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Twisted zastava and q-Whittaker functions

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To Valery Lunts on his 60th birthday

Abstract

We implement the program outlined in Section 7 of our earlier paper [*J. Amer. Math. Soc.* 27 (2014) 1147–1168] extending to the case of nonsimply laced simple Lie algebras the construction of solutions of q-difference Toda equations from geometry of quasimaps' spaces. To this end we introduce and study the twisted zastava spaces.

1. *Introduction*

In this note, we implement the program outlined in [**5**, Section 7] extending to the case of nonsimply laced simple Lie algebras the construction of solutions of q-difference Toda equations from geometry of quasimaps' spaces.

1.1. *Semiinfinite Borel–Weil–Bott*

Let G be an almost simple simply connected group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$; we shall denote by $\check{\mathfrak{g}}$ the Langlands dual algebra of $\mathfrak{g}$. We fix a Cartan torus and a Borel subgroup $T \subset B \subset G$. Also let $\mathcal{B}_{\mathfrak{g}}$ denote its flag variety. We have $H_2(\mathcal{B}_{\mathfrak{g}}, \mathbb{Z}) = \Lambda$, the coroot lattice of $\mathfrak{g}$. We shall denote by Λ_+ the sub-semigroup of positive elements in Λ.

Let $\mathbf{C} \simeq \mathbb{P}^1$ denote a (fixed) smooth connected projective curve (over $\mathbb{C}$) of genus 0; we are going to fix a marked point $\infty \in \mathbf{C}$, and a coordinate t on $\mathbf{C}$ such that $\mathsf{t}(\infty) = 0$. For each $\alpha \in \Lambda_+$ we can consider the space $\mathcal{M}_{\mathfrak{g}}^{\alpha}$ of maps $\mathbf{C} \to \mathcal{B}_{\mathfrak{g}}$ of degree α. This is a smooth quasi-projective variety. It has a compactification $\mathcal{QM}_{\mathfrak{g}}^{\alpha}$ by means of the space of *quasi-maps* from $\mathbf{C}$ to $\mathcal{B}_{\mathfrak{g}}$ of degree α. Set-theoretically this compactification can be described as follows:

$$\mathcal{QM}_{\mathfrak{g}}^{\alpha} = \bigsqcup_{0 \leqslant \beta \leqslant \alpha} \mathcal{M}_{\mathfrak{g}}^{\beta} \times \mathrm{Sym}^{\alpha-\beta}(\mathbf{C}) \tag{1.1}$$

where $\mathrm{Sym}^{\alpha-\beta}(\mathbf{C})$ stands for the space of 'colored divisors' of the form $\sum \gamma_i x_i$, where $x_i \in \mathbf{C}$, $\gamma_i \in \Lambda_+$ and $\sum \gamma_i = \alpha - \beta$. In particular, for $\beta \geqslant \alpha$ we have an embedding $\varphi_{\alpha,\beta}:\ \mathcal{QM}_{\mathfrak{g}}^{\alpha} \hookrightarrow \mathcal{QM}_{\mathfrak{g}}^{\beta}$ adding defect at the point $0 \in \mathbf{C}$ (such that $\mathsf{t}(0) = \infty$). The union of all $\mathcal{QM}_{\mathfrak{g}}^{\alpha}$ is an ind-projective scheme $\mathfrak{Q}_{\mathfrak{g}}$. To each weight $\check{\lambda} \in X^*(T)$ of G one associates a line bundle $\mathcal{O}(\check{\lambda})$ on $\mathfrak{Q}_{\mathfrak{g}}$.

Recall the notion of (global) Weyl modules $\mathcal{W}(\check{\lambda})$ over the current algebra $\mathfrak{g}[\mathsf{t}]$ (see, for example, [**8**]). The following version of the Borel–Weil–Bott theorem was proved in [**6**] in case $\mathfrak{g}$ is simply-laced. First, the higher cohomology $H^{>0}(\mathfrak{Q}_{\mathfrak{g}}, \mathcal{O}(\check{\lambda}))$ vanish identically. Second, in case $\check{\lambda}$ is *not* a dominant weight, the global sections $H^0(\mathfrak{Q}_{\mathfrak{g}}, \mathcal{O}(\check{\lambda}))$ vanish as well. Third, in case $\check{\lambda}$ is a dominant weight, the global sections $H^0(\mathfrak{Q}_{\mathfrak{g}}, \mathcal{O}(\check{\lambda}))$ are isomorphic to the *dual* global Weyl module $\mathcal{W}(\check{\lambda})^{\vee}$. In Section 5 of the present note we extend the Borel–Weil–Bott

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theorem to the case of arbitrary simple $\mathfrak{g}$, and also prove that the schemes $\mathcal{QM}_{\mathfrak{g}}^{\alpha}$ have rational singularities.

1.2. *The q-Whittaker functions*

Let $\check{G}$ denote the Langlands dual group of G with its maximal torus $\check{T}$. Let W be the Weyl group of (G, T). We recall the notion of q-Whittaker functions $\Psi_{\check{\lambda}}(q, z)$: W-invariant polynomials in $z \in T$ with coefficients in the field $\mathbb{C}(q)$ of rational functions in a variable q ($\check{\lambda} \in X^*(T)^+$ a dominant weight of G). The definition of $\Psi_{\check{\lambda}}(q, z)$ is as follows. In [**11**] and [**27**] the authors define (by adapting the so called Kostant–Whittaker reduction to the case of quantum groups) a homomorphism $\mathcal{M}: \mathbb{C}[T]^W \to \mathrm{End}_{\mathbb{C}(q)}\mathbb{C}(q)[\check{T}]$ called the quantum difference Toda integrable system associated with $\check{G}$. For each $f \in \mathbb{C}[T]^W$ the operator $\mathcal{M}_f := \mathcal{M}(f)$ is indeed a difference operator: it is a $\mathbb{C}(q)$-linear combination of shift operators $\mathbf{T}_{\check{\beta}}$ where $\check{\beta} \in X^*(T)$ and

$$\mathbf{T}_{\check{\beta}}(F(q, x)) = F(q, q^{\check{\beta}}x), \ x \in \check{T}.$$

In particular, the above operators can be restricted to operators acting in the space of functions on the lattice $X^*(T)$ by means of the embedding $X^*(T) \hookrightarrow \check{T}$ sending every $\check{\lambda}$ to $q^{\check{\lambda}}$. More precisely, we have a restriction morphism $\mathrm{res}\colon \mathbb{C}(q)[\check{T}] \to \mathbb{C}(q)[X^*(T)]$, $\mathrm{res}\, F(q, \check{\lambda}) := F(q, q^{\check{\lambda}})$, and a unique operator $\mathbf{T}_{\check{\beta}}\colon \mathbb{C}(q)[X^*(T)] \to \mathbb{C}(q)[X^*(T)]$ such that $\mathbf{T}_{\check{\beta}}\,\mathrm{res} = \mathrm{res}\,\mathbf{T}_{\check{\beta}}$. Namely, $\mathbf{T}_{\check{\beta}}(F(q, \check{\lambda})) = F(q, \check{\beta} + \check{\lambda})$. For any $f \in \mathbb{C}[T]^W$ we shall denote the corresponding operator on $\mathbb{C}(q)[X^*(T)]$ by $\mathcal{M}_f^{\mathrm{lat}}$.

There exists a collection of $\mathbb{C}(q)$-valued polynomials† of $\Psi_{\check{\lambda}}(q, z)$, $\check{\lambda} \in X^*(T)$, on T satisfying the following properties.

(a) $\Psi_{\check{\lambda}}(q, z) = 0$ if $\check{\lambda}$ is not dominant.
(b) $\Psi_0(q, z) = 1$.
(c) Let us consider all the functions $\Psi_{\check{\lambda}}(q, z)$ as one function $\Psi(q, z)\colon\ X^*(T) \to \mathbb{C}(q)$ depending on $z \in T$. Then for every $f \in \mathbb{C}[T]^W$ we have

$$\mathcal{M}_f^{\mathrm{lat}}(\Psi(q, z)) = f(z)\Psi(q, z).$$

There exists another definition of the q-Toda system using double affine Hecke algebras, studied for example in [**10**]. To be more specific, we restrict ourselves here to the double affine Hecke algebras of symmetric type in terminology of [**19**]. Since it is not clear to us how to prove *a priori* that the definition of q-Toda from [**10**] coincides with the definitions from [**11**] and [**27**], we shall denote the q-difference operators from [**10**] by $\mathcal{M}'_f$. Similarly we shall denote by $(\mathcal{M}_f^{\mathrm{lat}})'$ their 'lattice' version. We shall denote the corresponding polynomials by $\Psi'_{\check{\lambda}}(q, z)$, so that $(\mathcal{M}_f^{\mathrm{lat}})'(\Psi'(q, z)) = f(z)\Psi'(q, z)$. Their existence follows, for example, from the results of [**21**].

1.3. *Characters of twisted Weyl modules*

In case $\mathfrak{g}$ is simply laced, it was proved in [**6**] that $\Psi_{\check{\lambda}}(q, z)$ coincides with the character of the global Weyl module $\mathcal{W}(\check{\lambda})$ over $\mathfrak{g}[\mathfrak{t}] \rtimes \mathbb{C}^*$; and it was explained in Section 1.4 of [**6**] that such an equality does not hold in case of nonsimply laced $\mathfrak{g}$. In the nonsimply laced case we use the following remedy. We realize $\check{\mathfrak{g}}$ as a *folding* of a simple simply laced Lie algebra $\check{\mathfrak{g}}'$, that is, as invariants of an outer automorphism σ of $\check{\mathfrak{g}}'$ preserving a Cartan subalgebra $\check{\mathfrak{t}}' \subset \check{\mathfrak{g}}'$ and acting on the root system of $(\check{\mathfrak{g}}', \check{\mathfrak{t}}')$. In particular, σ gives rise to the same named automorphism of the Langlands dual Lie algebras $\mathfrak{g}' \supset \mathfrak{t}'$ (note that say, in case $\mathfrak{g}$ is of type B_n, $\mathfrak{g}'$ is of type

†Conjecturally, such a collection is unique. We do not use the uniqueness property in the present note. The existence follows, for example, from the results of the present note.

A_{2n-1}, while for $\mathfrak{g}$ of type C_n, $\mathfrak{g}'$ is of type D_{n+1}; in particular, $\mathfrak{g}\not\subset\mathfrak{g}'$). Let d stand for the order of σ. We choose a primitive root of unity ζ of order d. We consider an automorphism ς of $\mathfrak{g}'[\mathsf{t}]$ defined as the composition of two automorphisms: (a) σ of $\mathfrak{g}'$; (b) $\mathsf{t}\mapsto\zeta\mathsf{t}$ of $\mathbb{C}[\mathsf{t}]$. The subalgebra of invariants $\mathfrak{g}'[\mathsf{t}]^\varsigma$ is the twisted current algebra. The corresponding global twisted Weyl modules $\mathcal{W}^{\mathrm{twisted}}(\check\lambda)$ over $\mathfrak{g}'[\mathsf{t}]^\varsigma\rtimes\mathbb{C}^*$ (still numbered by the dominant $\mathfrak{g}$-weights $\check\lambda\in X^*(T)^+$) were introduced in [**9**].

In Section 4 of the present note we prove that the q-Whittaker function $\Psi_{\check\lambda}(q,z)$ coincides with the character of the global twisted Weyl module $\mathcal{W}^{\mathrm{twisted}}(\check\lambda)$ over $\mathfrak{g}'[\mathsf{t}]^\varsigma\rtimes\mathbb{C}^*$. The relation between the global and local twisted Weyl modules established in [**9**] then implies the following positivity property of $\Psi_{\check\lambda}(q,z)$. Let $d_i=1$ (respectively, $d_i=d$) for a short (respectively, long) simple coroot α_i of $\mathfrak{g}$. For $i\in I$: the set of simple coroots of $\mathfrak{g}$, we set $q_i:=q^{d_i}$. We set $\hat\Psi_{\check\lambda}(q,z):=\Psi_{\check\lambda}(q,z)\cdot\prod_{i\in I}\prod_{r=1}^{\langle\alpha_i,\check\lambda\rangle}(1-q_i^r)$. Then $\hat\Psi_{\check\lambda}(q,z)$ is a polynomial in z,q with nonnegative integral coefficients. Namely, $\hat\Psi_{\check\lambda}(q,z)$ is the character of the local twisted Weyl module.

