

Affine embeddings of homogeneous spaces

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Introduction

Throughout the paper G denotes a connected reductive algebraic group, unless otherwise specified, and H an algebraic subgroup of G . All groups and algebraic varieties considered are over an algebraically closed field \mathbb{K} of characteristic zero, unless otherwise specified. Let $\mathbb{K}[X]$ be the algebra of regular functions on an algebraic variety X and $\mathbb{K}(X)$ the field of rational functions on X provided X is irreducible. Our general references are [30] for algebraic groups and [56, 37, 29] for algebraic transformation groups and invariant theory.

Affine embeddings: definitions. Let us recall that an irreducible algebraic G -variety X is said to be an *embedding* of the homogeneous space G/H if X contains an open G -orbit isomorphic to G/H . We shall denote this relationship by $G/H \hookrightarrow X$. Let us say that an embedding $G/H \hookrightarrow X$ is *affine* if the variety X is affine. In many problems of invariant theory, representation theory and other branches of mathematics, only affine embeddings of homogeneous spaces arise. This is why it is reasonable to study specific properties of affine embeddings in the framework of a well-developed general embedding theory.

Which homogeneous spaces admit an affine embedding? It is easy to show that a homogeneous space G/H admits an affine embedding if and only if G/H is quasi-affine (as an algebraic variety). In this situation, the subgroup H is said to be *observable* in G . A closed subgroup H of G is observable if and only if there exist a rational finite-dimensional G -module V and a vector $v \in V$ such that the stabilizer G_v coincides with H . (This follows from the fact that any affine G -variety may be realized as a closed invariant subvariety in a finite-dimensional G -module [56, Th.1.5].) There is a nice group-theoretic description of

observable subgroups due to A. Sukhanov: a subgroup H is observable in G if and only if there exists a quasi-parabolic subgroup $Q \subset G$ such that $H \subset Q$ and the unipotent radical H^u is contained in the unipotent radical Q^u , see [63], [29, Th.7.3]. (Let us recall that a subgroup Q is said to be *quasi-parabolic* if Q is the stabilizer of a highest weight vector in some G -module V .)

It follows from Chevalley's theorem that any subgroup H without non-trivial characters (in particular, any unipotent subgroup) is observable. By Matsushima's criterion, a homogeneous space G/H is affine if and only if H is reductive. (For a simple proof, see [42] or [4]; a characteristic-free proof can be found in [57].) In particular, any reductive subgroup is observable. A description of affine homogeneous spaces G/H for non-reductive G is still an open problem.

Complexity of reductive group actions. Now we define the notion of complexity, which we shall encounter many times in the text. Let us fix the notation. By $B = TU$ denote a Borel subgroup of G with a maximal torus T and the unipotent radical U . By definition, the *complexity* $c(X)$ of a G -variety X is the codimension of a B -orbit of general position in X for the restricted action $B : X$. This notion firstly appeared in [45] and [70]. Now it plays a central role in embedding theory. By Rosenlicht's theorem, $c(X)$ is equal to the transcendence degree of the field $\mathbb{K}(X)^B$ of rational B -invariant functions on X . A normal G -variety X is called *spherical* if $c(X) = 0$ or, equivalently, $\mathbb{K}(X)^B = \mathbb{K}$. A homogeneous space G/H and a subgroup $H \subseteq G$ are said to be *spherical* if G/H is a spherical G -variety.

Rational representations, the isotypic decomposition and G -algebras. A linear action of G in vector space W is said to be *rational* if for any vector $w \in W$ the linear span $\langle Gw \rangle$ is finite-dimensional and the action $G : \langle Gw \rangle$ defines a representation of an algebraic group. Since any finite-dimensional representation of G is completely reducible, it is easy to prove that W is a direct sum of finite-dimensional simple G -modules. Let $\Xi_+(G)$ be the semigroup of dominant weights of G . For any $\lambda \in \Xi_+(G)$, denote by W_λ the sum of all simple submodules in W of highest weight λ . The subspace W_λ is called an *isotypic component* of W of weight λ , and the decomposition

$$W = \bigoplus_{\lambda \in \Xi_+(G)} W_\lambda$$

is called *the isotypic decomposition* of W .

If G acts on an affine variety X , the linear action $G : \mathbb{K}[X]$, $(gf)(x) :=$

$f(g^{-1}x)$, is rational [56, Lemma 1.4]. (Note that for irreducible X the action on rational functions $G : \mathbb{K}(X)$ defined by the same formula is not rational.) The isotypic decomposition

$$\mathbb{K}[X] = \bigoplus_{\lambda \in \Xi_+(G)} \mathbb{K}[X]_\lambda$$

and its interaction with the multiplicative structure on $\mathbb{K}[X]$ give important technical tools for the study of affine embeddings.

An affine G -variety X is spherical if and only if $\mathbb{K}[X]_\lambda$ is a simple G -module for any $\lambda \in \Xi_+(G)$ [32].

Suppose that \mathfrak{A} is a commutative associative algebra with unit over \mathbb{K} . If G acts on \mathfrak{A} by automorphisms and the action $G : \mathfrak{A}$ is rational, we say that \mathfrak{A} is a G -algebra. The algebra $\mathbb{K}[X]$ is a G -algebra for any affine G -variety X . Moreover, any finitely generated G -algebra without nilpotents arises in this way.

We conclude the introduction with a review of the contents of this survey.

One of the pioneering works in embedding theory was a classification of normal affine $SL(2)$ -embeddings due to V. L. Popov, see [52, 37]. In the same period (early seventies) the theory of toric varieties was developed. A toric variety may be considered as an equivariant embedding of an algebraic torus T . Such embeddings are described in terms of convex fans. Any cone in the fan of a toric variety X represents an affine toric variety. This reflects the fact that X has a covering by T -invariant affine charts. In 1972, V. L. Popov and E. B. Vinberg [55] described affine embeddings of quasi-affine homogeneous spaces G/H , where H contains a maximal unipotent subgroup of G . In Section 1 we discuss briefly these results together with a more recent one: a remarkable classification of algebraic monoids with a reductive group G as the group of invertible elements (E. B. Vinberg [71]). This is precisely the classification of affine embeddings of the space $(G \times G)/\Delta(G)$, where $\Delta(G)$ is the diagonal subgroup.

In Section 2 we consider connections of the theory of affine embeddings with Hilbert's 14th problem. Let H be an observable subgroup of G . By the Grosshans theorem, the following conditions are equivalent: 1) the algebra of invariants $\mathbb{K}[V]^H$ is finitely generated for any G -module V ; 2) the algebra of regular functions $\mathbb{K}[G/H]$ is finitely generated; 3) there exists a (normal) affine embedding $G/H \hookrightarrow X$ such that $\text{codim}_X(X \setminus (G/H)) \geq 2$ (such an embedding is called *the canonical embedding* of G/H).

It was proved by F. Knop that if $c(G/H) \leq 1$ then the algebra $\mathbb{K}[G/H]$ is finitely generated. This result provides a large class of subgroups with a positive solution of Hilbert's 14th problem. In particular, Knop's theorem together with Grosshans' theorem on the unipotent radical P^u of a parabolic subgroup $P \subset G$ includes almost all known results on Popov-Pommerening's conjecture (see 2.2). We study the canonical embedding of G/P^u from a geometric view-point. Finally, we mention counterexamples to Hilbert's 14th problem due to M. Nagata, P. Roberts, and R. Steinberg.

In Section 3 we introduce the notion of an *affinely closed space*, i.e. an affine homogeneous space admitting no non-trivial affine embeddings, and discuss the result of D. Luna related to this notion. (We say that an affine embedding $G/H \hookrightarrow X$ is *trivial* if $X = G/H$.) Affinely closed spaces of an arbitrary affine algebraic group are characterized and some elementary properties of affine embeddings are formulated.

Section 4 is devoted to affine embeddings with a finite number of orbits. We give a characterization of affine homogeneous spaces G/H such that any affine embedding of G/H contains a finite number of orbits. More generally, we compute the maximal number of parameters in a continuous family of G -orbits over all affine embeddings of a given affine homogeneous space G/H . The group of equivariant automorphisms of an affine embedding is also studied here.

Some applications of the theory of affine embeddings to functional analysis are given in Section 5. Let $M = K/L$ be a homogeneous space of a connected compact Lie group K , and $C(M)$ the commutative Banach algebra of all complex-valued continuous functions on M . The K -action on $C(M)$ is defined by the formula $(kf)(x) = f(k^{-1}x)$, $k \in K$, $x \in M$. We shall say that A is an *invariant algebra* on M if A is a K -invariant uniformly closed subalgebra with unit in $C(M)$. Denote by G and H the complexifications of K and L respectively. Then G is a reductive algebraic group with reductive subgroup H . There exists a correspondence between finitely generated invariant algebras on M and affine embeddings of G/F with some additional data, where F is an observable subgroup of G containing H . This correspondence was introduced by V. M. Gichev [25], I. A. Latypov [38], [39] and, in a more algebraic way, by E. B. Vinberg. We give a precise formulation of this correspondence and reformulate some facts on affine embeddings in terms of invariant algebras. Some results of this section are new and not published elsewhere.

The last section is devoted to G -algebras. It is easy to prove that any

subalgebra in the polynomial algebra $\mathbb{K}[x]$ is finitely generated. On the other hand, one can construct many non-finitely generated subalgebras in $\mathbb{K}[x_1, \dots, x_n]$ for $n \geq 2$. More generally, every subalgebra in an associative commutative finitely generated integral domain \mathfrak{A} with unit is finitely generated if and only if $\text{Kdim } \mathfrak{A} \leq 1$, where $\text{Kdim } \mathfrak{A}$ is the Krull dimension of \mathfrak{A} (Proposition 6.5). In Section 6 we obtain an equivariant version of this result. The problem was motivated by the study of invariant algebras in the previous section. The proof of the main result (Theorem 6.3) is based on a geometric method for constructing a non-finitely generated subalgebra in a finitely generated G -algebra and on properties of affine embeddings obtained above. In particular, the notion of an affinely closed space is crucial for the classification of G -algebras with finitely generated invariant subalgebras. The arguments used in this text are slightly different from the original ones [9]. A characterization of G -algebras with finitely generated invariant subalgebras for non-reductive G is also given in this section.

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1 Remarkable classes of affine embeddings

1.1 Affine toric varieties

We begin with some notation. Let T be an algebraic torus and $\Xi(T)$ the lattice of its characters. A T -action on an affine variety X defines a $\Xi(T)$ -grading on the algebra $\mathbb{K}[X] = \bigoplus_{\chi \in \Xi(T)} \mathbb{K}[X]_{\chi}$, where $\mathbb{K}[X]_{\chi} = \{f \mid tf = \chi(t)f \text{ for any } t \in T\}$. (This grading is just the isotypic decomposition, see the introduction.) If X is irreducible, then the set $L(X) = \{\chi \mid \mathbb{K}[X]_{\chi} \neq 0\}$ is a submonoid in $\Xi(T)$.

Definition 1.1. An *affine toric variety* X is a normal affine T -variety with an open T -orbit isomorphic to T .

Below we list some basic properties of T -actions:

- An action $T : X$ has an open orbit if and only if $\dim \mathbb{K}[X]_\chi = 1$ for any $\chi \in L(X)$. In this situation $\mathbb{K}[X]$ is T -isomorphic to the semigroup algebra $\mathbb{K}L(X)$.
- An action $T : X$ is effective if and only if the subgroup in $\Xi(T)$ generated by $L(X)$ coincides with $\Xi(T)$.
- Suppose that $T : X$ is an effective action with an open orbit. Then the following conditions are equivalent:
 - 1) X is normal;
 - 2) the semigroup algebra $\mathbb{K}L(X)$ is integrally closed;
 - 3) if $\chi \in \Xi(T)$ and there exists $n \in \mathbb{N}$, $n > 0$, such that $n\chi \in L(X)$, then $\chi \in L(X)$ (the saturation condition);
 - 4) there exists a solid convex polyhedral cone K in $\Xi(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $L(X) = K \cap \Xi(T)$.

In this situation, any T -invariant radical ideal of $\mathbb{K}[X]$ corresponds to the subsemigroup $L(X) \setminus M$ for a fixed face M of the cone K . This correspondence defines a bijection between T -invariant radical ideals of $\mathbb{K}[X]$ and faces of K .

The proof of these properties can be found, for example, in [23]. Summarizing all the results, we obtain

- Theorem 1.2.1)** *Affine toric varieties are in one-to-one correspondence with solid convex polyhedral cones in the space $\Xi(T) \otimes_{\mathbb{Z}} \mathbb{Q}$;*
- 2) *T -orbits on a toric variety are in one-to-one correspondence with faces of the cone.*

The classification of affine toric varieties will serve us as a sample for studying more complicated classes of affine embeddings. Generalizations of a combinatorial description of toric varieties were obtained for spherical varieties [45, 33, 17], and for embeddings of complexity one [68]. In this more general context, the idea that normal G -varieties may be described by some convex cones becomes rigorous through the method of U -invariants developed by D. Luna and Th. Vust. The essence of this method is contained in the following theorem (see [72, 37, 54, 29]).

Theorem 1.3. *Let \mathfrak{A} be a G -algebra and U a maximal unipotent subgroup of G . Consider the following properties of an algebra:*

- 1) *it is finitely generated;*
- 2) *it has no nilpotent elements;*
- 3) *it has no zero divisors;*

4) *it is integrally closed.*

If (P) is any of these properties, then the algebra \mathfrak{A} has property (P) if and only if the algebra \mathfrak{A}^U has property (P).

We try to demonstrate briefly some applications of the method of U -invariants in the following subsections.

1.2 Normal affine $SL(2)$ -embeddings

Suppose that the group $SL(2)$ acts on a normal affine variety X and there is a point $x \in X$ such that the stabilizer of x is trivial and the orbit $SL(2)x$ is open in X . We say in this case that X is a *normal $SL(2)$ -embedding*.

Let U_m be a finite extension of the standard maximal unipotent subgroup in $SL(2)$:

$$U_m = \left\{ \begin{pmatrix} \epsilon & a \\ 0 & \epsilon^{-1} \end{pmatrix} \mid \epsilon^m = 1, a \in \mathbb{K} \right\}.$$

Theorem 1.4 ([52]). *Normal non-trivial $SL(2)$ -embeddings are in one-to-one correspondence with rational numbers $h \in (0, 1]$. Furthermore,*

- 1) $h = 1$ corresponds to a (unique) smooth $SL(2)$ -embedding with two orbits: $X = SL(2) \cup SL(2)/T$;
- 2) if $h = \frac{p}{q} < 1$ and $(p, q) = 1$, then $X = SL(2) \cup SL(2)/U_{p+q} \cup \{pt\}$, and $\{pt\}$ is an isolated singular point in X .

The proof of Theorem 1.4 can be found in [52], [37, Ch. 3]. Here we give only some examples and explain what the number h (which is called the *height* of X) means in terms of the algebra $\mathbb{K}[X]$.

Example 1.5. 1) The group $SL(2)$ acts tautologically on the space \mathbb{K}^2 and by conjugation on the space $\text{Mat}(2 \times 2)$. Consider the point

$$x = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \in \text{Mat}(2 \times 2) \times \mathbb{K}^2$$

and its orbit

$$SL(2)x = \{(A, v) \mid \det A = -1, \text{tr } A = 0, Av = v, v \neq 0\}.$$

It is easy to see that the closure

$$X = \overline{SL(2)x} = \{(A, v) \mid \det A = -1, \text{tr } A = 0, Av = v\}$$

is a smooth $SL(2)$ -embedding with two orbits, hence X corresponds to $h = 1$.

