Limit Theorems for the Alloy-type Random Energy Model [*]

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Abstract: In this paper, we consider limit laws for the model, which is a generalisation of the random energy model (REM) to the case when the energy levels have the mixture distribution. More precisely, the distribution of the energy levels is assumed to be a mixture of two normal distributions, one of which is standard normal, while the second has the mean $\sqrt{n}a$ with some $a \in \mathbb{R}$, and the variance $\sigma \neq 1$. The phase space $(a, \sigma) \subset \mathbb{R} \times \mathbb{R}_+$ is divided onto several domains, where after appropriate normalisation, the partition function converges in law to the stable distribution. These domains are separated by the critical surfaces, corresponding to transitions from the normal distribution to $\alpha-$stable with $\alpha \in (1, 2)$, after to 1-stable, and finally to $\alpha-$stable with $\alpha \in (0, 1)$. The corresponding phase diagram is the central result of this paper.

Keywords and phrases: random energy model, normal mixtures, law of large numbers, central limit theorem, stable distribution, phase transition.

1. Introduction

In this paper, we study the limit theorems for the sums of random exponentials

$$\mathcal{S}_n(\beta) = \sum_{j=1}^{\lfloor e^n \rfloor} e^{\beta \sqrt{n} Z_j}, \tag{1}$$

where $Z_1, Z_2, ...$ are i.i.d. random variables with distribution equal to

$$F_{a,\sigma}(x) = \frac{1}{2}\Phi(x) + \frac{1}{2}\Phi\left(\frac{x - \sqrt{n}a}{\sigma}\right), \tag{2}$$

and by $\Phi(\cdot)$ we denote the probability distribution function of the standard normal random variable. The sum $\mathcal{S}_n(\beta)$ is the generalisation of the famous random energy model (REM) introduced by Derrida ([7], [8]) as the simplified version of the Sherrington-Kirkpatrick model of spin glass. Up to the technical detail ($\lfloor e^n \rfloor$ instead of 2^n), the system in the classical REM is determined by the partition function

$$S_n(\beta) = \sum_{i=1}^{\lfloor e^n \rfloor} e^{\beta \sqrt{n} \xi_i}, \qquad (3)$$

where ξ_i are i.i.d random variables with standard normal distribution. Physical interpretation assumes that n is the size of the system with $\lfloor e^n \rfloor$ energy levels $\sqrt{n} \xi_i$. On the other side, in this paper we give different interpretation of the model (1) in terms of the popular Anderson parabolic problem, see Section 2.

Mathematical study of the systems (1) and (3) is mainly concentrated on finding the free energy of the model, and on the consideration of the limit laws depending on the value of the parameter β. As for the free energy for the classical REM (3), Eisele [9], Olivieri and Picco [17] show that

$$P(\beta) = \lim_{n \to \infty} \frac{\ln S_n(\beta)}{n} = \begin{cases} 1 + \beta^2/2, & \beta \leq \sqrt{2}, \\ \sqrt{2}\beta, & \beta > \sqrt{2}. \end{cases}$$

In particular, the first line of the r.h.s. follows from the fact that for $\beta \leq \sqrt{2}$ the law of large numbers holds, that is,

$$\frac{S_n(\beta)}{\mathbb{E}[S_n(\beta)]} \xrightarrow{p} 1, \qquad n \to \infty,$$

Other limit laws for classical REM model were proven many years later, in the paper by Bovier, Kurkova, Löwe [5]. For instance, it was shown that if $\beta \leq \sqrt{2}/2$, then the central limit theorem holds, whereas for $\beta > \sqrt{2}/2$ the fluctuations of the sum $S_n(\beta)$ are non-Gaussian.

Returning to the model (1), it would be it would be important to note that the free energy for this model was recently studied by Grabchak and Molchanov [13]. The results are based on the observation that the free energy of the whole system is in fact the maximum

$$P(\beta) = \max \{P_1(\beta), P_2(\beta)\}, \qquad (4)$$

where P_1 and P_2 are the free energy functions of the system corresponding to $\lfloor e^n \rfloor$ energy levels with $\mathcal{N}(0,1)$ distribution, and to $\lfloor e^n \rfloor$ energy levels with $\mathcal{N}(\sqrt{n}a, \sigma^2)$ distribution respectively. Nevertheless, similar arguments cannot be applied for proving other limit laws such that the central limit theorem and convergence to the stable distributions.

In this paper, we aim to show the limiting laws for the model (1). It turns out that the limiting distributions are stable. The Lévy triplet $(\mu, 0, \nu)$ is such

that $\mu = \mu(\beta) \in \mathbb{R}_+$ is a drift and ν is a Lévy measure on $\mathbb{R}_+$ defined by

$$\nu((x, +\infty)) = \frac{1}{\sqrt{2\pi}} x^{-\alpha(\beta)}, \qquad \forall \, x > 0,$$

with $\alpha = \alpha(\beta) \in (0, 2]$. In this article we provide the exact forms of the parameters $\alpha(\beta)$ and $\mu(\beta)$ and show how these parameters change when the relation between a and σ varies. Our findings are illustrated by a phase diagram. It would be a worth mentioning that the explicit values of the parameters of the limiting stable distribution were not described previously even for more simple model (3), and below we also present the corresponding results for this case.

The rest of the paper is organised as follows. In the next section, we provide some physical motivation of the considered systems in terms of the Anderson parabolic problem. Next, in Section 3, we give a new formulation of the results for the sums (3), corresponding to the classical case of standard normal energy levels. Our main findings related to the case of energy levels with mixture distribution are given in Section 4. All proofs are collected in Section 5. For convenience, we also provide a statement of the main results from [13] in Appendix A.

2. Anderson parabolic problem

On the lattice $\mathbb{Z}^d$, let us consider the cube $Q_n = [-n, n]^d$ and the random Anderson Hamiltonian

$$H_n = \Delta + \beta V_n(x, \omega), \tag{5}$$

where β is the reciprocal temperature, $V_n(x, \omega), x \in Q_n$, is the random i.i.d. potential (on some probability space, "environment", $(\Omega, \mathcal{F}, \mathbb{P})$), and

$$\Delta \psi(x) = \sum_{x': |x'-x|=1} \psi(x')$$

is the lattice Laplacian on Q_n with the Dirichlet boundary condition

$$\psi(x) = 0, \qquad x \in \partial Q_n.$$

We assume that potential is "very strong": $V_n(x, \omega) = \sqrt{n} \xi(x, \omega)$, where $\xi(x, \omega)$ are i.i.d. r.v.'s with the law (5). Here the factor $\sqrt{n}$ is related to the Gaussian law. If, say, $\mathbb{P}\{\xi > a\} = \exp\{-a^\alpha/\alpha\}$, $\alpha > 1$ (Weibull's law), like in [1], instead of $\sqrt{n}$ one have to use $n^{1/\alpha}$.

Consider the parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= H_n u, & t \geq 0, \ x \in Q_n, \\ u(t, x) &= 0, & x \in \partial Q_n, \\ u(0, x) &= \delta_y(x), \end{aligned}$$

where $y \in Q_n$ is considered as a parameter. Then the fundamental solution of this problem is given by

$$u_n(t,x) = u_n(t,x,y) = \sum_{i=1}^{|Q_n|} e^{\lambda_{n,i}t} \psi_{n,i}(x)\psi_{n,i}(y), \tag{6}$$

where $\lambda_{n,i}, \psi_{n,i}$ are the eigenvalues and (normalised) eigenfunctions of the operator H_n, that is, $H_n\psi_{n,i} = \lambda_{n,i}\psi_{n,i}$. Now

$$\operatorname{Tr} e^{tH_n} = \sum_{x \in Q_n} u_n(t,x,x) = \sum_{i=1}^{|Q_n|} e^{t\lambda_{n,i}} \tag{7}$$

is the random exponential sum. The asymptotic analysis of this sum and related concepts of the intermittency and localisation were discussed in numerous works, e.g., [2], [3], [6], [10], [11], [12].