In fact, the above results are known if one replaces $\hat\Psi_{\check\lambda}(q,z)$ with the polynomials $\hat\Psi'_{\check\lambda}(q,z):=\Psi'_{\check\lambda}(q,z)\cdot\prod_{i\in I}\prod_{r=1}^{\langle\alpha_i,\check\lambda\rangle}(1-q_i^r)$ (these are often called q-Hermite polynomials in the literature). Namely, the above local twisted Weyl modules coincide by [**17**] with the level one Demazure module $D^{\mathrm{twisted}}(\check\lambda)$ over $\mathfrak{g}'[\mathsf{t}]^\varsigma\rtimes\mathbb{C}^*$. Now the characters of level one Demazure modules over dual untwisted affine Lie algebras were proved in [**21**] to coincide with the q-Hermite polynomials $\hat\Psi'_{\check\lambda}(q,z)$. Thus we obtain the following corollary:

Corollary 1.4. *We have $\Psi_{\check\lambda}(q,z)=\Psi'_{\check\lambda}(q,z)$. Hence the q-Toda systems of* [**11**], [**27**] *and of* [**10**] *are equivalent* (*they generate the same commutative subalgebras in the q-difference operators' rings*).

Let us note that the above proof of Corollary 1.4 is very roundabout. It would be nice to find a more direct argument.

1.5. *Twisted quasimaps*

Our proof of the properties Section 1.2(a,b,c) of the characters of the global twisted Weyl modules uses a twisted version of the semiinfinite Borel–Weil–Bott theorem of Section 1.1. Namely, the automorphism ς of $\mathfrak{g}'[\mathsf{t}]$ gives rise to the same named automorphism ς of the ind-projective scheme $\mathfrak{Q}_{\mathfrak{g}'}$ of Section 1.1. Its fixed point subscheme is denoted by $\mathfrak{Q}$. To each weight $\check\lambda\in X^*(T)$ of G one associates a line bundle $\mathcal{O}(\check\lambda)$ on $\mathfrak{Q}$. As in Section 1.1, we have $H^{>0}(\mathfrak{Q},\mathcal{O}(\check\lambda))=0$, while $H^0(\mathfrak{Q},\mathcal{O}(\check\lambda))=\mathcal{W}^{\mathrm{twisted}}(\check\lambda)^\vee$.

Now the q-difference equations of Section 1.2(c) for the characters of $H^0(\mathfrak{Q},\mathcal{O}(\check\lambda))$ are proved following the strategy of [**5**, **6**] provided we know some favourable geometric properties of the finite-type pieces $\mathcal{QM}^\alpha\subset\mathfrak{Q}$ (twisted quasimaps' spaces: the fixed point sets of the automorphism ς of certain quasimaps' spaces $\mathcal{QM}^\beta_{\mathfrak{g}'}$) and their local (based) analogues: twisted zastava spaces Z^α. The verification of these properties occupies the bulk of the present note, namely the central Section 3. Some properties, like irreducibility and normality of Z^α are proved similarly to their classical (nontwisted) counterparts, by reduction to the known properties of the twisted affine Grassmannian of $\mathfrak{g}'$. Some other, like the Cartier property of the (reduced) boundary, turn out harder to prove. Very roughly speaking, our derivation of the q-difference equations for the characters of twisted global Weyl modules from the semiinfinite Borel–Weil–Bott theorem is parallel to the derivation of the Weyl character formula from the classical BWB theorem via localization to the torus-fixed points. Finally, note that the previous results of [**5**, **6**] are formally contained in the results of the present note in case of trivial folding when $\sigma=1$ and $d=1$.

2. *Setup and notations*

2.1. *Root systems and foldings*

Let $\check{\mathfrak{g}}$ be a simple Lie algebra with the corresponding adjoint Lie group $\check{G}$. Let $\check{T}$ be a Cartan torus of $\check{G}$. We choose a Borel subgroup $\check{B} \supset \check{T}$. It defines the set of simple roots $\{\alpha_i,\ i \in I\}$. Let $G \supset T$ be the Langlands dual groups. We define an isomorphism $\alpha \mapsto \alpha^*$ from the root lattice of $(\check{G}, \check{T})$ to the root lattice of (G, T) in the basis of simple roots as follows: $\alpha_i^* := \check{\alpha}_i$ (the corresponding simple coroot). For two elements α, β of the root lattice of $(\check{G}, \check{T})$ we say $\beta \leqslant \alpha$ if $\alpha - \beta$ is a nonnegative linear combination of $\{\alpha_i,\ i \in I\}$. For such α we denote by z^{α^*} the corresponding character of T. As usually, q stands for the identity character of $\mathbb{G}_m$. We set $d_i = (\alpha_i, \alpha_i)/2$, and $q_i = q^{d_i}$.

We realize $\check{\mathfrak{g}}$ as a *folding* of a simple simply laced Lie algebra $\check{\mathfrak{g}}'$, that is, as invariants of an outer automorphism σ of $\check{\mathfrak{g}}'$ preserving a Cartan subalgebra $\check{\mathfrak{t}}' \subset \check{\mathfrak{g}}'$ and a Borel subalgebra $\check{\mathfrak{b}}' \supset \check{\mathfrak{t}}'$, and acting on the root system of $(\check{\mathfrak{g}}', \check{\mathfrak{t}}')$. Note that the unfolding $(\check{\mathfrak{g}}' \supset \check{\mathfrak{b}}' \supset \check{\mathfrak{t}}', \sigma)$ is defined uniquely up to an isomorphism. In particular, σ gives rise to the same named automorphism of the Langlands dual Lie algebras $\mathfrak{g}' \supset \mathfrak{t}'$. We choose a σ-invariant Borel subalgebra $\mathfrak{t}' \subset \mathfrak{b}' \subset \mathfrak{g}'$ such that $\mathfrak{b} = (\mathfrak{b}')^\sigma$. The corresponding set of simple roots is denoted by I'. We denote by Ξ the finite cyclic group generated by σ. We set $d := |\Xi|$. Note that $d_i \in \{1, d\}$. Let $G' \supset T'$ denote the simply connected Lie group and its Cartan torus with Lie algebras $\mathfrak{g}' \supset \mathfrak{t}'$. The *coinvariants* $X_*(T')_\sigma$ of σ on the coroot lattice $X_*(T')$ of $(\mathfrak{g}', \mathfrak{t}')$ coincide with the root lattice of $\check{\mathfrak{g}}$ and with the coroot lattice of $\mathfrak{g}$. We have an injective map $a:\ X_*(T')_\sigma \to X_*(T')^\sigma$ from coinvariants to invariants defined as follows: given a coinvariant α with a representative $\tilde{\alpha} \in X_*(T')$ we set $a(\alpha) := \sum_{\xi \in \Xi} \xi(\tilde{\alpha})$. The Weyl group W of $G \supset T$ coincides with the invariants $(W')^\sigma$ of σ on the Weyl group W' of $G' \supset T'$. We fix a primitive root of unity ζ of order d. We set $\mathcal{K} = \mathbb{C}((\mathsf{t})) \supset \mathcal{O} = \mathbb{C}[[\mathsf{t}]]$. We set $\mathbf{t} := \mathsf{t}^{-1}$.

2.2. *Ind-scheme* $\mathfrak{Q}$

We denote by Gr the twisted affine Grassmannian $G'(\mathcal{K})^\varsigma / G'(\mathcal{O})^\varsigma$: an ind-proper ind-scheme of ind-finite type, see [24, 29]. We consider the projective line $\mathbf{C}$ with coordinate $\mathbf{t}$, and with points $0 = 0_{\mathbf{C}},\ \infty = \infty_{\mathbf{C}}$ such that $\mathbf{t}(0_{\mathbf{C}}) = 0, \mathbf{t}(\infty_{\mathbf{C}}) = \infty$. We recall the setup of [6, Section 2] with $\mathfrak{g}'$ (respectively, $\mathbf{t}$) playing the role of $\mathfrak{g}$ (respectively, t) of *loc. cit.* In particular, $R = \mathbb{C}[[t^{-1}]]$ (respectively, $F = \mathbb{C}((t^{-1})))$ of *loc. cit.* is our $\mathcal{O} = \mathbb{C}[[\mathsf{t}]]$ (respectively, $\mathcal{K} = \mathbb{C}((\mathsf{t}))$). Furthermore, Λ_+ of *loc. cit.* is the cone in $X_*(T')$ generated over $\mathbb{N}$ by the simple coroots, while $\Lambda_+^\vee$ of *loc. cit.* is the cone in $X^*(T')$ generated over $\mathbb{N}$ by the fundamental weights. Given $\gamma \in \Lambda_+$, we consider the quasimaps' space $\mathcal{QM}_{\mathfrak{g}'}^\gamma$.

Recall the notations of Section 2.1. We consider the cone $Y_+ \subset Y = X_*(T')_\sigma$ generated over $\mathbb{N}$ by the classes of simple coroots of $\mathfrak{g}$. Given $\alpha \in Y_+$, we consider an automorphism ς of $\mathcal{QM}_{\mathfrak{g}'}^{a(\alpha)}$ defined as the composition of two automorphisms: (a) σ (arising from the same named automorphism of G'); (b) $\mathbf{t} \mapsto \zeta^{-1}\mathbf{t}$. We define $\mathcal{QM}^\alpha$ as the fixed point set $(\mathcal{QM}_{\mathfrak{g}'}^{a(\alpha)})^\varsigma$ equipped with the structure of reduced closed subscheme of $\mathcal{QM}_{\mathfrak{g}'}^{a(\alpha)}$.

For $\beta \geqslant \alpha \in Y_+$ (that is, $\beta - \alpha \in Y_+$), we consider the closed embedding $\varphi_{\alpha,\beta}:\ \mathcal{QM}^\alpha \hookrightarrow \mathcal{QM}^\beta$ adding the defect $a(\beta - \alpha) \cdot 0$ at the point $0 \in \mathbf{C}$. The direct limit of this system is denoted by $\mathfrak{Q}$.

2.3. *Infinite type scheme* $\mathbf{Q}$

We fix a collection of highest weight vectors $\mathsf{v}_{\check{\lambda}} \in V_{\check{\lambda}},\ \check{\lambda} \in \Lambda_+^\vee \subset X^*(T')$, satisfying the Plücker equations. We denote by $\sigma:\ V_{\check{\lambda}} \to V_{\sigma(\check{\lambda})}$ a unique isomorphism taking $\mathsf{v}_{\check{\lambda}}$ to $\mathsf{v}_{\sigma(\check{\lambda})}$ and intertwining $\sigma:\ G' \to G'$. We denote by $\widehat{\mathbf{Q}}$ the infinite type scheme whose $\mathbb{C}$-points are the collections of *nonzero* vectors $v_{\check\lambda}(\mathbf{t})\in V_{\check\lambda}\otimes\mathbb{C}[[\mathbf{t}^{-1}]],\ \check\lambda\in\Lambda^\vee_+$, satisfying the Plücker relations and the equation $\sigma(v_{\check\lambda})(\zeta^{-1}\mathbf{t})=v_{\sigma(\check\lambda)}(\mathbf{t})$. It is equipped with a free action of $T=(T')^\sigma$: if we view an element of T as a σ-invariant element $h\in(T')^\sigma$, then $h(v_{\check\lambda}(\mathbf{t}))=\check\lambda(h)v_{\check\lambda}(\mathbf{t})$. The quotient scheme $\mathbf{Q}=\widehat{\mathbf{Q}}/T$ is a closed subscheme in $\prod_{i\in I'}\mathbb{P}(V_{\check\omega_i}\otimes\mathbb{C}[[\mathbf{t}^{-1}]])$ where $\check\omega_i$ is a fundamental weight of $\mathfrak{g}'$. Any weight $\check\lambda\in\Lambda^\vee_\sigma=X^*(T')_\sigma=X$ gives rise to a line bundle $\mathcal{O}_{\check\lambda}$ on $\mathbf{Q}$.