2) Let $V_d = \langle x^d, x^{d-1}y, \dots, y^d \rangle$ be the $SL(2)$ -module of binary forms of degree d . It is possible to check that

$$X = \overline{SL(2)(x, x^2y)} \subset V_1 \oplus V_3$$

is a normal $SL(2)$ -embedding with the orbit decomposition $X = SL(2) \cup SL(2)/U_3 \cup \{pt\}$, hence X corresponds to $h = \frac{1}{2}$.

An embedding $SL(2) \hookrightarrow X$, $g \rightarrow gx$ determines the injective homomorphism $\mathfrak{A} = \mathbb{K}[X] \rightarrow \mathbb{K}[SL(2)]$ with $Q\mathfrak{A} = Q\mathbb{K}[SL(2)]$, where $Q\mathfrak{A}$ is the quotient field of \mathfrak{A} . Let U^- be the unipotent subgroup of $SL(2)$ opposite to U . Then

$$\mathbb{K}[SL(2)]^{U^-} = \{f \in \mathbb{K}[SL(2)] \mid f(ug) = f(g), g \in SL(2), u \in U^-\} = \mathbb{K}[A, B],$$

where $A \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a$ and $B \begin{pmatrix} a & b \\ c & d \end{pmatrix} = b$.

Below we list some facts ([37, Ch. 3]) that allow us to introduce the height of an $SL(2)$ -embedding X .

- If \mathfrak{C} is an integral F -domain, where F is a unipotent group, then $Q(\mathfrak{C}^F) = (Q\mathfrak{C})^F$. In particular, if $\mathfrak{C} \subseteq \mathfrak{A}$ and $Q\mathfrak{A} = Q\mathfrak{C}$, then $Q(\mathfrak{A}^{U^-}) = Q(\mathfrak{C}^{U^-})$.
- Suppose that $\lim_{t \rightarrow 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} x$ exists. Then $A \in \mathbb{K}[SL(2)] \subset \mathbb{K}(X)$ is regular on X .
- Let $\mathfrak{D} \subset \mathbb{K}[x, y]$ be a homogeneous integrally closed subalgebra in the polynomial algebra such that $Q\mathfrak{D} = \mathbb{K}(x, y)$ and $x \in \mathfrak{D}$. Then \mathfrak{D} is generated by monomials.

In our situation, the algebra $\mathfrak{D} = \mathfrak{A}^{U^-} \subset \mathbb{K}[A, B]$ is homogeneous because it is T -stable (since T normalizes U^-).

- There exists rational $h \in (0, 1]$ such that

$$\mathfrak{A}^{U^-} = \mathfrak{A}(h) = \langle A^i B^j \mid \frac{j}{i} \leq h \rangle.$$

Moreover, for any rational $h \in (0, 1]$ the subspace $\langle SL(2)\mathfrak{A}(h) \rangle \subset \mathbb{K}[SL(2)]$ is a subalgebra.

Remark . While normal $SL(2)$ -embeddings are parametrized by a discrete parameter h , there are families of non-isomorphic non-normal $SL(2)$ -embeddings over a base of arbitrary dimension [13].

Remark . A classification of $SL(2)$ -actions on normal three-dimensional affine varieties without open orbit can be found in [6], [5].

1.3 HV-varieties and S-varieties

In this subsection we discuss the results of V. L. Popov and E. B. Vinberg [55]. Suppose that G is a connected and simply connected semisimple group.

Definition 1.6. An *HV-variety* X is the closure of the orbit of a highest weight vector in a simple G -module.

Let $V(\lambda)$ be the simple G -module with highest weight λ and v_λ a highest weight vector in $V(\lambda)$. Denote by λ^* the highest weight of the dual G -module $V(\lambda)^*$.

- $X(\lambda) = \overline{Gv_{\lambda^*}}$ is a normal affine variety consisting of two orbits:
 $X(\lambda) = Gv_{\lambda^*} \cup \{0\}$.
- $\mathbb{K}[X(\lambda)] = \mathbb{K}[Gv_{\lambda^*}] = \bigoplus_{m \geq 0} \mathbb{K}[X(\lambda)]_{m\lambda}$, any isotypic component $\mathbb{K}[X(\lambda)]_{m\lambda}$ is a simple G -module, and

$$\mathbb{K}[X(\lambda)]_{m_1\lambda} \mathbb{K}[X(\lambda)]_{m_2\lambda} = \mathbb{K}[X(\lambda)]_{(m_1+m_2)\lambda}.$$

- The algebra $\mathbb{K}[X(\lambda)]$ is a unique factorization domain if and only if λ is a fundamental weight of G .

Example 1.7. 1) The quadratic cone $KQ_n = \{x \in \mathbb{K}^n \mid x_1^2 + \dots + x_n^2 = 0\}$ ($n \geq 3$) is an *HV-variety* for the tautological representation $SO(n) : \mathbb{K}^n$. (In fact, the group $SO(n)$ is not simply connected and we consider the corresponding module as a $\text{Spin}(n)$ -module.) It follows that KQ_n is normal and it is factorial if and only if $n \geq 5$.

2) The Grassmannian cone $KG_{n,m}$ ($n \geq 2$, $1 \leq m \leq n-1$) (i.e. the cone over the projective variety of m -subspaces in \mathbb{K}^n) is an *HV-variety* associated with the fundamental $SL(n)$ -representation in the space $\bigwedge^m \mathbb{K}^n$, hence it is factorial.

Definition 1.8. An irreducible affine variety X with an action of a connected reductive group G is said to be an *S-variety* if X has an open G -orbit and the stabilizer of a point in this orbit contains a maximal unipotent subgroup of G .

Any *S-variety* may be realized as $X = \overline{Gv}$, where $v = v_{\lambda_1^*} + \dots + v_{\lambda_k^*}$

is a sum of highest weight vectors $v_{\lambda_i^*}$ in some G -module V . We have the isotypic decomposition

$$\mathbb{K}[X] = \bigoplus_{\lambda \in L(X)} \mathbb{K}[X]_{\lambda},$$

where $L(X)$ is the semigroup generated by $\lambda_1, \dots, \lambda_k$, any $\mathbb{K}[X]_{\lambda}$ is a simple G -module, and $\mathbb{K}[X]_{\lambda} \mathbb{K}[X]_{\mu} = \mathbb{K}[X]_{\lambda+\mu}$. The last condition determines uniquely (up to G -isomorphism) the multiplicative structure on the G -module $\mathbb{K}[X]$. This shows that there is a bijection between S -varieties and finitely generated submonoids in $\Xi_+(G)$.

Consider the cone $K = \mathbb{Q}_+ L(X)$. As in the toric case, normality of X is equivalent to the saturation condition for the semigroup $L(X)$, and G -orbits on X are in one-to-one correspondence with faces of K . On the other hand, there are phenomena which are specific for S -varieties. For example, the complement to the open orbit in X has codimension ≥ 2 if and only if $\mathbb{Z}L(X) \cap \Xi_+(G) \subseteq \mathbb{Q}_+ L(X)$ (this is never the case for non-trivial toric varieties). For semisimple simply connected G , an S -variety X is factorial if and only if $L(X)$ is generated by fundamental weights.

Finally, we mention one more result on this subject. Say that an action $G : X$ on an affine variety X is *special* (or *horospherical*) if there is an open dense subset $W \subset X$ such that the stabilizer of any point of W contains a maximal unipotent subgroup of G .

Theorem 1.9 ([54]). *The following conditions are equivalent:*

- 1) *the action $G : X$ is special;*
- 2) *the stabilizer of any point on X contains a maximal unipotent subgroup;*
- 3) $\mathbb{K}[X]_{\lambda} \mathbb{K}[X]_{\mu} \subseteq \mathbb{K}[X]_{\lambda+\mu}$ *for any $\lambda, \mu \in \Xi_+(G)$.*

1.4 Algebraic monoids

The general theory of algebraic semigroups was developed by M. S. Putcha, L. Renner and E. B. Vinberg. In this subsection we recall briefly the classification results following [71].

Definition 1.10. An (*affine*) *algebraic semigroup* is an (*affine*) algebraic variety S with an associative multiplication

$$\mu : S \times S \rightarrow S,$$

which is a morphism of algebraic varieties. An algebraic semigroup S is *normal* if S is a normal algebraic variety.

Any algebraic group is an algebraic semigroup. Another example is the semigroup $\text{End}(V)$ of endomorphisms of a finite-dimensional vector space V .

Lemma 1.11. *An affine algebraic semigroup S is isomorphic to a closed subsemigroup of $\text{End}(V)$ for a suitable V . If S has a unit, one may assume that it corresponds to the identity map of V .*

Proof. The morphism $\mu : S \times S \rightarrow S$ induces the homomorphism $\mu^* : \mathbb{K}[S] \rightarrow \mathbb{K}[S] \otimes \mathbb{K}[S]$, $f(s) \mapsto F(s_1, s_2) := f(s_1 s_2)$. Hence $f(s_1 s_2) = \sum_{i=1}^n f_i(s_1) h_i(s_2)$. Consider the linear action $S : \mathbb{K}[S]$ defined by $(s * f)(x) = f(xs)$. One has $\langle Sf \rangle \subseteq \langle f_1, \dots, f_n \rangle$, i.e. the linear span of any ‘ S -orbit’ in $\mathbb{K}[S]$ is finite-dimensional and the linear action $S : \langle Sf \rangle$ defines an algebraic representation of S . Take as V any finite-dimensional S -invariant subspace of $\mathbb{K}[S]$ containing a system of generators of $\mathbb{K}[S]$.

Suppose that S is a monoid, i.e. a semigroup with unit. We claim that the action $S : V$ defines a closed embedding $\phi : S \rightarrow \text{End}(V)$. Indeed, there are $\alpha_{ij} \in \mathbb{K}[S]$ such that $s * f_i = \sum_j \alpha_{ij}(s) f_j$. The equalities $f_i(s) = (s * f_i)(e) = \sum_j \alpha_{ij}(s) f_j(e)$ show that the homomorphism $\phi^* : \mathbb{K}[\text{End}(V)] \rightarrow \mathbb{K}[S]$ is surjective.

The general case can be reduced to the previous one as follows: to any semigroup S one may add an element e with relations $e^2 = e$ and $es = se = s$ for any $s \in S$. Then $\tilde{S} = S \sqcup \{e\}$ is an algebraic monoid. \square

If $S \subseteq \text{End}(V)$ is a monoid, then any invertible element of S corresponds to an element of $GL(V)$. Conversely, if the image of s is invertible in $\text{End}(V)$, then it is invertible in S . Indeed, the sequence of closed subsets $S \supseteq sS \supseteq s^2S \supseteq s^3S \supseteq \dots$ stabilizes, and $s^kS = s^{k+1}S$ implies $S = sS$. Hence the group $G(S)$ of invertible elements is open in S and is an algebraic group. Suppose that $G(S)$ is dense in S . Then S may be considered as an affine embedding of $G(S)/\{e\}$ (with respect to left multiplication).

Proposition 1.12. *Let G be an algebraic group. An affine embedding $G/\{e\} \hookrightarrow S$ has a structure of an algebraic monoid with G as the group of invertible elements if and only if the G -equivariant G -action on the open orbit by right multiplication can be extended to S , or, equivalently, S is an affine embedding of $(G \times G)/\Delta(G)$, where $\Delta(G)$ is the diagonal in $G \times G$.*

Proof. If S is an algebraic monoid with $G(S) = G$ and $G(S)$ is dense in

S , then $G \times G$ acts on S by $((g_1, g_2), s) \mapsto g_1 s g_2^{-1}$ and the dense open $G \times G$ -orbit in S is isomorphic to $(G \times G)/\Delta(G)$.

For the converse, we give two independent proofs in their historical order.

Proof One (the reductive case). (E.B. Vinberg [71]) An algebraic monoid S is *reductive* if the group $G(S)$ is reductive and dense in S . The multiplication $\mu : G \times G \rightarrow G$ corresponds to the comultiplication $\mu^* : \mathbb{K}[G] \rightarrow \mathbb{K}[G] \otimes \mathbb{K}[G]$. Any $(G \times G)$ -isotypic component in $\mathbb{K}[G]$ is a simple $(G \times G)$ -module isomorphic to $V(\lambda)^* \otimes V(\lambda)$ for $\lambda \in \Xi_+(G)$ [37]. It coincides with the linear span of the matrix entries of the G -module $V(\lambda)$. This shows that μ^* maps an isotypic component to its tensor square, and for any $(G \times G)$ -invariant subspace $W \subset \mathbb{K}[G]$ one has $\mu^*(W) \subset W \otimes W$. Thus the spectrum S of any $(G \times G)$ -invariant finitely generated subalgebra in $\mathbb{K}[G]$ carries the structure of an algebraic semi-group. If the open $(G \times G)$ -orbit in S is isomorphic to $(G \times G)/\Delta(G)$, then $G(S) = G$. Indeed, G is dense in S and for any $s \in G(S)$ the intersection $sG \cap G \neq \emptyset$, hence $s \in G$.

Proof Two (the general case). (A. Rittatore [59]) If the multiplication $\mu : G \times G \rightarrow G$ extends to a morphism $\mu : S \times S \rightarrow S$, then μ is a multiplication because μ is associative on $G \times G$. It is clear that $1 \in G$ satisfies $1s = s1 = s$ for all $s \in S$. Consider the right and left actions of G given by

$$G \times S \rightarrow S, \quad gs = (g, 1)s,$$

$$S \times G \rightarrow S, \quad sg = (1, g^{-1})s.$$

These actions define coactions $\mathbb{K}[S] \rightarrow \mathbb{K}[G] \otimes \mathbb{K}[S]$ and $\mathbb{K}[S] \rightarrow \mathbb{K}[S] \otimes \mathbb{K}[G]$, which are the restrictions to $\mathbb{K}[S]$ of the comultiplication $\mathbb{K}[G] \rightarrow \mathbb{K}[G] \otimes \mathbb{K}[G]$. Hence the image of $\mathbb{K}[S]$ lies in

$$(\mathbb{K}[G] \otimes \mathbb{K}[S]) \cap (\mathbb{K}[S] \otimes \mathbb{K}[G]) = \mathbb{K}[S] \otimes \mathbb{K}[S],$$

and we have a multiplication on S . The equality $G(S) = G$ may be proved as above. \square

For the rest of this section we assume that G is reductive. For $\lambda_1, \lambda_2 \in \Xi_+(G)$, we denote by $\Xi(\lambda_1, \lambda_2)$ the set of $\lambda \in \Xi_+(G)$ such that the G -module $V(\lambda_1) \otimes V(\lambda_2)$ contains a submodule isomorphic to $V(\lambda)$. Since any $(G \times G)$ -isotypic component $\mathbb{K}[G]_{(\lambda^*, \lambda)}$ in $\mathbb{K}[G]$ is the linear span of the matrix entries corresponding to the representation of G in $V(\lambda)$, the

product $\mathbb{K}[G]_{(\lambda_1^*, \lambda_1)} \mathbb{K}[G]_{(\lambda_2^*, \lambda_2)}$ is the linear span of the matrix entries corresponding to $V(\lambda_1) \otimes V(\lambda_2)$. This shows that

$$\mathbb{K}[G]_{(\lambda_1^*, \lambda_1)} \mathbb{K}[G]_{(\lambda_2^*, \lambda_2)} = \bigoplus_{\lambda \in \Xi(\lambda_1, \lambda_2)} \mathbb{K}[G]_{(\lambda^*, \lambda)}.$$

Since every $(G \times G)$ -isotypic component in $\mathbb{K}[G]$ is simple, any $(G \times G)$ -invariant subalgebra in $\mathbb{K}[G]$ is determined by the semigroup of dominant weights that appear in its isotypic decomposition, and it is natural to classify reductive algebraic monoids S with $G(S) = G$ in terms of the semigroup that determines $\mathbb{K}[S]$ in $\mathbb{K}[G]$.