The limit theorems for the parabolic Anderson model become the subject of the studies only recently and the picture here is still not complete. In comparison with the setup discussed in [1] and [5], the main difficulty is the dependence of $\lambda_{n,i}(\omega)$. However, for the "very strong" potentials the situation is simplier. In this case, we can naturally use the parameter y instead of i, because for any y there exists the eigenfunction equal to the δ-function. More precisely, if $V_n(x,\omega) = \sqrt{n}\xi(x,\omega)$, $x \in Q_n$, then with high accurancy

$$\psi_{n,y}(x) = \delta_y(x), \qquad \lambda_{n,y}(\omega) = \sqrt{n}\xi(y,\omega).$$

One can estimate the errors using usual perturbation arguments, but we will not provide the caculations here. Let us simply state that at the level of physical intuition the sum

$$\sum_{y \in Q_n} e^{\beta\sqrt{n}\xi(y,\omega)} = S_n(\beta)$$

is close to sum (7) for $t = \beta$.

In this setting, the mixtures appear naturally. Assume that we have two highly disordered Hamiltonians with potentials

$$V_1(x,\omega) = \sqrt{n}\eta(x,\omega), \quad V_2(x,\omega) = \sqrt{n}\zeta(x,\omega), \qquad x \in Q_n,$$

where η, ζ are two independent systems of i.i.d. Gaussian r.v.'s with different parameters. Let us consider the new alloy-type potential

$$V_n(x,\omega) = \sqrt{n}\xi(x,\omega), \qquad \text{where} \quad \xi(x,\omega) = \begin{cases} \eta(x,\omega) & \text{with probability } 1/2, \\ \zeta(x,\omega) & \text{with probability } 1/2. \end{cases}$$

The trace of the parabolic Anderson problem with this potential is close to the sum (1).

3. Limit laws for the sums of normal exponentials

Before we will present our results for the model (1), we would like to shortly discuss the corresponding results for the classical REM model.

The limit laws for the sums (3) with standard normal r.v. ξ_i were firstly shown in [5] and later generalised in [1]. Below we formulate Proposition 3.1, which can be considered as a new version of the results from [5]. The main advantage of our version in comparison with the previously known facts (given in [5] and [15]) is that we provide the exact form of the limiting distribution. As follows from the next proposition, this distribution is in fact from the class of stable laws.

Proposition 3.1.   *(i) If $\beta < \sqrt{2}$, then the law of large numbers holds, that is,*

$$\frac{S_n(\beta)}{\mathbb{E}[S_n(\beta)]} \xrightarrow{p} 1, \qquad n \to \infty.$$

If $\beta = \sqrt{2}$, then

$$\frac{S_n(\beta)}{\mathbb{E}[S_n(\beta)]} \xrightarrow{p} 1/2, \qquad n \to \infty.$$

(ii) If $\beta < \sqrt{2}/2$, then the central limit theorem holds, that is,

$$\frac{S_n(\beta) - \mathbb{E}[S_n(\beta)]}{\sqrt{\mathrm{Var}(S_n(\beta))}} \xrightarrow{d} \mathcal{N}(0,1), \qquad n \to \infty.$$

If $\beta = \sqrt{2}/2$, then

$$\frac{S_n(\beta) - \mathbb{E}[S_n(\beta)]}{\sqrt{\mathrm{Var}(S_n(\beta))}} \xrightarrow{d} \mathcal{N}(0,1/2), \qquad n \to \infty.$$

(iii) For $\beta > \sqrt{2}/2$, it holds

$$\frac{S_n(\beta) - \delta_n(\beta)}{\gamma_n(\beta)} \xrightarrow{d} F_{\alpha(\beta),\mu(\beta)}, \qquad n \to \infty,$$

where $F_{\alpha(\beta),\mu(\beta)}$ stands for a stable distribution on $\mathbb{R}_+$, that is, an infinitely divisible distribution with Lévy triplet $(\mu, 0, \nu)$ such that $\mu = \mu(\beta) \in \mathbb{R}_+$ is a drift and ν is a Lévy measure on $\mathbb{R}_+$ defined by

$$\nu((x,+\infty)) = \frac{1}{\sqrt{2\pi}} x^{-\alpha(\beta)}, \qquad \forall\, x > 0.$$

Moreover,

$$\begin{aligned}
\alpha(\beta) &= \sqrt{2}/\beta, \\
\gamma_n(\beta) &= (2n)^{-\beta/(2\sqrt{2})} e^{\sqrt{2}\beta n},
\end{aligned}$$

and the choices of $\mu(\beta)$ and $\delta_n(\beta)$ are related to each other; for instance, they can be taken as follows:

$$(\mu(\beta), \delta_n(\beta)) = \begin{cases} \left(0, \mathbb{E}[S_n(\beta)]\right), & \text{if } \beta \in (\sqrt{2}/2, \sqrt{2}), \\ \left(0, \frac{1}{2}\mathbb{E}[S_n(\beta)]\right), & \text{if } \beta = \sqrt{2}, \\ \left(\frac{1}{\sqrt{\pi}(\beta - \sqrt{2})}, 0\right), & \text{if } \beta \in (\sqrt{2}, +\infty). \end{cases}$$

Proof. We provide the proof of this statement in Section 5.1. $\qquad\square$

4. Main results

Now let us return to the model (1), which can be rewritten as

$$\mathcal{S}_n(\beta) = \sum_{j=1}^{\nu_n^1} \exp\left\{\beta\sqrt{n}\xi_j^1\right\} + \sum_{j=1}^{\nu_n^2} \exp\left\{\beta\left(\sigma\sqrt{n}\xi_j^2 + an\right)\right\} := \mathcal{S}_n^1(\beta) + \mathcal{S}_n^2(\beta),$$

where ξ_j^1, ξ_j^2 are 2 independent sequences of i.i.d. standard normal r.v.'s, ν_n^1, ν_n^2 have binomial distribution with parameters $(\lfloor e^n \rfloor, 0.5)$, and satisfy $\nu_n^1 + \nu_n^2 = \lfloor e^n \rfloor$.

The next 3 theorems yield the values of β, for which the law of large numbers, the central limit theorem and the convergence to stable distributions hold.

Theorem 4.1 (Law of large numbers). *It holds*

$$\frac{\mathcal{S}_n(\beta)}{\mathbb{E}[\mathcal{S}_n(\beta)]} \xrightarrow{p} 1, \qquad n \to \infty, \tag{8}$$

provided that $\beta < \beta^+$, where

$$\beta^+ = \begin{cases} \sqrt{2}/\sigma, & \text{if } a > \left(1 - \sigma^2\right)/(\sqrt{2}\sigma), \\ \beta_\circ := \frac{2a}{1-\sigma^2}, & \text{if } \left(1 - \sigma^2\right)/\sqrt{2} < a < \left(1 - \sigma^2\right)/(\sqrt{2}\sigma), \\ \sqrt{2}, & \text{if } a < \left(1 - \sigma^2\right)/\sqrt{2}. \end{cases} \tag{9}$$

Theorem 4.2 (Central limit theorem). *It holds*

$$\frac{\mathcal{S}_n(\beta) - \mathbb{E}[\mathcal{S}_n(\beta)]}{\sqrt{\mathrm{Var}(\mathcal{S}_n(\beta))}} \xrightarrow{d} \mathcal{N}(0, 1), \qquad n \to \infty, \tag{10}$$

provided that $\beta < \beta^+/2$, where β^+ is defined by (9).