The construction of [6, 2.3] gives rise to the closed embedding $\mathfrak{Q}\hookrightarrow\mathbf{Q}$.

Finally, recall that the restriction of characters gives rise to a canonical isomorphism $X=X^*(T')_\sigma\xrightarrow{\sim}X^*(T)$. The T-torsor $\widehat{\mathbf{Q}}\to\mathbf{Q}$ defines, for any $\check\lambda\in X$, a line bundle $\mathcal{O}(\check\lambda)$ on $\mathbf{Q}$. Same notation for its restriction to $\mathfrak{Q}$.

2.4. *Twisted zastava*

The twisted quasimaps' space $\mathcal{QM}^\alpha=(\mathcal{QM}^{a(\alpha)}_{\mathfrak{g}'})^\varsigma$ has an open dense subvariety $'\mathcal{QM}^\alpha$ formed by the quasimaps without defect at $\infty\in\mathbf{C}$. We have an evaluation morphism $ev_\infty:\ '\mathcal{QM}^\alpha\to\mathcal{B}:=\mathcal{B}^\sigma_{\mathfrak{g}'}=(G'/B')^\sigma$. We define the twisted zastava space $Z^\alpha:=ev_\infty^{-1}(\mathfrak{b}_-)=(Z^{a(\alpha)}_{\mathfrak{g}'})^\varsigma$. Recall the factorization morphism $\pi:\ Z^{a(\alpha)}_{\mathfrak{g}'}\to\mathbb{A}^{a(\alpha)}:=(\mathbf{C}-\infty)^{a(\alpha)}$. We consider an automorphism ς of the coloured divisors' space $\mathbb{A}^{a(\alpha)}$ defined as the composition of two automorphisms: (a) σ on the set of colours; (b) $\mathbf{t}\mapsto\zeta^{-1}\mathbf{t}$ on $\mathbb{A}^1$. We have $(\mathbb{A}^{a(\alpha)})^\varsigma=\mathbb{A}^\alpha$; a few words about the meaning of the notation $\mathbb{A}^\alpha$ are in order. Let $\alpha=\sum_{i\in I}a_i\alpha_i$ where $I=I'/\Xi$ (the orbits of the cyclic group generated by σ) $=I_0\sqcup I_1$ where I_0 consists of one-point-orbits (fixed points), while I_1 consists of free orbits (so that α_i is a long (respectively, short) simple root of $(\check G,\check T)$ if $i\in I_0$ (respectively, $i\in I_1$)). Then $\mathbb{A}^\alpha=\prod_{i\in I_1}(\mathbf{C}-\infty)^{(a_i)}\times\prod_{i\in I_0}((\mathbf{C}-\infty)/(\mathbf{t}\mapsto\zeta^{-1}\mathbf{t}))^{(a_i)}$. Note that $(\mathbf{C}-\infty)/(\mathbf{t}\mapsto\zeta^{-1}\mathbf{t})\simeq\mathbb{A}^1$ with coordinate $\mathbf{t}^d$ (where $d=|\Xi|$, see Section 2.1). In particular, the diagonal stratification of $\mathbb{A}^{a(\alpha)}$ induces a *quasidiagonal* stratification of $\mathbb{A}^\alpha$: a point $\underline{w}\in\mathbb{A}^\alpha$ lies on a quasidiagonal if either of the following holds: (a) $w_{i,r}=w_{j,s}$ for $i,j\in I_0$ or $i,j\in I_1$ (and $1\leqslant r\leqslant a_i,\ 1\leqslant s\leqslant a_j$); (b) $w_{i,r}=w^d_{j,s}$ for $i\in I_0,\ j\in I_1$.

Now π commutes with ς, so that the following diagram commutes:

$$\begin{tikzcd} Z^\alpha \arrow[r] \arrow[d] & Z^{a(\alpha)}_{\mathfrak{g}'} \arrow[d, "\pi"'] \\ \mathbb{A}^\alpha \arrow[r] & \mathbb{A}^{a(\alpha)} \end{tikzcd} \tag{2.1}$$

We will denote the left vertical arrow by π as well. The commutativity of the diagram (2.1) implies that the factorization property holds for $\pi:\ Z^\alpha\to\mathbb{A}^\alpha$.

2.5. *An example*

We take $\mathfrak{g}'=\mathfrak{sl}(4)\supset(\mathfrak{g}')^\sigma=\mathfrak{sp}(4)$ (the invariants of the outer automorphism). We denote the simple coroots of $\mathfrak{g}\simeq\mathfrak{sp}(4)$ by α_1,α_2, and the simple coroots of $\mathfrak{g}'$ by β_1,β_2,β_3, so that $a(\alpha_1)=\beta_1+\beta_3$, and $a(\alpha_2)=2\beta_2$. We will exhibit an explicit system of equations defining the twisted zastava Z^α for $\alpha=\alpha_1+\alpha_2$.

To this end recall the fundamental representations of $\mathfrak{g}'$: $V=V_{\check\omega_1}$ with a base v_1,v_2,v_3,v_4; $\Lambda^2V=V_{\check\omega_2}$ with a base $v_{ij}:=v_i\wedge v_j,\ 1\leqslant i<j\leqslant 4$, and finally $\Lambda^3V=V_{\check\omega_3}$ with a base $v_{ijk}:=v_i\wedge v_j\wedge v_k,\ 1\leqslant i<j<k\leqslant 4$. The involutive outer automorphism σ takes V to Λ^3V, and Λ^2V to itself; its action in the above bases is as follows: $v_1\mapsto v_{123},\ v_2\mapsto v_{124},\ v_3\mapsto v_{134},\ v_4\mapsto v_{234};\ v_{12}\mapsto v_{12},\ v_{13}\mapsto v_{13},\ v_{24}\mapsto v_{24},\ v_{34}\mapsto v_{34},\ v_{14}\mapsto -v_{23},\ v_{23}\mapsto -v_{14}$.

Zastava space $Z^{(1,2,1)}_{\mathfrak{sl}(4)}$ is formed by the collections of $V_{\check\omega_i}$-valued polynomials of the form $(\mathbf{t}-a_1)v_1+a_2v_2+a_3v_3+a_4v_4,(\mathbf{t}-a_{123})v_{123}+a_{124}v_{124}+a_{134}v_{134}+a_{234}v_{234},(\mathbf{t}^2+b_{12}\mathbf{t}-a_{12})v_{12}+(b_{13}\mathbf{t}+a_{13})v_{13}+(b_{24}\mathbf{t}+a_{24})v_{24}+(b_{34}\mathbf{t}+a_{34})v_{34}+(b_{14}\mathbf{t}+a_{14})v_{14}+$

$(b_{23}\mathbf{t}+a_{23})v_{23}$ subject to the Plücker relations to be specified below. The twisted zastava space $Z^{(1,1)}\subset Z^{(1,2,1)}_{\mathfrak{sl}(4)}$ is cut out by the following invariance conditions: $a_{123}=-a_1,\ a_{124}=-a_2,\ a_{134}=-a_3,\ a_{234}=-a_4,\ b_{12}=b_{13}=b_{24}=b_{34}=0,\ b_{23}=b_{14},\ a_{23}=-a_{14}$.

When writing down the Plücker relations explicitly we will make use of the above invariance conditions to simplify the resulting equations. First, the $\mathfrak{sl}(4)$-invariant projection $V\otimes\Lambda^3V\to\mathbb{C}$ must annihilate our polynomials, that is $a_{234}-a_4=0$ and $a_3a_{124}+a_4a_{123}-a_1a_{234}-a_2a_{134}=0$. Substituting the invariance conditions we get $a_4=a_{234}=0$. Second, the $\mathfrak{sl}(4)$-invariant projection $\Lambda^2V\otimes\Lambda^2V\to\mathbb{C}$ must annihilate our polynomials, that is $a_{34}+b_{14}b_{23}=0,\ b_{14}a_{23}+b_{23}a_{14}=0,\ a_{14}a_{23}-a_{12}a_{34}-a_{13}a_{24}=0$. Third, the $\mathfrak{sl}(4)$-invariant projection $V\otimes\Lambda^2V\to\Lambda^3V$ must annihilate our polynomials, that is $a_3+b_{23}=0,\ a_4=0;\ a_{24}-a_2b_{14}=0,\ a_{34}-a_3b_{14}=0,\ a_4b_{23}=0,\ a_{23}-a_1b_{23}=0;\ a_1a_{23}+a_2a_{13}+a_3a_{12}=0,\ a_1a_{24}+a_2a_{14}+a_4a_{12}=0,\ a_1a_{34}+a_3a_{14}-a_4a_{13}=0,\ a_2a_{34}-a_3a_{24}+a_4a_{23}=0$.

All in all, we have $a_4=0,\ b_{23}=b_{14}=-a_3,\ a_{23}=-a_{14}$; substituting for a_{34},a_{24},a_{14} their values from the third group of equations, we are left with the variables $a_1,a_2,a_3,a_{12},a_{13}$ satisfying the *single* equation $a_3(a_1^2-a_{12})=a_2a_{13}$. The factorization projection $\pi:\ Z^\alpha\to\mathbb{A}^\alpha$ sends $(a_1,a_2,a_3,a_{12},a_{13})$ to (a_1,a_{12}). The boundary $\partial Z^\alpha=Z^\alpha\setminus\overset{\circ}{Z}{}^\alpha$ is given by a single equation $a_3=0$.

2.6. *Another example*

We take $\mathfrak{g}'=\mathfrak{so}(8)\supset(\mathfrak{g}')^\sigma$ (the invariants of the outer automorphism of order 3). We denote the simple coroots of $\mathfrak{g}$ of type G_2 by α_1,α_2, and the simple coroots of $\mathfrak{g}'$ by $\beta_1,\beta_2,\beta_3,\beta_4$, so that $a(\alpha_1)=3\beta_1$, and $a(\alpha_2)=\beta_2+\beta_3+\beta_4$. We will exhibit an explicit system of equations defining the twisted zastava Z^α for $\alpha=\alpha_1+\alpha_2$, so that $\beta=a(\alpha)=3\beta_1+\beta_2+\beta_3+\beta_4$.

Zastava space $Z^\beta_{\mathfrak{so}(8)}$ is formed by the collection of $V_{\check\omega_i}$-valued polynomials of the form $(\mathbf{t}^3-f''\mathbf{t}^2-f'\mathbf{t}-f)v_{\check\omega_1}+(e''\mathbf{t}^2+e'\mathbf{t}+e)v_{\check\omega_1-\check\alpha_1}+\dots,\quad(\mathbf{t}-a)v_{\check\omega_2}+bv_{\check\omega_2-\check\alpha_2}+cv_{\check\omega_2-\check\alpha_2-\check\alpha_1}+\dots$, and so on. One can show that the invariance conditions together with Plücker equations boil down to a single equation cutting out Z^α in $\mathbb{A}^5$ with coordinates (a,b,c,e,f), namely $c(a^3-f)=be$. The factorization projection $\pi\colon Z^\alpha\to\mathbb{A}^\alpha$ sends (a,b,c,e,f) to (a,f). The boundary ∂Z^α is given by a single equation $c=0$.

In effect, let $Y^\alpha\xrightarrow{\varpi}\mathbb{A}^\alpha$ denote the above hypersurface $c(a^3-f)=be$ in $\mathbb{A}^5$, and its projection $(a,b,c,e,f)\mapsto(a,f)$ to $\mathbb{A}^2$. Then the open subvarieties $\pi^{-1}(\mathbb{A}^2\setminus\{(0,0)\})\subset Z^\alpha$ and $\varpi^{-1}(\mathbb{A}^2\setminus\{(0,0)\})\subset Y^\alpha$ are isomorphic by Lemma 3.2 below and, for example, [**3**, 5.6]. This isomorphism extends to $Y^\alpha\simeq Z^\alpha$ due to normality of Z^α (Proposition 3.10 below).