Definition 1.13. A subsemigroup $L \subset \Xi_+(G)$ is *perfect* if it contains zero and $\lambda_1, \lambda_2 \in L$ implies $\Xi(\lambda_1, \lambda_2) \subset L$.

Let $\mathbb{Z}\Xi_+(G)$ be the group generated by the semigroup $\Xi_+(G)$. This group may be realized as the group of characters $\Xi(T)$ of a maximal torus of G .

Theorem 1.14 ([71]). *A subset $L \subset \Xi_+(G)$ defines an affine algebraic monoid S with $G(S) = G$ if and only if L is a perfect finitely generated subsemigroup generating the group $\mathbb{Z}\Xi_+(G)$.*

The classification of normal affine reductive monoids is more constructive. We fix some notation. The group $G = ZG'$ is an almost direct product of its center Z and the derived subgroup G' . Fix a Borel subgroup B_0 and a maximal torus $T_0 \subset B_0$ in G' . Then $B = ZB_0$ (resp. $T = ZT_0$) is a Borel subgroup (resp. a maximal torus) in G . By N (resp. N_0, N_1) denote \mathbb{Q} -vector space $\Xi(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $\Xi(T_0) \otimes_{\mathbb{Z}} \mathbb{Q}, \Xi(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$). Then $N = N_1 \oplus N_0$. The semigroup of dominant weights $\Xi_+(G)$ (with respect to B) is a subsemigroup in $\Xi(T) \subset N$. By $\alpha_1, \dots, \alpha_k \in N_1$ denote the simple roots of G with respect to B , and by $C \subset N$ (resp. $C_0 \subset N_0$) the positive Weyl chamber for the group G (resp. G') with respect to $\alpha_1, \dots, \alpha_k$.

Theorem 1.15 ([71]). *A subset $L \subset \Xi_+(G)$ defines a normal affine algebraic monoid S with $G(S) = G$ if and only if $L = \Xi_+(G) \cap K$, where K is a closed convex polyhedral cone in N satisfying the conditions:*

- 1) $-\alpha_1, \dots, -\alpha_k \in K$;
- 2) the cone $K \cap C$ generates N .

The monoid S has a zero if and only if:

- 3) the cone $K \cap N_1$ is pointed;

4) $K \cap C_0 = \{0\}$.

A characteristic-free approach to the classification of reductive algebraic monoids via the theory of spherical varieties was developed in [59]. Another interesting result of [59] is that any reductive algebraic monoid is affine. Recently A. Rittatore announced a proof of the fact that any algebraic monoid with an affine algebraic group of invertible elements is affine.

2 Connections with Hilbert's 14th Problem

2.1 Grosshans subgroups and the canonical embedding

Let H be a closed subgroup of $GL(V)$. Hilbert's 14th problem (in its modern version) may be formulated as follows: characterize subgroups H such that the algebra of polynomial invariants $\mathbb{K}[V]^H$ is finitely generated. It is a classical result that for reductive H the algebra $\mathbb{K}[V]^H$ is finitely generated. For non-reductive linear groups this problem seems to be very far from a complete solution.

Remark . Hilbert's original statement of the problem was the following:

For a field \mathbb{K} , let $\mathbb{K}[x_1, \dots, x_n]$ denote the polynomial ring in n variables over \mathbb{K} , and let $\mathbb{K}(x_1, \dots, x_n)$ denote its field of fractions. If K is a subfield of $\mathbb{K}(x_1, \dots, x_n)$ containing \mathbb{K} , is $K \cap \mathbb{K}[x_1, \dots, x_n]$ finitely generated over \mathbb{K} ?

Since $\mathbb{K}[V]^H = \mathbb{K}[V] \cap \mathbb{K}(V)^H$, our situation may be regarded as a particular case of the general one.

Let us assume that H is a subgroup of a bigger reductive group G acting on V . (For example, one may take $G = GL(V)$.) The intersection of a family of observable subgroups in G is an observable subgroup. Define the observable hull \hat{H} of H as the minimal observable subgroup of G containing H . The stabilizer of any H -fixed vector in a rational G -module contains \hat{H} . Therefore $\mathbb{K}[V]^H = \mathbb{K}[V]^{\hat{H}}$ for any G -module V , and it is natural to solve Hilbert's 14th problem for observable subgroups.

The following famous theorem proved by F. D. Grosshans establishes a close connection between Hilbert's 14th problem and the theory of affine embeddings.

Theorem 2.1 ([27, 29]). *Let H be an observable subgroup of a reductive group G . The following conditions are equivalent:*

- 1) for any G -module V the algebra $\mathbb{K}[V]^H$ is finitely generated;
- 2) the algebra $\mathbb{K}[G/H]$ is finitely generated;
- 3) there exists an affine embedding $G/H \hookrightarrow X$ such that

$$\text{codim}_X(X \setminus (G/H)) \geq 2.$$

Definition 2.2. 1) An observable subgroup H in G is said to be a *Grosshans subgroup* if $\mathbb{K}[G/H]$ is finitely generated.

2) If H is a Grosshans subgroup of G , then $G/H \hookrightarrow X = \text{Spec } \mathbb{K}[G/H]$ is called *the canonical embedding* of G/H , and X is denoted by $CE(G/H)$.

Note that any normal affine embedding $G/H \hookrightarrow X$ with $\text{codim}_X(X \setminus (G/H)) \geq 2$ is G -isomorphic to the canonical embedding [29]. A homogeneous space G/H admits such an embedding if and only if H is a Grosshans subgroup.

By Matsushima's criterion, H is reductive if and only if $CE(G/H) = G/H$. For non-reductive subgroups, $CE(G/H)$ is an interesting object canonically associated with the pair (G, H) . It allows us to reformulate algebraic problems concerning the algebra $\mathbb{K}[G/H]$ in geometric terms.

2.2 Popov-Pommerening's conjecture and Knop's theorem

Theorem 2.3 ([28, 19],[29, Th.16.4]). *Let P^u be the unipotent radical of a parabolic subgroup P of G . Then P^u is a Grosshans subgroup of G .*

Proof. Let $P = LP^u$ be a Levi decomposition and U_1 a maximal unipotent subgroup of L . Then $U = U_1P^u$ is a maximal unipotent subgroup of G , and $\mathbb{K}[G]^U = (\mathbb{K}[G]^{P^u})^{U_1}$. We know that $\mathbb{K}[G]^U$ is finitely generated (Theorem 1.3). On the other hand, Theorem 1.3 implies that the L -algebra $\mathbb{K}[G]^{P^u}$ is finitely generated if and only if $(\mathbb{K}[G]^{P^u})^{U_1}$ is, hence $\mathbb{K}[G]^{P^u}$ is finitely generated. (Another proof, using an explicit codimension 2 embedding, is given in [28].) \square

Let us say that a subgroup of a reductive group G is *regular* if it is normalized by a maximal torus in G . Generalizing Theorem 2.3, V. L. Popov and K. Pommerening conjectured that any observable regular subgroup is a Grosshans subgroup. At the moment a positive answer is known for groups G of small rank [64, 66, 65], and for some special classes of regular subgroups (for example, for unipotent radicals of parabolic subgroups of Levi subgroups of G [29]). Lin Tan [65] constructed explicitly canonical embeddings for regular unipotent subgroups in $SL(n)$, $n \leq 5$. A strong argument in favour of Popov-Pommerening's conjecture

is given in [14, Th.4.3] in terms of finite generation of induced modules, see also [29, § 23].

Another powerful method for checking that the algebra $\mathbb{K}[G/H]$ is finitely generated is provided by the following theorem of F. Knop.

Theorem 2.4 ([35, 29]). *Suppose that G acts on an irreducible normal unirational variety X . If $c(X) \leq 1$, then the algebra $\mathbb{K}[X]$ is finitely generated.*

Corollary . *If H is observable in G and $c(G/H) \leq 1$, then H is a Grosshans subgroup.*

2.3 The canonical embedding of G/P^u

Since the unipotent radical P^u of a parabolic subgroup P is a Grosshans subgroup of G , there exists a canonical embedding $G/P^u \hookrightarrow CE(G/P^u)$. Such embeddings provide an interesting class of affine factorial G -varieties, which was studied in [12]. Let us note that the Levi subgroup $L \subset P$ normalizes P^u , hence acts G -equivariantly on G/P^u and on $CE(G/P^u)$. By $V_L(\lambda)$ denote a simple L -module with the highest weight λ . Our approach is based on the analysis of the $(G \times L)$ -module decomposition of the algebra $\mathbb{K}[G/P^u]$ given by

$$\mathbb{K}[G/P^u] = \bigoplus_{\lambda \in \Xi_+(G)} \mathbb{K}[G/P^u]_\lambda,$$

where $\mathbb{K}[G/P^u]_\lambda \cong V(\lambda)^* \otimes V_L(\lambda)$ is the linear span of the matrix entries of the linear maps $V(\lambda)^{P^u} \rightarrow V(\lambda)$ induced by $g \in G$, considered as regular functions on G/P^u . (In fact, our method works for any affine embedding $G/P^u \hookrightarrow X$, where L acts G -equivariantly.) The multiplication structure looks like

$$\mathbb{K}[G/P^u]_\lambda \cdot \mathbb{K}[G/P^u]_\mu = \mathbb{K}[G/P^u]_{\lambda+\mu} \oplus \bigoplus_i \mathbb{K}[G/P^u]_{\lambda+\mu-\beta_i},$$

where $\lambda + \mu - \beta_i$ runs over the highest weights of all “lower” irreducible components in the L -module decomposition $V_L(\lambda) \otimes V_L(\mu) = V_L(\lambda + \mu) \oplus \dots$

Here we list the results from [12].

- Affine $(G \times L)$ -embeddings $G/P^u \hookrightarrow X$ are classified by finitely generated subsemigroups S of $\Xi_+(G)$ having the property that all highest weights of the tensor product of simple L -modules with highest

weights in S belong to S , too. Furthermore, every choice of the generators $\lambda_1, \dots, \lambda_m \in S$ gives rise to a natural G -equivariant embedding $X \hookrightarrow \text{Hom}(V^{P^u}, V)$, where V is the sum of simple G -modules of highest weights $\lambda_1, \dots, \lambda_m$. The convex cone Σ^+ spanned by S is precisely the dominant part of the cone Σ spanned by the weight polytope of V^{P^u} . In the case $X = CE(G/P^u)$, the semigroup S coincides with $\Xi_+(G)$ and Σ is the span of the dominant Weyl chamber by the Weyl group of L . In particular, if G is simply connected and semisimple then there is a natural inclusion

$$CE(G/P^u) \subset \bigoplus_{i=1}^l \text{Hom}(V(\omega_i)^{P^u}, V(\omega_i)),$$

where $\omega_1, \dots, \omega_l$ are the fundamental weights of G .

- The $(G \times L)$ -orbits in X are in bijection with the faces of Σ whose interiors contain dominant weights, the orbit representatives being given by the projectors onto the subspaces of V^{P^u} spanned by eigenvectors of eigenweights in a given face. For the canonical embedding, the $(G \times L)$ -orbits correspond to the subdiagrams in the Dynkin diagram of G such that no connected component of such a subdiagram is contained in the Dynkin diagram of L . We also compute the stabilizers of points in $G \times L$ and in G , and the modality of the action $G : X$.

- We classify smooth affine $(G \times L)$ -embeddings $G/P^u \hookrightarrow X$. In particular, the only non-trivial smooth canonical embedding corresponds to $G = SL(n)$, P is the stabilizer of a hyperplane in \mathbb{K}^n , and $CE(G/P^u) = \text{Mat}(n, n-1)$ with the G -action by left multiplication.

- The techniques used in the description of affine $(G \times L)$ -embeddings of G/P^u are parallel to those developed in [69] for the study of equivariant compactifications of reductive groups. An analogy with monoids becomes more transparent in view of the bijection between our affine embeddings $G/P^u \hookrightarrow X$ and a class of algebraic monoids M with the group of invertibles L , given by $X = \text{Spec } \mathbb{K}[G \times^P M]$.

- Finally, we describe the G -module structure of the tangent space of $CE(G/P^u)$ at the G -fixed point, assuming that G is **{simply connected and simple.}** This space is obtained from $\bigoplus_i \text{Hom}(V(\omega_i)^{P^u}, V(\omega_i))$ by removing certain summands according to an explicit algorithm. The tangent space at the fixed point is at the same time the minimal ambient G -module for $CE(G/P^u)$.

Author query: Does this mean sim. conn. (topologically) and simple (algebraically)?

2.4 Counterexamples

The famous counterexample of Nagata to Hilbert's 14th problem [49] yields a 13-dimensional unipotent subgroup H in $SL(32)$ acting naturally in $V = \mathbb{K}^{32}$ such that the algebra of invariants $\mathbb{K}[V]^H$ is not finitely generated. This shows that the algebra $\mathbb{K}[SL(32)/H]$ is not finitely generated, or, equivalently, the complement of the open orbit in any affine embedding $SL(32)/H \hookrightarrow X$ contains a divisor.

Nagata's construction was simplified by R. Steinberg. He proved that $\mathbb{K}[V]^H$ is not finitely generated for the following 6-dimensional commutative unipotent linear group:

$$H = \left\{ \left(\begin{array}{cccccc} 1 & c_1 & & & & \\ 0 & 1 & & & & 0 \\ & & \ddots & \ddots & & \\ & & & & & \\ & 0 & & & 1 & c_9 \\ & & & & 0 & 1 \end{array} \right), \sum_{j=1}^9 a_{ij}c_j = 0, i = 1, 2, 3 \right\},$$

where the nine points $P_j = (a_{1j} : a_{2j} : a_{3j})$ are nonsingular points on an irreducible cubic curve in the projective plane, their sum has infinite order in the group of the curve, and $V = \mathbb{K}^{18}$ (see [62] for details).