Theorem 4.3 (Convergence to the stable distribution).

(i) If $a < \sqrt{2}(1 - \sigma)$, then there exists a deterministic sequence $a_n^\sharp(\beta)$ such that

$$\frac{S_n(\beta) - a_n^\sharp(\beta)}{\gamma_n(\beta)} \xrightarrow{d} F_{\sqrt{2}/\beta, 0}, \qquad n \to \infty,$$

for any $\beta > \beta^{\sharp}$, where

$$\beta^{\sharp} = \begin{cases} \beta_{\diamond}/2, & \text{if } \sigma < 1 \text{ and } a > (1 - \sigma^2)/\sqrt{2}, \\ \sqrt{2}/2, & \text{otherwise}, \end{cases}$$

with $\beta_{\diamond} = \left((\sqrt{2} - a) - \sqrt{\left(\sqrt{2} - a\right)^2 - 2\sigma^2} \right) /\sigma^2$.

(ii) *If $a > \sqrt{2}(1 - \sigma)$, then there exists a deterministic sequence $\breve{a}_n(\beta)$ such that*

$$\frac{S_n(\beta) - \breve{a}_n(\beta)}{e^{\beta a n}\gamma_n(\beta\sigma)} \xrightarrow{d} F_{\sqrt{2}/(\beta\sigma),0}, \qquad n \to \infty,$$

for any $\beta > \breve{\beta}$, where

$$\beta^{\sharp} = \begin{cases} \beta_{*}/2, & \text{if } \sigma > 1 \text{ and } a < (1 - \sigma^2)/(\sqrt{2}\sigma), \\ \sqrt{2}/(2\sigma), & \text{otherwise}, \end{cases}$$

with $\beta_{*} = \left(\sigma\sqrt{2} + a\right) - \sqrt{\left(\sigma\sqrt{2} + a\right)^2 - 2}$.

Theorems 4.1, 4.2, 4.3 basically mean that there exist 6 essentially different types of relation between a and σ). Figure 1 illustrates the division of the area $(a, \sigma) \in \mathbb{R} \times \mathbb{R}_{+}$ into 6 subareas with different asymptotic behaviour of the sums $\mathcal{S}_n(\beta)$.

Remark 4.4. *Analogously to Proposition 3.1, for the critical values $\beta = \beta^{+}$ and $\beta = \beta^{+}/2$ the convergence in (24) and (25) holds, but the limits are equal to $1/2$ and $\mathcal{N}(0, 1/2)$ resp.*

5. Proofs

5.1. Proof of Proposition 3.1

The proof is based on the Proposition 3.1 from [18], which is in fact a combination of Theorem 1.7.3 from [14], Theorem 3.2.2 from [16], and a number of theorems given in Chapter IV from [19]. Below we provide the proof for (iii), because other parts of this proposition were shown in previous papers.

1. Choice of γ_n. First, we find a sequence γ_n such that the sum

$$\Sigma_n(x) := \sum_{i=1}^{\lfloor e^n \rfloor} \mathbb{P}\left\{ e^{\beta\sqrt{n}\xi_i} > \gamma_n x \right\}$$

has non-trivial limit as n tends to infinity. We get

$$\Sigma_n(x) = \lfloor e^n \rfloor \cdot \mathbb{P}\left\{\xi_1 > \kappa_n(x)\right\} = \lfloor e^n \rfloor \cdot (1 - \Phi\left(\kappa_n(x)\right)),$$
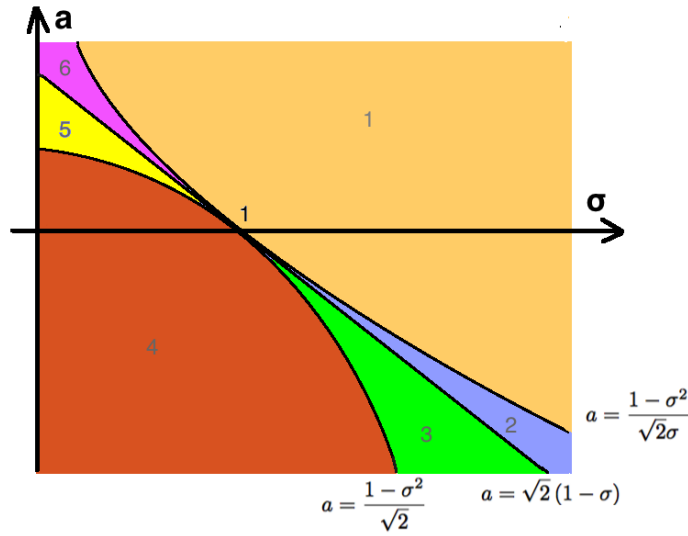
FIG 1. *Illustration of the asymptotic behavior of the sums $\mathcal{S}_n$ depending on a and σ:*
- Zone 1(orange), $a > \left(1 - \sigma^2\right)/(\sqrt{2}\sigma)$: CLT holds for any $\beta < \sqrt{2}/(2\sigma)$ whereas the sum converges to stable distribution for any $\beta > \sqrt{2}/(2\sigma)$; LLN holds for $\beta < \sqrt{2}\sigma$.
- Zone 4 (red), $a < \left(1 - \sigma^2\right)/\sqrt{2}$: CLT holds for any $\beta < \sqrt{2}/2$, whereas the sum converges to stable distribution for any $\beta > \sqrt{2}/2$; LLN holds for $\beta < \sqrt{2}$.
- Other zones, $\left(1 - \sigma^2\right)/\sqrt{2} < a < \left(1 - \sigma^2\right)/(\sqrt{2}\sigma)$: CLT holds for any $\beta < \beta_\circ/2$, LLN - for $\beta < \beta_\circ$, and the fluctuations are stable if
◁ Zone 2 (blue): $\sigma > 1$ and $\sqrt{2}(1 - \sigma) < a < \left(1 - \sigma^2\right)/(\sqrt{2}\sigma)$: $\beta > \beta_*/2$;
◁ Zone 3 (green): $\sigma > 1$ and $\left(1 - \sigma^2\right)/\sqrt{2} < a < \sqrt{2}(1 - \sigma)$: $\beta > \sqrt{2}/2$;
◁ Zone 5 (yellow): $\sigma < 1$ and $\left(1 - \sigma^2\right)/\sqrt{2} < a < \sqrt{2}(1 - \sigma)$: $\beta > \beta_\circ/2$;
◁ Zone 6 (purple): $\sigma < 1$ and $\sqrt{2}(1 - \sigma) < a < \left(1 - \sigma^2\right)/(\sqrt{2}\sigma)$: $\beta > \sqrt{2}/(2\sigma)$.