3. *Geometric properties of twisted quasimaps*

3.1. *Quasidiagonal fibers*

The factorization property of $\pi:\ Z^\alpha\to\mathbb{A}^\alpha$ implies that in order to describe the fibers of π it suffices to describe the quasidiagonal fibers $\mathcal{F}^\alpha_0:=\pi^{-1}(\alpha\cdot 0)$, and $\mathcal{F}^\alpha_1:=\pi^{-1}(\alpha\cdot 1)$ (note that $\mathcal{F}^\alpha_1$ is isomorphic to $\pi^{-1}(\alpha_0\cdot c^d+\alpha_1\cdot c)$ for any $c\neq 0$, where $\alpha_0:=\sum_{i\in I_0}a_i\alpha_i$, and $\alpha_1:=\sum_{i\in I_1}a_i\alpha_i$). Recall that the diagonal fiber $\pi^{-1}(\gamma\cdot c)\subset Z^\gamma_{\mathfrak{g}'}$ is denoted by $\mathcal{F}^\gamma_{\mathfrak{g}'}$ (these fibers are all canonically isomorphic for various choices of $c\in\mathbb{A}^1$); it is equidimensional of dimension $|\gamma|$. Let us choose a decomposition $a(\alpha)=\sum_{\xi\in\Xi}\xi(\tilde\alpha)$ as in Section 2.1 for $\tilde\alpha\in\Lambda_+\subset X_*(T')$.

Lemma 3.2. (a) $\mathcal{F}^\alpha_1\supset\mathcal{F}^{\tilde\alpha}_{\mathfrak{g}'}$.

(b) $\mathcal{F}^\alpha_1=(\bigcup_{\tilde\alpha}\mathcal{F}^{\tilde\alpha}_{\mathfrak{g}'})/\Xi$ (*the union over all the choices of* $\tilde\alpha\in\Lambda_+\subset X_*(T')$ *such that* $a(\alpha)=\sum_{\xi\in\Xi}\xi(\tilde\alpha)$).

(c) *In particular,* $\dim\mathcal{F}^\alpha_1=|\alpha|$.

Proof. Clear. $\square$

In order to describe the (quasi)diagonal fiber $\mathcal{F}_0^\alpha$ we need the twisted affine Grassmannian $\mathrm{Gr} = G'(\mathcal{K})^\varsigma / G'(\mathcal{O})^\varsigma$ of Section 2.2. The T-fixed points of Gr form the lattice Y. The attractor (respectively, repellent) of $2\rho(\mathbb{C}^*)$ to a fixed point μ is the orbit $N'(\mathcal{K})^\varsigma \cdot \mu =: S_\mu$ (respectively, $N'_-(\mathcal{K})^\varsigma \cdot \mu =: T_\mu$). According to [**28**, 3.3.2], $\mathrm{Gr} = \bigsqcup_{\mu \in Y} S_\mu = \bigsqcup_{\mu \in Y} T_\mu$.

LEMMA 3.3. (a) *The closure* $\overline{T}_\mu = \bigcup_{\nu \geqslant \mu} T_\nu$.
(b) *The closure* $\overline{S}_\mu = \bigcup_{\nu \leqslant \mu} S_\nu$.
(c) *There is an isomorphism* $\mathcal{F}_0^\alpha \simeq S_0 \cap \overline{T}_{-\alpha}$.

Proof. (a) and (b): same as [**23**, Proposition 3.1]. (c): same as [**7**, Theorem 2.7]. $\square$

LEMMA 3.4. $\dim \mathcal{F}_0^\alpha = |\alpha|$.

Proof. Same as [**23**, Theorem 3.2], provided we know the dimensions of $G'(\mathcal{O})^\varsigma$-orbits in the twisted Grassmannian: $\dim \mathrm{Gr}^\eta = 2|\eta|$ for $\eta \in Y^+$, according to, for example, [**25**, Corollary 2.10]. $\square$

COROLLARY 3.5. *Any fiber of* $\pi : Z^\alpha \to \mathbb{A}^\alpha$ *is equidimensional of dimension* $|\alpha|$.

Proof. Factorization. $\square$

3.6. *Irreducibility*

We consider the open subscheme $\mathring{Z}^\alpha := (\mathring{Z}^{a(\alpha)}_{\mathfrak{g}'})^\varsigma \subset Z^\alpha$ formed by the based twisted maps (as opposed to quasimaps). The smoothness of $\mathring{Z}^{a(\alpha)}_{\mathfrak{g}'}$ implies the smoothness of $\mathring{Z}^\alpha$.

PROPOSITION 3.7. $\mathring{Z}^\alpha$ *is connected.*

Proof. We argue as in [**4**, Proposition 2.25]. By induction in α and factorization, if there are more than one connected components, we may (and will) suppose that one of them, say K', has the property $\pi(K') \subset \Delta$ where $\Delta \subset \mathbb{A}^\alpha$ is the main quasidiagonal. By Corollary 3.5, $\dim K' \leqslant |\alpha| + 1$. By the same Corollary 3.5, there is another component K such that $\pi(K) = \mathbb{A}^\alpha$, and $\dim K = 2|\alpha|$. In the case $|\alpha| = 1$ (that is, α is a simple root of $(\check{G}, \check{T})$) we are reduced to one of the two situations: (a) $\mathfrak{g}' = \mathfrak{sl}_2$, and the degree $a(\alpha)$ is d (long root α); (b) $\mathfrak{g}' = \mathfrak{sl}_2^{\oplus d}$, and the degree $a(\alpha)$ is 1 along each factor (short root α). In both situations one checks immediately $Z^\alpha \simeq \mathbb{A}^2$. So we may assume $|\alpha| > 1$, and hence $\dim K > \dim K'$. This inequality will lead to a contradiction. For $\phi \in K$ we have $\dim K = \dim T_\phi \mathring{Z}^\alpha$. We have $T_\phi \mathring{Z}^\alpha = H^0(\mathbf{C}, \phi^* \mathcal{T}\mathcal{B}_{\mathfrak{g}'}(-\infty_{\mathbf{C}}))^\Xi$ where $\mathcal{T}\mathcal{B}_{\mathfrak{g}'}$ stands for the tangent bundle of the flag variety $\mathcal{B}_{\mathfrak{g}'} = G'/B'$. Since $\mathcal{T}\mathcal{B}_{\mathfrak{g}'}$ is generated by the global sections, $H^0(\mathbf{C}, \phi^* \mathcal{T}\mathcal{B}_{\mathfrak{g}'}(-\infty_{\mathbf{C}})) = 0$, and $\dim T_\phi \mathring{Z}^\alpha$ can be computed as the invariant part of the equivariant Euler characteristic of $\phi^* \mathcal{T}\mathcal{B}_{\mathfrak{g}'}(-\infty_{\mathbf{C}})$. By the Atiyah–Singer equivariant index formula [**2**], $\chi(\varsigma, \mathbf{C}, \phi^* \mathcal{T}\mathcal{B}_{\mathfrak{g}'}(-\infty_{\mathbf{C}}))$ is independent of ϕ, that is, it is the same for $\phi \in K$ and $\phi' \in K'$. Hence $\dim K = \dim K'$, a contradiction. $\square$

COROLLARY 3.8. Z^α *is irreducible.*

Proof. We have to prove that Z^α is the closure of $\overset{\circ}{Z}{}^\alpha$. The stratification $Z_{\mathfrak{g}'}^{a(\alpha)} = \bigsqcup_{\Lambda_+\ni\gamma\leqslant a(\alpha)} \overset{\circ}{Z}{}_{\mathfrak{g}'}^{\gamma} \times (\mathbf{C}-\infty)^{\alpha-\gamma}$ induces the stratification $Z^\alpha = \bigsqcup_{\beta\leqslant\alpha} \overset{\circ}{Z}{}^\beta \times \mathbb{A}^{\alpha-\beta}$. We argue as in [**4**, Theorem 10.2]. It suffices to prove that $(\phi,\underline{z}) \in \overset{\circ}{Z}{}^\beta \times \mathbb{A}^{\alpha-\beta}$ lies in the closure of $\overset{\circ}{Z}{}^\alpha$ for $\underline{z}$ lying away from all the quasidiagonals and distinct from $\pi(\phi)$. By factorization this reduces to the case of simple α. In this case $Z^\alpha \simeq \mathbb{A}^2$ is irreducible, as was explained in the proof of Proposition 3.7. □

3.9. *Normality*

Recall that each W-orbit in Y has a unique representative η such that $a(\eta) \in X_*^+(T')$ is a dominant coweight. We call such η dominant as well, and we denote by Y^+ the cone of all dominant elements. Thus $Y^+ \xrightarrow{\sim} Y/W \simeq G'(\mathcal{O})^\varsigma\backslash G'(\mathcal{K})^\varsigma/G'(\mathcal{O})^\varsigma$. We define the congruence subgroup $\mathbf{K}_{-1} \subset G'(\mathcal{K})^\varsigma$ as the kernel of the evaluation morphism $ev:\ G'(\mathbb{C}[\mathsf{t}^{-1}])^\varsigma \to (G')^\sigma$. Given $\eta \in Y^+$ we consider the orbit $\mathcal{W}_\eta := \mathbf{K}_{-1}\cdot\eta \subset \mathrm{Gr}$. For $\lambda \geqslant \eta \in Y^+$ we define the *transversal slice* $\overline{\mathcal{W}}{}_\eta^\lambda$ as the intersection $\overline{\mathrm{Gr}}{}^\lambda \cap \mathcal{W}_\eta$. It follows from [**24**, Theorem 8.4] that $\overline{\mathcal{W}}{}_\eta^\lambda$ is normal with rational singularities.

PROPOSITION 3.10. *Z^α is normal.*

Proof. As in [**5**, Theorem 2.8] we construct a $T\times\mathbb{G}_m$-equivariant morphism $s_\eta^\lambda:\ \overline{\mathcal{W}}{}_\eta^\lambda \to Z^\alpha$ for $\alpha = \lambda-\eta$ (note that in the present nonsimply laced situation, for the longest element $w_0 \in W$ we have $-w_0\alpha = \alpha$). More precisely, the desired morphism is just the restriction of the similar morphism of [**5**, Theorem 2.8] to ς-fixed points. Similarly to [**5**, Theorem 2.8] we show that s_η^λ induces an isomorphism $(s_\eta^\lambda)^*:\ \mathbb{C}[Z^\alpha] \to \mathbb{C}[\overline{\mathcal{W}}{}_\eta^\lambda]$ on functions of degree less than or equal to $n \in \mathbb{N}$ (with respect to the action of $\mathbb{G}_m$), provided η is big enough. Now one deduces the normality of Z^α from normality of $\overline{\mathcal{W}}{}_\eta^\lambda$ as in [**5**, Corollary 2.10]. □

3.11. *The boundary of Z^α*

Recall the stratification $Z^\alpha = \bigsqcup_{\beta\leqslant\alpha} \overset{\circ}{Z}{}^\beta \times \mathbb{A}^{\alpha-\beta}$. The closure of the stratum $\overset{\circ}{Z}{}^{\alpha-\gamma} \times \mathbb{A}^\gamma$ is denoted $\partial_\gamma Z^\alpha$. The union $\bigcup_{i\in I} \partial_{\alpha_i} Z^\alpha$ is denoted $\partial_1 Z^\alpha$ and is called the boundary of Z^α. More generally, the union $\bigcup_{|\gamma|\geqslant n} \partial_\gamma Z^\alpha$ is denoted $\partial_n Z^\alpha$ (with the reduced closed subscheme structure). The open subscheme $Z^\alpha \setminus \partial_2 Z^\alpha$ is denoted $\overset{\bullet}{Z}{}^\alpha$. By factorization and the calculations for $|\alpha| = 1$ (proof of Proposition 3.7), $\overset{\bullet}{Z}{}^\alpha$ is smooth. We are going to prove that $\partial_1 Z^\alpha \subset Z^\alpha$ with the reduced closed subscheme structure is a Cartier divisor. Recall the function $F_{a(\alpha)}$ on $Z_{\mathfrak{g}'}^{a(\alpha)}$ constructed in [**5**, Section 4].