Another method of obtaining counterexamples was proposed by P. Roberts [60]. Consider the polynomial algebra $R = \mathbb{K}[x, y, z, s, t, u, v]$ in 7 variables over a not necessarily algebraically closed field \mathbb{K} of characteristic zero with the grading $R = \bigoplus_{n \geq 0} R_n$ determined by assigning the degree 0 to x, y, z and the degree 1 to s, t, u, v . The elements s, t, u, v generate a free R_0 -submodule in R considered as R_0 -module. Choosing a natural number $m \geq 2$, Roberts defines an R_0 -module homomorphism on this submodule

$$f : R_0s \oplus R_0t \oplus R_0u \oplus R_0v \rightarrow R_0$$

given by $f(s) = x^{m+1}$, $f(t) = y^{m+1}$, $f(u) = z^{m+1}$, $f(v) = (xyz)^m$. The submodule $\text{Ker } f$ generates a subalgebra of R , which is denoted by A . It is proved in [60] that the \mathbb{K} -algebra $B = R \cap QA$ is not finitely generated. (Roberts shows how to construct an element in B of any given degree which is not in the subalgebra generated by elements of lower degree.) A linear action of a 12-dimensional commutative unipotent group on 19-dimensional vector space with the algebra of invariants isomorphic to the polynomial algebra in one variable over B is constructed in [1].

For a recent development in this direction, see [21, 22].

3 Some properties of affine embeddings

3.1 Affinely closed spaces and Luna's theorem

Definition 3.1. An affine homogeneous space G/H of an affine algebraic group G is called *affinely closed* if it admits only the trivial affine embedding $X = G/H$.

Assume that G is reductive; {then G/H affinely closed implies H Author reductive.} query By $N_G(H)$, $C_G(H)$ denote the normalizer and centralizer respectively of H in G , and by $W(H)$ denote the quotient $N_G(H)/H$. It is known that $N_G(H)^0 = H^0 C_G(H)^0$ and both $N_G(H)$ and $C_G(H)$ are reductive [44, Lemma 1.1].

Theorem 3.2 ([43]). *Let H be a reductive subgroup of a reductive group G . The homogeneous space G/H is affinely closed if and only if the group $W(H)$ is finite. Moreover, if G acts on an affine variety X and the stabilizer of a point $x \in X$ contains a reductive subgroup H such that $W(H)$ is finite, then the orbit Gx is closed in X .*

Remark . The last statement may be reformulated: if H is reductive, the group $W(H)$ is finite, and $H \subset H' \subset G$, where H' is observable, then H' is reductive and G/H' is affinely closed.

Remark . Let H be a Grosshans subgroup of G . The following conditions are equivalent:

- 1) H is reductive and $W(H)$ is finite;
- 2) H is reductive and for any one-parameter subgroup $\mu : \mathbb{K}^* \rightarrow C_G(H)$ one has $\mu(\mathbb{K}^*) \subseteq H$;
- 3) the algebra $\mathbb{K}[G/H]$ does not have non-trivial G -invariant ideals and does not admit non-trivial G -invariant \mathbb{Z} -gradings;
- 4) the algebra $\mathbb{K}[G/H]$ does not have non-trivial G -invariant ideals and the group of G -equivariant automorphisms of $\mathbb{K}[G/H]$ is finite.
- 5) no invariant subalgebra in $\mathbb{K}[G/H]$ admits a non-trivial G -invariant ideal.

Example 3.3. 1) Let $\rho : H \rightarrow SL(V)$ be an irreducible representation of a reductive group H . Then the space $SL(V)/\rho(H)$ is affinely closed ($W(\rho(H))$ is finite by the Schur Lemma).

2) If T is a maximal torus of G , then $W(T)$ is the Weyl group and G/T is affinely closed.

Proposition 3.4. *Let G be an affine algebraic group. The following conditions are equivalent:*

- 1) any monoid S with $G(S) = G$ and $\overline{G(S)} = S$ coincides with G ;
- 2) the group G/G^u is semisimple.

Proof. Let G be reductive. The space $(G \times G)/\Delta(G)$ is affinely closed if and only if the group $N_{G \times G}(\Delta(G))/\Delta(G)$ is finite. But this is exactly the case when the center of G is finite. The same arguments work for any G (Theorem 3.9). \square

We now give a proof of Theorem 3.2 in terms of so-called adapted (or optimal) one-parameter subgroups following G. Kempf [31, Cor.4.5].

We have to prove that if G/H' is a quasi-affine homogeneous space that is not affinely closed and $H \subset H'$ is a reductive subgroup, then there exists a one-parameter subgroup $\nu : \mathbb{K}^* \rightarrow C_G(H)$ such that $\nu(\mathbb{K}^*)$ is not contained in H . There is an affine embedding $G/H' \hookrightarrow X$ with a G -fixed point o , see 3.5. Denote by x the image of eH' in the open orbit on X . By the Hilbert-Mumford criterion, there exists a one-parameter subgroup $\gamma : \mathbb{K}^* \rightarrow G$ such that $\lim_{t \rightarrow 0} \gamma(t)x = o$. Moreover, there is a subgroup γ that moves x ‘most rapidly’ toward o . Such a γ is called *adapted to x* ; for the precise definition see [31, 56]. For adapted γ , consider the parabolic subgroup

$$P(\gamma) = \{g \in G \mid \lim_{t \rightarrow 0} \gamma(t)g\gamma(t)^{-1} \text{ exists in } G \}.$$

Then $P(\gamma) = L(\gamma)U(\gamma)$, where $L(\gamma)$ is the Levi subgroup that is the centralizer of $\gamma(\mathbb{K}^*)$ in G , and $U(\gamma)$ is the unipotent radical of $P(\gamma)$. By [31], [56, Th.5.5], the stabilizer $G_x = H'$ is contained in $P(\gamma)$. Hence there is an element $u \in U(\gamma)$ such that $uHu^{-1} \subset L(\gamma)$.

We claim that $\gamma(\mathbb{K}^*)$ is not contained in uHu^{-1} . In fact, γ is adapted to the element ux , too [31, Th.3.4], hence $\gamma(\mathbb{K}^*)$ is not contained in the stabilizer of ux . Thus $u^{-1}\gamma u$ is the desired subgroup ν .

Conversely, suppose that there exists $\nu : \mathbb{K}^* \rightarrow C_G(H)$ and $\nu(\mathbb{K}^*)$ is not contained in H . Consider the subgroup $H_1 = \nu(\mathbb{K}^*)H$. The homogeneous fiber space $G *_H \mathbb{K}$, where H acts on \mathbb{K} trivially and H_1/H acts on \mathbb{K} by dilation, is a two-orbit embedding of G/H . \square

3.2 Affinely closed spaces in arbitrary characteristic

In this subsection we assume that \mathbb{K} is an arbitrary algebraically closed field. Suppose that G acts on an affine variety X . In positive characteristic, the structure of an algebraic variety on the orbit Gx of a point $x \in X$ is not determined (up to G -isomorphism) by the stabilizer

$H = G_x$, and it is natural to consider the isotropy subscheme \tilde{H} at x , with H as the reduced part, identifying Gx and G/\tilde{H} . There is a natural bijective purely inseparable and finite morphism $\pi : G/H \rightarrow G/\tilde{H}$ [30, 4.3, 4.6]. The following technical proposition shows that this difficulty does not play an essential role for affinely closed spaces.

Proposition 3.5 ([9, Prop. 8]). *The homogeneous space G/H is affinely closed if and only if G/\tilde{H} is affinely closed.*

Definition 3.6. We say that an affinely closed homogeneous space G/H is *strongly affinely closed* if for any affine G -variety X and any point $x \in X$ fixed by H the orbit Gx is closed in X .

By Theorem 3.2, in characteristic zero any affinely closed space is strongly affinely closed.

The following notion was introduced by J.-P. Serre, c.f. [41].

Definition 3.7. A subgroup $D \subset G$ is called *G -completely reducible* (G -cr for short) if, whenever D is contained in a parabolic subgroup P of G , it is contained in a Levi subgroup of P .

A G -cr subgroup is reductive. For $G = GL(V)$ this notion agrees with the usual notion of complete reducibility. In fact, if G is any of the classical groups then the notions coincide, although for the symplectic and orthogonal groups this requires the assumption that $\text{char } \mathbb{K}$ is a good prime for G . The class of G -cr subgroups is wide. Some conditions which guarantee that certain subgroups satisfy the G -cr condition can be found in [41, 46].

The proof of Theorem 3.2 given above implies:

- if H is not contained in a proper parabolic subgroup of G , then G/H is strongly affinely closed;
- if there exists $\nu : \mathbb{K}^* \rightarrow C_G(H)$ such that $\nu(\mathbb{K}^*)$ is not contained in H , then G/H is not affinely closed;
- if H is a G -cr subgroup of G , then the following conditions are equivalent:

- 1) G/H is affinely closed;
- 2) G/H is strongly affinely closed;
- 3) for any one-parameter subgroup $\nu : \mathbb{K}^* \rightarrow C_G(H)$ one has $\nu(\mathbb{K}^*) \subseteq H$.

Example 3.8. The following example produced by George J. McNinch shows that the group $W(H)$ may be unipotent even for reductive H .

Let L be the space of $(n \times n)$ -matrices and H the image of $SL(n)$ in $G = SL(L)$ acting on L by conjugation.

If $p = \text{char } \mathbb{K} \mid n$, then L is an indecomposable $SL(n)$ -module with three composition factors, c.f. [46, Prop. 4.6.10, a)]. It turns out that $C_G(H)^0$ is a one-dimensional unipotent group consisting of operators of the form $\text{Id} + aT$, where $a \in \mathbb{K}$, and T is a nilpotent operator on L defined by $T(X) = \text{tr}(X)E$. The subgroup H is contained in a quasi-parabolic subgroup of G , hence G/H is not strongly affinely closed.

In the simplest case $n = p = 2$, we have $H \cong PSL(2) \subset SL(4)$, $N_G(H) = HC_G(H)$ (because H does not have outer automorphisms), $C_G(H)$ is connected, and $W(H) \cong (\mathbb{K}, +)$.

It would be very interesting to obtain a complete description of affinely closed spaces in arbitrary characteristic and to answer the following question: is it true that any affinely closed space is strongly affinely closed?

3.3 Affinely closed spaces of non-reductive groups

For non-reductive G the class of affinely closed homogeneous spaces is much wider. For example, it is well-known that an orbit of a unipotent group acting on an affine variety is closed, hence any homogeneous space of a unipotent group is affinely closed. Conversely, if any (quasi-affine) homogeneous space of an affine group G is affinely closed, then the connected component of the identity in G is unipotent [15, 10.1], [20, Th.4.2]. In this subsection we give a complete characterization of affinely closed homogeneous spaces of non-reductive groups.

Let us fix the Levi decomposition $G = LG^u$ of the group G in the semidirect product of a reductive subgroup L and the unipotent radical G^u . By ϕ denote the homomorphism $G \rightarrow G/G^u$. We shall identify the image of ϕ with L . Put $K = \phi(H)$.

Theorem 3.9 ([10, Th.2]). *The following conditions are equivalent:*

- (1) G/H is affinely closed;
- (2) L/K is affinely closed.

Proof. The subgroup H is observable in G if and only if the subgroup K is observable in L [63], [29, Th.7.3].

Suppose that L/K admits a non-trivial affine embedding. Then there are an L -module V and a vector $v \in V$ such that the stabilizer L_v equals K and the orbit boundary $Y = Z \setminus Lv$, where $Z = \overline{Lv}$, is nonempty. Let

$I(Y)$ be the ideal in $\mathbb{K}[Z]$ defining the subvariety Y . There exists a finite-dimensional L -submodule $V_1 \subset I(Y)$ that generates $I(Y)$ as an ideal. The inclusion $V_1 \subset \mathbb{K}[Z]$ defines L -equivariant morphism $\psi : Z \rightarrow V_1^*$ and $\psi^{-1}(0) = Y$. Then L -equivariant morphism $\xi : Z \rightarrow V_2 = V_1^* \oplus (V \otimes V_1^*)$, $z \rightarrow (\psi(z), z \otimes \psi(z))$ maps Y to the origin and is injective on the open orbit in Z . Hence we obtain an embedding of L/K in an L -module such that the closure of the image of this embedding contains the origin. Put $v_2 = \xi(v)$. By the Hilbert-Mumford Criterion, there is a one-parameter subgroup $\lambda : \mathbb{K}^* \rightarrow L$ such that $\lim_{t \rightarrow 0} \lambda(t)v_2 = 0$. Consider the weight decomposition $v_2 = v_2^{(i_1)} + \dots + v_2^{(i_s)}$ of the vector v_2 , where $\lambda(t)v_2^{(i_k)} = t^{i_k}v_2^{(i_k)}$. Here all i_k are positive.

By the identification $G/G^u = L$, one may consider V_2 as a G -module. Let W be a finite-dimensional G -module with a vector w whose stabilizer equals H . Replacing the pair (W, w) by the pair $(W \oplus (W \otimes W), w + w \otimes w)$, one may suppose that the orbit Gw intersects the line $\mathbb{K}w$ only at the point w . The weight decomposition shows that, for a sufficiently large N , in the G -module $W \otimes V_2^{\otimes N}$ one has $\lim_{t \rightarrow 0} \lambda(t)(w \otimes v_2^{\otimes N}) = 0$ ($\lambda(\mathbb{K}^*)$ may be considered as a subgroup of G). On the other hand, the stabilizer of $w \otimes v_2^{\otimes N}$ coincides with H . This implies that the space G/H is not affinely closed.

Conversely, suppose that G/H admits a non-trivial affine embedding. This embedding corresponds to a G -invariant subalgebra $A \subset \mathbb{K}[G/H]$ containing a non-trivial G -invariant ideal I . Note that the algebra $\mathbb{K}[L]$ may be identified with the subalgebra in $\mathbb{K}[G]$ of (left- or right-) G^u -invariant functions, $\mathbb{K}[G/H]$ is realized in $\mathbb{K}[G]$ as the subalgebra of right H -invariants, and $\mathbb{K}[L/K]$ is the subalgebra of left G^u -invariants in $\mathbb{K}[G/H]$. Consider the action of G^u on the ideal I . By the Lie-Kolchin Theorem, there is a non-zero G^u -invariant element in I . Thus the subalgebra $A \cap \mathbb{K}[L/K]$ contains the non-trivial L -invariant ideal $I \cap \mathbb{K}[L/K]$. If the space L/K is affinely closed then we get a contradiction with the following lemma.

Lemma 3.10. *Let L/K be an affinely closed space of a reductive group L . Then any L -invariant subalgebra in $\mathbb{K}[L/K]$ is finitely generated and does not contain non-trivial L -invariant ideals.*

Proof. Let $B \subset \mathbb{K}[L/K]$ be a non-finitely generated invariant subalgebra. For any chain $W_1 \subset W_2 \subset W_3 \subset \dots$ of finite-dimensional L -invariant submodules in $\mathbb{K}[L/K]$ with $\cup_{i=1}^{\infty} W_i = \mathbb{K}[L/K]$, the chain of subalgebras $B_1 \subset B_2 \subset B_3 \subset \dots$ generated by W_i does not stabi-

lize. Hence one may suppose that all inclusions here are strict. Let Z_i be the affine L -variety corresponding to the algebra B_i . The inclusion $B_i \subset \mathbb{K}[L/K]$ induces the dominant morphism $L/K \rightarrow Z_i$ and Theorem 3.2 implies that $Z_i = L/K_i$, $K \subset K_i$. But $B_1 \subset B_2 \subset B_3 \subset \dots$, and any K_i is strictly contained in K_{i-1} , a contradiction. This shows that B is finitely generated and, as proved above, L acts transitively on the affine variety Z corresponding to B . But any non-trivial L -invariant ideal in B corresponds to a proper L -invariant subvariety in Z . \square

Theorem 3.9 is proved. \square

Corollary . *Let G/H be an affinely closed homogeneous space. Then for any affine G -variety X and a point $x \in X$ such that $Hx = x$, the orbit Gx is closed.*

Proof. The stabilizer G_x is observable in G , hence $\phi(G_x)$ is observable in L . The subgroup $\phi(G_x)$ contains $K = \phi(H)$, and Theorems 3.2, 3.9 imply that the space $L/\phi(G_x)$ is affinely closed. By Theorem 3.9, the space G/G_x is affinely closed. \square

Corollary . *If X is an affine G -variety and a point $x \in X$ is T -fixed, where T is a maximal torus of G , then the orbit Gx is closed.*

A characteristic-free description of affinely closed homogeneous spaces for solvable groups is given in [67].