where $\Phi(x)$ stands for the distribution of the standard normal random variable and $\kappa_n(x) := \log(\gamma_n x)/(\beta\sqrt{n})$. Next, taking into account that

$$\Phi(x) = 1 - (\sqrt{2\pi}x)^{-1}e^{-x^2/2}(1 + o(1)), \qquad \text{as} \quad x \to \infty, \tag{11}$$

we conclude that

$$
\begin{aligned}
\Sigma_n(x) &= \frac{1}{\sqrt{2\pi}} \frac{\lfloor e^n \rfloor}{\kappa_n(x)} e^{-\kappa_n^2(x)/2} \\
&= \frac{1}{\sqrt{2\pi}} \frac{\lfloor e^n \rfloor}{\frac{\log \gamma_n}{\beta\sqrt{n}} + \frac{\log(x)}{\beta\sqrt{n}}} \cdot e^{-\frac{1}{2}\left(\frac{\log(\gamma_n)}{\beta\sqrt{n}}\right)^2 - \frac{\log(\gamma_n)}{\beta\sqrt{n}}\frac{\log(x)}{\beta\sqrt{n}} - \frac{1}{2}\left(\frac{\log(x)}{\beta\sqrt{n}}\right)^2}.
\end{aligned}
$$

Let us find γ_n in the form

$$\frac{\log(\gamma_n)}{\beta\sqrt{n}} = C\sqrt{n} + g(n),$$

where $g(n) = o(\sqrt{n})$. We get

$$\Sigma_n(x) = \frac{1}{\sqrt{2\pi}} \frac{\lfloor e^n \rfloor}{C\sqrt{n} + g(n) + \frac{\log(x)}{\beta\sqrt{n}}} \cdot e^{-\frac{1}{2}\left(C\sqrt{n} + g(n)\right)^2 - \frac{C}{\beta}\log(x)}(1 + o(1)).$$

Therefore, under the choice $C = \sqrt{2}$, $g(n) = -\log(\sqrt{2n})/\sqrt{2n}$, the sum Σ_n converges to a non-trivial limit, namely,

$$\Sigma_n(x) \to \frac{1}{\sqrt{2\pi}} x^{-\sqrt{2}/\beta}, \qquad \text{as} \quad n \to \infty.$$

Therefore, we conclude that the choice

$$\gamma_n = (2n)^{-\beta/(2\sqrt{2})} e^{\sqrt{2}\beta n}$$

yields the convergence of S_n/γ_n to a non-trivial limit.

2. Condition on the truncated moments. Let us analyse the asymptotic behaviour of the expression

$$J_n(s, \tau) := \lfloor e^n \rfloor \int_0^\tau x^s \mu_n(dx), \tag{12}$$

where μ_n is the distribution of $e^{\beta\sqrt{n}\xi_1}/\gamma_n$, $\tau > 0$ and $s = \{1, 2\}$. Our choice of γ_n yields

$$
\begin{aligned}
J_n(s, \tau) &= \frac{\lfloor e^n \rfloor}{\beta\sqrt{n}} \int_0^\tau x^{s-1} p\left(\frac{\log(\gamma_n x)}{\beta\sqrt{n}}\right) dx \\
&= \frac{\lfloor e^n \rfloor}{\gamma_n^s} \int_{-\infty}^{\frac{\log(\gamma_n \tau)}{\beta\sqrt{n}}} e^{\beta\sqrt{n}sy} p(y)\, dy \\
&= \frac{\lfloor e^n \rfloor e^{\beta^2 n s^2/2}}{\gamma_n^s} \int_{-\infty}^{\frac{\log(\gamma_n \tau)}{\beta\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-(y - \beta\sqrt{n}s)^2/2} dy \\
&= \frac{\lfloor e^n \rfloor e^{\beta^2 n s^2/2}}{\gamma_n^s} \Phi\left(\frac{\log(\gamma_n \tau)}{\beta\sqrt{n}} - \beta s\sqrt{n}\right), \tag{13}
\end{aligned}
$$

where p is the density of a standard normal random variable. Since $\log(\gamma_n) \asymp \sqrt{2}\beta n$, we get that

$$\frac{\log(\gamma_n \tau)}{\beta \sqrt{n}} - \beta s \sqrt{n} = \left(\sqrt{2} - \beta s + \frac{\log(\tau)}{\beta n} \right) \sqrt{n} - \frac{1}{2\sqrt{2}} \frac{\log(2n)}{\sqrt{n}}.$$

Finally, applying (11), we conclude that as $n \to \infty$

$$J_n(s,\tau) \asymp \begin{cases} \left(\sqrt{\pi} \left(\beta s - \sqrt{2} \right) \right)^{-1} \tau^{\beta s - \sqrt{2}}, & \text{if } \sqrt{2} - \beta s < 0, \\ \lfloor e^n \rfloor \cdot \mathbb{E}\left[e^{\beta s \sqrt{n} \xi_1} \right] \gamma_n^{-s}, & \text{if } \sqrt{2} - \beta s > 0, \\ \frac{1}{2} \lfloor e^n \rfloor \cdot \mathbb{E}\left[e^{\beta s \sqrt{n} \xi_1} \right] \gamma_n^{-s}, & \text{if } \sqrt{2} - \beta s = 0 \end{cases} \quad (14)$$

where $\mathbb{E}\left[e^{\beta s \sqrt{n} \xi_1} \right] = e^{\beta^2 n s^2 / 2}$, because of the fact that

$$\mathbb{E}e^{c\xi} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{cx} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(x-c)^2/2} dx \cdot e^{c^2/2} = e^{c^2/2}. \quad (15)$$

Therefore

$$\lim_{\tau \to 0} \lim_{n \to \infty} \left(J_n(2,\tau) - \frac{1}{\lfloor e^n \rfloor} \left(J_n(1,\tau) \right)^2 \right) = 0,$$

and therefore the condition (14) from [18] is fulfilled for any $\beta > \sqrt{2}/2$. To conlcude the proof, it is sufficient to note that $a_n = J_n(1,1) + o(1)$, and its asymptotical behaviour follows from (27).

5.2. Proof of Theorem 4.1

1. Denote

$$\widetilde{\mathcal{S}}_n(\beta) = \frac{\mathcal{S}_n(\beta)}{\mathbb{E}[\mathcal{S}_n(\beta)]} - 1 = \frac{\sum_{j=1}^{M} \left(e^{\beta \sqrt{n} Z_j} - \mathbb{E}e^{\beta \sqrt{n} Z_j} \right)}{M \cdot \mathbb{E}e^{\beta \sqrt{n} Z_1}},$$

where $M := \lfloor e^n \rfloor$. Our aim is to show that there exists a constant $r > 1$ such that $\mathbb{E}|\widetilde{\mathcal{S}}_n(\beta)|^r \to 0$ as $n \to \infty$. This will imply that $\widetilde{\mathcal{S}}_n(\beta) \xrightarrow{p} 0$, and therefore the result will follow.

Applying the Bahr-Esseen inequality for $r \in (1,2)$, see [20], we get that

$$\mathbb{E}\left[|\widetilde{\mathcal{S}}_n(\beta)|^r \right] \leq C_r \frac{\sum_{j=1}^{M} \mathbb{E}\left[\left| e^{\beta \sqrt{n} Z_j} - \mathbb{E}e^{\beta \sqrt{n} Z_j} \right|^r \right]}{\left(M \cdot \mathbb{E}e^{\beta \sqrt{n} Z_1} \right)^r} = C_r M^{1-r} \frac{\mathcal{M}_n(r)}{\left(\mathbb{E}e^{\beta \sqrt{n} Z_1} \right)^r}, \quad (16)$$

where C_r is some constant depending on r, and $\mathcal{M}_n(r) := \mathbb{E}|e^{\beta \sqrt{n} Z_1} - \mathbb{E}e^{\beta \sqrt{n} Z_1}|^r$. The further analysis consists in establishing the asymptotical behavior of the numerator and denominator of the last fraction in (16).