PROPOSITION 3.12. (a) *There is a function $F_\alpha \in \mathbb{C}[Z^\alpha]$ such that $F_\alpha^d = F_{a(\alpha)}|_{Z^\alpha}$.*
(b) *F_α is an equation of $\partial_1 Z^\alpha \subset Z^\alpha$.*

Proof. Let us denote $F_{a(\alpha)}|_{Z^\alpha}$ by f_α for short. Recall that $F_{a(\alpha)}$ has simple zeroes at any boundary component of $Z_{\mathfrak{g}'}^{a(\alpha)}$ [**5**, Lemma 4.2]. We first prove that f_α vanishes to the order exactly d at any boundary component $\partial_{\alpha_i} Z^\alpha,\ i \in I$. We start with $i \in I_0$ (notations of Section 2.4, a long simple root of $(\check{G},\check{T})$, that is, a Ξ-fixed point, say i', in I'). The corresponding simple coroot of (G',T') will be denoted by $\alpha'_{i'}$. Since Z^α is smooth at the generic point of $\partial_{\alpha_i} Z^\alpha$, and $Z_{\mathfrak{g}'}^{a(\alpha)}$ is smooth at the generic point of $\partial_{\alpha'_{i'}} Z_{\mathfrak{g}'}^{a(\alpha)}$, and set-theoretically $\partial_{\alpha_i} Z^\alpha = Z^\alpha \cap \partial_{\alpha'_{i'}} Z_{\mathfrak{g}'}^{a(\alpha)}$, we have to check that the multiplicity of intersection of Z^α with $\partial_{\alpha'_{i'}} Z_{\mathfrak{g}'}^{a(\alpha)}$ is generically equal to d. By factorization, we are reduced to the case $\mathfrak{g}' = \mathfrak{sl}_2,\ a(\alpha) = d$. Then

$Z^{a(\alpha)}_{\mathfrak{g}'}$ is the moduli space of pairs of polynomials $(P(\mathbf{t}), Q(\mathbf{t}))$, P monic of degree d, Q of degree less than d. Furthermore, $F_{a(\alpha)}$ is the resultant $\mathrm{Res}(P, Q)$. For the sake of definiteness, let $d = 3$. Then $Z^{a(\alpha)}_{\mathfrak{g}'} = \{(P = \mathbf{t}^3 + a_2\mathbf{t}^2 + a_1\mathbf{t} + a_0, Q = b_2\mathbf{t}^2 + b_1\mathbf{t} + b_0)\}$, and Z^α is cut out by the equations $a_2 = a_1 = b_2 = b_1 = 0$. Then we have $\mathrm{Res}(P, Q)|_{Z^\alpha} = b_0^3$. This takes care of the case of a long simple root α_i.

Now let $i \in I_1$ be a short simple root of $(\check{G}, \check{T})$ corresponding to a free Ξ-orbit, say i', i'', i''', in I' (again, for the sake of definiteness, we take $d = 3$). Then i', i'', i''' are all disjoint in the Dynkin diagram of $\mathfrak{g}'$, and the intersection $\partial_{\alpha'_{i'}} Z^{a(\alpha)}_{\mathfrak{g}'} \cap \partial_{\alpha'_{i''}} Z^{a(\alpha)}_{\mathfrak{g}'} \cap \partial_{\alpha'_{i'''}} Z^{a(\alpha)}_{\mathfrak{g}'}$ is generically transversal. Moreover, each of $\partial_{\alpha'_{i'}} Z^{a(\alpha)}_{\mathfrak{g}'}, \partial_{\alpha'_{i''}} Z^{a(\alpha)}_{\mathfrak{g}'}, \partial_{\alpha'_{i'''}} Z^{a(\alpha)}_{\mathfrak{g}'}$ is generically transversal to $Z^\alpha \subset Z^{a(\alpha)}_{\mathfrak{g}'}$, and generically $\partial_{\alpha_i} Z^\alpha = Z^\alpha \cap \partial_{\alpha'_{i'}} Z^{a(\alpha)}_{\mathfrak{g}'} = Z^\alpha \cap \partial_{\alpha'_{i''}} Z^{a(\alpha)}_{\mathfrak{g}'} = Z^\alpha \cap \partial_{\alpha'_{i'''}} Z^{a(\alpha)}_{\mathfrak{g}'} = Z^\alpha \cap \partial_{\alpha'_{i'}} Z^{a(\alpha)}_{\mathfrak{g}'} \cap \partial_{\alpha'_{i''}} Z^{a(\alpha)}_{\mathfrak{g}'} \cap \partial_{\alpha'_{i'''}} Z^{a(\alpha)}_{\mathfrak{g}'}$. This takes care of the case of a short simple root α_i.

We have $f_\alpha : \overset{\circ}{Z}{}^\alpha \to \mathbb{C}^*$, and $\sqrt[d]{f_\alpha}$ is well-defined on an unramified Galois covering $\widetilde{Z} \to \overset{\circ}{Z}{}^\alpha$ with Galois group Ξ. To show the existence of F_α we have to prove that this covering splits, that is, the corresponding class in $H^1(\overset{\circ}{Z}{}^\alpha, \Xi)$ vanishes. This is the subject of the following:

LEMMA 3.13. *There is a regular nonvanishing function $F_\alpha \in \mathbb{C}[\overset{\circ}{Z}{}^\alpha]$ such that $F_\alpha^d = f_\alpha$.*

Proof. Given a positive coroot α' of G' we consider the moduli stack $A^{\alpha'}_{\mathfrak{g}'}$ of B'-bundles over C equipped with trivialization at $\infty \in C$, such that the induced T'-bundle has degree α'. One can check that in case α' is dominant (as a coweight of G') $A^{\alpha'}_{\mathfrak{g}'} \simeq \mathbb{A}^{2|\alpha'|}$. In general, $A^{\alpha'}_{\mathfrak{g}'}$ is a quotient of an affine space, and the automorphism groups of all points are unipotent. The natural morphism $\overset{\circ}{Z}{}^{\alpha'}_{\mathfrak{g}'} \to A^{\alpha'}_{\mathfrak{g}'}$ is an affine open embedding with the image formed by all the B'-bundles $\phi_{B'}$ such that the induced G'-bundle $\phi_{G'}$ is trivial. The complement divisor $\mathfrak{D}^{\alpha'}_{\mathfrak{g}'} = A^{\alpha'}_{\mathfrak{g}'} \setminus \overset{\circ}{Z}{}^{\alpha'}_{\mathfrak{g}'}$ is irreducible, and $F^{-1}_{\alpha'}$ extends to a regular function $F'_{\alpha'}$ on $A^{\alpha'}_{\mathfrak{g}'}$ vanishing to the order 1 along $\mathfrak{D}^{\alpha'}_{\mathfrak{g}'}$.

In case $\alpha' = a(\alpha)$, the automorphism $\varsigma : \overset{\circ}{Z}{}^{a(\alpha)}_{\mathfrak{g}'} \to \overset{\circ}{Z}{}^{a(\alpha)}_{\mathfrak{g}'}$ extends to the same named automorphism $\varsigma : A^{a(\alpha)}_{\mathfrak{g}'} \to A^{a(\alpha)}_{\mathfrak{g}'}$, and we denote the connected component of the fixed point stack $(A^{a(\alpha)}_{\mathfrak{g}'})^\varsigma$ containing $\overset{\circ}{Z}{}^\alpha$ by A^α. One can check that in the appropriate coordinates of the covering affine space $\mathbb{A}^k \to A^{a(\alpha)}_{\mathfrak{g}'}$ the automorphism ς is linear, so that A^α is also a quotient stack of an affine space, and the automorphism groups of all points are unipotent as well. We denote the restriction $F'_{a(\alpha)}|_{A^\alpha}$ by f'_α for short. The same argument as in the first part of the proof of the proposition shows that f'_α vanishes to the order exactly d at the complementary divisor $\mathfrak{D}^\alpha := A^\alpha \setminus \overset{\circ}{Z}{}^\alpha$.

Finally, since $\pi_1(A^\alpha) = H_1(A^\alpha, \mathbb{Z}) = H_1(A^\alpha, \Xi) = 0$, the vanishing of the class in $H^1(\overset{\circ}{Z}{}^\alpha, \Xi)$ associated to $\sqrt[d]{f'_\alpha}$ follows by excision from the above local computations around $\mathfrak{D}^\alpha$. $\square$

So F_α is well-defined on $\overset{\circ}{Z}{}^\alpha$, and extends by zero through the generic points of the boundary divisor components $\partial_{\alpha_i} Z^\alpha$. Hence it is defined on the open subset whose complement has codimension 2, and extends to the whole of Z^α by normality of Z^α.

It remains to prove (b), that is to check that the zero-subscheme of F_α is reduced. In other words, given $f \in \mathbb{C}[Z^\alpha]$ vanishing at the boundary $\partial_1 Z^\alpha$ we have to check that f is divisible by F_α. The rational function f/F_α is regular at the generic points of all the boundary divisor components, so it is regular due to normality of Z^α. $\square$

PROPOSITION 3.14. *F_α is an eigenfunction of $T\times\mathbb{G}_m$ with the eigencharacter $q^{(\alpha,\alpha)/2}z^{\alpha^*}$ (notations of Section 2.1).*

Proof. Follows immediately from [5, Proposition 4.4] along with an observation that $d\cdot(\alpha,\alpha)=(a(\alpha),a(\alpha))$. □

REMARK 3.15. The invertible function $F_{a(\alpha)}|_{\overset{\circ}{Z}{}^{a(\alpha)}_{\mathfrak{g}'}}$ is constructed in [5, Section 4] as the ratio of two sections of the determinant line bundle lifted from $\mathrm{Bun}_{G'}(\mathbf{C})$ (the generator of its Picard group). The action of Ξ on G' gives rise to a group scheme $\mathcal{G}$ over $\mathbf{C}/\!\!/\Xi$ as in [20, Example (3)]. We have a natural morphism $\mathrm{Bun}_{\mathcal{G}}\to\mathrm{Bun}_{G'}(\mathbf{C})$, and the inverse image of the determinant line bundle on $\mathrm{Bun}_{G'}(\mathbf{C})$ is the determinant line bundle on $\mathrm{Bun}_{\mathcal{G}}$ (*not* its dth power), as follows from [20, Theorem 3] and [24, 10.a.1, (10.7)].

3.16. *Canonical class of Z^α*

Recall the smooth open subset $\overset{\bullet}{Z}{}^\alpha\subset Z^\alpha$ (notations of Section 3.11).