3.4 The Slice Theorem

The Slice Theorem due to D. Luna [42] is one of the most important technical tools in modern invariant theory. In this text we need only some corollaries of the Slice Theorem related to affine embeddings [42], [56].

- Let $G/H \hookrightarrow X$ be an affine embedding with a closed G -orbit isomorphic to G/F , where F is reductive. By the Slice Theorem, we may assume that $H \subseteq F$. Then there exists an affine embedding $F/H \hookrightarrow Y$ with an F -fixed point such that X is G -isomorphic to the homogeneous fiber space $G *_F Y$. This allows one to reduce many problems to affine embeddings with a fixed point. On the other hand, this gives us a G -equivariant projection of X onto G/F .

- Let $G/H \hookrightarrow X$ be a smooth affine embedding with closed G -orbit isomorphic to G/F . Then X is a homogeneous vector bundle over G/F .

In particular, if X contains a G -fixed point, then X is vector space with a linear G -action.

3.5 Fixed-point properties

Here we list some results concerning G -fixed points in affine embeddings.

- If G/H is a quasi-affine non affinely closed homogeneous space, then G/H admits an affine embedding with a G -fixed point [9, Prop.3].

- A homogeneous space G/H admits an affine embedding $G/H \hookrightarrow X$ such that $X = G/H \cup \{o\}$, where o is a G -fixed point, if and only if H is a quasi-parabolic subgroup of G [53, Th.4, Cor.5]. In this case the normalization of X is an HV -variety and the normalization morphism is bijective.

- Consider the canonical decomposition $\mathbb{K}[G/H] = \mathbb{K} \oplus \mathbb{K}[G/H]_G$, where the first term corresponds to the constant functions and $\mathbb{K}[G/H]_G$ is the sum of all nontrivial simple G -submodules in $\mathbb{K}[G/H]$. Suppose that H is an observable subgroup of G . The following conditions are equivalent [9, Prop.6]:

- (1) any affine embedding of G/H contains a G -fixed point;
- (2) H is not contained in a proper reductive subgroup of G ;
- (3) $\mathbb{K}[G/H]_G$ is an ideal in $\mathbb{K}[G/H]$.

If H is a Grosshans subgroup, then conditions (1)-(3) are equivalent to

- (4) $CE(G/H)$ contains a G -fixed point.

Example 3.11. Let G be a connected semisimple group and P a parabolic subgroup containing no simple components of G . For $H = P^u$ the properties (1)-(4) hold. In fact, (3) follows from the observation that $\mathbb{K}[G/P^u]_G$ is the positive part of a G -invariant grading on $\mathbb{K}[G/P^u]$ defined by the G -equivariant action of a suitable one-parameter subgroup in the centre of the Levi subgroup of P on G/P^u [9].

Proposition 3.12. *Let H be an observable subgroup of G .*

- 1) *If either G/H is affinely closed or H is a quasi-parabolic subgroup of G , then G/H admits only one normal affine embedding (up to G -isomorphism);*

- 2) *if $G = \mathbb{K}^*$ and H is finite, then there exist only two normal affine embeddings, namely \mathbb{K}^*/H and \mathbb{K}/H ;*

3) in all other cases there exists an infinite sequence

$$X_1 \xleftarrow{\phi_1} X_2 \xleftarrow{\phi_2} X_3 \xleftarrow{\phi_3} \dots$$

of pairwise nonisomorphic normal affine embeddings $G/H \hookrightarrow X_i$ and equivariant dominant morphisms ϕ_i .

Proof. The statements are obvious for affinely closed G/H and for $G = \mathbb{K}^*$. If H is a quasi-parabolic subgroup, then $\mathbb{K}[G/H]^U = \mathbb{K}[t]$. Suppose that $G/H \hookrightarrow X$ is a normal affine embedding. Then $\mathbb{K}[X]^U \subseteq \mathbb{K}[t]$ is a graded integrally closed subalgebra with $Q(\mathbb{K}[X]^U) = \mathbb{K}(t)$. This implies $\mathbb{K}[X]^U = \mathbb{K}[t]$ and $\mathbb{K}[X] = \mathbb{K}[G/H]$, hence X is G -isomorphic to the canonical embedding of G/H .

In all other cases there exists an integrally closed non-finitely generated invariant subalgebra \mathfrak{B} in $\mathbb{K}[G/H]$ with $Q\mathfrak{B} = \mathbb{K}(G/H)$; see Proposition 6.4. Let $f_1, f_2, \dots, f_n, f_{n+1}, \dots$ be a set of generators of \mathfrak{B} such that $\mathbb{K}(f_1, \dots, f_n) = \mathbb{K}(G/H)$. Define \mathfrak{B}_k as the integral closure of $\mathbb{K}\langle Gf_1, \dots, Gf_{n+k} \rangle$ in \mathfrak{B} . The varieties $X_k = \text{Spec } \mathfrak{B}_k$ are birationally isomorphic to G/H and hence $G/H \hookrightarrow X_k$. Infinitely many of the X_k are pairwise nonisomorphic. Renumbering, one may suppose that all X_k are nonisomorphic. The chain

$$\mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \mathfrak{B}_3 \dots$$

corresponds to the desired chain

$$X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \dots$$

□

4 Embeddings with a finite number of orbits

4.1 The characterization theorem

Spherical homogeneous spaces admit the following nice characterization in terms of equivariant embeddings.

Theorem 4.1 ([61, 45, 2]). *A homogeneous space G/H is spherical if and only if any embedding of G/H has finitely many G -orbits.*

To be more precise, F. J. Servedio proved that any affine spherical variety contains finitely many G -orbits, D. Luna, Th. Vust and D. N. Akhiezer extended this result to an arbitrary spherical variety and D. N. Akhiezer constructed a projective embedding with infinitely many G -orbits for any homogeneous space of positive complexity.

Now we are concerned with the following problem: characterize all quasi-affine homogeneous spaces G/H of a reductive group G with the property

(AF) *For any affine embedding $G/H \hookrightarrow X$, the number of G -orbits in X is finite.*

It follows from the results considered above that

- 1) spherical homogeneous spaces
- 2) affinely closed homogeneous spaces
- 3) homogeneous spaces of the group $SL(2)$

have property (AF). Our main result in some sense gives a unification of these three classes.

Theorem 4.2 ([11]). *For a reductive subgroup $H \subseteq G$, (AF) holds if and only if either $W(H) = N_G(H)/H$ is finite or any extension of H by a one-dimensional torus in $N_G(H)$ is spherical in G .*

Corollary . *For an affine homogeneous space G/H of complexity > 1 , (AF) holds if and only if G/H is affinely closed.*

Corollary . *An affine homogeneous space G/H of complexity 1 satisfies (AF) if and only if either $W(H)$ is finite, or $\text{rk} W(H) = 1$ and $N_G(H)$ is spherical.*

Corollary . *Let G be a reductive group with infinite center $Z(G)$ and H a reductive subgroup in G that does not contain $Z(G)^0$. Then property (AF) holds for G/H if and only if H is a spherical subgroup of G .*

The proof of Theorem 4.2 is based on the analysis of Akhiezer's construction [2] of projective embeddings and on some results of F. Knop. We give this proof in Section 4.2, obtaining a more general result, Theorem 4.7.

Our method applied to an arbitrary quasi-affine space G/H gives a necessary condition for property (AF) (see the remark on page 35 below), but a characterization of quasi-affine spaces with property (AF) is not obtained yet. Another open problem is to characterize Grosshans subgroups H of a reductive group G such that $CE(G/H)$ contains only a finite number of G -orbits [9].

4.2 Modality

The aim of this subsection is to generalize Theorem 4.2 following the ideas of [3], and to find the maximal number of parameters in a continuous family of G -orbits over all affine embeddings of a given affine space G/H .

Definition 4.3. Let $F : X$ be an algebraic group action. The integer

$$d_F(X) = \min_{x \in X} \operatorname{codim}_X Fx = \operatorname{tr.deg} \mathbb{K}(X)^F$$

is called the *generic modality* of the action. This is the number of parameters in the family of generic orbits. The *modality* of $F : X$ is the integer $\operatorname{mod}_F X = \max_{Y \subseteq X} d_F(Y)$, where Y runs through F -stable irreducible subvarieties of X .

An action of modality zero is just an action with a finite number of orbits. Note that $c(X) = d_B(X)$. E. B. Vinberg [70] proved that $\operatorname{mod}_B(X) = c(X)$ for any G -variety X . This means that if we pass from X to a B -stable irreducible subvariety $Y \subset X$, then the number of parameters for generic B -orbits does not increase. Simple examples show that the inequality $d_G(X) \leq \operatorname{mod}_G(X)$ can be strict. This motivates the following

Definition 4.4. With any G -variety X we associate the integer

$$m_G(X) = \max_{X'} \operatorname{mod}_G(X'),$$

where X' runs through all G -varieties birationally G -isomorphic to X .

For a homogeneous space G/H we have $m_G(G/H) = \max_X \operatorname{mod}_G(X)$, where X runs through all embeddings of G/H .

It is clear that for any subgroup $F \subset G$ the inequality $m_G(X) \leq m_F(X)$ holds. In particular, $m_G(X) \leq c(X)$. The next theorem shows that $m_G(X) = c(X)$.

Theorem 4.5 ([3]). *There exists a projective G -variety X' birationally G -isomorphic to X such that $\operatorname{mod}_G(X') = c(X)$.*

Now we introduce an affine counterpart of $m_G(X)$.

Definition 4.6. With any quasi-affine homogeneous space G/H we associate the integer

$$a_G(G/H) = \max_X \operatorname{mod}_G(X),$$

where X runs through all affine embeddings $G/H \hookrightarrow X$.

Theorem 4.7 ([7]). *Let H be a reductive subgroup of G .*

- (1) *If the group $W(H)$ is finite, then $a_G(G/H) = 0$;*
- (2) *If $W(H)$ is infinite, then*

$$a_G(G/H) = \max_{H_1} c(G/H_1),$$

where H_1 runs through all non-trivial extensions of H by a one-dimensional subtorus of $C_G(H)$. In particular, $a_G(G/H) = c(G/H)$ or $c(G/H) - 1$.

Proof. Step 1 – Affine cones. Consider the natural surjection $\kappa : N_G(H) \rightarrow W(H)$.

Proposition 4.8. *Let H be an observable subgroup of G . Suppose that there is a non-trivial one-parameter subgroup $\lambda : \mathbb{K}^* \rightarrow W(H)$ and put $H_1 = \kappa^{-1}(\lambda(\mathbb{K}^*))$. Then there exists an affine embedding $G/H \hookrightarrow X$ with $\text{mod}_G(X) \geq c(G/H_1)$.*

The idea of the proof is to apply Akhiezer's construction [3] to the homogeneous space G/H_1 and to consider the affine cone over a projective embedding $G/H_1 \hookrightarrow X'$ with $\text{mod}_G(X') = c(G/H_1)$

Lemma 4.9. *In the notation of Proposition 4.8, there exists a finite-dimensional G -module V and an H_1 -eigenvector $v \in V$ such that*

- 1) *the orbit $G\langle v \rangle$ of the line $\langle v \rangle$ in $\mathbb{P}(V)$ is isomorphic to G/H_1 ;*
- 2) *H fixes v ;*
- 3) *H_1 acts transitively on \mathbb{K}^*v ;*
- 4) *$\text{mod}_G(\overline{G\langle v \rangle}) = c(G/H_1)$.*

Proof (of Lemma 4.9). By Chevalley's theorem, there exist a G -module V' and a vector $v' \in V'$ having property 1). Let χ be the eigenweight of H at v' . Since H is observable in G , each finite-dimensional H -module can be embedded into a finite-dimensional G -module [47]. In particular, there exists a G -module V'' containing H -eigenvectors of the weight $-\chi$. Among them we can choose an H_1 -eigenvector v'' and set $V = V' \otimes V''$, $v = v' \otimes v''$. This pair has properties 1)-2).

If H_1 does not act transitively on \mathbb{K}^*v , then take an arbitrary G -module W containing a vector with stabilizer H . Take an H_1 -eigenvector in W^H with nonzero weight and replace V by $V \otimes W$ and v by $v \otimes w$. Conditions 1)-3) are now satisfied.

By a result of Akhiezer [3], we can find a pair (V', v') with properties 1) and 4). Then we proceed as above obtaining a pair (V, v) . The closure $\overline{G\langle v \rangle} \subseteq \mathbb{P}(V)$ lies in the image of the Segre embedding

$$\mathbb{P}(V') \times \mathbb{P}(V'') \times \mathbb{P}(W) \hookrightarrow \mathbb{P}(V),$$

and it projects G -equivariantly onto $\overline{G\langle v' \rangle} \subseteq \mathbb{P}(V')$. Now properties 1)-4) are satisfied for the pair (V, v) . \square

Remark . If H is reductive, then one can find v in Lemma 4.9 such that $G_v = H$. This is not possible for an arbitrary observable subgroup, see [11, Remark 2].

Proof (of Proposition 4.8). Let (V, v) be the pair from Lemma 4.9. Put $H' = G_v$ and $\tilde{X} = \overline{Gv}$. By properties 1)-3) and since H_1/H is isomorphic to \mathbb{K}^* , H' is a finite extension of H . By 3), the closure of the orbit Gv in V is a cone, therefore 4) implies the inequality $\text{mod}_G(\tilde{X}) \geq c(G/H_1)$.

Consider now the morphism $G/H \rightarrow G/H'$. It determines an embedding $\mathbb{K}[G/H'] \subseteq \mathbb{K}[G/H]$. Let A be the integral closure of the subalgebra $\mathbb{K}[\tilde{X}] \subseteq \mathbb{K}[G/H']$ in the field $\mathbb{K}(G/H)$. We have the following commutative diagrams:

$$\begin{array}{ccccc} A & \hookrightarrow & \mathbb{K}[G/H] & \hookrightarrow & \mathbb{K}(G/H) & & \text{Spec } A & \hookrightarrow & G/H \\ \uparrow & & \uparrow & & \uparrow & & \downarrow & & \downarrow \\ \mathbb{K}[\tilde{X}] & \hookrightarrow & \mathbb{K}[G/H'] & \hookrightarrow & \mathbb{K}(G/H') & & \tilde{X} & \hookrightarrow & G/H' \end{array}$$

The affine variety $X = \text{Spec } A$ with the natural G -action can be regarded as an affine embedding of G/H . The embedding $\mathbb{K}[\tilde{X}] \subseteq A$ defines a finite (surjective) morphism $X \rightarrow \tilde{X}$, therefore $\text{mod}_G(X) = \text{mod}_G(\tilde{X}) \geq c(G/H_1)$. \square

Step 2. Here we formulate several results due to F. Knop.