2. Note that for any $c \in \mathbb{R}$,

$$
\begin{aligned}
\mathbb{E}e^{cZ_1} &= \frac{1}{2}\mathbb{E}e^{c\xi} + \frac{1}{2} \cdot \mathbb{E}e^{c(\sqrt{n}a+\sigma\xi)} \\
&= \frac{1}{2} \cdot e^{c^2/2} + \frac{1}{2} \cdot e^{c\sqrt{n}a+c^2\sigma^2/2},
\end{aligned}
$$

where we use (15). Therefore, the denominator in (16) is equal to the $r-$ th power of

$$
\begin{aligned}
\mathbb{E}e^{\beta\sqrt{n}Z_1} &= \frac{1}{2}e^{(\beta^2/2)n} + \frac{1}{2} \cdot e^{(\beta a+\beta^2\sigma^2/2)n} \qquad (17) \\
&= \kappa e^{\gamma(\beta)n}\left(1 + o(1)\right), \quad n \to \infty.
\end{aligned}
$$

3. Next, we proceed with studying the asymptotics of $\mathcal{M}_n(r)$. Taking into account (15), we get

$$
\begin{aligned}
\mathcal{M}_n(r) &\leq 2^r \left(\mathbb{E}e^{r\beta\sqrt{n}Z_1} + \left(\mathbb{E}e^{\beta\sqrt{n}Z_1}\right)^r \right) \\
&\leq 2^r \left[\frac{1}{2} \cdot e^{(r^2\beta^2/2)n} + \frac{1}{2} \cdot e^{r\beta an+(r^2\beta^2\sigma^2/2)n} \right. \\
&\qquad \left. + \left(\frac{1}{2}e^{(\beta^2/2)n} + \frac{1}{2}e^{\beta an+(\beta^2\sigma^2/2)n} \right)^r \right] \\
&= 2^r \left(\frac{1}{2}e^{(r^2\beta^2/2)n} + \frac{1}{2}e^{(r\beta a+r^2\beta^2\sigma^2/2)n} \right)(1 + o(1)). \qquad (18)
\end{aligned}
$$

4. Finally, returning to (16), we conclude that

$$
\mathbb{E}\left[|\widetilde{\mathcal{S}}_n(\beta)|^r\right] \leq G_r \cdot \exp\left\{H_r(\beta)n\right\}(1 + o(1)), \qquad (19)
$$

where $G_r > 0$ is a constant depending on r,

$$
H_r(\beta) := (1 - r) + r\left(\lambda_r(\beta) - \gamma(\beta)\right),
$$

and

$$
\begin{aligned}
\lambda_r(\beta) &= \max(r\beta^2/2, \beta a + r\beta^2\sigma^2/2), \\
\gamma(\beta) &= \max(\beta^2/2, \beta a + \beta^2\sigma^2/2).
\end{aligned}
$$

For further analysis it would be convenient to consider 4 cases:

(i) $\sigma > 1, a > 0$: in this case, $\lambda_r(\beta) = \beta a + r\beta^2\sigma^2/2$, $\gamma(\beta) = \beta a + \beta^2\sigma^2/2$, and therefore the rhs in (19) tends to zero iff

$$
H_r(\beta) = (1 - r)\left(1 - r\frac{\beta^2\sigma^2}{2}\right) < 0.
$$

There exists an $r \in (1, 2)$ such that this inequality is fulfilled iff $\beta < \sqrt{2}/\sigma$.

(ii) $\sigma < 1, a < 0$: in this case, $\lambda_r(\beta) = r\beta^2/2$, $\gamma(\beta) = \beta^2/2$, and therefore

$$H_r(\beta) = \left(1 - r\right)\left(1 - r\frac{\beta^2}{2}\right) < 0$$

for some $r \in (1, 2)$ iff $\beta < \sqrt{2}$.

(iii) $\sigma > 1, a < 0$: it follows that

$$\lambda_r(\beta) = \begin{cases} r\beta^2/2, & \text{if } \beta \leq \beta_\circ/r, \\ \beta a + r\beta^2\sigma^2/2 & \text{if } \beta > \beta_\circ/r, \end{cases}$$

and

$$\gamma(\beta) = \begin{cases} \beta^2/2, & \text{if } \beta \leq \beta_\circ, \\ \beta a + \beta^2\sigma^2/2 & \text{if } \beta > \beta_\circ, \end{cases}$$

Therefore,

$$H_r(\beta) = \begin{cases} \left(1 - r\right)\left(1 - r\frac{\beta^2}{2}\right), & \text{if } \beta < \beta_\circ/r, \\ \left(1 - r\right) + r\left(\beta a + \left(r\sigma^2 - 1\right)\frac{\beta^2}{2}\right), & \text{if } \beta_\circ/r < \beta < \beta_\circ, \\ \left(1 - r\right)\left(1 - r\frac{\beta^2\sigma^2}{2}\right), & \text{if } \beta > \beta_\circ. \end{cases} \tag{20}$$

Taking into account the first and the third lines, we conclude that there exists some $r \in (1, 2)$ such that $H_r(\beta) < 0$ if $\beta < \min(\sqrt{2}, \beta_\circ)$ or $\beta_\circ < \beta < \sqrt{2}/\sigma$. Therefore, the further analysis depends on the order of numbers $\beta_\circ$ and $\sqrt{2}/\sigma < \sqrt{2}$. More precisely,

- If $a > \left(1 - \sigma^2\right)/(\sqrt{2}\sigma)$ (that is, $\beta_\circ < \sqrt{2}/\sigma < \sqrt{2}$), then the LLN holds for $\beta < \sqrt{2}/\sigma$.
- If $\left(1 - \sigma^2\right)/\sqrt{2} < a < \left(1 - \sigma^2\right)/(\sqrt{2}\sigma)$ (that is, $\sqrt{2}/\sigma < \beta_\circ < \sqrt{2}$), then the LLN holds for $\beta < \beta_\circ$.
- If $a < \left(1 - \sigma^2\right)/\sqrt{2}$ (that is, $\beta_\circ > \sqrt{2}$), then the LLN holds for $\beta < \sqrt{2}$. Note that in this case, $a < \sqrt{2}(1 - \sigma)$ and therefore the LLN is not fulfilled for any $\beta > \sqrt{2}$ due to the fact that $P(\beta) = P^1(\beta)$.

(iv) $\sigma < 1, a > 0$. The proof follows the same lines as in the previous case. In particular, we get that

$$H_r(\beta) = \begin{cases} \left(1 - r\right)\left(1 - r\frac{\beta^2\sigma^2}{2}\right), & \text{if } \beta < \beta_\circ/r, \\ \left(1 - r\right) + r\left(\frac{\beta^2}{2} - \beta a - \frac{\beta^2\sigma^2}{2}\right), & \text{if } \beta_\circ/r < \beta < \beta_\circ, \\ \left(1 - r\right)\left(1 - r\frac{\beta^2}{2}\right), & \text{if } \beta > \beta_\circ, \end{cases} \tag{21}$$

and the condition $H_r(\beta) < 0$ holds when $\beta < \min(\sqrt{2}/\sigma, \beta_\circ)$ or $\beta_\circ < \beta < \sqrt{2}$. To conclude the proof it is sufficient to consider 3 cases depending on the order of numbers $\beta_\circ$ and $\sqrt{2} < \sqrt{2}/\sigma$.