LEMMA 3.17. (a) *The canonical line bundle $\overset{\bullet}{\omega}_\alpha$ of $\overset{\bullet}{Z}{}^\alpha$ is trivial.*

(b) *The weight of its generating section with respect to the loop rotations is $\frac{(a(\alpha),a(\alpha))}{2}+|a(\alpha)|$.*

Proof. (a) For a simple coroot α_i, we have $Z^{\alpha_i}\simeq\mathbb{A}^2$; let us introduce coordinates w_i along $\mathbb{A}^{\alpha_i}$, and z_i along the fibers of $\pi_{\alpha_i}\colon Z^{\alpha_i}\to\mathbb{A}^{\alpha_i}$. For arbitrary $\alpha=\sum a_i\alpha_i$, the factorization gives rise to the rational coordinates $w_{i,r},z_{i,r},\ i\in I,\ 1\leqslant r\leqslant a_i$ on $\underline{Z}^\alpha:=Z^\alpha\times_{\mathbb{A}^\alpha}\mathbb{A}^{|\alpha|}$. Here the coordinates $z_{i,r}$ are defined away from the quasidiagonal divisor in $\mathbb{A}^{|\alpha|}$ (see Section 2.4). Let us orient the Dynkin graph of $\mathfrak{g}$ so that the arrows go from I_0 to I_1. We define a rational section of the canonical line bundle of $\overset{\bullet}{\underline{Z}}{}^\alpha$ by the following formula $s_\alpha:=\prod_{i\in I}^{1\leqslant r<s\leqslant a_i}(w_{i,r}-w_{i,s})^2\cdot\prod_{i\Rightarrow j}^{1\leqslant r\leqslant a_i,\ 1\leqslant s\leqslant a_j}(w_{i,r}-w_{j,s}^d)^{-1}\cdot\prod_{i\to j}^{1\leqslant r\leqslant a_i,\ 1\leqslant s\leqslant a_j}(w_{i,r}-w_{j,s})^{-1}\cdot\prod_{i\in I}^{1\leqslant r\leqslant a_i}dw_{i,r}dz_{i,r}$. Here $i\Rightarrow j$ means this arrow belongs to the orientation chosen above and $i\in I_0,\ j\in I_1$; and $i\to j$ means this arrow belongs to the orientation and $i,j\in I_0$ or $i,j\in I_1$. This section extends regularly through the generic points of the irreducible components of the quasidiagonal divisor due to examples in Section 2.5, Section 2.6 (for a divisor $w_{i,r}=w_{j,s},\ i\Rightarrow j$); due to [3, 5.6] (for a divisor $w_{i,r}=w_{j,s},\ i\to j$); due to [3, 5.5] (for a divisor $w_{i,r}=w_{i,s}$). Moreover, this section is $\mathfrak{S}_\alpha$-invariant, hence it descends to a rational section $\overline{s}_\alpha$ of $\overset{\bullet}{\omega}_\alpha$ that is regular nonvanishing at the generic points of the irreducible components of the quasidiagonal divisor (again due to [3, 5.5]). Hence $\overline{s}_\alpha$ trivializes $\overset{\bullet}{\omega}_\alpha$.

(b) The weights of $w_{i,r},z_{i,r},dw_{i,r},dz_{i,r}$ with respect to the loop rotations are all equal to 1 if $i\in I_1$, and to d if $i\in I_0$. The explicit formula for $\overline{s}_\alpha$ implies the desired result. □

3.18. *Rational singularities*

PROPOSITION 3.19. *Z^α is a Gorenstein (hence, Cohen–Macaulay) scheme with canonical (hence rational) singularities.*

Proof. We follow closely the proof of [5, Proposition 5.1], and use freely the notations thereof. There we have considered the Kontsevich resolution $\pi:\ M^{a(\alpha)}_{\mathfrak{g}'}\to Z^{a(\alpha)}_{\mathfrak{g}'}$, and computed its discrepancy divisor. Now we consider the (smooth) fixed point stack $(M^{a(\alpha)}_{\mathfrak{g}'})^\Xi$ (see [26, especially Proposition 3.7] for the basics on fixed-point stacks with respect to the finite groups' actions); more precisely, its irreducible component M^α which is the closure of $\mathring{Z}^\alpha \subset \mathring{Z}^{a(\alpha)}_{\mathfrak{g}'} \subset M^{a(\alpha)}_{\mathfrak{g}'}$[†]. Note that there are other irreducible components of $(M^{a(\alpha)}_{\mathfrak{g}'})^\Xi$, for example, the loop rotation invariant stable maps $(M^{a(\alpha)}_{\check{\mathfrak{g}}})^{\mathbb{G}_m}$ (recall that $\mathcal{B} = \mathcal{B}^\sigma_{\mathfrak{g}'}$ is isomorphic to σ-fixed points in the flag variety of $\check{\mathfrak{g}}'$ since $\mathfrak{g}'$ is simply laced and hence isomorphic to $\check{\mathfrak{g}}'$. Hence $\mathcal{B}$ is isomorphic to the flag variety $\mathcal{B}_{\check{\mathfrak{g}}}$ of $\check{\mathfrak{g}}$, and $a(\alpha) \in H_2(\mathcal{B}, \mathbb{Z}) = H_2(\mathcal{B}_{\mathfrak{g}'}, \mathbb{Z})^\sigma = X_*(T')^\sigma$). In notations of [**5**, proof of Proposition 5.1] the latter component consists of stable maps such that $C = C_h \cup C_v$ where $\deg C_h = (1, 0)$, and $\phi(C_h \cap C_v) = (0, \mathfrak{b}_-)$. This component is isomorphic to the substack of based stable maps in the moduli stack of stable maps $\overline{M}_{0,1}(\mathcal{B}, a(\alpha))$, and has dimension $2|a(\alpha)| - 2$. Note also that the fixed point stack $(M^{a(\alpha)}_{\mathfrak{g}'})^\Xi$ is *not* a closed substack of $M^{a(\alpha)}_{\mathfrak{g}'}$: the natural morphism $(M^{a(\alpha)}_{\mathfrak{g}'})^\Xi \to M^{a(\alpha)}_{\mathfrak{g}'}$ has finite fibers over the points with nontrivial automorphisms.

The complement $M^\alpha \setminus \mathring{Z}^\alpha$ is a union of smooth irreducible divisors $D_{\beta'}$ numbered by all $\beta' \in \Lambda_+$ (notations of Section 2.2) such that $\sum_{\xi\in\Xi} \xi(\beta') \leqslant a(\alpha)$. The generic point of $D_{\beta'}$ parametrizes the pairs (C, ϕ) such that $C = C_h \cup C_v$, the degree of $\phi|_{C_h}$ equals $(1, a(\alpha) - \sum_{\xi\in\Xi} \xi(\beta'))$, and C_v consists of irreducible components $C^\xi_v,\ \xi \in \Xi,\ \deg C^\xi_v = (0, \xi(\beta'))$ (Ξ-invariance implies in particular that the set of points $\{C^\xi_v \cap C_h\}_{\xi\in\Xi} \subset C_h \simeq \mathbb{P}^1$ is Ξ-invariant). Among those divisors, $D_{\beta'}$ for *simple* β' project generically one-to-one onto the boundary divisors of Z^α. The remaining divisors are exceptional.

The discrepancy of $\pi :\ M^\alpha \to Z^\alpha$ equals $\sum_{\beta':\ \sum_{\xi\in\Xi} \xi(\beta') \leqslant a(\alpha)} m_{\beta'} D_{\beta'}$, and we have to show $m_{\beta'} \geqslant 0$. As in [**5**, Proposition 5.1], by factorization it suffices to consider the components $D_{\beta'}$ such that $\sum_{\xi\in\Xi} \xi(\beta') = a(\alpha)$. The fixed point stack $D^{\mathbb{G}_m}_{\beta'}$ with respect to the action of the loop rotations contains all the pairs (C, ϕ) such that C consists of $2 + d$ irreducible components $C_h,\ C^0_v,\ C^\xi_v,\ \xi \in \Xi,\ \deg C_h = (1, 0),\ \deg C^\xi_v = (0, \xi(\beta')),\ \deg C^0_v = (0, 0)$, with the following intersection pattern. The horizontal component C_h intersects C^0_v at the point $0 \in C_h \simeq \mathbb{P}^1$. The component C^ξ_v intersects only C^0_v, and Ξ acts on C preserving C_h, C^0_v, and permuting the components $C^\xi_v,\ \xi \in \Xi$. Note that the codimension of $D^{\mathbb{G}_m}_{\beta'}$ in $D_{\beta'}$ is one.

We will prove $m_{\beta'} = |\beta'| + [(\beta', \beta')/2] - 2$ (cf. [**5**, Lemma 5.2]). We will distinguish between the following two cases: (a) *invariant case*, when β' is Ξ-fixed; then the group of automorphisms of generic point of $D^{\mathbb{G}_m}_{\beta'}$ is equal to Ξ; (b) *noninvariant case*, when $\beta' \neq \xi\beta'$ for a nontrivial element $\xi \in \Xi$; then the group of automorphisms of generic point of $D^{\mathbb{G}_m}_{\beta'}$ is trivial.

We first consider the noninvariant case. Let $(C, \phi) \in D_{\beta'}$ be a general point, and let $p_\xi := C^\xi_v \cap C_h$. Then the fiber of the normal bundle $\mathcal{N}_{D_{\beta'}/M^\alpha}$ at the point (C, ϕ) equals $(\bigoplus_{\xi\in\Xi} T_{p_\xi} C^\xi_v \otimes T_{p_\xi} C_h)^\Xi$. As $p_\xi \in C_h$ tends to $0 \in C_h$, this tends to the fiber of $\mathcal{N}_{D_{\beta'}/M^\alpha}$ at a point $('C, \phi')$ of $D^{\mathbb{G}_m}_{\beta'}$ equal to $(\bigoplus_{\xi\in\Xi} T_{p_\xi}\ 'C^\xi_v \otimes T_0 C_h)^\Xi$ where p_ξ is the intersection point of the components $'C^\xi_v$ and $'C^0_v$. The group $\mathbb{G}_m$ acts on this fiber via the character q^{-1} (cf. [**5**, proof of Lemma 5.2]). On the other hand, the fiber of $\mathcal{N}_{D^{\mathbb{G}_m}_{\beta'}/D_{\beta'}}$ at the point $('C, \phi')$ equals $T_0 C^0_v \otimes T_0 C_h$, and $\mathbb{G}_m$ acts on this fiber via the character q^{-1} as well. Finally, $T_{('C,\phi')} D^{\mathbb{G}_m}_{\beta'}$ is nothing but Ξ-invariants in the similar tangent space described in *loc. cit.* From this description it follows that $\mathbb{G}_m$ acts trivially on these invariants. All in all, $\mathbb{G}_m$ acts on $\det T_{('C,\phi')} M^\alpha$ via the character q^{-2}, and on the fiber of the canonical bundle ω_{M^α} at $('C, \phi')$ via the character q^2. Now the same argument as in [**5**, proof of Lemma 5.2] yields $m_{\beta'} = |\beta'|[(\beta', \beta')/2] - 2$.

In the invariant case, due to the presence of the automorphism group Ξ, repeating the above argument, we obtain that $\mathbb{G}_m$ acts on the fiber of $\mathcal{N}_{D_{\beta'}/M^\alpha}$ at $('C, \phi')$ via the character q^{-d}, and on the fiber of ω_{M^α} at $('C, \phi')$ via the character q^{2d}. From this we deduce again $m_{\beta'} = |\beta'| + [(\beta', \beta')/2] - 2$.

[†]It is easy to see that $(M^{a(\alpha)}_{\mathfrak{g}'})^\Xi$ is actually a special case of the moduli space of twisted stable maps defined in [**1**].

Now we finish the proof of the proposition the same way as in [**5**, proof of Proposition 5.1]. $\square$

3.20. *Cohomology vanishing*

Recall the notations of Section 2.2. We will consider the global quasimaps' spaces $\mathcal{QM}^\alpha$, and the corresponding ind-scheme $\mathfrak{Q}$. We will generalize the results of [**6**, Section 3] on cohomology of the line bundles $\mathcal{O}_{\check\lambda}$ to the twisted case. We denote by $\widetilde{H}^n(\mathfrak{Q},\mathcal{O}_{\check\lambda})$ the subspace of $\mathbb{G}_m$-finite vectors in $H^n(\mathfrak{Q},\mathcal{O}_{\check\lambda})$. Finally, given $\check\lambda\in X$, we define a cofinal subsystem $Y_+^{\check\lambda}\subset Y_+$ formed by α such that $\alpha^*+\check\lambda$ is dominant.

Proposition 3.21. (1) *For $n>0$ and $\alpha\in Y_+^{\check\lambda}$ we have $H^n(\mathcal{QM}^\alpha,\mathcal{O}_{\check\lambda})=0$.*
(2) *For $n>0$ and $\check\lambda\in X$ we have $\widetilde{H}^n(\mathfrak{Q},\mathcal{O}_{\check\lambda})=0$.*
(3) *For $\check\lambda\notin X^+$ we have $\widetilde{H}^0(\mathfrak{Q},\mathcal{O}_{\check\lambda})=0$.*

Proof. (3) is clear, and (2) follows from (1). We prove (1).