Lemma 4.10 ([34, 7.3.1], see also [11, Lemma 3]). *Let X be an irreducible G -variety, and v a G -invariant valuation of $\mathbb{K}(X)$ over \mathbb{K} with residue field $\mathbb{K}(v)$. Then $\mathbb{K}(v)^B$ is the residue field of the restriction of v to $\mathbb{K}(X)^B$.*

Definition 4.11 ([36, §7]). Let X be a normal G -variety. A discrete \mathbb{Q} -valued G -invariant valuation of $\mathbb{K}(X)$ is said to be *central* if it vanishes on $\mathbb{K}(X)^B \setminus \{0\}$. A *source* of X is a non-empty G -stable subvariety $Y \subseteq X$ that is the center of a central valuation of $\mathbb{K}(X)$.

The following lemma is an easy consequence of [36]; for more details see [11, Lemma 4].

Lemma 4.12. *If X is a normal affine G -variety containing a proper source, then there exists a one-dimensional torus $S \subseteq \text{Aut}_G(X)$ such that $\mathbb{K}(X)^B \subseteq \mathbb{K}(X)^S$. (Here $\text{Aut}_G(X)$ is the group of G -equivariant automorphisms of X).*

Step 3. Assertion (1) of Theorem 4.7 follows from Theorem 3.2. To prove (2) we use Proposition 4.8. Since H is reductive, the group $W(H)$ is reductive and contains a one-dimensional subtorus $\lambda(\mathbb{K}^*)$. Hence $a_G(G/H) \geq c(G/H_1) \geq c(G/H) - 1$. If there exists a one-dimensional torus in $W(H)$ such that $c(G/H) = c(G/H_1)$, we obtain an affine embedding of G/H of modality $c(G/H)$.

Conversely, let $G/H \hookrightarrow X$ be an affine embedding of modality $c(G/H)$. We have to find a one-dimensional subtorus $\lambda(\mathbb{K}^*) \subseteq W(H)$ such that $c(G/H_1) = c(G/H)$. By the definition of modality, there exists a proper G -invariant subvariety $Y \subset X$ such that the codimension of a generic G -orbit in Y is $c(G/H)$, hence $c(Y) = c(G/H)$. Consider a G -invariant valuation v of $\mathbb{K}(X)$ with centre Y . For the residue field $\mathbb{K}(v)$ we have $\text{tr.deg } \mathbb{K}(v)^B \geq \text{tr.deg } \mathbb{K}(Y)^B$, therefore $\text{tr.deg } \mathbb{K}(v)^B = \text{tr.deg } \mathbb{K}(X)^B$. If the restriction of v to $\mathbb{K}(X)^B$ is non-trivial, then, by Lemma 4.10, $\text{tr.deg } \mathbb{K}(v)^B < \text{tr.deg } \mathbb{K}(X)^B$, a contradiction. Thus, v is central and Y is a source of X . Lemma 4.12 provides a one-dimensional subtorus $S \subseteq \text{Aut}_G(X) \subseteq \text{Aut}_G(G/H) = W(H)$ that yields an extension of H of the same complexity. \square

Note that Theorem 4.2 is a particular case of Theorem 4.7 with $a_G(G/H) = 0$.

Remark . If H is an observable subgroup and $W(H)$ contains a non-trivial subtorus, then the formula $a_G(G/H) = \max_{H_1} c(G/H_1)$ can be obtained by the same arguments. In particular, Corollary 4.1 holds for observable H . But for non-reductive H the group $W(H)$ can be unipotent [11]: this is the case when $G = SL(3) \times SL(3)$ and

$$H = \left\{ \left(\begin{pmatrix} 1 & a & b + \frac{a^2}{2} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b & a + \frac{b^2}{2} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{K} \right\}.$$

For such subgroups our proof yields only the inequality $a_G(G/H) \leq c(G/H) - 1$.

Let us mention an application of Theorem 4.7 which may be regarded as its algebraic reformulation. Let G be a connected semisimple group. Note that, for the action by left multiplication, one has $c(G) = \frac{1}{2}(\dim G - \text{rk } G)$ and $c(G/S) = \frac{1}{2}(\dim G - \text{rk } G) - 1$, where S is a one-dimensional subtorus in G . Applying Theorem 4.7 to the case $H = \{e\}$, we obtain

Theorem 4.13 ([8]). *Let $A \subset \mathbb{K}[G]$ be a left G -invariant finitely generated subalgebra and $I \subset A$ a G -invariant prime ideal. Then*

$$\mathrm{tr.deg} (Q(A/I))^G \leq \frac{1}{2} (\dim G - \mathrm{rk} G) - 1. \quad (1)$$

Moreover, there exist a subalgebra A and an ideal I such that (1) is an equality.

Example 4.14. The closure of an $SL(3)$ -orbit in an algebraic $SL(3)$ -variety X may contain at most a 3-parameter family of $SL(3)$ -orbits. If X is affine then the maximal number of parameters equals 2.

4.3 Equivariant automorphisms and symmetric embeddings

The group $\mathrm{Aut}_G(G/H)$ of G -equivariant automorphisms of G/H is isomorphic to $W(H)$. The action $W(H) : G/H$ is induced by the action $N_G(H) : G/H$ by right multiplication, i.e. $n * gH = gn^{-1}H$. Let $G/H \hookrightarrow X$ be an embedding. The group $\mathrm{Aut}_G X$ preserves the open orbit, and may be considered as a subgroup of $W(H)$.

Definition 4.15. An embedding $G/H \hookrightarrow X$ is said to be *symmetric* if $W(H)^0 \subseteq \mathrm{Aut}_G(X)$. If $\mathrm{Aut}_G(X) = W(H)$, we say that X is *very symmetric*.

Lemma 4.16. *The following affine embeddings are very symmetric:*

- 1) *an affine embedding of a spherical homogeneous space;*
- 2) *the canonical embedding $CE(G/H)$;*
- 3) *an affine monoid M considered as the embedding $G(M)/\{e\} \hookrightarrow M$.*

Proof. 1) Let G/H be a quasi-affine spherical homogeneous space. By the Schur Lemma, the group $W(H)$ acts on any isotypic component of $\mathbb{K}[G/H]$ by dilation. Hence any G -invariant subspace of $\mathbb{K}[G/H]$ is also $W(H)$ -invariant.

2) The group $W(H)$ acts on G/H and on $\mathbb{K}[G/H]$, thus on $\mathrm{Spec} \mathbb{K}[G/H]$.

3) The group $W(H) \cong G(M)$ acts on M by right multiplication. \square

Proposition 4.17. *Let H be a reductive subgroup of G . The following conditions are equivalent:*

- (1) *there exists a unique symmetric embedding $X = G/H$;*
- (2) *$W(H)^0$ is a semisimple group.*

Proof. The existence of a non-trivial affine embedding $G/H \hookrightarrow X$ with $\dim \operatorname{Aut}_G(X) = \dim W(H)$ means that G/H as a $(G \times W(H)^0)$ -homogeneous space is not affinely closed. Denote by L the $(G \times W(H)^0)$ -stabilizer of the point eH . Then $L = \{(n, nH) \mid n \in \kappa^{-1}(W(H)^0)\}$ and the group $N_{G \times W(H)^0}(L)/L$ is finite if and only if $W(H)$ is semisimple. \square

Proposition 4.17 implies that in the case of affine $SL(2)$ -embeddings only the trivial embedding $X = SL(2)$ is symmetric. In fact, in all other cases with normal X the group $\operatorname{Aut}_{SL(2)} X$ is a Borel subgroup of $SL(2)$ [37, III.4.8, Satz 1]. The theorem below is a partial generalization of this result.

Theorem 4.18 ([12]). *Let $G/H \hookrightarrow X$ be an affine embedding with a finite number of G -orbits and with a G -fixed point. Then the group $\operatorname{Aut}_G(X)^0$ is solvable.*

We begin the proof with the following

Lemma 4.19. *Let X be an affine variety with an action of a connected semisimple group S . Suppose that there is a point $x \in X$ and a one-parameter subgroup $\gamma : \mathbb{K}^* \rightarrow S$ such that $\lim_{t \rightarrow 0} \delta(t)x$ exists in X for any subgroup δ conjugate to γ . Then x is a $\gamma(\mathbb{K}^*)$ -fixed point.*

Proof. Let T be a maximal torus in S containing $\gamma(\mathbb{K}^*)$. One can realize X as a closed S -stable subvariety in V for a suitable S -module V . Let $x = x_{\lambda_1} + \cdots + x_{\lambda_n}$ be the weight decomposition (with respect to T) of x with weights $\lambda_1, \dots, \lambda_n$. One-parameter subgroups of T form the lattice $\Xi_*(T)$ dual to the character lattice $\Xi(T)$. The existence of $\lim_{t \rightarrow 0} \gamma(t)x$ in X means that all pairings $\langle \gamma, \lambda_i \rangle$ are non-negative. Let $\gamma_1, \dots, \gamma_m$ be all the translates of γ under the action of the Weyl group $W = N_S(T)/T$. By assumption, $\langle \gamma_j, \lambda_i \rangle \geq 0$ for any $i = 1, \dots, n, j = 1, \dots, m$, hence $\langle \gamma_1 + \cdots + \gamma_m, \lambda_i \rangle \geq 0$. Since $\gamma_1 + \cdots + \gamma_m = 0$, one has $\langle \gamma_j, \lambda_i \rangle = 0$ for all i, j . This shows that the points x_{λ_i} (and x) are $\gamma(\mathbb{K}^*)$ -fixed. \square

The next proposition is a generalization of [28, Th.4.3].

Proposition 4.20. *Suppose that $G/H \hookrightarrow X$ is an affine embedding with a non-trivial G -equivariant action of a connected semisimple group S . Then the orbit $S * x$ is closed in X , for every $x \in G/H$.*

Proof. We may assume $x = eH$. If $S * x$ is not closed, then, by [31, Th.1.4], there is a one-parameter subgroup $\gamma : \mathbb{K}^* \rightarrow S$ such that the

limit

$$\lim_{t \rightarrow 0} \gamma(t) * x$$

exists in X and does not belong to $S * x$. Replacing S by a finite cover, we may assume that S embeds in $N_G(H)$ (and thus in G) with a finite intersection with H . By the definition of $*$ -action, one has $\gamma(t) * x = \gamma(t^{-1})x$. For any $s \in S$ the limit

$$\lim_{t \rightarrow 0} (s\gamma(t)) * x = \lim_{t \rightarrow 0} \gamma(t^{-1})s^{-1}x$$

exists. Hence $\lim_{t \rightarrow 0} s\gamma(t^{-1})s^{-1}x$ exists too. This shows that for any one-parameter subgroup δ of S , conjugate to $-\gamma$, $\lim_{t \rightarrow 0} \delta(t)x$ exists in X . Lemma 4.19 implies that $x = \lim_{t \rightarrow 0} \gamma(t) * x$, and this contradiction proves Proposition 4.20. \square

Proof (of Theorem 4.18). Suppose that $\text{Aut}_G(X)^0$ is not solvable. Then there is a connected semisimple group S acting on X G -equivariantly. By Proposition 4.20, any $(S, *)$ -orbit in the open G -orbit of X is closed in X .

Let X_1 be the closure of a G -orbit in X . Since G has a finite number of orbits in X , the variety X_1 is $(S, *)$ -stable. Applying the above arguments to X_1 , we show that any $(S, *)$ -orbit in X is closed. But in this case all $(S, *)$ -orbits have the same dimension $\dim S$. On the other hand, a G -fixed point is an $(S, *)$ -orbit, a contradiction. \square

Corollary (of the proof). *Let X be an affine G -variety with an open G -orbit. Suppose that*

- (1) *a semisimple group S acts on X effectively and G -equivariantly;*
- (2) *the dimension of a closed G -orbit in X is less than $\dim S$.*

Then the number of G -orbits in X is infinite.

Corollary . *Let M be a reductive algebraic monoid with zero. Then the number of left (right) $G(M)$ -cosets in M is finite if and only if M is commutative.*

The following corollary gives a partial answer to a question posed in Subsection 4.1.

Corollary . *The number of G -orbits in $CE(G/P^u)$ is finite if and only if either $P \cap G_i = G_i$ or $P \cap G_i = B \cap G_i$ for each simple factor $G_i \subseteq G$.*

In many cases, Theorem 4.18 may be used to show that the group $\text{Aut}_G(X)$ cannot be very big. On the other hand, the group $\text{Aut}_G(X)$ may be finite (trivial), in particular, for $X = G/H$ with affinely closed G/H . Answering a question from [11], I. V. Losev proposed an example of an observable non-reductive subgroup H in $SL(n)$, where $W(H)$ is finite. (This example is included in the electronic version of [12].) Note that any affine embedding of $SL(n)/H$ gives an example of a locally transitive non-transitive reductive group action on an affine variety with a finite group of equivariant automorphisms.

Finally, we give a variant of Theorem 4.2 for symmetric embeddings.

Theorem 4.21 ([11, Prop.2]). *Let H be a reductive subgroup of G . Every symmetric affine embedding of G/H has finitely many G -orbits if and only if either (AF) holds or $W(H)^0$ is semisimple.*

5 Application One: Invariant algebras on homogeneous spaces of compact Lie groups

5.1 Invariant algebras and self-conjugate algebras

For any compact topological space M the set $C(M)$ of all continuous \mathbb{C} -valued functions on M is a commutative Banach algebra with respect to pointwise addition, multiplication, and the uniform norm. We shall consider the case where $M = K/L$ is a homogeneous space of a compact connected Lie group K . Let us recall that A is an *invariant algebra* on M if A is a K -invariant uniformly closed subalgebra with unit in $C(M)$. In this section G, H denote the complexifications of K, L respectively. The group G is a complex reductive algebraic group with a reductive subgroup H .

The main problem is to describe all invariant algebras on a given space M and to study their properties. Let us start with a particular class of invariant algebras.

Definition 5.1. An invariant algebra A is *self-conjugate* if $f \in A$ implies $\bar{f} \in A$, where the bar denotes the complex conjugation.

The classification of self-conjugate invariant algebras is based on the Stone-Weierstrass Theorem. Here we follow [39].

The Stone-Weierstrass Theorem. *Let R be a compact topological space and A a subalgebra with unit in $C(R)$ such that*

- 1) A separates points on R , i.e. for any $x_1 \neq x_2 \in R$ there exists $f \in A$ such that $f(x_1) \neq f(x_2)$;
- 2) A is invariant under complex conjugation.

Then A is dense in $C(R)$.

Given a self-conjugate invariant algebra A , define an equivalence relation on M : $x \sim y$ if and only if $f(x) = f(y)$ for any $f \in A$. The space M' of equivalence classes is a homogeneous K -space, hence $M' = K/L'$, where L' is a closed subgroup containing L . By construction, the self-conjugate algebra A separates points on M' and thus $A = C(M')$. Conversely, for any $L \subseteq L' \subseteq K$ the inverse image of $C(K/L')$ under the projection $K/L \rightarrow K/L'$ determines a self-conjugate invariant algebra on M . This shows that self-conjugate invariant algebras on M are in one-to-one correspondence with closed subgroups L' , $L \subseteq L' \subseteq K$.