5.3. *Proof of Theorem 4.2*

Let us show 2 methods, which lead to the proof of this theorem. The first one is rather classical and is essentially based on the Lyapounov condition. The second proof is based on the third part of Proposition 3.1 from [18].

Method 1. Let us check that the Lyapounov condition holds (see (27.16) from [4]): there exists $\delta > 0$ such that

$$\Omega_n := \frac{\mathcal{M}_n(2+\delta)}{(\lfloor e^n \rfloor)^{\delta/2} \left(\mathrm{Var}(e^{\beta\sqrt{n}Z_1})\right)^{1+\delta/2}} \to 0, \qquad \text{as} \quad n \to \infty.$$

Applying (17) and (18),

$$
\begin{aligned}
\Omega_n \quad &\asymp \quad \frac{2^{2+\delta}\left(\frac{1}{2}e^{((2+\delta)^2\beta^2/2)n} + \frac{1}{2}\cdot e^{((2+\delta)\beta a + (2+\delta)^2\beta^2\sigma^2/2)n}\right)}{(\lfloor e^n \rfloor)^{\delta/2}\left(\frac{1}{2}e^{2\beta^2 n} + \frac{1}{2}\cdot e^{2(\beta a + \beta^2\sigma^2)n}\right)^{1+\delta/2}} \\
&\asymp \quad 2^{2\Delta-1}\exp\left\{\left[1 - \Delta + \Delta\left(g_\Delta(\beta) - h(\beta)\right)\right]n\right\},
\end{aligned}
$$

where $\Delta = 1 + \delta/2$, and

$$
\begin{aligned}
g_\Delta(\beta) &= \max\left(2\Delta\beta^2, \quad 2\beta a + 2\Delta\beta^2\sigma^2\right), \\
h(\beta) &= \max\left(2\beta^2, \quad 2\beta a + 2\beta^2\sigma^2\right),
\end{aligned}
$$

There, one should find the condition on the existence of $\Delta > 1$ such that

$$G_\Delta(\beta) := 1 - \Delta + 2\Delta\left(g_\Delta(\beta) - h(\beta)\right) < 0. \tag{22}$$

Note that this task was in fact previously considered on Step 4 of the proof of Theorem 4.1. The only difference is that β should be changed to 2β everywhere. This observation completes the proof.

Method 2. Alternatively, let us show how the proof can be derived from the CLT for the summands. From the proof of Proposition 3.1, we get the following 2 statements.

1. For any sequence γ_n, the distribution μ_n of the random variable $e^{\beta\sqrt{n}\xi_1}/\gamma_n$ satisfies

$$
\begin{aligned}
J_n(s,\tau) &:= \lfloor e^n \rfloor \int_0^\tau x^s \mu_n(dx) \\
&= \frac{\lfloor e^n \rfloor e^{\beta^2 n s^2/2}}{\gamma_n^s}\Phi\left(\frac{\log(\gamma_n\tau)}{\beta\sqrt{n}} - \beta s\sqrt{n}\right),
\end{aligned}
$$

where $\tau > 0$, $s = \{1,2\}$. Assume now that

$$\gamma_n = e^{n(1+2\beta^2)/2}R(n) \tag{23}$$

with some $R(n)$. Then

- if $R(n) \asymp c$ for some $c > 0$, then $J_n(2, \tau) \asymp c^{-s}$ for any $\beta < \sqrt{2}/2$;
- if $R(n) \to \infty$ as $n \to \infty$ then $J_n(2, \tau) \to 0$.

2. If the sequence γ_n is such that

$$\lim_{n \to \infty} \frac{\gamma_n}{e^{\sqrt{2}\beta n}} = +\infty, \tag{24}$$

then

$$\sum_{i=1}^{\lfloor e^n \rfloor} \mathbb{P}\left\{ e^{\beta\sqrt{n}\xi_i} > \gamma_n x \right\} \to 0.$$

Similar outcomes are valid also for the second distribution:

1. for any sequence γ_n, the distribution $\tilde{\mu}_n$ of $e^{\beta(\sigma\sqrt{n}\xi_i + an)}/\gamma_n$ satisfies

$$
\begin{aligned}
\tilde{J}_n(s, \tau) \quad &:= \quad \lfloor e^n \rfloor \int_0^\tau x^s \tilde{\mu}_n(dx) \\
&= \quad \frac{\lfloor e^n \rfloor e^{\beta ans + \beta^2 n s^2 \sigma^2/2}}{\gamma_n^s} \Phi\left(\frac{\log(\gamma_n \tau) - \beta an}{\beta\sigma\sqrt{n}} - \beta\sigma s\sqrt{n} \right),
\end{aligned}
$$

where $\tau > 0$, $s = \{1, 2\}$; in particular, from here it follows then if

$$\gamma_n = e^{n(1 + 2\sigma^2\beta^2 + 2\beta a)/2} R(n)$$

with some $R(n)$, then

- if $R(n) \asymp c$ for some $c > 0$, then $\tilde{J}_n(2, \tau) \asymp c^{-s}$ when $\beta < \sqrt{2}/2$;
- if $R(n) \to \infty$, then $\tilde{J}_n(2, \tau) \to 0$;

2. if the sequence γ_n is such that

$$\lim_{n \to \infty} \frac{\gamma_n}{e^{(\sqrt{2}\sigma + a)\beta n}} = +\infty, \tag{25}$$

then

$$\sum_{i=1}^{\lfloor e^n \rfloor} \mathbb{P}\left\{ e^{\beta(\sigma\sqrt{n}\xi_i + an)} > \gamma_n x \right\} \to 0.$$

Consider now the mixture of 2 distributions. Our aim is to find a normalizing sequence $\breve{\gamma}_n$ such that both condition of Proposition 3.1 from [18] (part 3) are fulfilled. Natural candidate is $\breve{\gamma}_n := \sqrt{\operatorname{Var}(\mathcal{S}_n(\beta))}$. Note that

$$
\begin{aligned}
\breve{\gamma}_n = \left(\sum_{j=1}^{\lfloor e^n \rfloor} \operatorname{Var} e^{\beta\sqrt{n}Z_j} \right)^{1/2} \quad &= \quad \left(\lfloor e^n \rfloor \cdot \left(\mathbb{E}e^{2\beta\sqrt{n}Z_1} - \left(\mathbb{E}e^{\beta\sqrt{n}Z_1} \right)^2 \right) \right)^{1/2} \\
&\asymp \quad \left(\frac{1}{2} \lfloor e^n \rfloor \cdot e^{2\beta \max\{\beta, a + \beta\sigma^2\}n} \right)^{1/2},
\end{aligned}
$$

since for any $c \in \mathbb{R}$,

$$\mathbb{E}e^{cZ_1} = \left(e^{c^2/2} + e^{c\sqrt{n}a + c^2\sigma^2/2} \right)/2.$$

