set $X(u) \subseteq X$ of semistable points is said to be *stable*, if its orbit $T \cdot x$ is closed in $X(u)$ and of maximal dimension. If the set $X(u)$ consists of stable points, then the fibres of the quotient map $X(u) \to Y(u)$ are precisely the T-orbits of $X(u)$.

Corollary 2.2. *Let M be a lattice, and let A be an M-graded, finitely generated, integral $\mathbb{K}$-algebra. Given $\lambda \in \Lambda(A)$, consider the (finitely generated) algebra*

$$A' \quad := \quad \bigoplus_{u \in \lambda \cap M} A_u.$$

Then the corresponding action of the torus $T = \mathrm{Spec}(\mathbb{K}[M])$ on the affine variety $X' = \mathrm{Spec}(A')$ has the following properties.

(i) *The GIT-fan $\Lambda(A')$ associated to A' is the fan of faces of the cone $\lambda \in \Lambda(A)$.*
(ii) *The union $W \subseteq X'$ of all T-orbits of maximal dimension is a set of semistable points, and every $x \in W$ is stable.*

Proof. To see (i), note first that $\Lambda(A')$ subdivides $\omega(A') = \lambda$. Moreover, Proposition 2.1 (iii) implies that two weights $u, v \in \lambda$ lie in a common cone of $\Lambda(A')$ if and only if they lie in a common cone of $\Lambda(A)$.

For (ii), note that the dimension of an orbit cone $\omega(x)$ equals that of the orbit $\dim(T \cdot x)$. Since $\lambda \in \Lambda(A')$ is the only cone of maximal dimension, we obtain

$$W \;=\; \{x \in X;\ \omega(x) = \lambda\} \;=\; X'(u)$$

for any u from the relative interior of λ. Since all orbits in W have the same dimension, each of them is closed in W. □

The next step is a geometric characterization of the GIT-fan. It is given in terms of the map $\kappa\colon Y(u+v) \to Y(u) \times Y(v)$ introduced in the diagram 1.4.1.

Proposition 2.3. *Let $u, v \in \omega(A) \cap M$ belong to a common GIT-cone $\lambda \in \Lambda(A)$. Then, in the setting of 1.4.1, the following statements are equivalent:*

(i) *The pair $u, v \in \omega(A) \cap M$ is generating.*
(ii) *The map $\kappa\colon Y(u+v) \to Y(u) \times Y(v)$ is a closed embedding.*

Proof. Recall that the quotient spaces $Y(w) = \mathrm{Proj}(A(w))$ are projective over $Y_0 = \mathrm{Spec}(A_0)$. Moreover, denoting by $q\colon X(w) \to Y(w)$ the quotient map, we obtain for $n \in \mathbb{Z}_{\ge 0}$ a sheaf on $Y(w)$, namely

$$\mathcal{L}_{nw} \;:=\; \left(q_*\mathcal{O}_{X(w)}\right)_{nw} \;=\; \mathcal{O}_{Y(w)}(n).$$

Replacing u with a large multiple, we may assume that $A(u)$ is generated as an A_0-algebra by the component A_u, and that for any $n \in \mathbb{Z}_{\ge 1}$ the canonical maps

$$\imath_{nu}\colon A_{nu} \quad\to\quad \Gamma(Y(u), \mathcal{L}_{nu})$$

are surjective, see [4, Exercise II.5.9]. Note that then $\mathcal{L}_u$ is an ample invertible sheaf on $Y(u)$. Of course, we may arrange the same situation for v and $u + v$.

On $Y(u) \times Y(v)$ we have the ample invertible sheaves $\mathcal{E}_n := \pi_u^*\mathcal{L}_{nu} \otimes \pi_v^*\mathcal{L}_{nv}$. We claim that the natural map

$$\Gamma(Y(u), \mathcal{L}_{nu}) \otimes \Gamma(Y(v), \mathcal{L}_{nv}) \quad\to\quad \Gamma(Y(u) \times Y(v), \mathcal{E}_n)$$

is an isomorphism. Indeed, using the projection formula, we obtain canonical isomorphisms

$$\Gamma(Y(u) \times Y(v), \mathcal{E}_n) \;\cong\; \Gamma(Y(u), \pi_{u*}\mathcal{E}_n) \;\cong\; \Gamma(Y(u), \mathcal{L}_{nu} \otimes \pi_{u*}\pi_v^*\mathcal{L}_{nv}).$$

We look a bit closer at $\pi_{u*}\pi_v^*\mathcal{L}_{nv}$. Given an open subset $U \subseteq Y(u)$, we denote by $\pi_v^U\colon U \times Y(v) \to Y(v)$ the restricted projection. Then we have

$$\Gamma(U, \pi_{u*}\pi_v^*\mathcal{L}_{nv}) \;=\; \Gamma(U \times Y(v), \pi_v^*\mathcal{L}_{nv}) \;\cong\; \Gamma(Y(v), \mathcal{L}_{nv} \otimes {\pi_v^U}_*\mathcal{O}_{U \times Y(v)}).$$

Likewise, one obtains ${\pi_v^U}_*\mathcal{O}_{U \times Y(v)} \cong \Gamma(U, \mathcal{O}_U) \otimes \mathcal{O}_{Y(v)}$ for any affine open set $U \subseteq Y(u)$. Consequently, we have a canonical isomorphism

$$\Gamma(U, \pi_{u*}\pi_v^*\mathcal{L}_{nv}) \;\cong\; \Gamma(U, \mathcal{O}_U) \otimes \Gamma(Y(v), \mathcal{L}_{nv}).$$