5.2 Spherical functions

The space $M = K/L$ may be considered as a compact subset of the affine homogeneous space $X_0 = G/H$. Moreover, M is a real form of X_0 in the natural sense. In particular, the restriction of polynomial functions to M determines an embedding $\mathbb{C}[X_0] \hookrightarrow C(M)$. Denote the image of this embedding by $\mathbb{C}[M]$.

Definition 5.2. A function $f \in C(M)$ is called *spherical* if the linear span $\langle Kf \rangle$ is finite-dimensional. More generally, for a linear action of a Lie group K on vector space V , a vector $v \in V$ is *spherical* if $\dim \langle Kv \rangle < \infty$.

Denote by V_{sph} the subspace of all spherical vectors in V .

Proposition 5.3. *The algebra $\mathbb{C}[M]$ coincides with $C(M)_{sph}$.*

Proof. Any regular function is contained in a finite-dimensional invariant subspace. Conversely, any complex finite-dimensional representation of K is completely reducible and any irreducible component may be considered as a simple G -module. Hence the matrix entries of such a module are in $\mathbb{C}[M]$. If $f \in C(M)$ is spherical and $V = \langle Kf \rangle$, then f is a linear combination of the matrix entries of the dual representation $K : V^*$. Indeed, let f_1, \dots, f_k be a basis in V . For any $f \in V$, $g \in K$ one has $f_i(g^{-1}eL) = \sum a_{ij}(g)f_j(eL)$ and $f_i(gL) = \sum c_j a_{ij}(g^{-1})$, where $c_j = f_j(eL)$ are constants. \square

By the Peter-Weyl Theorem, the matrix entries (with respect to some orthonormal basis) over all irreducible finite-dimensional representations of K form an orthonormal basis in space $L^2(K)$. Spherical functions are finite linear combinations of the basic elements. They form a uniformly dense subspace in $C(K)$. The following generalization of this result plays a key role in this section.

Proposition 5.4 ([51, Th.5.1], [48, 2.16]). *Given a continuous linear representation of a compact Lie group K in a Fréchet space E , the subspace E_{sph} is dense in E .*

In particular, in any invariant algebra, spherical functions form a dense subalgebra. Moreover, if S is K -invariant subspace in $C(M)_{sph}$ and \overline{S} is its uniform closure in $C(M)$, then $\overline{S} \cap C_{sph}(M) = S$. (For the proof see [26, Lemma 14].) Finally, we get

Theorem 5.5. *There is a natural bijection ψ between invariant algebras on the space M and invariant subalgebras in $\mathbb{C}[M]$. More precisely, $\psi(A) = \mathfrak{A} = A_{sph} = A \cap \mathbb{C}[M]$ and $\psi^{-1}(\mathfrak{A}) = \overline{\mathfrak{A}}$.*

This result provides nice connections between functional and algebraic problems. To make this link really useful we need to reformulate functional properties in algebraic terms and conversely. For this purpose we are going to use the geometric language of affine embeddings.

5.3 Finitely generated invariant algebras and affine embeddings

Definition 5.6. An invariant algebra A is *finitely generated* if it is generated (as a Banach algebra) by a K -invariant finite-dimensional subspace.

An invariant algebra A is finitely generated if and only if A_{sph} is a finitely generated algebra. It is clear that $C(M)$ is finitely generated. As follows from the discussion above, any self-conjugate invariant algebra is finitely generated. The question as to when any invariant subalgebra in $\mathbb{C}[M]$ is finitely generated will be considered in the last section.

Any finitely generated subalgebra $\mathfrak{A} \subset \mathbb{C}[G/H]$ defines an affine G -variety $X = \text{Spec } \mathfrak{A}$ with an open orbit isomorphic to G/F , where F is an observable subgroup containing H . The inclusion $\mathfrak{A} \subset \mathbb{C}[G/H]$ defines the morphism $\phi : G/H \rightarrow X$ and the base point $x_0 = \phi(eH)$. If

we look at \mathfrak{A} as at an abstract G -algebra, then there may exist different equivariant inclusion homomorphisms $\mathfrak{A} \rightarrow \mathbb{C}[G/H]$ with the same image. Two different base points $x_0 \in X$ and $x'_0 \in X$ determine the same subalgebra $\mathfrak{A} \subset \mathbb{C}[G/H]$ if and only if there exists $n \in \text{Aut}_G(X)$ such that $x_0 = nx'_0$. (Corresponding inclusions $\mathfrak{A} \subset \mathbb{C}[G/H]$ differ by a G -equivariant automorphism of \mathfrak{A} .) Let us denote the subalgebra \mathfrak{A} by $\mathfrak{A}(X, x_0)$ and the corresponding invariant algebra $\overline{\mathfrak{A}(X, x_0)}$ by $A(X, x_0)$. We have proved:

Theorem 5.7. *Invariant finitely generated algebras on the space $M = K/L$ are in one-to-one correspondence with the following data:*

- 1) *an affine embedding $G/F \hookrightarrow X$, where $F \subseteq G$ is an observable subgroup containing H ;*
- 2) *an H -fixed point x_0 in the open G -orbit on X , which is defined up to the action of $\text{Aut}_G(X)$.*

It is natural to classify invariant algebras up to some equivalence. The group of K -equivariant automorphisms of M is the group $N = N_K(L)/L$, acting as $n * kL = kn^{-1}L$. This action defines a K -equivariant action $N : C(M)$. The group N acts transitively on the set M^L .

Definition 5.8. Two invariant algebras A_1 and A_2 on M are *equivalent* if there exists $n \in N$ such that $n * A_1 = A_2$.

Clearly, this equivalence preserves all reasonable properties of invariant algebras. In terms of Theorem 5.7, it is reasonable to expect that base points from the same K -orbit in X determine equivalent invariant algebras.

Definition 5.9. Two invariant algebras $A(X, x_0)$ and $A(X', x'_0)$ on M are *weakly equivalent* if $X \cong_G X'$ and there exist $n \in \text{Aut}_G(X)$ and $k \in K$ such that $x_0 = n * kx'_0$.

An invariant algebra A on M may be regarded as an invariant algebra \tilde{A} on K such that every element $f \in \tilde{A}$ is fixed by right L -multiplication. Two such subalgebras A_1 and A_2 are weakly equivalent if A_1 may be shifted to A_2 by the map $R(k) : f(x) \rightarrow f(xk)$ for some $k \in K$.

Clearly, equivalent invariant algebras are weakly equivalent, but the converse is not always true. One may suppose that $x_0 = kx'_0$ ($\text{Aut}_G(X)$ -action does not change the subalgebra). Consider the subgroups $L_1 = K_{x_0}$, $L_2 = K_{x'_0}$, and the map $\phi : K/L \rightarrow X$, $\phi(eL) = x_0$. Denote by $\text{Aut}(X, x_0)$ the subgroup of $\text{Aut}_G(X)$ that preserves Kx_0 . (In fact, $\text{Aut}(X, x_0) \subset N_K(L_1)/L_1$.)

Definition 5.10. A closed subgroup $L \subset K$ is an A -subgroup if any two weakly equivalent finitely generated invariant algebras on $M = K/L$ are equivalent.

Proposition 5.11. A subgroup $L \subset K$ is an A -subgroup if and only if for any affine embedding $G/F \hookrightarrow X$, $H \subset F$, and any base point $x_0 \in (G/F)^H$ one has $\text{Aut}(X, x_0)\phi((K/L)^L) = (Kx_0)^L$.

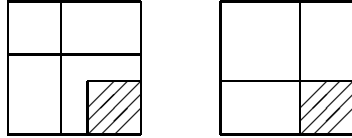
Proof. Let $x'_0 = kx_0$ be an L -fixed point. The equivalence of invariant algebras $\mathfrak{A}(X, x_0)$ and $\mathfrak{A}(X, x'_0)$ means that there is an element $n \in N_K(L)$ such that $\mathfrak{A}(X, nx_0) = \mathfrak{A}(X, x'_0)$, i.e. nx_0 and x'_0 are in the same $\text{Aut}_G(X)$ -orbit. If $m \in \text{Aut}_G(X)$ and $m * nx_0 = x'_0$, then $m \in \text{Aut}(X, x_0)$. But the set of points nx_0 , $n \in N_K(L)$, coincides with $\phi((K/L)^L)$. \square

If for any $L \subseteq L_1$ the natural map $(K/L)^L \rightarrow (K/L_1)^L$ is surjective, then L is an A -subgroup. In particular, the unit subgroup and any maximal subgroup in K are A -subgroups.

Corollary . If L is an A -subgroup, two subgroups L_1 and L_2 contain L and are K -conjugate, then they are $N_K(L)$ -conjugate.

Proof. On K/L_1 any point fixed by L has the form $m * nL_1$, where $m \in N_K(L_1)$ and $n \in N_K(L)$. In particular, for $L_2 = kL_1k^{-1}$, $k \in K$, one has $kL_1 = m * nL_1$ and $L_2 = nm^{-1}L_1mn^{-1} = nL_1n^{-1}$. \square

Example 5.12. Put $K = SU(5)$, $L = \{e\} \times \{e\} \times \{e\} \times SU(2)$, $L_1 = SU(2) \times SU(3)$, $L_2 = SU(3) \times SU(2)$ as shown on the picture. Here L_1 and L_2 are K -conjugate, contain L , but are not $N_K(L)$ -conjugate. This proves that L is not an A -subgroup.



5.4 Some classes of invariant algebras

The results of Subsection 5.1 and Theorem 5.5 imply:

Proposition 5.13 ([39]). An invariant algebra $A = A(X, x_0)$ is self-conjugate if and only if $X = Gx_0$ and G_{x_0} is the complexification of K_{x_0} .

Remark. There is one more characterization of this class of K -orbits obtained by V. M. Gichev and I. A. Latypov. Consider any G -equivariant embedding of X into a G -module V . Then the conditions of Proposition 5.13 are equivalent to the polynomial convexity of the orbit Kx_0 in V ; see [26] for details.

The following theorem due to I. A. Latypov may be regarded as a variant of Luna's theorem (see 3.1) for compact groups.

Theorem 5.14 ([38]). *Any invariant algebra on M is self-conjugate if and only if the group $N = N_K(L)/L$ is finite.*

In this case any invariant algebra on M is finitely generated. It follows from the results of Section 6 that any invariant algebra on M is finitely generated if and only if either N is finite or $K = U(1)$. (Here we assume that the action $K : M$ is effective.)

Now we introduce a class of invariant algebras, which are in some sense opposite to self-conjugate algebras.

Definition 5.15. An invariant algebra A is said to be *antisymmetric* if the set $\{f \in A \mid \bar{f} \in A\}$ coincides with the set of constant functions.

It is easy to see that antisymmetry is equivalent to either of the following conditions:

- 1) any real-valued function in A is a constant;
- 2) A contains no non-trivial self-conjugate invariant subalgebra.

Hence an invariant algebra $A = A(X, x_0)$ is antisymmetric if and only if there exists no G -equivariant map $\phi : X \rightarrow G/H'$, where G/H' is an affine homogeneous space of positive dimension and $G_{\phi(x_0)}$ is the complexification of $K_{\phi(x_0)}$. In particular, if X contains a G -fixed point, then $A(X, x_0)$ is antisymmetric.

Example 5.16. Let $K = SU(2)$, $G = SL(2)$, and $L = H = \{e\}$. Consider $X = SL(2)/T$. Any point $x_0 \in X$ may be regarded as a base point for some invariant algebra $A(X, x_0)$ on $M = K$. If the stabilizer of x_0 contains a torus from K , then $A(X, x_0)$ is self-conjugate, and any two such invariant algebras are equivalent. Other base points determine antisymmetric algebras: we obtain a 1-parameter family of mutually non-equivalent antisymmetric invariant algebras on $SU(2)$. In particular, this example shows that the property ' $A(X, x_0)$ separates points on M ' depends on the choice of the base point x_0 on X . For more information on invariant algebras on $SU(2)$, see [40].

Finally we consider one more natural class of invariant algebras.

Definition 5.17. An invariant algebra A on M is called a *Dirichlet algebra* if the real parts of functions from A are uniformly dense in the algebra of real-valued continuous functions on M .

Any Dirichlet algebra separates the points of M , but the converse is not true. Some results on Dirichlet invariant algebras on compact groups can be found in [58]. In particular, it is proved there that there exists a biinvariant antisymmetric Dirichlet algebra on K if and only if K is connected and commutative. It would be interesting to characterize Dirichlet algebras $A(X, x_0)$ in terms of affine embeddings.

5.5 Biinvariant algebras and invariant algebras on spheres.

A *biinvariant* algebra on K is a uniformly closed subalgebra with unit in $C(K)$ invariant with respect to both left and right translations (here $M = (K \times K)/\Delta(K)$).

Suppose that F is a subgroup in $G \times G$ containing $\Delta(G)$. Then the subgroup $F_0 = \{g \in G \mid (g, e) \in F\}$ is normal in G . This shows that F is the preimage of $\Delta(\tilde{G})$ for the homomorphism $G \times G \rightarrow \tilde{G} \times \tilde{G}$, where $\tilde{G} = G/F_0$. Moreover, $\Delta(G)$ -fixed points in $(\tilde{G} \times \tilde{G})/\Delta(\tilde{G})$ correspond to central elements of \tilde{G} . These elements form an orbit of the center $Z(\tilde{G})$, and $Z(\tilde{G})$ acts $(\tilde{G} \times \tilde{G})$ -equivariantly on any affine embedding of $(\tilde{G} \times \tilde{G})/\Delta(\tilde{G})$. Hence different base points on such embeddings define the same invariant algebras. An affine embedding of the space $(\tilde{G} \times \tilde{G})/\Delta(\tilde{G})$ is nothing else but an algebraic monoid \tilde{S} with $G(\tilde{S}) = \tilde{G}$ (Proposition 1.12).

Let us summarize all these observations in the following one-to-one correspondences (all biinvariant algebras are supposed to be finitely generated):

- { self-conjugate biinvariant algebras on K } \iff { quotient groups \tilde{G} of the group G };
- { biinvariant algebras on K } \iff { algebraic monoids \tilde{S} with $G(\tilde{S}) = \tilde{G}$ };
- { biinvariant algebras separating points on K } \iff { algebraic monoids S with $G(S) = G$ };
- { antisymmetric biinvariant algebras on K } \iff { algebraic monoids \tilde{S} with zero and $G(\tilde{S}) = \tilde{G}$ }.

To explain the last equivalence, we note that \tilde{S} has a zero if and only

if the closed $(\tilde{G} \times \tilde{G})$ -orbit in \tilde{S} is a point. Embeddings with a G -fixed point correspond to antisymmetric invariant algebras (see 5.4). If the closed orbit has positive dimension, it is isomorphic to $(\tilde{G}_1 \times \tilde{G}_1)/\Delta(\tilde{G}_1)$ for a non-trivial quotient \tilde{G}_1 of the group \tilde{G} , and the corresponding projection (see 3.4) determines a non-trivial self-conjugate subalgebra in our invariant algebra.

Theorem 5.14 (or Proposition 3.4) shows that any biinvariant algebra on K is self-conjugate if and only if K is semisimple. This result was proved by R. Gangolli [24] and J. Wolf [73].