Therefore, $\breve{\gamma}_n$ has different asymptotics in the cases when $\beta(1 - \sigma^2) > a$ is fulfilled or not. In the first case, $\breve{\gamma}_n \asymp 2^{-1/2} e^{n(1+2\beta^2)/2}$. Under this choice of $\breve{\gamma}_n$, (24) trivially holds, and moreover from

$$
\lim_{n \to \infty} \frac{\gamma_n}{e^{(\sqrt{2}\sigma + a)\beta n}} \geq 2^{-1/2} \lim_{n \to \infty} \frac{e^{(1 + 2\beta a + 2\beta^2 \sigma^2)n/2}}{e^{(\sqrt{2}\sigma + a)\beta n}}
$$
$$
= 2^{-1/2} \lim_{n \to \infty} e^{n(1 - \sqrt{2}\beta\sigma)^2/2} = +\infty,
$$

we conclude that (25) also holds. Therefore,

$$
\begin{aligned}
\breve{\Sigma}_n &= \lfloor e^n \rfloor \cdot \mathbb{P}\left\{ e^{\beta\sqrt{n}Z_1} \geq \breve{\gamma}_n x \right\} \\
&= \frac{1}{2}\lfloor e^n \rfloor \cdot \mathbb{P}\left\{ e^{\beta\sqrt{n}\xi_1} \geq \breve{\gamma}_n x \right\} + \frac{1}{2}\lfloor e^n \rfloor \cdot \mathbb{P}\left\{ e^{\beta(\sigma\sqrt{n}\xi_1 + an)} \geq \breve{\gamma}_n x \right\} \to 0.
\end{aligned}
$$

On another hand,

$$\lfloor e^n \rfloor \int_0^\tau x^s m_n(dx) = \frac{1}{2}J_n(s, \tau) + \frac{1}{2}\tilde{J}_n(s, \tau), \tag{26}$$

where m_n stands for the distribution of $e^{\beta\sqrt{n}Z_1}/\gamma_n$. Note that $J_n(2, \tau) \asymp 2$ if $\beta < \sqrt{2}/2$, and moreover $\tilde{J}_n(2, \tau) \to 0$. Therefore, we conclude that the sequence $\mathcal{S}_n(\beta)/\breve{\gamma}_n$ converges to a standard normal random variable, and

$$
\begin{aligned}
a_n &= \frac{1}{2}J_n(1, 1) + \frac{1}{2}\tilde{J}_n(1, 1) \\
&= \frac{\lfloor e^n \rfloor}{\gamma_n}\left(\frac{1}{2}\mathbb{E}\left[e^{\beta s\sqrt{n}\xi_1} \right] + \frac{1}{2}\mathbb{E}\left[e^{\beta s(\sigma\sqrt{n}\xi_i + an)} \right] \right) = \mathbb{E}[\mathcal{S}_n(\beta)].
\end{aligned}
$$

Finally, we conclude that the CLT holds if β belongs to one of the following areas:

$$
\begin{aligned}
\mathcal{B}_1 &:= \left\{ \beta : \quad \beta(1 - \sigma^2) > a, \ \beta < \sqrt{2}/2 \right\}, \\
\mathcal{B}_2 &:= \left\{ \beta : \quad \beta(1 - \sigma^2) < a, \ \beta < \sqrt{2}/(2\sigma) \right\}
\end{aligned}
$$

(the proof for $\mathcal{B}_2$ follows the same lines). Consideration of these areas depending on the relations between a and σ concludes the proof.

5.4. Proof of Theorem 4.3

Due to Proposition 3.1 (iii), we get that for $\beta > \sqrt{2}/2$, it holds

$$\frac{\sum_{i=1}^{\lfloor e^n \rfloor} e^{\beta\sqrt{n}\xi_i} - \delta_n}{\gamma_n} \xrightarrow{d} F_{\alpha(\beta), \mu(\beta)}, \qquad n \to \infty.$$

From this result, it follows that

$$\frac{\sum_{i=1}^{\lfloor e^n \rfloor} e^{\beta\left(\sigma\sqrt{n}\xi_i + an\right)} - \tilde{\delta}_n(\beta)}{\tilde{\gamma}_n(\beta)} \xrightarrow{d} F_{\alpha(\beta\sigma),\mu(\beta\sigma)}, \qquad n \to \infty,$$

where $\tilde{\gamma}_n(\beta) = e^{\beta an}\gamma_n(\beta\sigma)$, $\tilde{\delta}_n(\beta) = e^{\beta an}\delta_n(\beta\sigma)$.

As for the mixtures, we should find some $\breve{\gamma}_n = \breve{\gamma}_n(\beta)$ such that the sum

$$
\begin{aligned}
\breve{\Sigma}_n &= \lfloor e^n \rfloor \cdot \mathbb{P}\left\{ e^{\beta\sqrt{n}Z_1} \geq \breve{\gamma}_n x \right\} \\
&= \frac{1}{2}\lfloor e^n \rfloor \cdot \mathbb{P}\left\{ e^{\beta\sqrt{n}\xi_1} \geq \breve{\gamma}_n x \right\} + \frac{1}{2}\lfloor e^n \rfloor \cdot \mathbb{P}\left\{ e^{\beta\left(\sigma\sqrt{n}\xi_1 + an\right)} \geq \breve{\gamma}_n x \right\},
\end{aligned}
$$

converges to a non-trivial limit. Note that if $\gamma_n(\beta) \lesssim \tilde{\gamma}_n(\beta)$ (equivalently, $a > \sqrt{2}(1-\sigma)$) then the choice $\breve{\gamma}_n = \tilde{\gamma}_n$ yields the second summand in $\breve{\Sigma}_n$ converges to $\left(2\sqrt{2\pi}\right)^{-1} x^{-\sqrt{2}/(\beta\sigma)}$, while the first tends to 0:

$$
\begin{aligned}
\frac{1}{2}\lfloor e^n \rfloor \cdot \mathbb{P}\left\{ e^{\beta\sqrt{n}\xi_1} \geq \breve{\gamma}_n x \right\} &= \frac{1}{2}\lfloor e^n \rfloor \cdot \mathbb{P}\left\{ e^{\beta\sqrt{n}\xi_1} \geq \gamma_n\left(x\frac{\tilde{\gamma}_n}{\gamma_n} \right) \right\} \\
&\asymp \frac{1}{2\sqrt{2\pi}}\left(x\frac{\tilde{\gamma}_n}{\gamma_n} \right)^{-\sqrt{2}/\beta} \to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

Now let us consider formula (26) with this choice of $\breve{\gamma}_n$. The asymptotic behaviour of the second summand follows from (27), namely,

$$
J_n(s,\tau) \asymp \begin{cases}
\frac{1}{\sqrt{\pi}\left(\beta\sigma s - \sqrt{2}\right)}\tau^{\beta\sigma s - \sqrt{2}}, & \text{if } \sqrt{2} - \beta\sigma s < 0, \\
\exp\left\{ n(\sqrt{2} - \beta\sigma s)^2/2 \right\}, & \text{if } \sqrt{2} - \beta\sigma s > 0, \\
\frac{1}{2}\exp\left\{ n(\sqrt{2} - \beta\sigma s)^2/2 \right\}, & \text{if } \sqrt{2} - \beta\sigma s = 0,
\end{cases}
$$

and therefore $\lim_{\tau\to 0} \limsup_{n\to\infty} J_n(2,\tau) = 0$ iff $\beta > \sqrt{2}/(2\sigma)$. As for $J_n(s,\tau)$, we get

$$J_n(s,\tau) = \frac{\lfloor e^n \rfloor e^{\beta^2 n s^2/2}}{\breve{\gamma}_n^s}\Phi\left(\frac{\log(\breve{\gamma}_n\tau)}{\beta\sqrt{n}} - \beta s\sqrt{n} \right)$$