We will use the self-evident notation $\partial_{\alpha_i}\mathcal{QM}^\alpha$ for the boundary divisors of $\mathcal{QM}^\alpha$. We consider a divisor $\Delta:=\sum_{i\in I}\partial_{\alpha_i}\mathcal{QM}^\alpha$. We introduce the open subvariety $\overset{\circ}{\mathcal{QM}}{}^\alpha\subset\mathcal{QM}^\alpha$ formed by all the twisted quasimaps without defect at $\infty\in\mathbf{C}$, and the evaluation morphism $ev_\infty:\ \overset{\circ}{\mathcal{QM}}{}^\alpha\to\mathcal{B}=(G'/B')^\sigma$. It is a fibration with the fibers isomorphic to Z^α. We have $ev_\infty^*\omega_{\mathcal{B}}=\mathcal{O}_{-2\check\rho}$. It follows from Lemma 3.17 that $K_{\overset{\circ}{\mathcal{QM}}{}^\alpha}+\Delta-ev_\infty^*K_{\mathcal{B}}=0$ (here K stands for the canonical class). According to Proposition 3.19, Z^α is Gorenstein with rational singularities; but $\mathcal{QM}^\alpha$ is locally in étale topology isomorphic to $Z^\alpha\times\mathcal{B}$, hence $\mathcal{QM}^\alpha$ is Gorenstein with rational singularities as well. We conclude that the canonical bundle $\omega^\alpha:=\omega_{\mathcal{QM}^\alpha}\simeq\mathcal{O}_{\mathcal{QM}^\alpha}(-\Delta)\otimes\mathcal{O}_{-2\check\rho}$. We have the following analogue of [**6**, Lemma 4]:

Lemma 3.22. $\omega^\alpha\simeq\mathcal{O}_{-\alpha^*-2\check\rho}$.

Proof. As in the proof of [**6**, Lemma 4] we see that there is $\check\mu\in X$ such that $\omega^\alpha\simeq\mathcal{O}_{\check\mu}$. We have to check $\check\mu=-\alpha^*-2\check\rho$. We will do this on an open subvariety $\overset{\bullet}{\mathcal{QM}}{}^\alpha\subset\mathcal{QM}^\alpha$ with the complement of codimension 2. Namely, $\overset{\bullet}{\mathcal{QM}}{}^\alpha$ is formed by all the twisted quasimaps of defect at most a simple coroot $\alpha_i,\ i\in I$ (or no defect at all). Note that $\Delta\cap\overset{\bullet}{\mathcal{QM}}{}^\alpha$ is a disjoint union of smooth divisors $\partial_{\alpha_i}\overset{\bullet}{\mathcal{QM}}{}^\alpha$. Moreover, $\overset{\bullet}{\mathcal{QM}}{}^\alpha$ itself is smooth, and the Kontsevich resolution $K^\alpha\to\mathcal{QM}^\alpha$ (cf. proof of Proposition 3.19) is an isomorphism over $\overset{\bullet}{\mathcal{QM}}{}^\alpha$. Let us fix a quasimap without defect $\phi\in\mathcal{QM}^{\alpha-\alpha_i}$, choose a representative $\tilde\alpha_i$ of α_i, and consider a map $p:\ \mathbf{C}\to\partial_{\alpha_i}\overset{\bullet}{\mathcal{QM}}{}^\alpha$ sending $\mathbf{t}\in\mathbf{C}$ to $\phi(\sum_{r=1}^d\sigma^r\tilde\alpha_i\cdot\zeta^{-r}\mathbf{t})$ (twisting ϕ by a defect in $\mathbf{C}^{a(\alpha_i)}$). Clearly, if $i\in I_1$ (α_i is a short root of $(\check G,\check T)$), then p is a closed embedding; and if $i\in I_0$ (α_i a long root of $(\check G,\check T)$), then p factors through $\mathbf{C}\to\mathbf{C}/\!\!/\Xi\hookrightarrow\partial_{\alpha_i}\overset{\bullet}{\mathcal{QM}}{}^\alpha$. We will denote the categorical quotient $\mathbf{C}/\!\!/\Xi$ (a projective line) by $\overline{\mathbf{C}}$, and its closed embedding into $\partial_{\alpha_i}\overset{\bullet}{\mathcal{QM}}{}^\alpha$ by $\overline{p}$. In both cases, the image of $\mathbf{C}$ in $\partial_{\alpha_i}\overset{\bullet}{\mathcal{QM}}{}^\alpha$ will be denoted by C_i^ϕ. It is easy to see that $\deg\mathcal{O}_{\check\omega_j}|_{C_i^\phi}=\delta_{ij}=\langle\alpha_i,\check\omega_j\rangle$. Hence it remains to check that $\deg(\omega^\alpha|_{C_i^\phi})=-\langle\alpha_i,\alpha^*+2\check\rho\rangle$. To this end recall that $\omega^\alpha\simeq\mathcal{O}_{\mathcal{QM}^\alpha}(-\Delta)\otimes\mathcal{O}_{-2\check\rho}$, and the Kontsevich resolution $K^\alpha\to\mathcal{QM}^\alpha$ is an isomorphism over $\overset{\bullet}{\mathcal{QM}}{}^\alpha$. Thus we have to compute the degree of the normal line bundle $\mathcal{N}_{\partial_{\alpha_i}\overset{\bullet}{K}{}^\alpha/K^\alpha}|_{C_i^\phi}$ restricted to C_i^ϕ, and prove $\deg\mathcal{N}_{\partial_{\alpha_i}\overset{\bullet}{K}{}^\alpha/K^\alpha}|_{C_i^\phi}=\langle\alpha_i,\alpha^*\rangle$.

We follow the argument of [**15**, proof of Proposition 4.4], and consider first the case $i\in I_1$. The universal stable map $(\mathcal{C},\varphi)$ over $C_i^\phi\subset K^\alpha$ looks as follows. For $\mathbf{t}\in\mathbf{C}\setminus\{0,\infty\}$ (recall that $C_i^\phi \simeq \mathbf{C}$) the curve $\mathcal{C}_\mathbf{t}$ has components $C_h = \mathbf{C}, C_v^r,\ 1 \leqslant r \leqslant d$, and $\varphi_\mathbf{t}|_{C_h} = (\mathrm{id}, \phi)$, while $\deg(\varphi_\mathbf{t}|_{C_v^r}) = (0, \sigma^r \tilde{\alpha}_i)$. The intersection point $C_v^r \cap C_h$ is $\zeta^{-r}\mathbf{t}$. For $\mathbf{t} = 0$ (respectively, ∞), the curve $\mathcal{C}_\mathbf{t}$ has components $C_h = \mathbf{C}, C_v^0, C_v^r,\ 1 \leqslant r \leqslant d$, and $\varphi_\mathbf{t}|_{C_h} = (\mathrm{id}, \phi)$, while $\deg(\varphi_\mathbf{t}|_{C_v^0} = (0,0)$, and $\deg(\varphi_\mathbf{t}|_{C_v^r}) = (0, \sigma^r \tilde{\alpha}_i)$. The intersection points of the components all lie on C_v^0, and $C_v^0 \cap C_h = 0$ (respectively, ∞).

The description of the normal bundle $\deg \mathcal{N}_{\partial_{\alpha_i} \overset{\bullet}{K}{}^\alpha / K^\alpha}$ given in the proof of Proposition 3.19 implies $\deg \mathcal{N}_{\partial_{\alpha_i} \overset{\bullet}{K}{}^\alpha / K^\alpha}|_{C_i^\phi} = 2 + \langle \alpha_i, \alpha^* - \alpha_i^* \rangle = \langle \alpha_i, \alpha^* \rangle$. The argument in the case $i \in I_0$ is similar. □

Returning to the proof of the Proposition, it is finished the same way as the one of [**6**, Theorem 3.2]. □

4. *Fermionic formula and q-Whittaker functions*

4.1. *Fermionic formula*

Recall the setup of Section 2.1. In particular, an isomorphism $\alpha \mapsto \alpha^*$ from the root lattice of $(\check{G}, \check{T})$ to the root lattice of (G, T) defined in the basis of simple roots as follows: $\alpha_i^* := \check{\alpha}_i$ (the corresponding simple coroot). For an element α of the root lattice of $(\check{G}, \check{T})$, we denote by z^{α^*} the corresponding character of T. As usual, q stands for the identity character of $\mathbb{G}_m$, and $q_i = q^{d_i}$. For $\gamma = \sum_{i \in I} c_i \alpha_i$, we set $(q)_\gamma := \prod_{i \in I} \prod_{s=1}^{c_i} (1 - q_i^s)$.

According to [**12**, Theorem 3.1], the recurrence relations

$$\mathcal{J}_\alpha = \sum_{0 \leqslant \beta \leqslant \alpha} \frac{q^{(\beta,\beta)/2} z^{\beta^*}}{(q)_{\alpha-\beta}} \mathcal{J}_\beta \tag{4.1}$$

uniquely define a collection of rational functions $\mathcal{J}_\alpha,\ \alpha \geqslant 0$, on $T \times \mathbb{G}_m$, provided $\mathcal{J}_0 = 1$. Moreover, these functions are nothing but the Shapovalov scalar products of the weight components of the Whittaker vectors in the universal Verma module over the corresponding quantum group.

Theorem 4.2. *$\mathcal{J}_\alpha$ equals the character of $T \times \mathbb{G}_m$-module $\mathbb{C}[Z^\alpha]$.*

Proof. We have to prove that the collection of characters of $T \times \mathbb{G}_m$-modules $\mathbb{C}[Z^\alpha]$ satisfies the recursion relation (4.1). Given the geometric preparations undertaken in Section 3, the proof is the same as the one of [**5**, Theorem 1.5]. □

We organize all $\mathcal{J}_\alpha$ into a generating function $J_\mathfrak{g}^{\mathrm{twisted}}(z, x, q) = \sum_{\alpha \in \Lambda_+} x^\alpha \mathcal{J}_\alpha$, the equivariant twisted K-theoretic J-function of $\mathcal{B}_{\mathfrak{g}'}$. The same way as [**5**, Corollaries 1.6,1.8] follow from [**5**, Theorem 1.5], Theorem 4.2 implies the following

Corollary 4.3. *The equivariant twisted K-theoretic J-function $J_\mathfrak{g}^{\mathrm{twisted}}$ of $\mathcal{B}_{\mathfrak{g}'}$ is equal to the Whittaker matrix coefficient of the universal Verma module of $U_q(\check{\mathfrak{g}})$; it is an eigen-function of the quantum difference Toda integrable system associated with $\check{\mathfrak{g}}$.* □

4.4. *Twisted Weyl modules and q-Whittaker functions*

The notions of the local (respectively, global) Weyl modules over the twisted current algebra $(\mathfrak{g}'[\mathrm{t}])^\varsigma$ were introduced in [**17**] (respectively, [**9**, Section 9]). Recall the notations of Section 2.4. Given a dominant G-weight $\check{\lambda} = \sum_{i \in I} \langle \alpha_i, \check{\lambda} \rangle \check{\omega}_i$ we define $\mathbb{A}^{\check{\lambda}} := \prod_{i \in I_1} (\mathbf{C} -$

$\infty)^{(\langle\alpha_i,\check\lambda\rangle)}\times\prod_{i\in I_0}((\mathbf{C}-\infty)/(\mathbf{t}\mapsto\zeta^{-1}\mathbf{t}))^{(\langle\alpha_i,\check\lambda\rangle)}$. The character of $\mathbb{C}[\mathbb{A}^{\check\lambda}]$ with respect to the natural action of $\mathbb{C}^*$ is equal to $\prod_{i\in I}\prod_{r=1}^{\langle\alpha_i,\check\lambda\rangle}(1-q_i^r)^{-1}$. According to [9, Section 9] there exists an action of $\mathbb{C}[\mathbb{A}^{\check\lambda}]$ on the global twisted Weyl $(\mathfrak{g}'[\mathsf{t}])^\varsigma$-module $\mathcal{W}^{\mathrm{twisted}}(\check\lambda)$ such that

(1) this action commutes with $(G'[\mathsf{t}])^\varsigma\rtimes\mathbb{C}^*$;
(2) $\mathcal{W}^{\mathrm{twisted}}(\check\lambda)$ is finitely generated and free over $\mathbb{C}[\mathbb{A}^{\check\lambda}]$;
(3) the fiber of $\mathcal{W}^{\mathrm{twisted}}(\check\lambda)$ at $\check\lambda\cdot 0\in\mathbb{A}^{\check\lambda}$ is the local twisted Weyl module $D^{\mathrm{twisted}}(\check\lambda)$ of [17].