This in turn shows $\pi_{u*}\pi_v^*\mathcal{L}_{nv} \cong \mathcal{O}_{Y(u)} \otimes \Gamma(Y(v), \mathcal{L}_{nv})$, and our claim follows. Thus, we arrive at a commutative diagram

$$\begin{array}{ccc}
A_{nu} \otimes A_{nv} & \xrightarrow{\quad \mu_n \quad} & A_{nu+nv} \\
\cong \big\downarrow & & \big\downarrow \cong \\
\Gamma(Y(u) \times Y(v), \mathcal{E}_n) & \xrightarrow[\quad \kappa_n^* \quad]{} & \Gamma(Y(u+v), \mathcal{L}_{nu+nv})
\end{array}$$

where the upper horizontal arrow is the multiplication map we are interested in, and the lower horizontal arrow is the canonical pullback map

$$\begin{array}{rcl}
\kappa_n^*\colon \Gamma(Y(u) \times Y(v), \mathcal{E}_n) & \to & \Gamma(Y(u+v), \mathcal{L}_{nu+nv}) \\
\pi_u^* f \otimes \pi_v^* g & \mapsto & \kappa_u^* f \cdot \kappa_v^* g.
\end{array}$$

Now, note that the morphism $\kappa\colon Y(u+v) \to Y(u) \times Y(u)$ is induced from the multiplication map, because we have

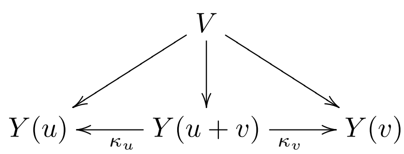
$$Y(u) \times Y(v) \;=\; \mathrm{Proj}\left(\bigoplus_{n \ge 0} A_{nu} \otimes A_{nv}\right), \qquad Y(u+v) \;=\; \mathrm{Proj}\left(\bigoplus_{n \ge 0} A_{nu+nv}\right).$$

Thus, the assertion follows from the basic fact that κ is a closed embedding if and only if there is an $l > 1$ such that μ_{ln} are surjective for any $n > 0$. $\square$

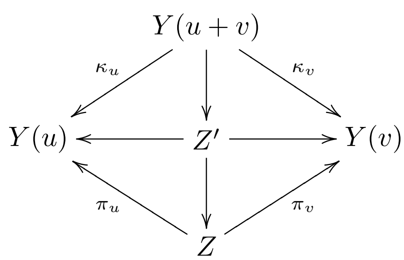
Proof of Theorem 1.1. If $u, v \in \omega \cap M$ is a generating pair, then the algebra $A(u, v)$ is large in $A(u+v)$. Thus, the first assertion follows from Proposition 2.1. To see the second one, note that both, u and $u+v$, lie in the relative interior λ° of the GIT-cone $\lambda \in \Lambda(A)$. Thus, $Y(u+v) \to Y(u)$ is an isomorphism, and the statement follows from Proposition 2.3. □

Proof of Theorem 1.5. First note that the set $W := W(A) \subseteq X$ consisting of all $x \in X$ with orbit cone $\omega(x) = \omega(A)$ admits a geometric quotient $V := W/T$ and that for any $w \in \omega(A)^\circ$, the inclusion $W \subseteq X(w)$ induces an open embedding $V \to Y(w)$ of the quotient spaces. Since $W \subseteq X$ has a complement of codimension at least two in X, the same must hold for the image of V in $Y(w)$. Moreover, as a good quotient space of a normal variety, $Y(w)$ is normal. Thus, $V \to Y(w)$ is a V-embedding in the sense of [1, Sec. 2].

To proceed, consider the morphisms of 1.4.1. Clearly, $\kappa_u \colon Y(u+v) \to Y(u)$ and $\kappa_v \colon Y(u+v) \to Y(v)$ are morphisms of V-embeddings, that means that we have a commutative diagram



Now consider the map $\kappa \colon Y(u+v) \to Y(u) \times Y(v)$ of 1.4.1, and denote its image by $Z := Z(u, v)$. Then κ lifts to the normalization $Z' \to Z$, and we obtain a commutative diagram



Lifting $V \to Y(u+v) \to Z$ to Z' defines a V-embedding $V \to Z'$. According to [1, Prop. 2.3], there is an open T-invariant subset $W' \subseteq X$ with good quotient $W' \to Z'$ by the T-action such that $V \to Z'$ is induced by the inclusion $W \subseteq W'$.

Moreover, the map $Y(u+v) \to Z'$ as well as the maps $Z' \to Y(u)$ and $Z' \to Y(v)$ are morphisms of V-embeddings. Thus, [1, Prop. 2.4] tells us that they are induced by inclusions of sets of semistable points

$$X(u+v) \subseteq W', \qquad W' \subseteq X(u), \qquad W' \subseteq X(v).$$

By Proposition 2.1, we have $X(u+v) = X(u) \cap X(v)$. This shows $W' = X(u+v)$. Thus, the map $Y(u+v) \to Z'$ is an isomorphism. From this we see that the map $\kappa \colon Y(u+v) \to Y(u) \times Y(v)$ is a closed embedding if and only if Z is normal. The assertion then follows from Proposition 2.3. □

Acknowledgements

We would like to thank D. Timashev for fruitful discussions and the referee for careful reading and helpful comments.

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