Taking into account that $\log(\breve{\gamma}_n) \asymp (a + \sqrt{2}\sigma)\beta n$, and

$$\frac{\log(\breve{\gamma}_n\tau)}{\beta\sqrt{n}} - \beta s\sqrt{n} = \left(a + \sqrt{2}\sigma - \beta s + \frac{\log(\tau)}{\beta n} \right)\sqrt{n} - \frac{\sigma}{2\sqrt{2}}\frac{\log(2n)}{\sqrt{n}},$$

we conclude that

$$
J_n(s,\tau) \asymp \begin{cases}
c_1 n^{r_1} e^{r_2 n}, & \text{if } a + \sqrt{2}\sigma - \beta s < 0, \\
\lfloor e^n \rfloor \cdot \mathbb{E}\left[e^{\beta s\sqrt{n}\xi_1} \right]\breve{\gamma}_n^{-s}, & \text{if } a + \sqrt{2}\sigma - \beta s > 0, \\
\frac{1}{2}\lfloor e^n \rfloor \cdot \mathbb{E}\left[e^{\beta s\sqrt{n}\xi_1} \right]\breve{\gamma}_n^{-s}, & \text{if } a + \sqrt{2}\sigma - \beta s = 0.
\end{cases}
$$

with $r_2 = 1 - \left(a + \sqrt{2}\sigma\right)^2/2 < 0$ and some r_1. Note that in the second case $(a + \sqrt{2}\sigma - \beta s > 0)$,

$$\log(J_n(2,\tau)) \asymp \left(2\beta^2 - 2(a + \sqrt{2}\sigma)\beta + 1\right)n,$$

where the quadratic form has 2 roots,

$$\beta_{\pm} = \frac{\left(a + \sqrt{2}\sigma\right) \pm \sqrt{\left(a + \sqrt{2}\sigma\right)^2 - 2}}{2},$$

where $\beta_- = \beta_*/2$ and $\beta_- \leq (a + \sqrt{2}\sigma)/2 \leq \beta_+$, and therefore $J_n(2,\tau) \to 0$ iff $\beta > \beta_*/2$. Finally, we conclude that the convergence to a stable law holds if and only if

$$\beta > \max\left\{\frac{\beta_*}{2}, \frac{\sqrt{2}}{2\sigma}.\right\}$$

In the considered area $\left[a > \sqrt{2}(1 - \sigma)\right]$, the first item is larger then the second if and only if $\sigma > 1$ and $a < (1 - \sigma^2)/(\sqrt{2}\sigma)$. This observation concludes the proof.

Appendix A: Previous research

As it was already mentioned in the introduction, the free energy of the considered system was recently studied by Grabchak, Molchanov [13]. In this appendix, we shortly discuss the main results from that research.

Free energy of the system with $\lfloor e^n \rfloor$ standard normal energy levels is equal to

$$P_1(\beta) = \lim_{n\to\infty} \frac{\ln S_n^1(\beta)}{n} = \begin{cases} 1 + \beta^2/2, & \beta \leq \sqrt{2}, \\ \sqrt{2}\beta, & \beta > \sqrt{2}. \end{cases},$$

whereas the free energy of system with $\lfloor e^n \rfloor$ energy levels having $\mathcal{N}(\sqrt{n}a, \sigma^2)$ distribution equals

$$P_2(\beta) = \lim_{n\to\infty} \frac{\ln S_n^2(\beta)}{n} = \begin{cases} 1 + \beta a + \beta^2\sigma^2/2, & \beta \leq \sqrt{2}/\sigma \\ \left(\sigma\sqrt{2} + a\right)\beta, & \beta > \sqrt{2}/\sigma. \end{cases}$$

It turns out that the free energy of the system with mixture distribution of energy levels is equal to $P(\beta) = \max\{P_1(\beta), P_2(\beta)\}$. This observation leads to the conclusion, that $P(\beta)$ essentially differs between 6 possible types of relations between a and σ, see Figure 2:

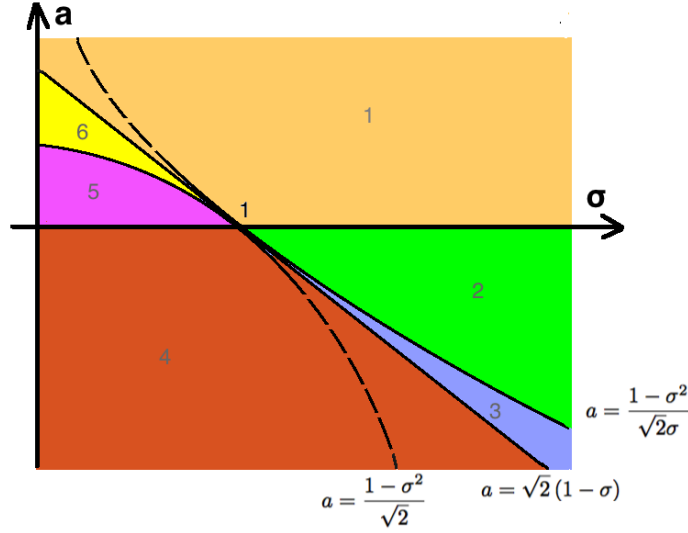1. Zone 1 (orange), $a > 0$ and $a > \sqrt{2}(1 - \sigma)$: $P(\beta) = P_2(\beta)$.

FIG 2. *Illustration of the asymptotic behavior of the free energy depending on the parameters a and σ.*

2. Zone 2 (green), $a < 0$ and $a > (1 - \sigma^2)/(\sqrt{2}\sigma)$:

$$P(\beta) = \begin{cases} P_1(\beta), & \beta \leq \beta_\circ \\ P_2(\beta), & \beta > \beta_\circ. \end{cases}$$

with $\beta_\circ = 2a/(1 - \sigma^2)$.

3. Zone 3 (blue), $a < 0$ and $\sqrt{2}(1 - \sigma) < a < (1 - \sigma^2)/(\sqrt{2}\sigma)$:

$$P(\beta) = \begin{cases} P_1(\beta), & \beta \leq \beta_* \\ P_2(\beta), & \beta > \beta_*. \end{cases}$$

with $\beta_* = \left(\sigma\sqrt{2} + a\right) - \sqrt{\left(\sigma\sqrt{2} + a\right)^2 - 2}$.

4. Zone 4 (red), $a < 0$ and $a < \sqrt{2}(1 - \sigma)$: $P(\beta) = P_2(\beta)$.

5. Zone 5 (purple), $a > 0$ and $(1 - \sigma^2)/\sqrt{2} < a < \sqrt{2}(1 - \sigma)$:

$$P(\beta) = \begin{cases} 1 + a\beta + \sigma^2\beta^2/2, & \beta \leq \beta_\diamond, \\ \sqrt{2}\beta, & \beta > \beta_\diamond, \end{cases}$$

where $\beta_\diamond = \left((\sqrt{2} - a) - \sqrt{\left(\sqrt{2} - a\right)^2 - 2\sigma^2}\right)/\sigma^2$.

6. Zone 6 (yellow): $a > 0$ and $a < (1 - \sigma^2)/\sqrt{2}$:

$$P(\beta) = \begin{cases} 1 + a\beta + \sigma^2 \beta^2/2, & \beta \le \beta_\circ, \\ 1 + \beta^2/2, & \beta_\circ < \beta < \sqrt{2}, \\ \sqrt{2}\beta, & \beta > \sqrt{2}. \end{cases}$$

It would be a worth mentioning that these zones do not completely coincide with the zones on Figure 1. Nevertheless, the critical values β_* and $\beta_\diamond$ appears under the same assumptions on the relation between a and σ.

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