Now the local twisted Weyl modules $D^{\mathrm{twisted}}(\check\lambda)$ coincide by [17] with the level one Demazure modules over $\mathfrak{g}'[\mathsf{t}]^\varsigma\rtimes\mathbb{C}^*$. And the characters of level one Demazure modules over dual untwisted affine Lie algebras were proved in [21] to coincide with the q-Hermite polynomials $\hat\Psi'_{\check\lambda}(q,z)$ (see Section 1.3).

On the other hand, recall q-Whittaker functions $\Psi_{\check\lambda}(q,z)$ and $\hat\Psi_{\check\lambda}(q,z):=\Psi_{\check\lambda}(q,z)\cdot\prod_{i\in I}\prod_{r=1}^{\langle\alpha_i,\check\lambda\rangle}(1-q_i^r)$ of [6, Theorem 1.2]. Given the geometric preparations undertaken in Section 3, the following theorem is proved the same way as [6, Theorem 1.3]:

THEOREM 4.5. *The characters of $T\times\mathbb{C}^*$-modules $\mathcal{W}^{\mathrm{twisted}}(\check\lambda)$ and $D^{\mathrm{twisted}}(\check\lambda)$ are given by the corresponding q-Whittaker functions: $\chi(\mathcal{W}^{\mathrm{twisted}}(\check\lambda))=\Psi_{\check\lambda}(q,z)$; $\chi(D^{\mathrm{twisted}}(\check\lambda))=\hat\Psi_{\check\lambda}(q,z)$.* □

Also, the same argument as the one for [6, Theorem 1.5] establishes the following version of the Borel–Weil theorem for the dual global and local twisted Weyl modules:

THEOREM 4.6. *There is a natural isomorphism $\Gamma((G'[[\mathsf{t}]]/T'\cdot U'_-[[\mathsf{t}]])^\varsigma,\mathcal{O}(\check\lambda))\simeq\Gamma(\mathfrak{Q},\mathcal{O}(\check\lambda))\simeq\mathcal{W}^{\mathrm{twisted}}(\check\lambda)^\vee$. Similarly, $\Gamma((G'[[\mathsf{t}]]/B'_-[[\mathsf{t}]])^\varsigma,\mathcal{O}(\check\lambda))\simeq D^{\mathrm{twisted}}(\check\lambda)^\vee$.*

5. *Nontwisted non simply laced case*

5.1. *Quasimaps: rational singularities*

Recall that $\mathfrak{g}$ is a nonsimply laced simple Lie algebra, and $Z^\alpha_{\mathfrak{g}}$ is the corresponding Zastava space.

PROPOSITION 5.2. *$Z^\alpha_{\mathfrak{g}}$ has rational singularities.*

Proof. We are going to apply [14, Corollary 7.7]. Recall [14, Definition 3.7] that an effective divisor Δ is called a *boundary* on a variety X if $K_X+\Delta$ is a $\mathbb{Q}$-Cartier divisor. We will take $X=Z^\alpha_{\mathfrak{g}}$, and $\Delta=\sum_{i\in I}\partial_{\alpha_i}Z^\alpha_{\mathfrak{g}}$ (the sum of boundary divisors $\partial_{\alpha_i}Z^\alpha_{\mathfrak{g}}$ with multiplicity one). Recall the symplectic form Ω on $\mathring{Z}^\alpha_{\mathfrak{g}}$ constructed in [16], and let $\Lambda^{|\alpha|}\Omega$ be the corresponding regular nonvanishing section of $\omega_{\mathring{Z}^\alpha_{\mathfrak{g}}}$. According to [16], $\Lambda^{|\alpha|}\Omega$ has a pole of the first order at each boundary divisor component $\partial_{\alpha_i}Z^\alpha_{\mathfrak{g}}\subset\overset{\bullet}{Z}{}^\alpha_{\mathfrak{g}}$. Here $\overset{\bullet}{Z}{}^\alpha_{\mathfrak{g}}\subset Z^\alpha_{\mathfrak{g}}$ is an open smooth subvariety with codimension 2 complement formed by all the quasimaps with defect of degree at most a simple coroot. Recall a function $F_\alpha\in\mathbb{C}[Z^\alpha_{\mathfrak{g}}]$ [5, 4.1]. According to [5, Lemma 4.2], F_α has a zero of order $d_i=(\alpha_i,\alpha_i)/2$ at $\partial_{\alpha_i}Z^\alpha_{\mathfrak{g}}$. Hence $F_\alpha\Lambda^{|\alpha|}\Omega$ is a regular section of $\omega_{\overset{\bullet}{Z}{}^\alpha_{\mathfrak{g}}}$ nonvanishing at the boundary divisors $\partial_{\alpha_i}Z^\alpha_{\mathfrak{g}}$ for a short coroot α_i, and with a zero of order d_i-1 for a long coroot α_i. We conclude that $\omega_{\overset{\bullet}{Z}{}^\alpha_{\mathfrak{g}}}\simeq\mathcal{O}_{\overset{\bullet}{Z}{}^\alpha_{\mathfrak{g}}}(\sum_{i\in I}(d_i-1)\partial_{\alpha_i}Z^\alpha_{\mathfrak{g}})$, and $K_{\overset{\bullet}{Z}{}^\alpha_{\mathfrak{g}}}+\sum_{i\in I}\partial_{\alpha_i}Z^\alpha_{\mathfrak{g}}$

is the divisor of F_α. So indeed $\sum_{i\in I}\partial_{\alpha_i}Z^\alpha_{\mathfrak g}$ is a boundary on $Z^\alpha_{\mathfrak g}$ in the sense of [**14**, Definition 3.7].

Recall [**5**, Proof of Proposition 5.1] the Kontsevich resolution $\pi:\ M^\alpha\to Z^\alpha_{\mathfrak g}$. According to [**14**, Definition 3.8], the *log relative canonical divisor* $K^\Delta_{M^\alpha/Z^\alpha_{\mathfrak g}}:=K_{M^\alpha}+\Delta_M-\pi^*(K_{Z^\alpha_{\mathfrak g}}+\Delta)$ where Δ_M is the proper transform of Δ on M^α. According to [**14**, Corollary 7.7], if $K^\Delta_{M^\alpha/Z^\alpha_{\mathfrak g}}$ is a sum of exceptional divisors of M^α with positive multiplicities, then $Z^\alpha_{\mathfrak g}$ has rational singularities. So we have to compute the multiplicities in $K^\Delta_{M^\alpha/Z^\alpha_{\mathfrak g}}$. We use freely the notations of [**5**, Proof of Proposition 5.1]. As in *loc. cit.*, by factorization it suffices to compute the single multiplicity m_α of D_α. In case $\alpha=\alpha_i$ is simple, we have $m_{\alpha_i}=0$ by the definition of $K^\Delta_{M^\alpha/Z^\alpha_{\mathfrak g}}$ since D_{α_i} is not exceptional (note that this zero multiplicity is *not* given by the formula of [**5**, Lemma 5.2]). In case α is *not* simple, the divisor D_α *is* exceptional, and the argument in the proof of [**5**, Lemma 5.2] goes through word for word, giving the result $m_\alpha=|\alpha|+[(\alpha,\alpha)/2]-2>0$. This completes the proof of the proposition. □

5.3. *Quasimaps: cohomology vanishing*

In this section we follow the notations of [**6**]. In particular, we will consider the global quasimaps' spaces $\mathcal{QM}^\alpha_{\mathfrak g}$, and the corresponding ind-scheme $\mathfrak{Q}_{\mathfrak g}$. We will generalize the results of [**6**, Section 3] on cohomology of the line bundles $\mathcal O(\check\lambda)$ to the case of nonsimply laced G.

PROPOSITION 5.4. (1) *For $n>0$ and $\alpha\in\Lambda^{\check\lambda}_+$ we have $H^n(\mathcal{QM}^\alpha_{\mathfrak g},\mathcal O(\check\lambda))=0$.*

(2) *For $n>0$ and $\check\lambda\in\Lambda^\vee$ we have $\widetilde H^n(\mathfrak Q_{\mathfrak g},\mathcal O(\check\lambda))=0$.*

(3) *For $\check\lambda\notin\Lambda^\vee_+$ we have $\widetilde H^0(\mathfrak Q_{\mathfrak g},\mathcal O(\check\lambda))=0$.*

Proof. (3) is clear, and (2) follows from (1). We prove (1).

We will use the self-evident notation $\partial_{\alpha_i}\mathcal{QM}^\alpha_{\mathfrak g}$ for the boundary divisors of $\mathcal{QM}^\alpha_{\mathfrak g}$. We define the boundary $\Delta_Q:=\sum_{i\in I}\partial_{\alpha_i}\mathcal{QM}^\alpha_{\mathfrak g}$. Recall the open subvariety $\overset{\circ}{\mathcal{QM}}{}^\alpha_{\mathfrak g}\subset\mathcal{QM}^\alpha_{\mathfrak g}$ formed by all the quasimaps without defect at $\infty\in\mathbf C$, and the evaluation morphism $ev_\infty:\ \overset{\circ}{\mathcal{QM}}{}^\alpha_{\mathfrak g}\to\mathcal B_{\mathfrak g}$. It is a fibration with the fibers isomorphic to $Z^\alpha_{\mathfrak g}$. We have $ev^*_\infty\omega_{\mathcal B_{\mathfrak g}}=\mathcal O(-2\check\rho)$. The proof of Proposition 5.2 implies $K_{\overset{\circ}{\mathcal{QM}}{}^\alpha_{\mathfrak g}}+\Delta_Q-ev^*_\infty K_{\mathcal B_{\mathfrak g}}=0$.

Now we have $\mathcal O(K_{\overset{\circ}{\mathcal{QM}}{}^\alpha_{\mathfrak g}}+\Delta_Q)=\mathcal O(-\alpha^*-2\check\rho)$. In effect, the proof of [**6**, Lemma 4] goes through word for word: first it suffices to check the equality on the open subvariety $\overset{\bullet}{\mathcal{QM}}{}^\alpha_{\mathfrak g}\subset\mathcal{QM}^\alpha_{\mathfrak g}$ formed by all the quasimaps with defect at most a simple root since the complement $\mathcal{QM}^\alpha_{\mathfrak g}\setminus\overset{\bullet}{\mathcal{QM}}{}^\alpha_{\mathfrak g}$ has codimension 2. Second, it suffices to calculate the degree of the normal bundle $\mathcal N_{\partial_{\alpha_i}\mathcal{QM}^\alpha_{\mathfrak g}/\overset{\bullet}{\mathcal{QM}}{}^\alpha_{\mathfrak g}}$ restricted to the curve C^ϕ_i defined in *loc. cit.* Third, the equality $\deg\mathcal N_{\partial_{\alpha_i}\mathcal{QM}^\alpha_{\mathfrak g}/\overset{\bullet}{\mathcal{QM}}{}^\alpha_{\mathfrak g}}|_{C^\phi_i}=\langle\alpha_i,\alpha^*+2\check\rho\rangle$ is proved in [**15**, Proposition 4.4].

Finally, for $\alpha\in\Lambda^{\check\lambda}_+$ the line bundle $\mathcal L=\mathcal O(\check\lambda)\otimes\mathcal O(-K_{\mathcal{QM}^\alpha_{\mathfrak g}}-\Delta_Q)$ on $\mathcal{QM}^\alpha_{\mathfrak g}$ is very ample. The vanishing of $H^{>0}(\mathcal{QM}^\alpha_{\mathfrak g},\mathcal O(\check\lambda))=H^{>0}(\mathcal{QM}^\alpha_{\mathfrak g},\mathcal L\otimes\mathcal O(K_{\mathcal{QM}^\alpha_{\mathfrak g}}+\Delta_Q))$ follows from [**18**, Theorem 2.42] which in turn is an immediate corollary of [**22**, Corollary 1.3]. □

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