Our final remark concerns invariant algebras on spheres S^n . The classification of transitive actions of compact Lie groups on spheres was obtained by A. Borel, D. Montgomery and H. Samelson (see [50]). All corresponding homogeneous spaces are spherical with a unique exception: there is a transitive action of the group $Sp(n) = GL(n, \mathbb{H}) \cap U(2n)$ on S^{4n-1} with stabilizer $Sp(n-1)$ and the complexification of $Sp(n)/Sp(n-1)$ is a homogeneous space of complexity one. (This is the reason why the classification of invariant algebras on spheres was not completed in this case only, see [39].)

The complexification of $Sp(n)/Sp(n-1)$ satisfies the conditions of Theorem 4.2. This implies the following general result: the number of radical invariant ideals in any invariant algebra on a sphere (with respect to any transitive action) is finite.

6 Application Two: G -algebras with finitely generated invariant subalgebras

6.1 The reductive case

In this section by \mathfrak{A} we denote a finitely generated G -algebra without zero divisors. Let us introduce three special types of G -algebras.

Type C. Here \mathfrak{A} is a finitely generated domain of Krull dimension $\text{Kdim } \mathfrak{A} = 1$ (i.e. the transcendence degree of the quotient field $Q\mathfrak{A}$ equals one) with any (for example, trivial) G -action. Such algebras may be considered as the algebras of regular functions on irreducible affine curves.

Type HV. Let λ be a dominant weight of the group G (with respect to some fixed Borel subgroup) and $V(\lambda)$ be a simple finite-dimensional G -module with highest weight λ . Let λ^* be the highest weight of the dual module $V(\lambda)^*$. Consider a subsemigroup P in the additive semigroup of

non-negative integers (it is automatically finitely generated), and put

$$\mathfrak{A}(P, \lambda) = \bigoplus_{p \in P} V(p\lambda).$$

There exists a unique structure (up to G -isomorphism) of a G -algebra on $\mathfrak{A}(P, \lambda)$ such that $V(p\lambda)V(m\lambda) = V((p+m)\lambda)$. In fact, consider the closure $X(\lambda) = \overline{Gv}$ of the orbit of a highest weight vector v in $V(\lambda^*)$. The algebra $\mathbb{K}[X(\lambda)]$ of regular functions on $X(\lambda)$ as a G -module has the isotypic decomposition

$$\mathbb{K}[X(\lambda)] = \bigoplus_{k \geq 0} \mathbb{K}[X(\lambda)]_{k\lambda},$$

any $\mathbb{K}[X(\lambda)]_{k\lambda}$ is a simple G -module, and

$$\mathbb{K}[X(\lambda)]_{k\lambda} \mathbb{K}[X(\lambda)]_{m\lambda} = \mathbb{K}[X(\lambda)]_{(k+m)\lambda},$$

see 1.3. This allows us to realize $\mathfrak{A}(P, \lambda)$ as a subalgebra in $\mathbb{K}[X(\lambda)]$. The proof of uniqueness of such multiplication is left to the reader. Further we shall say that the algebra $\mathfrak{A}(P, \lambda)$ is an algebra of type HV.

Example 6.1. Let $G = SL(n)$ and $\omega_1, \dots, \omega_{n-1}$ be its fundamental weights. The natural linear action $G : \mathbb{K}^n$ induces an action on regular functions

$$G : \mathfrak{A} = \mathbb{K}[x_1, \dots, x_n], \quad (gf)(v) := f(g^{-1}v).$$

The homogeneous polynomials of degree m form an (irreducible) isotypic component corresponding to the weight $m\omega_{n-1}$. The algebra \mathfrak{A} is of type HV with $\lambda = \omega_{n-1}$ and $P = \mathbb{Z}_+$. The variety $X(\omega_{n-1})$ is the original space \mathbb{K}^n .

Type N. Let H be a closed subgroup of G and

$$\begin{aligned} \mathfrak{A}(H) &= \mathbb{K}[G]^H = \mathbb{K}[G/H] \\ &= \{f \in \mathbb{K}[G] \mid f(gh) = f(g) \text{ for any } g \in G, h \in H\}. \end{aligned}$$

If H is reductive, then $\mathfrak{A}(H)$ is finitely generated. We say that a G -algebra \mathfrak{A} is of type N if there exists a reductive subgroup $H \subset G$ with $|N_G(H)/H| < \infty$ and \mathfrak{A} is G -isomorphic to $\mathfrak{A}(H)$.

Example 6.2. The algebra $\mathfrak{A}(T) = \{f \in \mathbb{K}[G] \mid f(gt) = f(g) \text{ for any } t \in T\}$ is a G -algebra of type N with respect to the left G -action.

Now we are ready to formulate the main result.

Theorem 6.3 ([9]). *Let \mathfrak{A} be a finitely generated G -algebra without zero divisors. Then any G -invariant subalgebra of \mathfrak{A} is finitely generated if and only if \mathfrak{A} is an algebra of one of the types C, HV or N.*

We start the proof of Theorem 6.3 with a method of constructing a non-finitely generated subalgebra. Let X be an irreducible affine algebraic variety and Y a proper closed irreducible subvariety. Consider the subalgebra

$$\mathfrak{A}(X, Y) = \{f \in \mathbb{K}[X] \mid f(y_1) = f(y_2) \text{ for any } y_1, y_2 \in Y\} \subset \mathfrak{A} = \mathbb{K}[X].$$

Proposition 6.4. *The algebra $\mathfrak{A}(X, Y)$ is finitely generated if and only if Y is a point.*

Proof. If Y is a point, then $\mathfrak{A}(X, Y) = \mathbb{K}[X]$. Suppose that Y has positive dimension and $\mathfrak{J} = \mathfrak{J}(Y) = \{f \in \mathbb{K}[X] \mid f(y) = 0 \text{ for any } y \in Y\}$. Then $\mathfrak{A}/\mathfrak{J}$ is infinite-dimensional vector space. By the Nakayama Lemma, we can find $i \in \mathfrak{J}$ such that in the local ring of Y the element i is not in \mathfrak{J}^2 . Then for any $a \in k[X] \setminus \mathfrak{J}$ the element ia is in $\mathfrak{J} \setminus \mathfrak{J}^2$. Hence the space $\mathfrak{J}/\mathfrak{J}^2$ has infinite dimension.

On the other hand, suppose that f_1, \dots, f_n are generators of $\mathfrak{A}(X, Y)$. Subtracting constants, one may assume that all f_i are in \mathfrak{J} . Then $\dim \mathfrak{A}(X, Y)/\mathfrak{J}^2 \leq n + 1$, a contradiction. \square

Proposition 6.5. *Let \mathfrak{A} be a finitely generated domain. Then any subalgebra in \mathfrak{A} is finitely generated if and only if $\text{Kdim } \mathfrak{A} \leq 1$.*

Proof. If $\text{Kdim } \mathfrak{A} \geq 2$, then the statement follows from the previous proposition. The case $\text{Kdim } \mathfrak{A} = 0$ is obvious. It remains to prove that if $\text{Kdim } \mathfrak{A} = 1$, then any subalgebra is finitely generated. By taking the integral closure, one may suppose that \mathfrak{A} is the algebra of regular functions on a smooth affine curve C_1 . Let C be the smooth projective curve such that $C_1 \cong C \setminus \{P_1, \dots, P_k\}$. The elements of \mathfrak{A} are rational functions on C that may have poles only at points P_i . Let \mathfrak{B} be a subalgebra in \mathfrak{A} . By induction on k , we may suppose that the subalgebra $\mathfrak{B}' \subset \mathfrak{B}$ consisting of functions regular at P_1 is finitely generated, say $\mathfrak{B}' = \mathbb{K}[s_1, \dots, s_m]$. (Functions that are regular at any point P_i are constants.) Let $v(f)$ be the order of the zero/pole of $f \in \mathfrak{B}$ at P_1 . The set $V = \{v(f), f \in \mathfrak{B}\}$ is an additive subsemigroup of integers. Such a subsemigroup is finitely generated. Let f_1, \dots, f_n be elements of \mathfrak{B} such that the $v(f_i)$ generate V . Then for any $f \in \mathfrak{B}$ there exists a polynomial $P(y_1, \dots, y_n)$ with $v(f - P(f_1, \dots, f_n)) \geq 0$, and thus $f - P(f_1, \dots, f_n) \in \mathfrak{B}'$. This shows that \mathfrak{B} is generated by $f_1, \dots, f_n, s_1, \dots, s_m$. \square

Let \mathfrak{A} be a finitely generated G -algebra with $\text{Kdim} \mathfrak{A} \geq 2$. Consider the affine variety $X = \text{Spec} \mathfrak{A}$. The action $G : \mathfrak{A}$ induces a regular action $G : X$.

Suppose that there exists a proper irreducible closed invariant subvariety $Y \subset X$ of positive dimension. Then $\mathfrak{A}(X, Y)$ is an invariant subalgebra, which is not finitely generated. In particular, this is the case if G acts on X without a dense orbit. Hence we may assume that either

- (i) the action $G : X$ is transitive, or
- (ii) X consists of an open orbit and a G -fixed point p .

In case (i), $X = G/H$ and H is reductive. If G/H is not affinely closed then there exists a non-trivial affine embedding $G/H \hookrightarrow X'$, and the complement in X to the open affine subset G/H is a union of irreducible divisors. Let Y be one of these divisors. The algebra $\mathfrak{A}(X', Y)$ is a non-finitely generated invariant subalgebra in $\mathbb{K}[X']$ and the inclusion $G/H \hookrightarrow X'$ defines an embedding $\mathbb{K}[X'] \subset \mathbb{K}[X] = \mathfrak{A}$. On the other hand,

Lemma 6.6. *If $X = G/H$ is affinely closed, i.e. \mathfrak{A} is of type N, then any invariant subalgebra in \mathfrak{A} is finitely generated.*

Proof. Suppose that there exists an invariant subalgebra $\mathfrak{B} \subset \mathfrak{A}$ that is not finitely generated. Let f_1, f_2, \dots be a system of generators of \mathfrak{B} . Consider the finitely generated subalgebras $\mathfrak{B}_i = \mathbb{K}[\langle Gf_1, \dots, Gf_i \rangle]$. Infinitely many of them are pairwise different. For the corresponding varieties $X_i := \text{Spec} \mathfrak{B}_i$ one has natural dominant G -morphisms

$$\begin{array}{c} X \\ \swarrow \downarrow \searrow \\ X_1 \longleftarrow X_2 \longleftarrow X_3 \longleftarrow \dots \end{array}$$

By Theorem 3.2, any X_i is an affine homogeneous space G/H_i , $H \subseteq H_i$. The infinite sequence of algebraic subgroups

$$H_1 \supset H_2 \supset H_3 \supset \dots$$

leads to a contradiction. \square

Remark . As is obvious from what has been said, any invariant subalgebra in the algebra $\mathfrak{A}(H)$ of type N has the form $\mathfrak{A}(H')$, where $H \subseteq H' \subseteq G$ and also has type N. Algebras of type N can be characterized by the following equivalent properties:

- (P1) any invariant subalgebra contains no proper invariant ideals;
(P2) the algebra contains no proper invariant ideals and the group of equivariant automorphisms is finite.

Now consider case (ii). Let us recall the following theorem due to F. Bogomolov.

Theorem 6.7 ([16], see also [29, Th.7.6]). *Let X be an irreducible affine variety with a non-trivial G -action and with a unique closed orbit, which is a G -fixed point. Then there exists a G -equivariant surjective morphism $\phi : X \rightarrow X(\mu)$ for some dominant weight $\mu \neq 0$.*

In our case the preimage $\phi^{-1}(0)$ is the point p , and thus all fibres of ϕ are finite. This shows that X is a spherical variety of rank one (see [18] for definitions), i.e.

$$\mathbb{K}[X] = \bigoplus_{m \geq 0} \mathbb{K}[X]_{m\lambda},$$

where $\mathbb{K}[X]_{m\lambda}$ is either zero or irreducible, and $\mu = k\lambda$ for some $k > 0$. On the other hand, the stabilizer of any point on $X(\mu)$ contains a maximal unipotent subgroup of G , and the same is true for X . By Theorem 1.9, this implies $\mathbb{K}[X]_{m_1\lambda} \mathbb{K}[X]_{m_2\lambda} = \mathbb{K}[X]_{(m_1+m_2)\lambda}$. Hence $\mathfrak{A} = \mathbb{K}[X]$ is an algebra of type HV.

Conversely, any subalgebra of the $\mathfrak{A}(P, \lambda)$ is finitely generated because it corresponds to some subsemigroup $P' \subset P$ and P' is finitely generated. This completes the proof of Theorem 6.3. \square

6.2 The non-reductive case

Let us classify affine G -algebras with finitely generated invariant subalgebras for a non-reductive affine group G with the Levi decomposition $G = LG^u$. Surprisingly, the result in this case is simpler than in the reductive case.

In the previous subsection we assumed that a G -algebra \mathfrak{A} has no zero divisors. In fact, this restriction is inessential.

Lemma 6.8 ([10]). *Let $\text{rad}(\mathfrak{A})$ be the ideal of all nilpotents in \mathfrak{A} . The following conditions are equivalent:*

- (1) any G -invariant subalgebra in \mathfrak{A} is finitely generated;
- (2) any G -invariant subalgebra in $\mathfrak{A}/\text{rad}(\mathfrak{A})$ is finitely generated and $\dim \text{rad}(\mathfrak{A}) < \infty$.

Proof. Any finite-dimensional subspace in $\text{rad}(\mathfrak{A})$ generates a finite-dimensional subalgebra in \mathfrak{A} . Hence if $\dim \text{rad}(\mathfrak{A}) = \infty$, then the subalgebra generated by $\text{rad}(\mathfrak{A})$ is not finitely generated. On the other hand, the preimage in \mathfrak{A} of any non-finitely generated subalgebra in $\mathfrak{A}/\text{rad}(\mathfrak{A})$ is not finitely generated.

Conversely, assume that (2) holds. Then any subalgebra in \mathfrak{A} is generated by elements whose images generate the image of this subalgebra in $\mathfrak{A}/\text{rad}(\mathfrak{A})$, and by a basis of the radical of the subalgebra. \square

If \mathfrak{A} contains non-nilpotent zerodivisors, then the proof of Theorem 6.3 goes through with small technical modifications, see [10]. The same proof also goes well for a non-reductive G . The only difference is that case HV is excluded by the result of V. L. Popov.

Proposition 6.9 ([53, Th.3]). *If G acts on an affine variety X with an open orbit, and*

- (1) *the induced action $G^u : X$ is non-trivial and*
- (2) *the complement to the open G -orbit in X does not contain a component of positive dimension,*

then the action $G : X$ is transitive.

These arguments prove

Theorem 6.10 ([10, Th.3]). *Let \mathfrak{A} be a G -algebra without nilpotents with the non-trivial induced G^u -action. The following conditions are equivalent:*

- (1) *any G -invariant subalgebra in \mathfrak{A} is finitely generated;*
- (2) *any G -invariant subalgebra in \mathfrak{A} does not contain non-trivial G -invariant ideals;*
- (3) *any L -invariant subalgebra in \mathfrak{A}^{G^u} does not contain non-trivial L -invariant ideals;*
- (4) *$\mathfrak{A} = \mathbb{K}[G/H]$, where G/H is an affinely closed homogeneous space.*

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